CASCADES IN THE DYNAMICS OF AFFINE INTERVAL EXCHANGE TRANSFORMATIONS.

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Abstract. We describe in this article the dynamics of a 1-parameter family of affine interval exchange transformations. It amounts to studying the directional foliations of a particular affine surface introduced in [DFG], the *Disco surface*. We show that this family displays various dynamical behaviours: it is generically *dynamically trivial*, but for a Cantor set of parameters the leaves of the foliations accumulate to a (transversely) Cantor set. This is done by analysing the dynamics of the Veech group of this surface, combined with an original use of the Rauzy induction in the context of affine interval exchange transformations. Finally, this analysis allows us to completely compute the Veech group of the Disco surface.

1. Introduction.

We consider the map $F : D \longrightarrow D$, where $D = [0, 1[$, defined the following way:

$$\begin{array}{llll} \text{if } x \in \left[0, \frac{1}{6}\right[& \text{then} & F(x) = & 2x + \frac{1}{6} \\ \text{if } x \in \left[\frac{1}{6}, \frac{1}{2}\right[& \text{then} & F(x) = & \frac{1}{2}(x - \frac{1}{6}) \\ \text{if } x \in \left[\frac{1}{2}, \frac{5}{6}\right[& \text{then} & F(x) = & \frac{1}{2}(x - \frac{1}{2}) + \frac{5}{6} \\ \text{if } x \in \left[\frac{5}{6}, 1\right[& \text{then} & F(x) = & 2(x - \frac{5}{6}) + \frac{1}{2} \end{array}$$

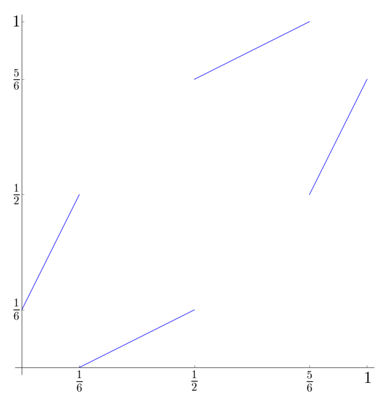

Figure 1. The graph of F.

The map F is an *affine interval exchange transformation* (AIET) and one easily verifies that for all $x \in D$, $F^2(x) = x$. Its dynamical behaviour is therefore as simple as can be. We consider a family $(F_t)_{t\in S^1}$, parametrised by $S^1 = \mathbb{R}/\mathbb{Z}$, defined by

$$F_t = F \circ r_t$$

where $r_t : [0,1[\longrightarrow [0,1[$ is the translation by t modulo 1.

This article aims at highlighting that this one-parameter family of AIETs displays rich and various dynamical behaviours. The analysis developed in it, using tools borrowed from the theory of geometric structures on surfaces, leads to the following theorems.

Theorem 1. *For Lebesgue-almost all $t \in S^1$, F_t is dynamically trivial.*

We use the terminology of Liousse (see [Lio95]) who introduced it and proved that this dynamically trivial behaviour is topologically generic for transversely affine foliations on surfaces. As such, this theorem is somewhat a strengthening of Liousse's theorem for this 1-parameter family of AIETs. Indeed, we prove that this genericity is also of measure theoretical nature. We say that F_t is dynamically trivial if there exists two periodic points $x^+, x^- \in D$ of orders $p, q \in \mathbb{N}$ such that

- $(F_t^p)'(x^+) < 1$
- $(F_t^q)'(x^-) > 1$
- for all $z \in D$ which is not in the orbit of x^-, the ω-limit of z is equal to $\mathcal{O}(x^+)$ the orbit of x^+.

Roughly speaking, it means that the map F_t has two periodic orbits, one of which is attracting all the other orbits but the other periodic orbit which is repulsive.

A remarkable feature of this family is that there also are a lot of parameters for which the dynamics has an attracting *exceptional minimal set*. The existence of such a dynamical behavior for AIETs has been known since the work of Levitt [Lev87] (see also [CG97], [BHM10] and [MMY10] for further developments on this construction).

Theorem 2.

- *For all t in a Cantor set of parameters in S^1 there exists a Cantor set $\mathcal{C}_t \subset D$ for which for all $x \in D$, the F_t ω-limit of x is equal to $\mathcal{C}_t$.*
- *There are countably infinitely many parameters $t \in S^1$ for which F_t is completely periodic, that is there exists $p(t) \in \mathbb{N}^*$ such that $F_t^{p(t)} = \mathrm{Id}$.*

The remaining parameters form a Cantor set $\Lambda_\Gamma \subset S^1$ which is the limit set of a Zariski dense subgroup $\Gamma < \mathrm{SL}(2,\mathbb{R})$ of infinite covolume. For the parameters in Λ_Γ, we only have partial results in section 5. However, numerical experiments suggest the following conjecture:

Conjecture 1. *For every parameter $t \in \Lambda_\Gamma$, F_t is minimal.*

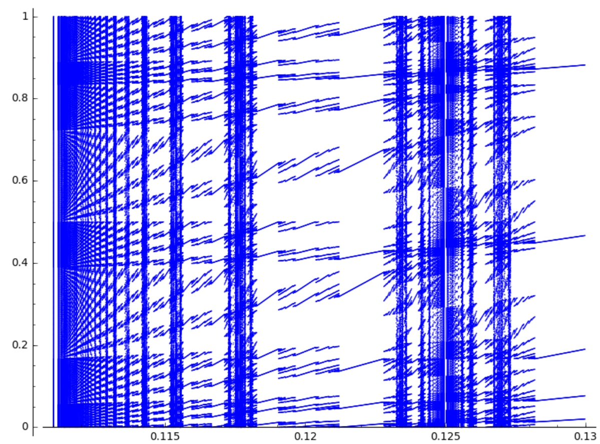

FIGURE 2. The ω-limit of a random point depending on F_t, for $0.11 \leq t \leq 0.13$.

The associated affine surface Σ. We associate to the family $(F_t)_{t \in S^1}$ the affine surface Σ obtained by the following gluing:

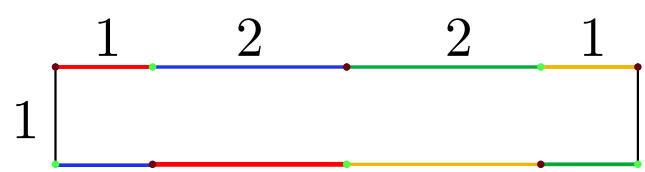

FIGURE 3. The surface Σ.

Those are linked by the fact that the directional foliation in direction θ admits F_t as their first return map on a cross-section, for $t = \frac{6}{\tan \theta}$. In particular they share the exact same dynamical properties and the study of the family F_t reduces to the study of the directional foliations of Σ.

The Veech group of Σ. The major outcome of this change of viewpoint is the appearance of hidden symmetries. Indeed, the surface Σ has a large group of affine symmetries which is called its Veech group and that we denote by $\mathbf{V}_\Sigma$. It is precisely the group of diffeomorphisms which writes in coordinates as an element of $\mathrm{SL}(2, \mathbb{R})$. As a consequence we get that directions θ which are $\mathrm{SL}(2, \mathbb{R})$-equivalent through $\mathbf{V}_\Sigma$ have same dynamical behaviour and this remark allow us to reduce considerably the number of parameters θ (equivalently t) that we need to analyse.

The group $\mathbf{V}_\Sigma$ seen as a subgroup of $\mathrm{SL}(2, \mathbb{R})$ is discrete and contains the following group, see Section 3

$$\Gamma = \left\langle \begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \frac{3}{2} & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle.$$

The projection of Γ on $\mathrm{PSL}(2,\mathbb{R})$ is a discrete group of infinite covolume. It is also a Schottky group of rank 2, and the analysis of its action on $\mathbb{RP}^1$ the set of directions of $\mathbb{R}^2$ leads to the following.

- There is a Cantor set $\Lambda_\Gamma \subset \mathbb{RP}^1$ of zero measure on which Γ acts minimally (Λ_Γ is the limit set of Γ). The directions $\theta \in \Gamma$ are *conjecturally* minimal.
- The action of Γ on $\Omega_\Gamma = \mathbb{RP}^1 \setminus \Lambda_\Gamma$ is properly discontinuous and the quotient is homeomorphic to a circle (Ω_Γ is the discontinuity set of Γ). It allows us to identify a narrow fundamental domain $I \subset \mathbb{RP}^1$ such that the understanding of the dynamics of foliations in directions $\theta \in I$ implies the understanding for the whole Ω_Γ (which is an open set of full measure).

Affine Rauzy-Veech induction. A careful look at the Figure 3 gives that the study of the dynamics of the directional foliations for $\theta \in I$ reduces to the understanding of the dynamics of contracting affine 2-intervals transformations. We use in Section 4 an adapted form of Rauzy-Veech induction to show that

- there is a Cantor set of zero measure of parameters $\theta \in \Omega_\Gamma$ for which the associated foliation accumulates on a (transversely) Cantor set;
- the rest of directions in Ω_Γ are dynamically trivial.

A remarkable corollary of the understanding of the dynamics of the directions in I is the complete description of $\mathbf{V}_\Sigma$:

Theorem 3. *The Veech group of Σ is exactly Γ.*

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2. Affine surfaces.

We introduce in this section the various objects and concepts which will play a role in this paper, and recall or prove few structural results about affine surfaces.

2.1. **Affine surfaces.**

Definition 1. An *affine surface* is a surface Σ together with a finite set $S \subset \Sigma$ and an atlas of charts $\mathcal{A} = (U_i, \varphi_i)_{i \in I}$ on $\Sigma \setminus S$ whose charts take values in $\mathbb{C}$ such that

- the transition maps are locally restriction of elements of $\mathrm{Aff}_{\mathbb{R}_+^*}(\mathbb{C}) = \{z \mapsto az + b \mid a \in \mathbb{R}_+^*, b \in \mathbb{C}\}$;
- each point of S has a punctured neighborhood which is affinely equivalent to a punctured neighborhood of the cone point of a euclidean cone of angle a multiple of 2π.

