Lecture notes on circle diffeomorphisms

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CHAPTER 1

Flows on surfaces

In this chapter we make the standing assumption that all vector fields we consider have isolated critical points.

1. Flows on the sphere are tame

In this paragraph we consider the dynamics of flows on the sphere S^2. For this entire section, $\vec{X}$ is a vector field of class $\mathcal{C}^1$ on the sphere S^2 *i.e.* a section of $\mathrm{T}S^2$. We are interested in the differential equation

$$\frac{df}{dt}(x) = \vec{X}(x) \tag{1}$$

Because $\vec{X}$ is of class $\mathcal{C}^1$, local solutions to this equation given an initial condition exist and are unique, by Cauchy-Lipschitz. Because the sphere is compact, such solutions are defined for all time; such a vector field $\vec{X}$ therefore defines a flow

$$\phi_X : S^2 \times \mathbb{R} \longrightarrow S^2. \tag{2}$$

We are going to discuss the following question

QUESTION 1. *What is the topological dynamics of ϕ_X? In other words, given $x_0 \in S^2$, can we describe its ω and α-limits?*

1.1. Flow boxes.

We start by discussing the local structure of a flow on a surface at a point where $\vec{X} \neq \vec{0}$. We have the following fundamental lemma

LEMMA 2 (Flow box). *Let X be a $\mathcal{C}^1$ vector field on a smooth manifold M of dimension n. Consider a point $x \in X$ such that $\vec{X}(x) \neq 0$. Then there exists a neighbourhood U of x and a chart $\varphi : U \to]-1,1[\times B^{n-1}$ such that*

$$\varphi_*(\vec{X}) = \partial x_1$$

where x_1 is the coordinate on the factor $]-1,1[$ in the product $]-1,1[\times B^{n-1}$. In particular, φ maps integral curves of the flow of $\vec{X}$ onto straight lines of the form $\{p\}\times]-1,1[$ with $p \in B^{n-1}$.

PROOF. Exercise.

□

1.2. Examples. We give here a few examples of flows on S^2.

Example 1 : Flow with a parabolic fixed point. Consider a vector field $\vec{X}$ that vanishes at a point p. Up to a choice of coordinate, we can assume that $p = 0 \in \mathbb{R}^2$. The flow of $\vec{X}$ fixes the point $p = 0$. We say that this fixed point is parabolic if $\vec{X}$ is locally of the form

$$\vec{X}(x, y) = (x^2 - y^2)\partial x + (xy)\partial y.$$

In complex coordinates $z = x + iy$, X is just $z \mapsto z^2$. The figure below describes a flow on the sphere with a unique parabolic fixed point.

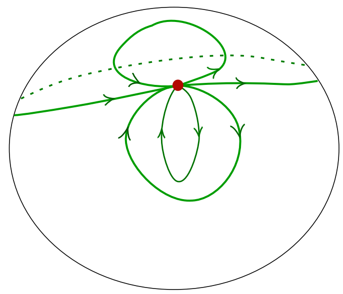

FIGURE 1. A flow with a parabolic fixed point

Example 2 : North-South dynamics (not spiraling). A *North-South* flow is a flow satisfying the following conditions.

- It has exactly two critical points S and N which are respectively (locally) of the form $\vec{X}(x, y) = x\partial x + y\partial y$ and the second $\vec{X}(x, y) = -x\partial x - y\partial y$.
- For every regular point p, $\omega(p) = N$ and $\alpha(p) = S$.

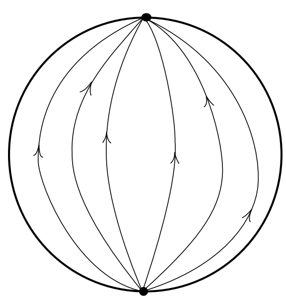

FIGURE 2. A North-South flow

We leave it to the reader to check as an exercise that the flow defined on $\mathbb{R}^2$ by the equation $\vec{X} = (x, y)$ extends to $S^2 = \mathbb{R}^2 \cup \{\infty\}$ to a North-South flow.

Example 3 : Completely periodic flow. Consider $\mathbb{R}^2$ with polar coordinates (r, θ). We consider the flow

$$\vec{X}(r, \theta) = r\partial\theta.$$

One easily checks the following properties:

- $\vec{X}$ extends to S^2 to a smooth vector field;
- $\vec{X}$ has exactly two critical points 0 and ∞;
- integral curves of $\vec{X}$ are exactly circle centred at 0.

Thus every trajectory of $\vec{X}$ is periodic.

1.3. Poincaré-Bendixson theorem. We introduce some terminology in order to be able to state the main theorem of this chapter.

- A *singular (or critical) point* is a point x at which $\vec{X}(x) = \vec{0}$.
- A *connection* is a trajectory whose α-limit is a singular point and whose ω-limit is also a singular point. In particular, the closure of a connection is equal to itself union the two limiting singular point. Note that these limiting points can be equal.
- A *cycle* is a connected and closed union of singular points and connections.

THEOREM 3 (Poincaré-Bendixson). *Let $\vec{X}$ be a vector field of class $\mathcal{C}^1$ on the 2-dimensional sphere S^2 and consider $x_0 \in S^2$. Then the ω-limit or α-limit of x_0 for the flow of $\vec{X}$ is either*

(1) a closed orbit;
(2) a singular point;
(3) a cycle.

