Midterm

Model theory of valued fields

February 15-19 2021

You can always use a previous question when answering a later question, even if you did not prove it.

- Let $\mathfrak{L}$ be the language with three sorts $\mathbf{K}$ (with the ring language), $\mathbf{RV}$ (with the ring language) and $\mathbf{\Gamma}$ (with the ordered group language), a function $\mathrm{rv} : \mathbf{K} \to \mathbf{RV}$ and a function $\mathrm{v} : \mathbf{RV} \to \mathbf{\Gamma}$.
- Any valued field (K, v) can be made into an $\mathfrak{L}$-structure by interpreting K as the ring K, $\mathbf{\Gamma}$ as vK as an ordered monoid — with $-$ as the inverse on $vK^\times$ and $-\infty = \infty$ — and $\mathbf{RV}$ as $K/1+\mathfrak{m}$ with its multiplicative structure, $0 = \mathrm{rv}(0)$, $+$ is interpreted as the trace of addition when it is well-defined and 0 otherwise, and $-\mathrm{rv}(x) = \mathrm{rv}(-x)$.
- Let T denote the $\mathfrak{L}$-structure of algebraically closed non trivially valued fields.

Problem 1. Let $M \vDash T$.

1. Show that for every $\xi, \upsilon, \zeta \in \mathbf{RV}(M)$, $\zeta \cdot (\xi + \upsilon) = (\zeta \cdot \xi) + (\zeta \cdot \upsilon)$.
2. Show that for every $(\xi_i)_{i<n} \in \mathbf{RV}(M)$, $\sum_i \xi_i \in \bigoplus_i \xi_i$ and $0 \in \bigoplus_i \xi_i$ if and only if, for some permutation σ of n, $\sum_i \xi_{\sigma(i)} = 0$.
3. Let $\alpha \in \mathbf{RV}(M)$ and $P \in \mathbf{K}(M)[x]$ such that $0 \in \mathrm{rv}(P)(\alpha)$. Show that there exists $a \in \mathbf{K}(M)$ such that $\mathrm{rv}(a) = \alpha$ and $P(a) = 0$.
4. Let $A \leqslant M$, $\alpha \in \mathbf{RV}(A)$, $P \in \mathbf{K}(A)[x]$ minimal such that $0 \in \mathrm{rv}(P)(\alpha)$, $a \in \mathbf{K}(M)$ such that $\mathrm{rv}(a) = \alpha$ and $P(a) = 0$ and C be the structure generated by Aa. Show that $\mathbf{RV}(C) = \mathbf{RV}(A)$.
5. Show that T eliminates quantifiers.
   [Hint : You can consider a maximal $\mathfrak{L}$-embedding $f : C \leqslant M \to N$. Start by showing that $\mathbf{RV}(C) = \mathrm{rv}(\mathbf{K}(C))$.]

Problem 2. Let $M \vDash T$ and $A \leqslant M$.

1. Let $a, b \in \mathbf{K}(M)$ with $\mathrm{v}(\mathrm{rv}(a)) = \mathrm{v}(\mathrm{rv}(b)) \notin \mathbb{Q} \cdot \mathrm{v}(\mathbf{RV}(A))$. Show that $\mathrm{tp}(a/A) = \mathrm{tp}(b/A)$.

Let $f : \mathbf{\Gamma} \to \mathbf{RV}$ be $\mathcal{L}(A)$-definable.

2. Show that $\mathrm{v}(f(\mathbf{\Gamma})) \subseteq \mathbb{Q} \cdot \mathbf{\Gamma}(A)$.
3. Show that $\mathrm{v} \circ f$ has finite image.
4. Show that f has finite image.

Problem 3. Let (K, v) be a valued field $\gamma \in vK^\times_{\geqslant 0}$ and $(\xi_i)_{i<n} \in \mathrm{RV}_\gamma = K/1 + \gamma\mathfrak{m}$. Show that $\{\sum_{i<n} x_i : \mathrm{rv}_\gamma(x_i) = \xi_i\}$ is an open ball and give its radius.