We call an element of the set S a *singularity* of the affine surface Σ.

This definition is rather formal, and the picture one has to have in mind is that an affine surface is what you get when you take a euclidean polygon, and glue pairs of parallel sides along the unique affine transformation that sends one on the other. An other interesting equivalent formulation of this definition is to say that an affine surface if the datum of

- a representation $\rho : \pi_1\Sigma \longrightarrow \mathrm{Aff}_{\mathbb{R}_+^*}(\mathbb{C}) = \{z \mapsto az + b \mid a \in \mathbb{R}_+^*, b \in \mathbb{C}\}$;
- a non constant holomorphic map $\mathrm{dev} : \widetilde{\Sigma} \longrightarrow \mathbb{C}$ which is equivariant with respect to the above representation.

The map dev is called the *developing map* of the affine surface and the representation ρ its *holonomy.* One recovers the initial definition by restricting dev to sufficiently small open sets to get the charts of the affine structure, the equivariance of dev under the action of ρ ensuring that the transition maps belong to $\mathrm{Aff}_{\mathbb{R}_+^*}(\mathbb{C})$. The singular points are the projection of the critical points of dev. This formalism borrowed from the theory of (G, X)-structures is often convenient to overcome minor technical issues.

2.2. **Foliations and saddle connections.** Together with an affine surface comes a natural family of foliations. Fix an angle $\theta \in S^1$ and consider the trivial foliation of $\mathbb{C}$ by straight lines directed by θ. This foliation being invariant by the action of $\mathrm{Aff}_{\mathbb{R}_+^*}(\mathbb{C})$, it is well defined on $\Sigma \setminus S$ and extends at points of S to a singular foliation on Σ such that its singular type at a point of S is saddle-like.

A *saddle connection* on Σ is a singular leave that goes from a singular point to another, that is which is finite in both directions. The set of saddle connections of an affine surface is countable hence so is the set of directions having saddle connections.

2.3. **Cylinders.** If $\theta \in]0, 2\pi[$, the quotient of the angular sector based at 0 in $\mathbb{C}$ by the action by multiplication of $\lambda > 1$ is called the *affine cylinder* of angle θ and of multiplier λ which we denote by $C_{\lambda,\theta}$. This definition extends to arbitrarily large values of θ by considering sufficiently large ramified cover of $\mathbb{C}$, see [DFG, Section 2.2] for a precise construction.

An affine cylinder is an affine surface whose boundary is totally geodesic, homeomorphic to a topological cylinder, with no singular points. For the sake of convenience, we will consider that euclidean cylinders, which are the affine surfaces one gets by identifying two opposite sides of a rectangle are a degenerated case of affine cylinder. In the sequel when we will refer to a *cylinder* we will mean an affine or euclidean cylinder, unless explicitly mentioned.

We say an affine surface has a *cylinder in direction* θ if there exists a regular closed leave of the foliation in direction θ. The *cylinder in direction* θ is therefore the cylinder to which this closed leave belongs.

Proposition 1. *A closed affine surface of genus g has at most g cylinders in a given direction θ.*

Proof. It suffices to notice that two homotopic simple closed geodesics α and β on a surface Σ which are disjoint must bound an affine or a flat cylinder. The fact that α and β are homotopic implies that they bound a cylinder C in Σ. This cylinder has totally geodesic boundary. A Euler characteristic argument implies that, since ∂C is totally geodesic, C has no singular point in its interior. Such a cylinder is characterised by the linear holonomy $\lambda \in \mathbb{R}_+^*$ of a generator of its fundamental group that is isomorphic to $\mathbb{Z}$: because of the local rigidity of affine structures, the local isomorphism with a cylinder $C_{\lambda,\infty}$ of holonomyλ (if $\lambda \neq 1$) extends to the whole C. This proves that C is affinely isomorphic to $C_{\lambda,\theta}$ for a certain θ or to a flat cylinder if $\lambda = 1$. Moreover, if α and β lie in the same directional foliation and if $\lambda \neq 1$, θ must be a multiple of π.

□

2.4. **Affine interval exchange transformations.** An *affine interval exchange transformation* (or AIET) is a bijective piecewise continuous map $f : [0,1] \longrightarrow [0,1]$ with a finite number of discontinuity points, which is affine on each of its continuity intervals. A convenient way to represent an AIET is to draw on a first copy of $[0,1]$ its continuity intervals and on a second copy their images.

2.5. **The Veech group of an affine surface.** We recall in this paragraph basic definitions. An appendix to this article (see Appendix A) gives further details on its construction and on the Veech group of the Disco surface (to be formally defined in the next paragraph). Let Σ be an affine surface. We say $g \in \mathrm{Diff}^+(\Sigma)$ is an *affine automorphism* of Σ if in coordinates, g writes likes the action of an element of $\mathrm{GL}^+(2,\mathbb{R}) \ltimes \mathbb{R}^2$ with the standard identification $\mathbb{C} \simeq \mathbb{R}^2$. We denote by $\mathrm{Affine}(\Sigma)$ the subgroup of $\mathrm{Diff}^+(\Sigma)$ made of affine automorphisms. The linear part in coordinates of an element of $\mathrm{Affine}(\Sigma)$ is well defined up to multiplication by a constant $\lambda \in \mathbb{R}_+^*$, we have therefore a well defined morphism:

$$\rho : \mathrm{Affine}(\Sigma) \longrightarrow \mathrm{SL}(2,\mathbb{R})$$

which associate to an affine automorphism its normalised linear part. We call ρ the *Fuchsian representation*. Its image $\rho(\mathrm{Affine}(\Sigma))$ is called the *Veech group* of Σ and is denoted by $\mathbf{V}_\Sigma$.

2.6. **The 'Disco' surface.** The surface we are about to define will be the main object of interest of this text. It is the surface obtained after proceeding to the gluing below:

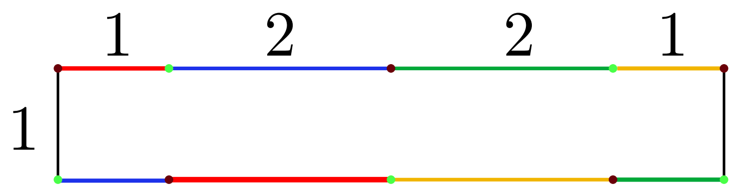


FIGURE 4. The surface Σ.

We call the resulting surface the 'Disco' surface. In the following Σ will denote this particular surface.

- This is a genus 2 affine surface, which has two singular points of angle 4π. They correspond to the vertices of the polygon drawn in Figure 4. Green ones project onto one singular point and brown ones project onto the other.
- Line segments joining the middle of the two blue (red, yellow and green respectively) sides project onto four closed simple curves that we call a_1, b_1, a_2 and b_2 respectively. If oriented 'from the bottom to the top', they form a symplectic basis of $\mathrm{H}_1(\Sigma,\mathbb{Z})$.
- With the orientation as above, we have that the linear holonomy of ρ of Σ satisfies $\rho(a_1) = \rho(b_2) = 2$ and $\rho(a_2) = \rho(b_1) = \frac{1}{2}$.

2.7. **About the Veech group of Σ.**

Flat cylinders. A *flat cylinder* is the affine surface you get when gluing two opposite sides of a rectangle. The *height* of the cylinder is the length of the sides glued and the*diameter* of the cylinder is the length of the non-glued sides, that is the boundary components of the resulting cylinder. Of course, only the ratio of these two quantities is actually a well-defined invariant of the flat cylinder, seen as an affine surface. More precisely, we define

$$m = \frac{\text{diameter}}{\text{height}}$$

and call this quantity the *modulus* of the associated flat cylinder. The set of flat cylinders (seen as affine surfaces) is parametrised by the modulus.

If C is a cylinder of modulus m, there is an element of $f \in \text{Affine}(C)$ which is has the following properties:

- f is the identity on ∂C;
- f acts has a the unique Dehn twist of C;
- the matrix associated to f is $\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$, if ∂C is assumed to be in the horizontal direction.

Decomposition in flat cylinders and parabolic elements of the Veech group. We say an affine surface Σ has a *decomposition in flat cylinders* in a given direction(say the horizontal one) if there exists a finite number of saddle connections in this direction whose complement in Σ is a union of flat cylinders. If additionally, the flat cylinders of have commensurable moduli, the Veech group of Σ contains the matrix

$$\begin{pmatrix} 1 & m' \\ 0 & 1 \end{pmatrix}$$

where m' is the smallest common multiple of all the moduli of the cylinders appearing in the cylinder decomposition. If the decomposition is in an another direction θ, the Veech group actually contains the conjugate of this matrix by a rotation of angle θ. Moreover, the affine diffeomorphism realising this matrix is a Dehn twist along the multicurve made of all the simple closed curves associated to each of the cylinders of the decomposition.

Calculation of elements of the Veech group of Σ. The above paragraph allows us to bring to light two parabolic elements in $\mathbf{V}_\Sigma$. Indeed, Σ has two cylinder decompositions in the horizontal and vertical direction.

- The decomposition in the horizontal direction has one cylinder of modulus 6, in green in the Figure 16 below.

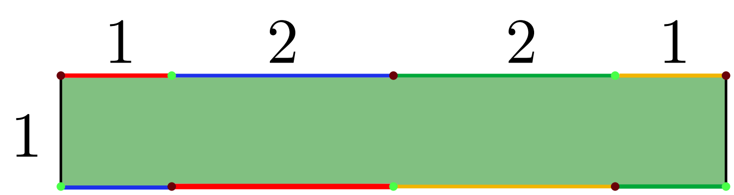


FIGURE 5. Cylinder decomposition in the horizontal direction.