1.4. Jordan curve. The key fact, very specific to the sphere and the dimension 2, is the Jordan curve theorem. That is the main ingredient of the proof of Poincaré-Bendixson theorem. In its most general version (continuous embeddings of the circle), it is a difficult theorem. We will only need the piecewise $\mathcal{C}^1$ version which is much easier.

THEOREM 4 (Piecewise $\mathcal{C}^1$ Jordan curve). *Let $\gamma : S^1 \longrightarrow S^2$ a piecewise $\mathcal{C}^1$-embedding. Then $\gamma(S^1)$ separates S^2 into two connected components both homeomorphic to a disc.*

We do not give a proof here.

1.5. Proof of Theorem 3. The main idea is the following: consider any forward trajectory starting off at a point x. Because S^2 is compact, it is going to accumulate somewhere. Consider a regular accumulation point $p \in S^2$ and a small arc γ at p that is transverse to the flow. Because p is an accumulation point of the forward orbit of x the latter is going to intersect γ infinitely many times. The main point is that a given trajectory can only intersect such a transverse arc following a simple combinatorial pattern: consecutive intersections come in increasing (or decreasing) order on γ. Indeed a very long trajectory that comes back close to p can be closed using a small segment on γ. By the Jordan curve theorem (Theorem 4), this closed curve bounds a topological disc in S^2 (which we call D) that the trajectory enters after its second intersection with γ (see Figure 3 below).

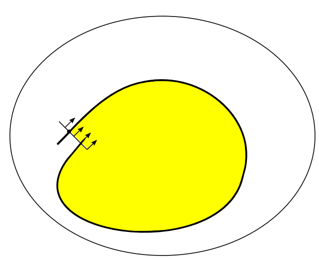

FIGURE 3. A trajectory coming back close to its initial point

We notice that from that moment onwards, the trajectory is trapped within this disc as its boundary is the union of a bit of a trajectory and a transverse segment at which the vector field $\vec{X}$ is pointing inwards. It therefore cannot cross the small arc γ on the side of p that is outside D (the left side on Figure 3).

This proves that future intersections of the trajectory with γ come in increasing or decreasing order. Altogether this discussion proves the

LEMMA 5. *Let $x \in S^2$ and let γ a smooth arc transverse to the flow (an arc is transverse to the flow if at any point of γ, $\vec{X}$ is not is the direction of the tangent vector at γ). Then $\omega(x) \cap \gamma$ consists of at most one point.*

This was the key step, and we can now use this lemma to conclude. We will make use of the fact that the ω-limit of a point x is always connected (see Exercise 6).

Case 1. The ω-limit of x contains no regular point. Since it is connected, non-empty and that the set of critical point is discrete, it must be a single singular point.

Case 2. The ω-limit of x contains no regular point. Since it is non-empty, we consider a point $p \in \omega(x)$ and γ a small transverse arc at p. By Lemma 5, $\gamma \cap \omega(x) = p$. We want to show that the orbit of p is periodic. Because $p \in \omega(x)$, $\mathcal{O}^+(p) \subset \omega(x)$ and in particular does not contain any singular point. Take an accumulation point p' of the forward orbit of p, it is regular. But because the forward orbit of p is contained in the ω-limit of x, it intersects a transverse arc at p' only in p'. Because p' is an accumulation point, we have that $p' \in \mathcal{O}^+(p)$ and that p' (and thus p) is periodic. We have obtained that in that case the ω-limit of x is a union of isolated periodic orbit. By connectedness, it is a unique periodic orbit.

Case 3. The ω-limit of x contains both regular and singular point. Consider p a regular point in $\omega(x)$. Its forward orbit cannot be periodic because otherwise the whole ω-limit would be this periodic orbit. We claim that the forward orbit of x accumulates to a critical point. If it were not the case, since it is not periodic, it would accumulate to a regular point that is not on its orbit, contradicting Lemma 5. Same applies with the forward orbit. We have thus proved that the ω-limit is a connected union of singular points and connections, which concludes.

1.6. In higher dimensions. Poincaré-Bendixson theorem does not generalise.

(1) It does not generalise to higher dimensions : there exist flows on the 3-sphere which have complicated dynamics. The most famous example is given by the Lorenz equations which give birth to the *Lorenz attractor*.
(2) It does not generalise to higher genus surfaces. We treat the case of the torus in the next section and provide simple examples which do not satisfy the conclusion of Poincaré-Bendixson theorem.
(3) It does not generalise to singular foliations on the sphere.

2. Flows on the torus

In this section we discuss flows on the torus without singular points *i.e.* for all $x \in \mathbb{T}^2$, $\vec{X}(x) \neq 0$.

2.1. Examples.

Linear flows. Let $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the 2-torus. We consider the vector field

$$\vec{X}_\theta = (\cos\theta) \cdot \partial x + (\sin\theta) \cdot \partial y.$$

PROPOSITION 6.
- *If* $\tan(\theta)$ *is rational,* $\vec{X}$ *is completely periodic (i.e. every single orbit is periodic).*
- *If* $\tan(\theta)$ *is irrational,* $\vec{X}$ *is minimal.*

PROOF. Exercise. □

Hopf torus. Consider $\mathbb{R}^2 \setminus \{0\}$ and the vector field

$$\vec{X}(x,y) = (x^2 + y^2)\partial x.$$

This vector field is invariant by the action of $(x,y) \mapsto 2(x,y)$ and thus defines a smooth vector field on

$$\mathbb{R}^2 \setminus \{0\}/\big((x,y) \sim (2x,2y)\big) \simeq \mathbb{T}^2.$$

PROPOSITION 7.
- $\vec{X}$ *has exactly two periodic orbits.*
- *One of the two periodic orbit is the* ω*-limit of every non-periodic point and the other is the* α*-limit of every non-periodic point.*

PROOF. Exercise.