  Applying the discussion of the last paragraph gives, we get that the matrix $\begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix}$ belongs to $\mathbf{V}_\Sigma$.

- The decomposition in the vertical direction has two cylinders, both of modulus $\frac{3}{2}$, in repectively in blue and pink in the Figure 16 below.

  Again, we get that the matrix $\begin{pmatrix} 1 & 0 \\ \frac{3}{2} & 1 \end{pmatrix}$ belongs to $\mathbf{V}_\Sigma$.

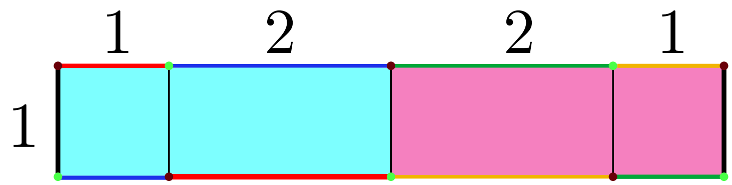


FIGURE 6. Cylinder decomposition in the vertical direction.

Finally, remark that both the polygon and the gluing pattern we used to build Σ is invariant by the rotation of angle π, which implies that the matrix $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ is realised by an involution in $\mathrm{Affine}(\Sigma)$. Putting all the pieces together we get the

Proposition 2. *The group*

$$\left\langle \begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \frac{3}{2} & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle$$

is a subgroup of $\mathbf{V}_\Sigma$.

2.8. **A word about the directional foliations of Σ.** We have seen in the introduction that we have built the surface Σ in order that its directional foliations capture the dynamical properties of the family of AIETs $\{\varphi_t \mid t \in [0,1]\}$. Precisely, we have that the foliation in direction $\theta \in]0, \frac{\pi}{2}[$ is the suspension of the AIET φ_t for

$$t = \frac{6}{\tan\theta}.$$

3. THE VEECH GROUP OF Σ.

3.1. **The Veech group acting on the set of directional foliations.** We have just seen in the previous section that an affine surface comes a family of oriented foliation, called **directional foliation** parametrised by the circle S^1 and denoted by $\mathcal{F}_\theta$. Any affine diffeomorphism of the surface acts on the set of directional foliations. It is locally described by an affine map of $\mathbb{R}^2$:

$$\begin{pmatrix} x \\ y \end{pmatrix} \to A \begin{pmatrix} x \\ y \end{pmatrix} + B$$

where $A \in \mathrm{GL}_2(\mathbb{R})$ and B is a vector of $\mathbb{R}^2$. Such a map sends the θ-directional foliation on the $[A](\theta)$-directional foliation, where $[A]$ is the projective action of the matrix A on the circle S^1. Therefore (as a global object) an element of the Veech group also acts on the set of foliations $(\mathcal{F}_\theta)_{\theta \in S^1}$. Such an affine diffeomorphism conjugates one foliation and her foliations which are image of each other one by an element of the Veech group are $\mathcal{C}^\infty$-conjugate and therefore have the same dynamical behaviour. Recall that the goal of this article is to describe the dynamics of all the foliations $\mathcal{F}_\theta$ of the affine surface Σ. The remark above allows us to reduce the study to the quotient of S^1 by the action of Veech group on the set of foliations. We will find an interval (in projective coordinates) which is a fundamental domain for the action of the Veech group on its discontinuity set. The study of the dynamic for parameters θ in this interval will be performed in Section 4.

3.2. **The subgroup Γ.** We found in Section 2 three elements of the Veech group of the surface Σ :

$$\left\langle A = \begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix} \quad , B = \begin{pmatrix} 1 & 0 \\ \frac{3}{2} & 1 \end{pmatrix}, -\mathrm{Id} \right\rangle \subset \mathbf{V}_\Sigma$$

The presence of the matrix $-\mathrm{Id}$ in $\mathbf{V}_\Sigma$ indicates that directional foliations on the surface Σ are invariant by reversing time. This motivates the study of the Veech group action on $\mathbb{RP}^1 := S^1 /_{-\mathrm{Id}}$ instead of S^1.

We will often identify $\mathbb{RP}^1$ to the the interval $[-\frac{\pi}{2}, \frac{\pi}{2})$ by using projective coordinates :

$$\begin{array}{rcl} \mathbb{RP}^1 & \to & [-\frac{\pi}{2}, \frac{\pi}{2}] \\ \left[\begin{pmatrix} x \\ y \end{pmatrix}\right] & \mapsto & \arctan\left(\frac{x}{y}\right) \end{array}$$

At the level of the Veech group it means to project it to $\mathrm{PSL}_2(\mathbb{R})$ by the canonical projection π. Let us denote by $\Gamma \subset \pi(\mathbf{V}_\Sigma) \subset \mathrm{PSL}_2(\mathbb{R})$ the group generated by the two elements :

$$\Gamma := \left\langle A = \begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix} \quad , B = \begin{pmatrix} 1 & 0 \\ \frac{3}{2} & 1 \end{pmatrix} \right\rangle$$

We will study the group Γ as a Fuchsian group, that is a **discrete** group of isometries of the real hyperbolic plane $\mathbb{H}^2$. For the action of a Fuchsian group Φ on $\mathbb{RP}^1$, there are two invariant subsets which we will distinguish:

- one called its *limit set* on which Φ acts minimally and that we will denote by $\Lambda_\Phi \subset \mathbb{RP}^1$;
- the complement of Λ_Φ which is called its *discontinuity set*, on which Φ acts properly and discontinuously and which we will denote by Ω_Φ.

We will give precise definitions in Section 3.4. In restriction to the discontinuity set, one can form the quotient by the action of the group. The topological space $\Omega_\Phi /_\Phi$ is a manifold of dimension one : a collection of real lines and circles.
We will show in Proposition 5 that for the group Γ this set is a single circle, and therefore a fundamental domain I for the action of the group Γ can be taken to be an single interval (we will make it explicit: $I = [\arctan(1), \arctan(4)] \subset [0, \pi[\simeq \mathbb{RP}^1$). The dynamic of the directional foliations in the directions θ belonging to the interior of the interval I will be studied in Section 4.

Remark 1. *We will prove in Section 5 that the group Γ is actually equal to the full Veech group of the surface Σ.*

3.3. **The action of the group Γ on $\mathbb{H}$.** Two hyperbolic isometries $A, B \in \mathrm{Isom}_+(\mathbb{H}^2)$ are said to be in **Schottky position** if the following condition holds :

$$\text{There exists four disjoints domains } D_i,\ 1 \leq i \leq 4 \text{ which satisfy}$$
$$A(D_1^c) = D_2 \qquad\qquad B(D_3^c) = D_4$$
$$\text{where } D_i^c \text{ denotes the complementary set of } D_i.$$

A group generated by two elements in Schottky position is also called a **Schottky group**.

Proposition 3. *The group Γ is Schottky. Moreover the surface M_Γ is a three punctured sphere with two cusps and one end of infinite volume.*

Proof. Viewed in the upper half plane model of $\mathbb{H}^2$ the action is easily shown to be Schottky. In fact the action of A *(resp.* B) becomes $z \mapsto z+6$ *(resp.* $z \mapsto \frac{z}{\frac{3z}{2}+1}$). The two matrices are parabolic and fix ∞ and 0 respectively. Moreover, we observe that $A(-3) = 3$ and $B(-1) = 2$. Figure 7 below shows that the two matrices A and B are in Schottky position with associated domains D_i for $1 \leq i \leq 4$.

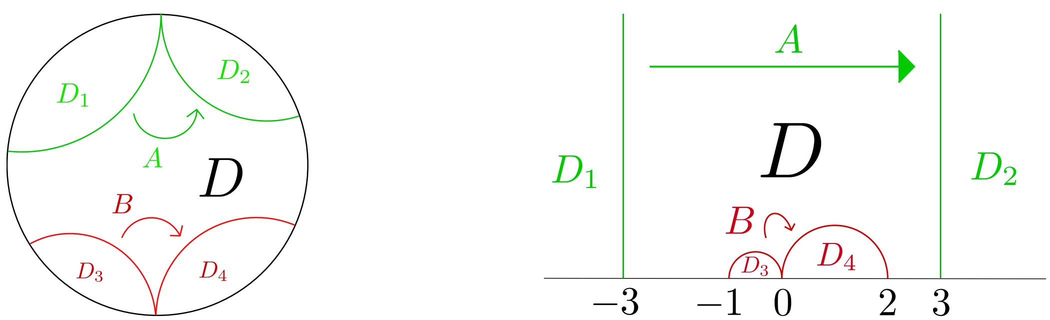


FIGURE 7. A fundamental domain for the action of the group Γ acting on the hyperbolic plane.

The domain D of Figure 7 is a fundamental domain for the action of Γ. The isometry A identifies the two green boundaries of D together and B the two red ones. The quotient surface is homeomorphic to a three punctured sphere. $\square$

3.4. **The limit set and the discontinuity set.** The following notion will play a key role in our analysis of the affine dynamics of the surface Σ :

Definition 2. The **limit set** $\Lambda_\Phi \subset S^1$ of a Fuchsian group Φ is the set of accumulation points in $\mathbb{D} \cup S^1$ of any orbit $\Phi \cdot \{z_0\}$, $z_0 \in \mathbb{D}^2$ where $\mathbb{D} \subset \mathbb{C}$ is the disk model for the hyperbolic plane $\mathbb{H}^2$.