□

2.2. Flows with a periodic orbit. In this paragraph, we deal with flows having at least one periodic orbit.

PROPOSITION 8. *Assume that* $\vec{X}$ *has one periodic orbit. Then the* ω*-limit or* α*-limit of every point is a periodic orbit.*

PROOF. This Proposition is a corollary of Poincaré-Bendixson theorem. We assume $\vec{X}$ has a periodic orbit. This periodic orbit is not trivial in homology otherwise there would be a critical point in the interior of the disc it would bound (see Exercise 3). This is impossible since we have assume that $\vec{X}$ vanishes nowhere. The complement of this periodic orbit is a cylinder which is invariant by the flow.

We can thus cut along this periodic orbit to get a flow on a cylinder such that its two boundary components are periodic orbits. We embed this cylinder in a sphere in such a way that the complement of the two boundary components of the cylinder is the union of the interior of the cylinder and two discs. We extend the flow to those two discs to a completely periodic flow restricted to those discs (for instance take the flow $r \cdot \partial\theta$ on the disc of radius in 1 in $\mathbb{R}^2$ with (r,θ) standard polar coordinates). There is a slight difficulty at this point: one has to show that $\vec{X}$ can be extended to be $\mathcal{C}^1$ on the whole of S^2 (which is not obvious along the two periodic orbits where we do the gluing). We leave it as an exercise to show that this can actually be done, using standard bump functions in coordinates.

It remains to apply Poincaré-Bendixson to this flow. Take any point within the cylinder, by Poincaré-Bendixson its ω-limit (or α-limit) is either a cycle or a periodic orbit. Because the cylinder is invariant under the flow, the α-limit is contained in the (closed cylinder) and thus does not contain any singular point. Therefore it is a periodic orbit. Since the dynamics of the extended flow restricted to the invariant cylinder is the same as the initial one, it concludes.
□

2.3. Flows without periodic orbits. Finally, we show how we can reduce the study of the dynamics of a flow without a periodic orbit to that of a circle diffeomorphism.

PROPOSITION 9. *If $\vec{X}$ does not have a periodic orbit, then there exists a smooth, simple closed curve γ that is everywhere transverse to the flow.*

PROOF. Consider any point x such that $x \in \omega(x)$ (see Exercise 4). The proof is essentially contained in the Figure below. Because $\vec{X}$ does not have critical point, x is regular. Consider an arc γ at x. We consider a *strip* that is a union of trajectories issued from γ. Because x is recurrent we eventually get (up to possibly inverting left and right) the following configuration:

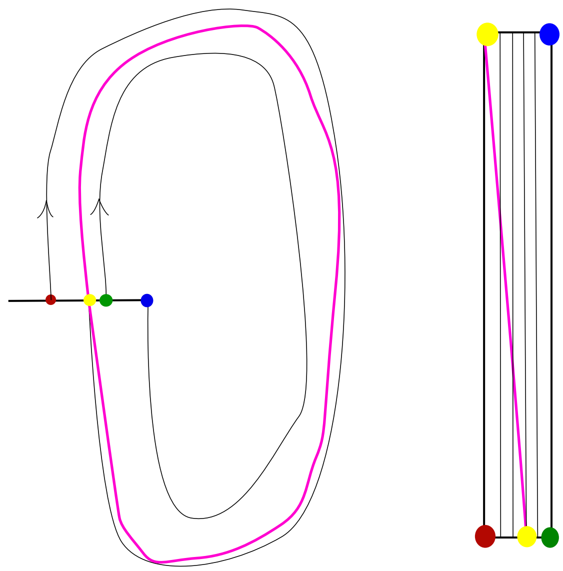

FIGURE 4. The violet curve is transverse to the flow

The closed curve in violet is the transverse curve we are seeking. □

Reduction to a circle diffeomorphism.

PROPOSITION 10. *A flow on a torus with no periodic orbit is the suspension of a circle diffeomorphism which is as regular as the flow is.*

PROOF. We have proved that there exists a curve that is transverse to the flow. What we want to show is that the first return map on this curve realises this suspension. We split the proof into three steps.

Step 1 A transverse curve does not bound a disc. Otherwise such a disc could be cut out and extended to a flow on the sphere with the following extra property that any trajectory entering the disc never leaves. The ω-limit of any point in the disc must be a periodic orbit (because $\vec{X}$ does not have critical point). But by Exercise 3, the interior of the disc bounded by this periodic orbit should contain a critical point, which is a contradiction.

Step 2 This transverse curve is therefore essential and its complement in $\mathbb{T}^2$ is a cylinder. We can cut along this curve to get a flow with two boundary components. At one boundary component the flow is pointing inwards and at the other it is pointing outwards. What we want to show is that any trajectory issued from the boundary component where the flow is pointing inwards eventually intersects the other boundary component. To prove it, we extend in an arbitrary fashion the flow to a flow on the sphere by adjoining to the two boundary components two discs. The ω-limit of a point x on the boundary component where the flow is pointing inwards is either a cycle or a periodic orbit. Since we have assumed that the initial flow has neither periodic orbits nor critical points, then the ω-limit of this point is not fully contained within the cylinder. It cannot intersect the disc that caps the boundary component where the flow is pointing inwards, thus it intersect the other disc. Which implies that the trajectory of x must intersect the other boundary component.