The complementary set of the limit set is the good tool to understand the infinite volume part of such a surface.

Definition 3. The complementary set $\Omega_\Phi := S^1 \backslash \Lambda_\Phi$ is the **set of discontinuity** of the action of Γ on the circle.

The group Φ acts properly and discontinuously on the set of discontinuity one can thus form the quotient space ${}^{\Omega_\Phi}/_{\Phi}$ which is a manifold of dimension one : a collection of circles and real lines. These sets are very well understood for Schottky groups thanks to the ping-pong Lemma, for further details and developments see [Dal11] chapter 4.

Proposition 4 (ping-pong lemma)**.** *A Schottky group is freely generated by any two elements in Schottky position and its limit set is homeomorphic to a Cantor set.*

The idea of the proof is to code points in the limit set. As it is shown in the following figure a point in the limit set is uniquely determined by a sequence in the alphabet $\{A, B, A^{-1}, B^{-1}\}$ such that two consecutive elements are not inverse from the other. The first letter of the sequence will determines in which domain D_i the point is, the second one in which sub-domain and so on :

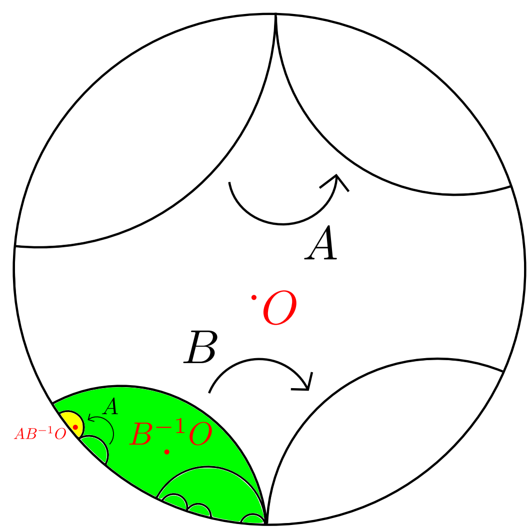


FIGURE 8. In this example the first latter is A and the point as to be on the boundary of the green part. The second one being B^{-1} it has also to lie in the yellow one too.

The intersection of all these domains will give the standard construction of a Cantor set by cutting some part of smaller intervals repetitively.
The following theorem will be used in 5 to prove our main Theorem 1

Theorem 4 (Ahlfors, [Ahl66])**.** *A finitely generated Fuchsian group satisfies the following alternative :*

(1) either its limit set is the full circle S^1 ;
(2) or its limit set is of zero Lebesgues measure

In our setting, it is clear that the limit set is not the full circle, thus the theorem implies that the limit set of Γ is of zero Lebesgue measure.

3.5. **The action on the discontinuity set and the fundamental interval.** The following proposition is the ultimate goal of this section.

Proposition 5. *The quotient space*

$$\Omega_\Gamma / \Gamma$$

is a circle. A fundamental domain for the action of Γ on Ω_Γ corresponds to the interval of slopes $I = [\arctan\left(\frac{1}{4}\right), \arctan(1)]$

Foliations defined by slopes which belong to this precise interval will be studied in section 4. To prove this proposition we will use the associated hyperbolic surface M_Γ and link its geometrical and topological properties to the action of the group Γ on the circle.

The definition of the limit set itself implies that it is invariant by the Fuchsian group. One can therefore seek a geometric interpretation of such a set on the quotient surface. We will consider the smallest convex set (for the hyperbolic metric) which contains all the geodesics which start and end in the limit set Λ_Γ. We denote it by $C(\Lambda_\Gamma)$. Because the group Γ is a group of isometries it preserves $C(\Lambda_\Gamma)$.

Definition 4. The **convex hull** of a hyperbolic surface M_Γ, denoted by $C(M_\Gamma)$ is defined as :

$$C(M_\Gamma) := C(\Lambda_\Gamma) / \Gamma$$

This is a subset of the surface M_Γ as a quotient of a Γ-invariant subset of $\mathbb{H}^2$. The convex hull of a Fuchsian group is a surface with geodesic boundary, moreover if the group is finitely generated the convex hull has to be of finite volume. As a remark, a Fuchsian group is a lattice if and only if we have the equality $C(M_\Gamma) = M_\Gamma$. In the special case of the Schottky group Γ the convex hull is a surface whose boundary is a single closed geodesic as it is shown on Figure 9. For a finitely generated group we will see that we have a one to one correspondence between connected components of the boundary of the convex hull and connected component of the quotient of the discontinuity set by the group.

The following lemma is the precise formulation of what we discussed above

Lemma 1. *Let Γ a finitely generated Fuchsian group. Any connected component I_0 of the discontinuity set Ω_Γ is stabilised by a cyclic group generated by an hyperbolic isometry γ_0. Moreover ∂I_0 is composed of the two fixed points of the isometry γ_0.*

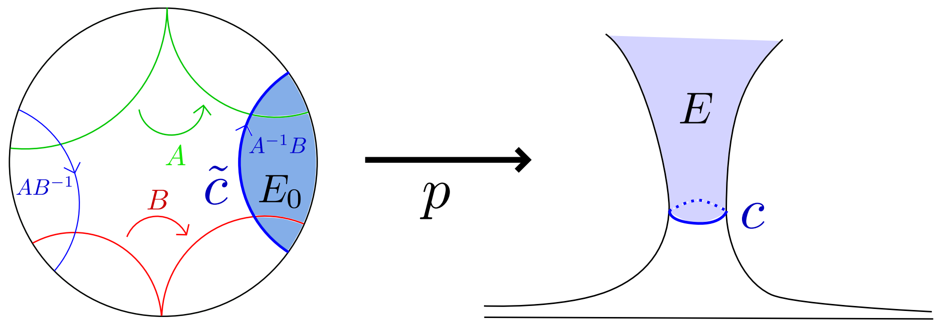


FIGURE 9. The closed geodesic c cuts the surface M_Γ into two pieces. The blue part is the only infinite volume end and its complementary is the convex hull. The choice of a lift $\tilde{c}$ of the geodesic c made on the picture allows us to describe the isometry which translate along $\tilde{c}$ in terms of the generators of the group Γ. Indeed the pairing of the edges of the fundamental domain given by the action of the group Γ shows that the geodesic $\tilde{c}$ is the translation axes of the matrix $A^{-1}B$.

We will keep the notations introduced with Figure 9. We start by showing that for any choice of a lift $\tilde{c}$ in the universal cover of a geodesic c in the boundary of the convex hull one can associate an isometry verifying the properties of Lemma 1. c a closed geodesic consisting of a connected component of the boundary of the convex hull of M_Γ. One can choose a lift $\tilde{c}$ of such a geodesic in the universal cover, the geodesic $\tilde{c}$ is the axe of some hyperbolic isometry Φ, whose fixed points are precisely the intersection of $\tilde{c}$ with the circle. As an element of the boundary of C_Γ it cuts the surface M_Γ into two pieces : C_Γ and an end E. One can check that this isometry Φ is exactly the stabiliser of the connected component E_0 of $p^{-1}(E)$ whose boundary is the geodesic $\tilde{c}$. Therefore such an isometry stabilises the connected component of the discontinuity set given by the endpoints of the geodesic $\tilde{c}$. We have shown that given a boundary component of C_Γ one can associate an element (in fact a conjugacy class) of the group Γ which stabilises a connected component of Ω_Γ. We will not show how to associate a geodesic in the boundary of the convex hull to a connected component of the discontinuity set. However we want to put the emphasis on the fact the assumption that the group is finitely generated will be used here. The key point is the geometric finiteness theorem [Kat92, Theorem 4.6.1] which asserts that any finitely generated group is also **geometrically finite**. It means that the action of such a group admits a polygonal fundamental domain with finitely many edges. It is not difficult to exhibit from such a fundamental domain the desired geodesic by looking at the pairing induced by the group, as it is done in Figure 9 for our Schottky group.

Remark 2. *The proof of the lemma 1 also shows that any point of the boundary of Ω_Γ is exactly the set of fixed points of hyperbolic isometries in Γ.*

Corollary 1. *Connected components of* ${}^{\Omega_\Gamma}/_{\Gamma}$ *are in one to one correspondence with infinite volume end of the surface* ${}^{\mathbb{H}^2}/_{\Gamma}$*.*

We now have all the materials needed to prove the Proposition 5.

Proof of Proposition 5. Because the surface ${}^{\mathbb{H}^2}/_{\Gamma}$ has only one end of infinite volume the corollary 1 gives immediately that ${}^{\Omega_\Gamma}/_{\Gamma}$is a single circle. Proof of the second part of Proposition 1 consists in a simple matrix computation. Figure 9 gives explicitly the elements of the group Γ which stabilise a connected component of the discontinuity set. We then have to prove :

$$\left[AB^{-1}\begin{pmatrix}1\\1\end{pmatrix}\right] = \left[\begin{pmatrix}4\\1\end{pmatrix}\right]$$

where $[X]$ is the projective class of the vector X. The computation is easy :

$$\begin{aligned}AB^{-1}\begin{pmatrix}1\\1\end{pmatrix} &= \begin{pmatrix}-8 & 6\\ -\frac{3}{2} & 1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}\\ &= \begin{pmatrix}-2\\ -\frac{1}{2}\end{pmatrix}\end{aligned}$$

$\square$

Remark 3.

1. *The fact that the boundary of the discontinuity set could be identified with the set of fixed points of hyperbolic isometries is not general. For example the limit set of fundamental group of any hyperbolic compact surface is the full circle, but this group being countable there is only a countable set of fixed points.*
2. *As a corollary we also have that hyperbolic fixed points are dense in the limit set.*

4. Generic directions and Rauzy induction.

This section aims at describing the dynamics of the Disco surface in an open interval of directions from which we will deduce in the next section its generic dynamics thanks to the study of the group Γ we performed in the previous section.

The boundary of $\mathbb{H}$ is canonically identified with $\mathbb{RP}^1$ through the natural embedding $\mathbb{H} \to \mathbb{CP}^1$. Recall that the action of $\mathrm{PGL}(2, \mathbb{C})$ by Möbius transformations on $\mathbb{C}$ is induced by matrix multiplication on $\mathbb{CP}^1$ after identification with $\mathbb{C}$ by the affine chart $z \to [z : 1]$. Thus the action of matrices of the Veech group on the set of directions corresponds to the action of these matrices as homographies on the boundary of $\mathbb{H}$.

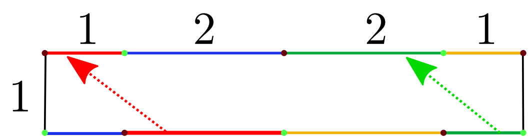


Figure 10. Attractive leaf in red, repulsive in green

We notice straightaway that for directions $[t : 1]$ with t between 1 and 2, there is an obvious attractive leaf of dilatation parameter 1/2 (see Figure 10). It corresponds to the set of directions of an affine cylinder (see Section **??**). There is also a repulsive closed leaf in this direction. This will always be the case since $-Id$ is in the Veech group, sending attractive closed leaves to repulsive closed leaves.

In the following we will describe dynamics of the directional foliation for t between 2 and 4. According to Section 2.7 the interval of direction $[1, 4]$ is a fundamental domain for the action of Γ on Ω_Γ its discontinuity set. Moreover this discontinuity set has full Lebesgue measure in the set of directions, thus understanding the dynamical behaviour of a typical direction therefore amounts to understanding it for $t \in [1, 4]$. Further discussion on what happens in other directions will be done in the next section.

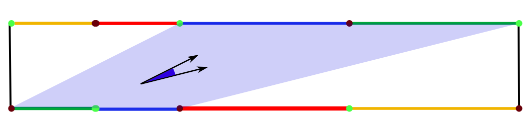


Figure 11. The stable sub-surface and its directions

4.1. **Reduction to an AIET.** The directions for $t \in [2, 4]$ have an appreciable property; they correspond to the directions of a *stable sub-surface* in the Disco surface represented in Figure 11. Every leaf in the the given angular set of directions that enters the sub-surface will stay trapped in it thereafter. Moreover the foliation in it will have a hyperbolic dynamics. We therefore seek attractive closed leaves in this subset. To do so, take a horizontal interval overlapping exactly this *stable sub-surface* and consider the first return map on it. It has a specific form, which is close to an affine interval exchange, and which we will study in this section.

In the following, an affine interval exchange transformation (AIET) [1] is a piecewise affine function on an interval. For any $m, n \in \mathbb{N}$, let $\mathcal{I}(m,n)$ be the set of AIETs defined on $[0,1]$ with two intervals on which it is affine and such that the image of the left interval is an interval of its length divided by 2^n which rightmost point is 1, and that the image of its right interval is an interval of its length divided by 2^m which leftmost point is 0 (see Figure 12 for such an IET defined on $[0,1]$). When representing an AIET, we will color the intervals on which it is affine in different colors, and represent a second interval on which we color the image of each interval with the corresponding color; this will be sufficient to characterise the map. The geometric representation motivates the fact that we call the former and latter set of intervals the *top* and *bottom intervals*.

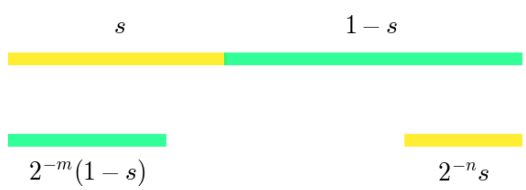


FIGURE 12. Geometric representation of an element of $\mathcal{I}(m,n)$

Remark that the above cross-sections are in $\mathcal{I}(1,1)$. We will study the dynamical behaviour of this family of AIET.

4.2. **Rauzy-Veech induction.** Let T be an AIET and D be its interval of definition. The first return map on a subinterval $D' \subset D$, $T' : D' \to D'$ is defined for every $x \in D'$, as

$$T'(x) = T^{n_0}(x) \text{ where } n_0 = \inf\{n \geq 1 | T^n(x) \in D'\}$$

Since we have no information on the recursivity of an AIET this first return map is a priori not defined on an arbitrary subinterval. Nonetheless generalizing a wonderful idea of Rauzy [Rau79] on IETs, we get a family of subinterval on which this first return map is well defined. Associating an AIET its first return map on this well-chosen smaller interval will be called the Rauzy-Veech induction.

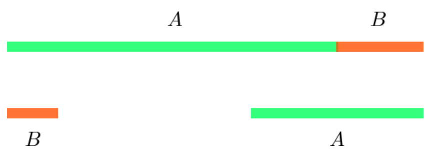
[1] In classical definitions of (affine) interval exchanges the maps are also assumed to be bijective, here they will be injective but not surjective.

The general idea in the choice of this interval is to consider the smallest of the two top and bottom intervals at one end of D (left or right) the interval of definition. We then consider the first return map on D minus this interval.

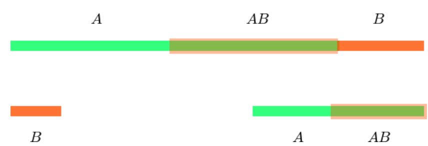
In the following we describe explicitly the induction for a simple family $\mathcal{I}(m,n)$. A general and rigorous definition of Rauzy-Veech induction here is certainly possible with a lot of interesting questions emerging but is beyond the scope of this article.

Assume now that T is an element of $\mathcal{I}(m,n)$, let $A, B \subset D$ be the left and right top intervals of T, and λ_A, λ_B their length. Several distinct cases can happen,

(1) (a) $B \subset T(A)$ *i.e.* $\lambda_B \leq 2^{-n}\lambda_A$.



(A) Example of such AIET

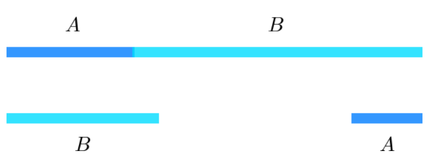


(B) Right Rauzy-Veech induction

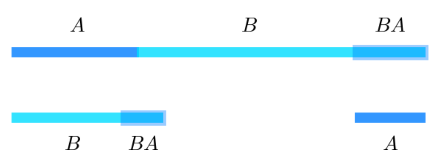
We consider the first return map on $D' = D - B$. $T^{-1}(B)$ of length $2^n\lambda_B$ has no direct image by T in D' but $T(B) \subset D'$. Thus for the first return map, this interval will be sent directly to $T(B)$ dividing its length by 2^{n+m}. We call this a right Rauzy-Veech induction of our AIET. The new AIET is in $\mathcal{I}(m+n,n)$, and its length vector (λ'_A, λ'_B) satisfies

$$\begin{pmatrix} \lambda'_A \\ \lambda'_B \end{pmatrix} = \overbrace{\begin{pmatrix} 1 & -2^n \\ 0 & 2^n \end{pmatrix}}^{R_{m,n}} \begin{pmatrix} \lambda_A \\ \lambda_B \end{pmatrix}$$

(b) If $A \subset T(B)$ *i.e.* $2^{-m}\lambda_B \geq \lambda_A$.



(A) AIET satisfying $A \subset T(B)$



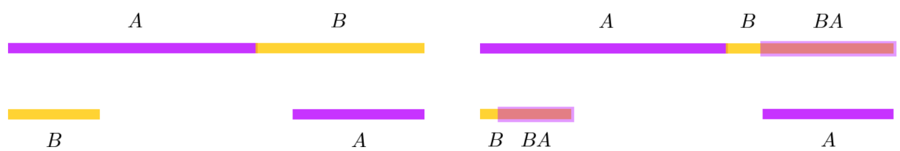
(B) Right Rauzy-Veech induction

In this case, the right Rauzy-Veech induction will not yield an AIET of the form of the $\mathcal{I}(m,n)$ family therefore we consider the first return map on $D' = D - A$ which we call the left Rauzy-Veech induction of our AIET. We obtain a new AIET in $\mathcal{I}(m, n+m)$ and its length vector (λ'_A, λ'_B) satisfies,

$$\begin{pmatrix} \lambda'_A \\ \lambda'_B \end{pmatrix} = \overbrace{\begin{pmatrix} 2^m & 0 \\ -2^m & 1 \end{pmatrix}}^{L_{m,n}} \begin{pmatrix} \lambda_A \\ \lambda_B \end{pmatrix}$$

(2) $T(A) \subset B$ *i.e.* $\lambda_B \geq 2^{-n}\lambda_A$ and $T(B) \subset A$ *i.e.* $2^{-m}\lambda_B \leq \lambda_A$.

We consider the first return map on the subinterval $D' = D - T(A)$. Then A has no direct image by T in D' but $T^2(A) \subset T(B) \subset D'$. Thus in the first return map, this interval will be sent directly to $T^2(A)$ dividing its length by 2^{n+m}.



(A) AIET satisfying $T(B) \subset A$ (B) Right Rauzy-Veech induction

Then $T^2(A) \subset A$ thus it has an attractive fixed point of derivative 2^{-n-m}.

Remark 4. *The set of length for which we apply left or right Rauzy-Veech induction in the above trichotomy is exactly the set on which lengths λ'_A and λ'_B implied by the above formulas are both positive.*

More precisely, $0 \leq \lambda_B \leq 2^{-n}\lambda_A \iff R_{m,n} \cdot \begin{pmatrix} \lambda_A \\ \lambda_B \end{pmatrix} \geq 0,$

and $0 \leq 2^{-m}\lambda_B \leq \lambda_A \iff L_{m,n} \cdot \begin{pmatrix} \lambda_A \\ \lambda_B \end{pmatrix} \geq 0.$

This will be useful latter on, to describe the set of parameters which corresponds to the sequence of induction we apply.