Step 3 Going back to the initial flow on the torus, this discussion implies that the first return map

$$T : \gamma \simeq S^1 \longrightarrow \gamma$$

is well-defined. Because we know that through a point passes only one trajectory it is injective. The same argument applied to $-\vec{X}$ shows that T is surjective and that its differential is invertible at any point (this is because the flow of a differential equation at a given time is a diffeomorphism). T is as regular as the flow of $\vec{X}$ which itself is as regular as $\vec{X}$. We have thus realised the flow of $\vec{X}$ as the suspension of a circle diffeomorphism.

□

2.4. Higher genus surfaces. Flows on surfaces of higher genus always have critical points. An similar analysis can be carried out, but it is combinatorially much more complicated.

3. Exercises

EXERCISE 1. *Prove the flow box lemma (Lemma 2).*
(Hint : work in coordinates, and use the flow map to construct the inverse of the flow box chart.)

EXERCISE 2. *Show that any solution to a differential equation*

$$\frac{df}{dt}(x) = \vec{X}(x)$$

on the sphere S^2, with $\vec{X}$ of class $\mathcal{C}^1$ can be extended to a solution defined for all $t \in \mathbb{R}$.

EXERCISE 3.
Let Σ be a compact surface and let $\vec{X}$ be a smooth flow on this surface. Assume Γ is a non-trivial periodic orbit of $\vec{X}$ which bounds a topological disk D.

1. *Show that the interior of D contains a periodic orbit or a critical point.*
2. *Using Zorn's lemma, show that there is a critical point in D.*
   (Hint: proceed by contradiction and consider concentric periodic orbits.)
3. *Show that any smooth vector field on S^2 has a critical point.*

EXERCISE 4. *Show that any vector field on a compact manifold has at least one recurrent point (i.e. a point x such that $x \in \omega(x)$).*

EXERCISE 5. *Show that there is no simple closed curve that is transverse to the flow on the Hopf torus defined in 2.1.*

EXERCISE 6. *Show that there ω-limit of a point under the action of a smooth flow on a compact surface is always connected.*

CHAPTER 2

Circle homeomorphisms and diffeomorphisms

1. The circle

1.1. Definition.

Definition 1. *The circle* S^1 *is the topological space* $\mathbb{R}/\mathbb{Z}$ *which is formally the quotient of* $\mathbb{R}$ *by the action of* $x \mapsto x+1$.

We collect below basic properties on the circle.

(1) S^1 is homeomorphic to the Euclidean circle of radius 1 in $\mathbb{R}^2$.
(2) It is naturally endowed with a structure of analytic manifold.
(3) The metric dx^2 on $\mathbb{R}$ is invariant by the translation $x \mapsto x+1$ and therefore passes to the quotient $\mathbb{R}/\mathbb{Z}$.

The following theorem somewhat justifies the central place of the circle in one-dimensional dynamics.

Theorem 11. *The circle is the only connected closed topological* 1*-manifold. Moreover, it carries a unique (up to homeomorphism) structure of analytic manifold.*

1.2. Intervals and cyclic order. The circle is an orientable manifold, by default we will work with the counter-clockwise orientation. This orientation allows us to define *intervals* on S^1. Given two points a and b, the complement in S^1 of their union consists of two intervals. We will denote these two intervals (a,b) and (b,a). (a,b) is the interval for which a is on the left with respect to the counter-clockwise orientation and (b,a) the other. Closed intervals will be denoted by $[a,b]$.

A sequence of point $p_1, \cdots, p_n$ with $n \geq 3$ is said to be *cyclically ordered* if $p_2, \cdots, p_{n-1}$ belong to the interval $[p_1, p_n]$ and if on this interval $p_2 \leq p_3 \leq \cdots \leq p_{n-1}$. This relation defines on S^1 what is called a *cyclic order*.

2. Circle homeomorphisms

Definition 2. *A circle homeomorphism is an homeomorphism of the topological space* S^1.

2.1. Lift. Let $T : S^1 \to S^1$ be a circle homeomorphism. Because $\mathbb{R}$ is the universal cover of S^1, there exists a map $\tilde{T} : \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$T \circ \pi = \pi \circ \tilde{T}$$

where $\pi : \mathbb{R} \longrightarrow S^1$ is the natural projection. We call $\tilde{T}$ the *lift* of T to $\mathbb{R}$. We make the following comments:

(1) $\tilde{T}$ is a homeomorphism of $\mathbb{R}$ and in particular it is strictly monotone. More precisely, if T preserve the orientation of S^1, then $\tilde{T}$ will be increasing whereas if T reverses the orientation of S^1, $\tilde{T}$ will be decreasing.
(2) Because $\tilde{T}$ is the lift of a circle homeomorphism, it satifies the following equation if

$$\forall x \in \mathbb{R}, \tilde{T}(x+1) = \tilde{T}(x) + 1$$

(3) Such a lift $\tilde{T}$ is not unique. Two different lifts differ by a translation by an element of $\mathbb{Z}$.

EXERCISE 7. *Give formal proofs of the above statements.*

PROPOSITION 12. *The set of all circle homeomorphisms is a topological space when endowed with the topology of the uniform convergence.*

2.2. Rotations. A fundamental class of examples of circle diffeomorphisms is the set of rotation. Let $\alpha \in [0, 1[$; the rotation of angle α is the map

$$\begin{array}{rcccc} r_\alpha & := & \mathbb{R}/\mathbb{Z} & \longrightarrow & \mathbb{R}/\mathbb{Z} \\ & & x & \longmapsto & x+\alpha \end{array}$$

PROPOSITION 13. *(1) If α is rational, i.e. $\alpha = \frac{p}{q}$ with p and q co-prime integers, then r_α is periodic of period q.*
(2) If α is irrational, then r_α is minimal.