The algorithm. We define in what follows an algorithm using the Rauzy induction that will allows us to determine if an element of $\mathcal{I}(1,1)$ has an attractive periodic orbit; and if so the length of its periodic orbit (or equivalently the dilatation coefficient of the associated leaf in Σ).

The algorithm goes the following way:

The entry is an element of $\mathcal{I}(m, n)$,

(1) If the entry is in case (1), perform in case (a) the right Rauzy induction R or in case (b) the left Rauzy induction L to obtain an element of $\mathcal{I}(m+n,n)$ or $\mathcal{I}(m,n+m)$ respectively. Repeat the loop with this new element.

(2) If it is in case (2), it means that the first return map on a well chosen interval has a periodic attractive point of derivative $2^{-m_l-n_l}$. The algorithm stops.

Alongside the procedure comes a sequence of symbols R and L keeping track of whether we have performed the Rauzy induction on the left or on the right at the n^{th} stage. This sequence is finite if and only if the algorithm describe above finishes. An interesting phenomenon will happen for AIET for which the induction never stops, and will be described latter.

4.3. **Directions with attractive closed leaf.** In the directions of Figure 11, we consider the first return map of the directional foliation on the interval given by the two length 1 horizontal interval at the bottom of the rectangle. We have chosen directions such that the first return map is well defined although it is not bijective, and it belongs to $\mathcal{I}(1,1)$. The ratio of the two top intervals' length will vary smoothly between 0 and ∞ depending on the direction we choose. We parametrise this family of AIET by $s \in I := [0,1]$, where $(s, 1-s)$ is the length vector of the element of $\mathcal{I}(1,1)$ we get. Our purpose here is to characterise the subspace $H \subset I$ for which the above algorithm stops, in particular they correspond to AIET with a periodic orbit. The case of $I - H$ will be settled in the next subsection.

We describe for any finite word in the alphabet $\{L, R\}$, $w = w_1 \dots w_{l-1}$, the subset of parameters $H(w) \subset H \subset I$ for which the algorithm stops after the sequence w of Rauzy-Veech inductions.

We associate to w the sequences $n_1 = 1, \dots, n_l$, $m_1 = 1, \dots, m_l$ and $M_1 = Id, \dots, M_l$ defined by the recursive properties,

$$m_{i+1} = \begin{cases} m_i & \text{if } w_i = L \\ n_i + m_i & \text{if } w_i = R \end{cases}, \qquad n_{i+1} = \begin{cases} n_i + m_i & \text{if } w_i = L \\ n_i & \text{if } w_i = R \end{cases},$$

$$M_{i+1} = \begin{cases} L_{m_i,n_i} \cdot M_i & \text{if } w_i = L \\ R_{m_i,n_i} \cdot M_i & \text{if } w_i = R \end{cases}$$

Let $s \in I$ such that we can apply Rauzy-Veech inductions corresponding to w to the element of $\mathcal{I}(1,1)$ of lengths $(s, 1-s)$. The induced AIET after all the steps of the induction is in $\mathcal{I}(m_l, n_l)$ and its length vector is

$$M_l \cdot \begin{pmatrix} s \\ 1-s \end{pmatrix} := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} s \\ 1-s \end{pmatrix} = \begin{pmatrix} (a-b)s+b \\ (c-d)s+d \end{pmatrix}$$

Following Remark 4, the property of s that we can apply all the Rauzy-Veech inductions corresponding to w to the initial AIET in $\mathcal{I}(1,1)$ is equivalent to $(a-b)s+b \geq 0$ and $(c-d)s+d \geq 0$. A simple recurrence on M_i shows that it is an integer matrix with $a,d \geq 0$, $b,c \leq 0$, hence $s \in \left[\frac{-b}{a-b}, \frac{d}{d-c}\right] =: I(w)$. $H(w)$ will be the central subinterval of $I(w)$ for which the induced AIET in $\mathcal{I}(m_l, n_l)$ is in case (2).

If we consider the sets

$$H_k := \bigcup_{|w| \leq k} H(w) \quad \text{and} \quad H = \bigcup_k H_k$$

we remark that H has the same construction as the complement of the Cantor triadic set; each H_k is constructed from H_{k-1} by adding a central part of each interval which is a connected component of $I - \bigcup_{j<k} H_j$.

The rest of the subsection aims now at proving the following lemma,

Lemma 2. *$H \subset I$ has full Lebesgue measure.*

Proof. We will prove in the following that for any non-empty word w,

$$\frac{|H(w)|}{|I(w)|} \geq \delta \tag{1}$$

for some $\delta > 0$. Thus at each step k, H_k is at least a δ-proportion larger in Lebesgue measure than H_{k-1}. This implies that the Lebesgue measure

$$\lambda(H) \geq \lambda(H_k) \geq 1 - \delta^k \text{ for any } k$$

□

We now show Inequality 1. Let w be any finite word in the alphabet $\{L, R\}$. For convenience we normalise the interval $I(w)$ for such that it is $[0,1]$. We denote by $(\lambda_A(s), \lambda_B(s))$ the length vector of the AIET induced by the sequence w of Rauzy-Veech inductions. These two lengths are linear functions of s, λ_A is zero at the left end of the interval and λ_B is zero at the right end. As a consequence, these two functions have the form $\lambda_A(s) = \alpha s$ and $\lambda_B(s) = \beta(1-s)$ for $s \in [0,1]$, where α and β are the maximal values of λ_A and λ_B respectively equal to according to the previous computations

$$\alpha = (a-b)\frac{d}{d-c} + b = \frac{ad - bd + bd - bc}{d-c} = \frac{det(M_l)}{d-c}$$

and

$$\beta = (c-d)\frac{-b}{a-b} + d = \frac{-bc + bd + da - db}{a-b} = \frac{det(M_l)}{a-b}$$

We see that $\lambda_A(s) = 2^{-m}\lambda_B(s) \iff 2^m\alpha s = \beta(1-s) \iff s = \frac{\beta}{2^m\alpha+\beta}$ and similarly $\lambda_B(s) = 2^{-n}\lambda_A(s) \iff 2^n\beta(1-s) = \alpha s \iff s = \frac{2^n\beta}{\alpha+2^n\beta}$. Hence

$$\lambda_A(s) \leq 2^{-m}\lambda_B(s) \iff s \in \left[0, \frac{\beta}{2^m\alpha+\beta}\right]$$

and

$$\lambda_B(s) \leq 2^{-n}\lambda_A(s) \iff s \in \left[\frac{\beta}{2^{-n}\alpha+\beta}, 1\right]$$

thus

$$H(w) = \left[\frac{\beta}{2^m\alpha+\beta}, \frac{\beta}{2^{-n}\alpha+\beta}\right].$$

If we denote by $x = \frac{\alpha}{\beta} = \frac{a-b}{d-c}$,

$$\frac{|H(w)|}{|I(w)|} = \frac{1}{1+2^{-n}x} - \frac{1}{1+2^m x}$$

We prove the following technical lemma

Lemma 3. *For any word w in $\{R, L\}$, if $M(w) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have,*

$$2^{-1} \leq \frac{a-b}{d-c} \leq 2$$

This implies directly that

$$\frac{|H(w)|}{|I(w)|} \geq \frac{1}{1+2^{-n+1}} - \frac{1}{1+2^{m-1}}$$

Hence for w not empty, either $n \geq 2$ or $m \geq 2$ thus either

$$\frac{|H(w)|}{|I(w)|} \geq \frac{2}{3} - \frac{1}{2} = \frac{1}{6} \text{ or } \frac{|H(w)|}{|I(w)|} \geq \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

Proof. The proof goes by induction on the length of w. Let us assume that $2^{-1} = \frac{a-b}{d-c} = 2$ for some w. We denote by

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = R_{m,n} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a-2^n c & b-2^n d \\ 2^n c & 2^n d \end{pmatrix}$$

Thus $\frac{a'-b'}{d'-c'} = 2^{-n}\frac{a-b}{d-c} + 1$ from which the inequality follows.

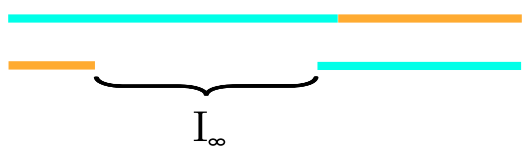
$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = L_{m,n} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 2^m a & 2^m b \\ c-2^m a & d-2^m b \end{pmatrix}$$

The inequality is similar to the previous one. $\square$

4.4. **AIET with infinite Rauzy-Veech induction.** We focus in this subsection on what happens for AIETs on which we apply Rauzy-Veech induction infinitely many times. First, remark that if we apply the induction on the same side infinitely many times, the length of the top interval of the corresponding side on the induced AIET is multiplied each time by a positive power of 2, therefore it goes to infinity. Yet the total length of the subinterval is bounded by 1 the length of the definition interval from which we started the induction. Thus the length of the interval has to be zero; this corresponds to the case where there is a saddle connection and it is included in the closed orbit case, since we chose to take $H(w)$ closed.