PROOF. Exercise. □

We will see that rotations play a central role in the theory. An important fact is that rotations altogether form a group with respect to the composition; and it is equal to the group of orientation-preserving isometries of the flat metric of S^1.

2.3. Blow-up of a rotation. We describe in this paragraph a general construction of circle homeomorphisms displaying an interesting dynamical behaviour. Consider r_α the rotation of angle α and choose α irrational. Consider an arbitrary x_0 and define $x_n = r_\alpha^n(x_0)$ for all $n \in \mathbb{Z}$. Because we have assumed α irrational the sequence (x_n) is dense in S^1.

- The first thing we do is that we "blow-up" the orbit at x_0. This means that we construct a new topological space by deleting each x_n and replacing it by an entire closed interval I_n. We get this way a new topological space $\mathcal{S}$ which is still a topological circle and a degree 1 map $\varphi : \mathcal{S} \longrightarrow S^1$ which maps each I_n to x_n and all other points in $\mathcal{S}$ to their initial position. This map φ is locally increasing and constant on the intervals I_n.
- The rotation r_α straightforwardly extends to the complement in $\mathcal{S}$ of the union of the I_ns to a map T. By choosing orientation-preserving homeomorphisms $I_n \longrightarrow I_{n+1}$, it is possible to extend T to a homeomorphism

$$T : \mathcal{S} \longrightarrow \mathcal{S}.$$

- By construction, T satisfies the following

$$\varphi \circ T = R_\alpha \circ \varphi.$$

One says that φ is a *semi-conjugation* between T and r_α.

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Proposition 14. *Let T, (x_n) and (I_n) be as above.*

1. *J_0, the interior or I_0 is a wandering interval, meaning that its iterated images $T^n(J_0)$ for $n \in \mathbb{Z}$ are pairwise disjoint.*
2. *The complement of $\bigcup T^n(J_0)$, which we denote by $\mathcal{C}$, is a Cantor set.*
3. *T restricted to $\mathcal{C}$ is minimal.*

Proof. (1) The first point is true by construction.

(2) $\mathcal{C}$ is closed because it is the complement of an open set. A point in $\mathcal{C}$ is either a point that was initially in the complement of the orbit of x_0 or an end point of on the I_n. At any rate it is accumulated by points that are in $S^1 \setminus \mathcal{O}(x_0)$ thus $\mathcal{C}$ has no isolated points. Moreover, the orbit of x_0 in S_1 was dense so any point in S^1 is accumulated by elements of $\mathcal{O}(x_0)$ which implies that any point in $\mathcal{C}$ is accumulated by intervals I_n; this in turn implies that $\mathcal{C}$ has empty interior. $\mathcal{C}$ is therefore a Cantor set.

(3) T is minimal restricted to $\mathcal{C}$ by construction, as R_α is minimal.

□

3. Groups of circle homeomorphisms

3.1. Regularity. We have endowed S^1 with the structure of an oriented analytic Riemannian manifold (meaning that S^1 is endowed with a structure of analytic manifold together with an analytic Riemannian metric). For any function $f : S^1 \longrightarrow S^1$ it therefore make sense to say that

1. f is differentiable;
2. f is of class $\mathcal{C}^r$ for any $r \in \mathbb{N}$;
3. f is a local diffeomorphism;
4. f is a global diffeomorphism;
5. f is an orientation preserving homeomorphism;
6. f is α-Hölder for any $\alpha > 0$;
7. f is Lipschitz.

Definition 3. *A homeomorphism of a $\mathcal{C}^1$-manifold M of finite dimension is a diffeomorphism of M satisfying the following*

1. *it is of class $\mathcal{C}^1$;*
2. *at any point its derivative is invertible.*

Note that an homeomorphism of class $\mathcal{C}^1$ is not necessarily a diffeomorphism. The basic example is

$$x \longmapsto x^3$$

which is a homeomorphism of $\mathbb{R}$ but it is not a diffeomorphism since its derivative at 0 vanishes.

The derivative of a circle diffeomorphism. Formally, the derivative of a diffeomorphism $T : S^1 \longrightarrow S^1$ at a point $p \in S^1$ is a linear map

$$DT(p) : \mathrm{T}_p S^1 \longrightarrow \mathrm{T}_{T(p)} S^1.$$

But thanks to the Riemannian metric on S^1 and the orientation on S^1, each tangent space $\mathrm{T}_p S^1$ identifies in a unique way to $\mathbb{R}$. With these identifications, $DT(p)$ is the multiplication by a number $T'(p)$. This way the derivative of can be thought of as a map

$$T' : S^1 \longrightarrow \mathbb{R}.$$

DEFINITION 4. *For $r \geq 2$, a homeomorphism is said to be a $\mathcal{C}^r$-diffeomorphism if it is a diffeomorphism and that its derivative is of class $\mathcal{C}^{r-1}$.*

3.2. Groups. Finally, we define the following subsets :

(1) $\mathrm{Homeo}(S^1) = \{$circle homeomorphisms$\}$;
(2) $\mathrm{Homeo}^+(S^1) = \{$orientation-preserving circle homeomorphisms$\}$;
(3) $\mathrm{Diff}_r(S^1) = \{\mathcal{C}^r -$ circle diffeomorphisms$\}$;
(4) $\mathrm{Diff}_r^+(S^1) = \{$orientation-preserving $\mathcal{C}^r -$ circle diffeomorphisms$\}$.

We leave it to the reader to verify that this sets together with the composition law are topological groups. The topology on $\mathrm{Homeo}(S^1)$ is that of the uniform convergence (as S^1 is a metric space) whereas that on $\mathrm{Diff}_r(S^1)$ is that of the uniform convergence of all consecutive derivatives up to rank r.

Exercises

EXERCISE 8. *Show that a Möbius transformation with a periodic orbit of order larger or equal to* 2 *is conjugate in* $\mathrm{PSL}(2, \mathbb{R})$ *to a rotation of finite order.*

EXERCISE 9. *Let g_1 and g_2 be two Riemannian metric on S^1. Show that g_1 and g_2 are isometric if and only if* $\mathrm{Vol}(g_1) = \mathrm{Vol}(g_2)$*.*

EXERCISE 10. *Prove Proposition 22.*

CHAPTER 3

The rotation number

1. Continued fractions

In this paragraph we discuss rational approximations of real numbers. The connection with circle dynamics is the following: if α is an irrational number, times q such that r_α^q is very close to the identity correspond to rational numbers $\frac{p}{q}$ such that $|\alpha - \frac{p}{q}|$ is very small. Unless explicitly mentioned, we will always assume that rational numbers are written in reduced form, *i.e.* $\frac{p}{q}$ with p and q coprime.

1.1. Rational approximations.

Definition 5. *A rational approximation of a number α is a rational $\frac{p}{q}$ such that*

$$|\alpha - \frac{p}{q}| < \frac{1}{q^2}.$$

We have the following Proposition

Proposition 15. *Any irrational number α has infinitely many rational approximations.*

Proof. Consider an integer $n \geq 1$ and subdivide the interval $[0,1]$ into n subintervals $[\frac{k}{n}, \frac{k+1}{n}]$. Consider the sequence $\{j\alpha\}$. By Dirichlet principle, there are two distinct integers q_1 and q_2 such that $\{q_1\alpha\}$ and $\{q_2\alpha\}$ belong to the same interval $[\frac{k}{n}, \frac{k+1}{n}]$. In particular there exists $p \in \mathbb{N}$ such that

$$|(q_1 - q_2)\alpha - p| \leq \frac{1}{n}.$$

Set $q = q_1 - q_2$ and one gets that $|\alpha - \frac{p}{q}| \leq \frac{1}{qn}$. In particular since $q \leq n$ we get

$$|\alpha - \frac{p}{q}| \leq \frac{1}{q^2}.$$

The last thing we need to show is that we get by this mean infinitely many such $\frac{p}{q}$. Since α is irrational $|\alpha - \frac{p}{q}| \neq 0$. As n gets larger, since $|\alpha - \frac{p}{q}| \leq \frac{1}{qn}$ the quantity $|\alpha - \frac{p}{q}|$ tends to zero which implies that this process produces infinitely many rational approximations.

□

1.2. Continued fractions. We describe hereafter the *continued fraction algorithm* which is a powerful tool to construct rational approximations. We define the *Gauss map* by

$$G := x \longmapsto \{\frac{1}{x}\} = \frac{1}{x} - [\frac{1}{x}].$$

Let $a := x \mapsto [\frac{1}{x}]$. For any irrational number α we define the sequence $a_0 = [\alpha]$ and $a_{i+1} = a(G^i(\{\alpha\}))$ for $i \geq 1$. We could have tried to define the sequence (a_i) for α rational but for some n we would have had $G^n(\alpha) = 0$ and been unable to continue the construction of the a_i beyond the step n.

DEFINITION 6. *Let α be a real number. When well-defined, the **partial quotient of order** n of α is*

$$\frac{p_n}{q_n} = [a_0, a_1, \cdots, a_n] = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\cdots + \frac{1}{a_n}}}}.$$

We have the following important identities

PROPOSITION 16.
- $\forall n \geq 2,\ p_n = a_n p_{n-1} + p_{n-2}$ *with* $p_0 = a_0$ *and* $p_1 = a_0 a_1 + 1$*;*
- $\forall n \geq 2,\ q_n = a_n q_{n-1} + q_{n-2}$ *with* $q_0 = 1$ *and* $q_1 = a_1$*;*
- $q_n p_{n-1} - p_n q_{n-1} = (-1)^n$.

PROOF. Exercise. Proceed by induction on n for general partial quotients where the a_is are allowed to be real.

□

One of the reason why we care so much about partial quotients is the following

PROPOSITION 17. *Let α be an irrational number. Then its partial quotients $\frac{p_n}{q_n} = [a_0, a_1, \cdots, a_n]$ are rational approximations of α.*

PROOF. Note that for any α,

$$\alpha = [a_0, a_1, \cdots, a_n + G^n(\{\alpha\})].$$

Recall that $\frac{p_n}{q_n} = [a_0, a_1, \cdots, a_n]$. By induction we have

$$\alpha = \frac{p_n + G^n(\{\alpha\}) p_{n-1}}{q_n + G^n(\{\alpha\}) q_{n-1}}.$$

Thus we obtain

$$\alpha - \frac{p_n}{q_n} \leq \frac{(-1)^n}{q_n(\frac{1}{G^n(\{\alpha\})} q_n + q_{n-1})}.$$

By definition we have $a_{n+1} \leq \frac{1}{G^n(\{\alpha\})}$ thus we get

$$|\alpha - \frac{p_n}{q_n}| \leq \frac{1}{q_n q_{n+1}} \leq \frac{1}{q_n^2}.$$

□

We make one short comment about the last line of the proof. A slightly more careful computation would have given

$$|\alpha - \frac{p_n}{q_n}| \leq \frac{1}{q_n q_{n+1}} \leq \frac{1}{a_{n+1} q_n^2}.$$

This shows us that effectively, how good an approximation of α the partial quotient $\frac{p_n}{q_n}$ is is controlled by the value of a_{n+1}. The bigger a_{n+1} the better the approximation is.