In consequence, for an IET T with parameter in $I - H$, we apply Rauzy-Veech induction infinitely many times, and the sequence of inductions we apply is not constant after a finite number of steps. Now let as above D be the interval of definition of the given AIET, and $A, B \subset D$ be its two consecutive domain of continuity. Remark that the induction keeps the right end of $T(B)$ and the left end of $T(A)$ unchanged. Moreover the induction divides the length of one of the bottom interval (depending on which Rauzy-Veech induction we apply) by at least two. As a consequence, if we consider I_n to be the open subinterval of D on which we consider the first return map after the n-th induction, the limit of these nested intervals is

$$I_\infty := \bigcap_{n=1}^{\infty} I_n = D - \{T(A) \cup T(B)\}$$



By definition, this interval is disjoint from $T(D)$, and therefore

$$\forall x \in D \text{ and } n \in \mathbb{N},\ T^n(x) \notin I_\infty$$

Moreover, our definition of Rauzy-Veech induction implies that any point outside of the subinterval on which we consider the first return will end up in this subinterval in finite time. Thus

$$\forall x \in D \text{ and } n \in \mathbb{N}, \exists k \in \mathbb{N} \text{ such that } T^k(x) \in I_n$$

This implies that the orbit of any point of D accumulates on ∂I_∞.

Let us introduce the complementary of all the images of I_∞,

$$\Omega := D - \bigcup_{n=0}^{\infty} T^n I_\infty$$

The measure of I_∞ is $1/2$, taking the image by T divides the measure of any interval by two and any iterated image of this set is disjoint, since its image is disjoint from itself and T is injective. Hence the measure of Ω is $1/2\cdot(1+1/2+1/2^2+\dots) = 1$. As we remarked, the orbit of any point of D accumulates to ∂I_∞ and thus to any image of it, hence to any point of $\partial\Omega$. As $\bigcup_{n=0}^{\infty} T^n I_\infty$ has full measure, Ω has zero Lebesgue measure and thus has empty interior. To conclude, Ω is the limit set of any orbit of T.

Now Ω is closed set with empty interior. Moreover if we take a point in Ω, any neighborhood contains an interval and thus its boundary. Hence no point is isolated, and Ω is a Cantor set. Which leads to the following proposition,

Proposition 6. *In the space of directions $[1,4]$, there is a set H, which complementary set is a Cantor set of zero measure which satisfies,*

- *$\forall\theta \in \overline{H}$ the foliation $\mathcal{F}_\theta$ concentrates to an attractive leaf.*
- *$\forall\theta \in \partial H$ the foliation $\mathcal{F}_\theta$ concentrates to a closed saddle connection.*
- *$\forall\theta \in [1,4]\setminus\overline{H}$ the foliation $\mathcal{F}_\theta$ concentrates on a stable Cantor set of zero measure in the foliation.*

5. The global picture.

Gathering all materials developed in the previous sections, we prove here the main theorems announced in the introduction.

Proposition 7. *Assume that the foliation $\mathcal{F}_\theta$ of Σ has a closed attracting leaf F^+. Then it has a unique repulsing leaf F^- and any leaf which is different from F^- and regular accumulates on F^+.*

This proposition ensures that in all the cases where we have already found an attracting leaf, the dynamics of the foliation is as simple as can be.

Proof. The image of F^+ by the action of the diffeomorphism of Affine(Σ) whose image by the Fuchsian representation is $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is the repulsive leaf announced. Both are *quasiminimal sets* of $\mathcal{F}_\theta$. According to [Lev87, p.93], an orientable foliation on a genus g surface has at most g quasiminimal sets, so F^+ and F^- are the only quasiminimal sets of $\mathcal{F}_\theta$. The ω-limit of any regular leaf is a quasiminimal set. F^- being repulsive, any regular leaf accumulates on F^+.

□

We say that a direction having such a dynamical behavior is *dynamically trivial.*

Theorem 5. *The set of dynamically trivial directions in S^1 is open and has full measure.*

Proof. Because $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ belongs to the Veech group of Σ, the foliations $\mathcal{F}_\theta$ and $\mathcal{F}_{-\theta}$ have the same dynamical behavior. We will therefore consider parameters θ in $\mathbb{RP}^1$ instead of in S^1. We denote then by $\mathcal{T} \subset \mathbb{RP}^1$ the set of dynamically trivial directions in $\mathcal{T}$. We have proved in Section 4 that the trace of $\mathcal{T}$ on

$J = \{[1:t] \mid t \in [1,4]\} \subset \mathbb{RP}^1$ is the complement of a Cantor set and that $\mathcal{T} \cap J$ has full measure.

Also J is a fundamental domain for the action of Γ on Ω_Γ the discontinuity set of Γ. Since $\Gamma < \mathbf{V}_\Sigma$, two directions in $\mathbb{RP}^1$ in the same orbit for the action of θ induce conjugated foliations on Σ and therefore have same dynamical behavior. This implies that $\mathcal{T} \cap \Omega_\Gamma$ is open and has full measure in Ω_Γ. Since Ω_Γ has itself full measure in $\mathbb{RP}^1$, $\mathcal{T}$ has full measure in $\mathbb{RP}^1$. Its openness is a consequence of the stability of dynamically trivial foliations for the $\mathcal{C}^\infty$ topology, see [Lio95] for instance.

□

Relying on a similar argument exploiting in a straightforward manner the action of the Veech group and the depiction of the dynamics made in Section 4, we get the

Theorem 6. *There exists a Cantor set $\mathcal{K} \subset S^1$ such that for all $\theta \in \mathcal{K}$, the foliation $\mathcal{F}_\theta$ accumulates to a set which is transversely a Cantor set of zero Haussdorf measure.*

We believe it is worth pointing out that the method we used to find these *'Cantor like'* directions is new compared to the one used in [CG97], [BHM10] and [MMY10]. Also, our Cantor sets have zero Haussdorf dimension. It would be interesting to compare with the aforementioned examples.

Theorem 7. *The Veech group of Σ is exactly Γ.*

Proof. We cut the proof into four steps:

1. proving that any element in $\mathbf{V}_\Sigma$ preserves Λ_Γ;
2. proving that Γ has finite index in $\mathbf{V}_\Sigma$;
3. proving that the group Γ is normal in $\mathbf{V}_\Sigma$;
4. concluding.

(1) Let us prove the first point. Because of the description of the dynamics of the directional foliations we have achieved, one can show the limit set of the Veech group is the same than the limit set of Γ. If not, there must be a point of $\Lambda_{\mathbf{V}_\Sigma}$ in the fundamental interval I. But since the group $\mathbf{V}_\Sigma$ is non elementary it implies that we have to find in I infinitely many copies of a fundamental domain for the action of Γ on the discontinuity set. In particular infinitely many disjoint intervals corresponding to directions where the foliation has an attracting leave of dilatation parameter 2. But by the study performed in the above section the only sub-interval of I having this property is $]1,2[$.

(2) The second point follows from that the projection

$$\Gamma\backslash\mathbb{H} \longrightarrow \mathbf{V}_\Sigma\backslash\mathbb{H}$$

induces an isometric orbifold covering

$$C(\Gamma\backslash\mathbb{H}) \longrightarrow C(\mathbf{V}_\Sigma\backslash\mathbb{H}).$$

Since $C(\Gamma\backslash\mathbb{H})$ has finite volume and because $[\Gamma : \mathbf{V}_\Sigma] = \frac{\mathrm{vol}(C(\Gamma\backslash\mathbb{H}))}{\mathrm{vol}(C(\mathbf{V}_\Sigma\backslash\mathbb{H}))}$, this ratio must be finite and hence Γ has finite index in $\mathbf{V}_\Sigma$.

(3) Remark that Γ is generated by two parabolic elements A and B and that these define the only two conjugacy class in Γ of parabolic elements. We are going to prove that any element $g \in \mathbf{V}_\Sigma$ normalise both A and B. Since Γ has finite index in $\mathbf{V}_\Sigma$, there exists $n \geq 1$ such that $(gAg^{-1})^n \in \Gamma$. There are but two classes of conjugacy of parabolic elements in Γ which are the ones of A and B. If $n \geq 2$, this implies that $\mathbf{V}_\Sigma$ contains a strict divisor of A, which would make the limit set of $\mathbf{V}_\Sigma$ larger that Λ_Γ (consider the eigenvalues of the matrix AB^{-1} which determine points in the boundary on the limit set, see Lemma 1). Therefore gAg^{-1} belongs to Γ. A similar argument shows that $gBg^{-1} \in \Gamma$ and since A and B generate Γ, g normalises Γ. Hence Γ is normal in $\mathbf{V}_\Sigma$.

(4) Any $g \in \mathbf{V}_\Sigma$ thus acts on the convex hull $C(\Gamma\backslash\mathbb{H})$ of the surface $C(\Gamma\backslash\mathbb{H})$. In particular it has to preserve the boundary of $C(\Gamma\backslash\mathbb{H})$, which is a single geodesic by Proposition 5. At the universal cover it means that g has to fix a lift of the geodesic c, thus the isometry g permutes two fixed points of a hyperbolic element h of Γ. Two situations can occur:

- g is an elliptic element. His action on $\Gamma\backslash\mathbb{H}$ cannot permute the two cusps because they correspond to two essentially different cylinder decompositions on Σ. It therefore fixes the two cusps and hence must be trivial.
- g is hyperbolic and fixes the two fixed points of h. Moreover, by Proposition 1, it acts on the fundamental interval I and as we discussed above such an action has to be trivial because of our study of the associated directional foliations, the translation length of g is then the same than h. But g is fully determined by its fixed points and its translation length, which shows that $g = h$ and thus $g \in \Gamma$.

Any element of $\mathbf{V}_\Sigma$ therefore belongs to Γ and the theorem is proved.

□

Appendix A. On the Veech group of Σ.

We aim at discussing the different definitions of the Veech group, and how they relate to each other. Classically there are two ways two define the Veech group of a translation surface: either by saying that it is the group the derivatives of the (real) affine diffeomorphisms of the surface Σ or that it is the stabiliser of Σ for the $\mathrm{SL}(2,\mathbb{R})$-action on a certain moduli space.