2. Continued fractions, dynamical viewpoint

In this section we explain another way to think of continued fractions. It is the good point of view in view of generalising this construction to circle homeomorphisms. For the rest of the paragraph, α is a number in $]0,1[$. Consider the rotation of angle α

$$r_\alpha := x \mapsto x + \alpha.$$

Consider any point $x \in S^1$. The algorithm outputs a sequence of positive integers $a_1, a_2, \cdots, a_n, \cdots$ which is linked to special times q_ks such that r_α^k is very close to the identity. In more arithmetic term, integers q_k such that $d(q_k \cdot \alpha, \mathbb{N})$ is very small.

- **Step 1** We look at the sequence $x, r_\alpha(x), r_\alpha^2(x), \cdot, r_\alpha^k(x), \cdots$. Let a_1 be largest integer such that $x, r_\alpha(x), r_\alpha^2(x), \cdot, r_\alpha^{a_1}(x)$ are cyclically ordered on S^1.
  $\frac{1}{a_1}$ should be thought of a first order rational approximation of α. At the dynamical level, it is characterised by the property that *for any* y, it is the largest integer such that $y, r_\alpha(y), r_\alpha^2(y), \cdot, r_\alpha^{a_1}(y)$; and by the fact that $r_\alpha^{a_1}$ is close to the identity. We now want to understand what happens at the second order. We have the following configuration

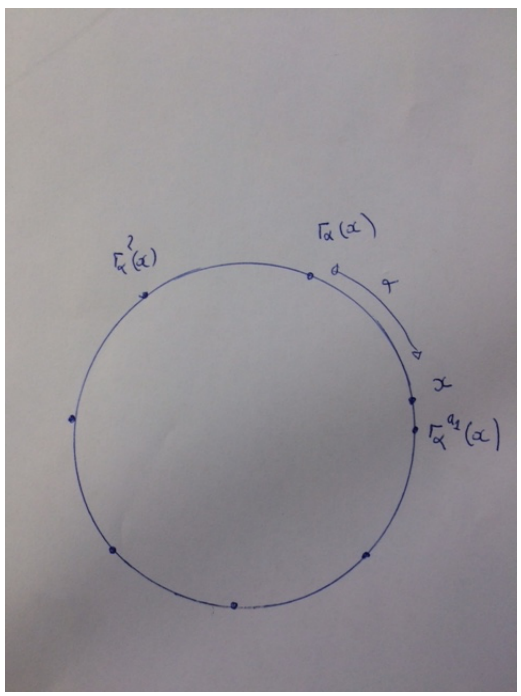

FIGURE 1. Combinatorial configuration of the orbit

- **Step 2** We are seeking to construct the "second order" rational, dynamical approximation of α. We already know the structure of orbits up to time $q_1 = a_1$, we are thus going to speed up the process by considering $r_\alpha^{q_1}$. If one were to start with $y = r_\alpha(x)$, by definition of a_1 we would have that

$$r_\alpha^{q_1}\big(y = r_\alpha(x)\big) \in [x, r_\alpha(x)].$$

By iterating the process, we obtain a sequence $y, r_\alpha^{q_1}(y), r_\alpha^{2q_1}(y), \cdots, r_\alpha^{a_2 \cdot q_1}(y)$ coming in decreasing order; with a_2 defined as the largest integer such that $r_\alpha^{a_2 \cdot q_1}(y) \in [x, r_\alpha(x)]$, see Figure below.

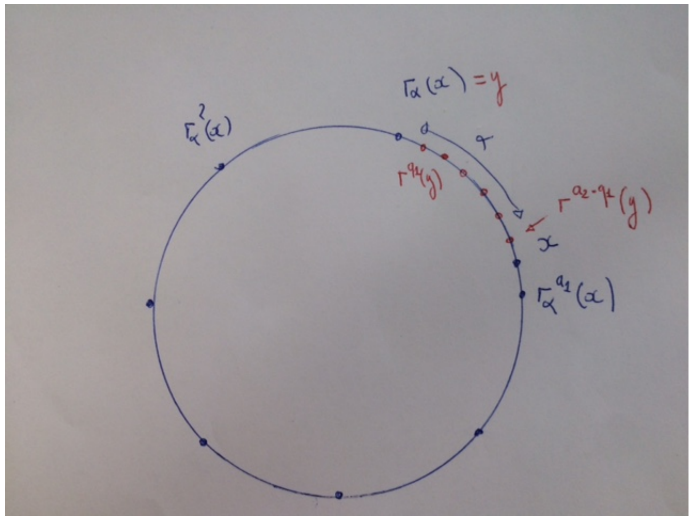

FIGURE 2. Combinatorial configuration of the orbit, step 2

Thus $r_\alpha^{a_2 \cdot q_1}(y) = r_\alpha^{a_2 \cdot q_1 + 1}(x)$ has come very close to x.[1] We set $q_2 = a_2 \cdot q_1 + 1$.