It is often convenient to consider the affine diffeomorphism realising a certain element of $\mathrm{SL}(2,\mathbb{R})$ and to know about its topological properties, namely which type of class in the mapping class group of Σ it represents.

In order to extricate the subtleties between these different points of view, we make the following definitions:

(1) $\mathrm{Affine}(\Sigma)$ is the subgroup of $\mathrm{Diff}^+(\Sigma)$ made of affine automorphisms;

(2) $\rho : \mathrm{Affine}(\Sigma) \longrightarrow \mathrm{SL}(2,\mathbb{R})$ is the natural projection obtained the following way: any element $f \in \mathrm{Affine}(\Sigma)$ writes down in coordinate charts as an element of $\mathrm{GL}^+(2,\mathbb{R}) \ltimes \mathbb{R}^2$. Its linear part in $\mathrm{GL}^+(2,\mathbb{R})$ only changes under a different choice of charts by a multiplication by a certain $\lambda \in \mathbb{R}_+^*$. Its class in $\mathrm{GL}^+(2,\mathbb{R})/\mathbb{R}_+^* = \mathrm{SL}(2,\mathbb{R})$ is therefore well defined and is by definition $\rho(f)$. We call ρ the *Fuchsian representation*.

(3) $\tau : \mathrm{Affine}(\Sigma) \longrightarrow \mathrm{MCG}(\Sigma)$ is the natural projection to the mapping class group.

Remark 5. *It is important to understand that the fact that the image of* ρ *lies in* $\mathrm{SL}(2,\mathbb{R})$ *is somewhat artificial and that the space it naturally lies in is* $\mathrm{GL}^+(2,\mathbb{R})/\mathbb{R}_+^*$*. In particular, when an element of the Veech group is looked at in coordinate charts, there is no reason the determinant of its derivative should be equal to* 1*, however much natural the charts are.*

We first make the following elementary but important remark: the kernel of ρ need not to be trivial. Consider $\widehat{\Sigma}$ any ramified cover of Σ together with the pulled-back affine structure. The Galois group of the cover acts non-trivially as elements of $\mathrm{Affine}(\widehat{\Sigma})$ whose image under ρ are trivial, by definition. Conversely, the quotient of an affine surface Σ by $\mathrm{Ker}(\rho)$ is still an affine surface over which Σ is a ramified cover, and whose Fuchsian representation has trivial kernel. The *Veech group* $\mathbf{V}_\Sigma$ of Σ is by definition $\rho(\Sigma) \subset \mathrm{SL}(2,\mathbb{R})$. We also say an affine surface is *irreducible* if its Fuchsian representation has trivial kernel or equivalently if it is not a non-trivial ramified cover over another affine surface.

A.1. **Thurston's theorem on multi-twists.** We recall in this subsection a theorem of Thurston allowing the understanding of the topological type of the elements of a subgroup of $\mathrm{MCG}(\Sigma)$ generated by a couple of multi-twists. Let α and β be two multicurve on Σ. We say that

- α and β are *tight* if they intersect transversely and that their number of intersection is minimal in their isotopy class;
- α and β fill up Σ if $\Sigma \setminus (\alpha \cup \beta)$ is a union of cells.

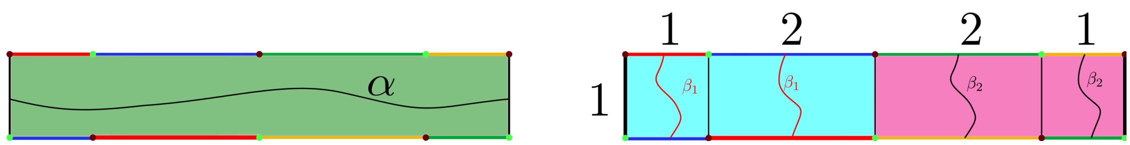
Denote by $\alpha_1, \cdots, \alpha_k$ and $\beta_1, \cdots, \beta_l$ the components of α and β respectively. We form the $k \times l$ matrix $N = (i(\alpha_i, \beta_j))_{1 \leq i \leq k,\ 1 \leq j \leq l}$. One easily checks that $\alpha \cup \beta$ is connected if and only if a power of $N^t N$ is positive. Under this assumption, $N^t N$ has a unique positive eigenvector V of eigenvalue $\mu > 0$. We also denote by T_α (resp.T_β) the Dehn twist along α (resp. along β).

Theorem 8 (Theorem 7 of [Thu88])**.** *Let* α *and* β *two multicurves which are tight and which fill up* Σ*, and assume that* $\alpha \cup \beta$ *is connected. Denote by* $G(\alpha,\beta)$ *the subgroup of* $\mathrm{MCG}(\Sigma)$ *generated by* T_α *and* T_β*. There is a representation* $\rho : G(\alpha,\beta) \longrightarrow \mathrm{PSL}(2,\mathbb{R})$ *defined by*

$$\rho(T_\alpha) = \begin{pmatrix} 1 & \mu^{\frac{1}{2}} \\ 0 & 1 \end{pmatrix} \ \textit{and}\ \rho(T_\beta) = \begin{pmatrix} 1 & 0 \\ -\mu^{\frac{1}{2}} & 1 \end{pmatrix}$$

such that g is of finite order, reductive or pseudo-Anosov according to whether $\rho(g)$ is elliptic, parabolic or pseudo-Anosov.

A.2. **Topological type of elements of the Veech group of Σ.** We apply Thurston's theorem to the Veech group of Σ. Let us narrow our focus on the elements of Affine(Σ) acting as $\begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ \frac{3}{2} & 1 \end{pmatrix}$ that we identified in Section 2.7. They are the Dehn twists along the curves α and β drawn in Figure 16.



(A) The curve α.

(B) The multi-curve $\beta = \beta_1 \cup \beta_2$.

FIGURE 16. Definition of α and β

One checks that:
- $\alpha \cup \beta$ is connected;
- α and β are tight since they can both be realised as geodesics of Σ;
- α and β are filling up Σ.

With an appropriate choice of orientation for α and β, we have that $i(\alpha, \beta_1) = i(\alpha, \beta_2) = 2$. The intersection matrix associated is therefore $N = \begin{pmatrix} 2 & 2 \end{pmatrix}$ and $N^t N = \begin{pmatrix} 8 \end{pmatrix}$. The parameter μ is then equal to 8 and $\sqrt{\mu} = 2\sqrt{2}$. We are left with two representations

$$\rho_1, \rho_2 : G(\alpha, \beta) \longrightarrow \mathrm{PSL}(2, \mathbb{R}).$$

(1) ρ_1 is the restriction of the Fuchsian representation to $G(\alpha, \beta) < \mathrm{Affine}(\Sigma)$ composed with the projection onto $\mathrm{PSL}(2, \mathbb{R})$.
(2) ρ_2 is the representation given by Thurston's theorem.

By definition of these two representations, ρ_1 maps T_α to $\begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix}$ and ρ_2 maps it to $\begin{pmatrix} 1 & 2\sqrt{2} \\ 0 & 1 \end{pmatrix}$; and ρ_1 maps T_β to $\begin{pmatrix} 1 & 0 \\ \frac{3}{2} & 1 \end{pmatrix}$ and ρ_2 maps it to $\begin{pmatrix} 1 & 0 \\ -2\sqrt{2} & 1 \end{pmatrix}$.

Proposition 8. *For all $g \in G(\alpha, \beta)$, $\rho_1(g)$ and $\rho_2(g)$ have same type.*

Proof.
- ρ_1 and ρ_2 are faithful;
- ρ_1 and ρ_2 are Schottky subgroups of $\mathrm{PSL}(2, \mathbb{R})$ of infinite covolume;
- ρ_1 and ρ_2 send T_α and T_β to two parabolic elements;

As a consequence of these three facts, the quotient of $\mathbb{H}$ by the respective actions of $G(\alpha, \beta)$ through ρ_1 and ρ_2 respectively are both a sphere S with two cusps and a funnel. No element of $\rho_1(G(\alpha, \beta))$ or $\rho_2(G(\alpha, \beta))$ is elliptic, and the image of $g \in$

$G(\alpha, \beta)$ is parabolic in $\rho_1(G(\alpha, \beta))$ or $\rho_2(G(\alpha, \beta))$ if and only if the corresponding element in $\pi_1(S)$ is in the free homotopy class of a simple closed curve circling a cusp. Which proves the proposition.

□

There is little needed to complete the topological description of the elements of the Veech group of Σ. Indeed, Proposition 8 above together with Thurston's theorem ensures that the topological type of $g \in G(\alpha, \beta) \subset \mathrm{Affine}(\Sigma)$ is determined by (the projection to $\mathrm{PSL}(2, \mathbb{R})$ of) its image by the Fuchsian representation (namely g has finite order if $\rho_1(g)$ is elliptic, g is reductive if $\rho_1(g)$ is parabolic and pseudo-Anosov if $\rho_1(g)$ is hyperbolic).
The group $G(\alpha, \beta)$ has index 2 in $\mathbf{V}_\Sigma$. The involution $i \in \mathrm{Affine}(\Sigma)$ acting as $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ preserves the multi-curves α and β and therefore commutes to the whole $G(\alpha, \beta)$. In particular, any element of $\mathbf{V}_\Sigma$ writes $g \cdot i$ with $g \in G(\alpha, \beta)$. The type of $g \cdot i$ being the same as the type of g, we get the following

Theorem 9. *For all $f \in \mathrm{Affine}(\Sigma)$, f is of finite order, reductive or pseudo-Anosov according to whether its image by the Fuchsian representation in* $\mathrm{SL}(2, \mathbb{R})$ *is elliptic, parabolic or hyperbolic.*

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