- **Step 3 (the inductive step)** We are now ready to defined the sequences $(a_n)_{n \in \mathbb{N}^*}$ and $(q_n)_{n \in \mathbb{N}}$. Set $q_0 = 1$, $q_1 = a_1$ and $q_2 = a_1 \cdot q_1 + q_0$.
  - The q_ns will satisfy the following property $r_\alpha^{q_n}$ is closer to the identity than any other r_α^k with $k \leq n$.
  - a_n is a measure of how good the approximation of the identity by $r_\alpha^{q_{n-1}}$ is. The larger a_n, the better the approximation.

  We have initiated the induction. Assume that a_{n-1} and q_{n-1} have been defined and that at step n we have the following configuration if n is even

  and a similar one (where the ordering has been reverted) if n is odd. We assume, without loss of generality, that n is even. We consider the interval $[x, r_\alpha^{q_{n-1}}(x)]$. $r_\alpha^{q_n}$ is a rotation by a small negative angle. We start iterating $r_\alpha^{q_n}$ starting from $y = r_\alpha^{q_{n-1}}(x)$. We get a decreasing sequence $y, r_\alpha^{q_n}(y), r_\alpha^{2 \cdot q_n}(y), \cdots, r_\alpha^{a_{n+1} \cdot q_n}(y)$ of points of the interval $[x, r_\alpha^{q_{n-1}}(x)]$. $r_\alpha^{q_n}$, where a_{n+1} is defined to be the largest integer such that $r_\alpha^{a_{n+1} \cdot q_n}(y)$ belongs to $[x, r_\alpha^{q_{n-1}}(x)]$. $r_\alpha^{q_n}$. We see that a_{n+1} is large if the angle of $r_\alpha^{q_n}$ is small compared to the angle of $r_\alpha^{q_{n-1}}$. Because

$$r_\alpha^{a_{n+1} \cdot q_n}(y) = r_\alpha^{a_{n+1} \cdot q_n + q_{n-1}}(x)$$

  we define $q_{n+1} = a_{n+1} q_n + q_{n-1}$.
  One easily checks that the iterated $r^{q_n}(x)$ define configurations described by Figure 1.

To sum up, we have defined sequences $(a_n)_{n \in \mathbb{N}^*}$ and $(q_n)_{n \in \mathbb{N}}$ having the following properties.

[1]Actually, one can prove that $a_2 \cdot q_1 + 1$ can be defined by the property that $r_\alpha^{a_2 \cdot q_1 + 1}$ is closer to the identity than any other r_α^k for $k \leq a_2 \cdot q_1 + 1$.

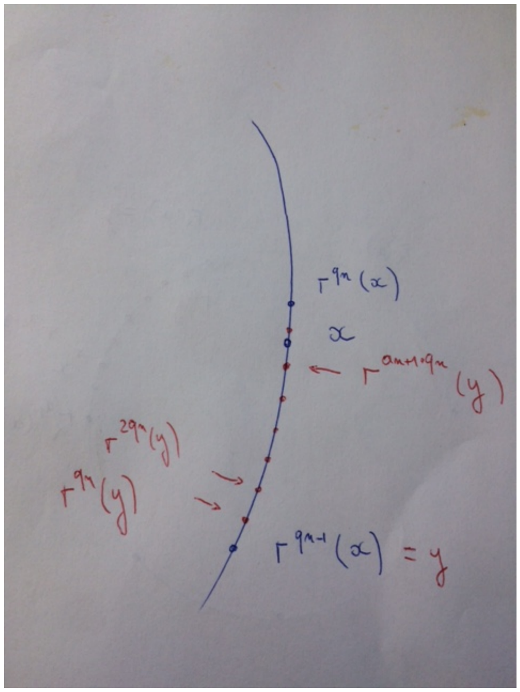

FIGURE 3. Combinatorial configuration of the orbit, step n

(1) $r_\alpha^{q_n}$ is a "best" approximation of the identity, in the sense that it is closer to the identity that any shorter iterate of r_α.
(2) a_{n+1} quantifies how good the n-th approximation is compared to the $(n-1)$-th.
(3) The sequence (q_n) is completely determined by (a_n).
(4) One can completely reconstructs the combinatorics of an orbit of r_α from the a_ns. By combinatorics of an orbit we mean the way it distributes itself on the circle with respect to the cyclic order of S^1.

The rational case. In the discussion above we have implicitly ignored the case where α is rational; $\alpha = \frac{p}{q}$. This case causes the algorithm to stop. What happens is that there exists n such that $q_n = q$ and therefore $r^{q_n}(x) = x$ which makes it impossible to continue the algorithm.

PROPOSITION 18. *The sequence (a_n) and (q_n) defined dynamically for r_α are the same as those defined by the continued fraction algorithm for α.*

PROOF. Exercise. □

3. Periodic orbits

From now onwards all circle homeomorphisms will be assumed to be orientating preserving (unless explicitely mentioned).

PROPOSITION 19. *Assume that T has a periodic whose combinatorics is that of the rational number $\frac{p}{q}$. Then*

- *all periodic orbits are of type $\frac{p}{q}$;*
- *the ω-limit of any point is a periodic orbit;*
- *the α-limit of any point is a periodic orbit.*

PROOF. We start with the case $\frac{p}{q}=1$, in other words T has a fixed point. T therefore induces an homeomorphism of $[0,1]$ fixing both 0 and 1. We show now that for any such homeomorphism f, the ω/α-limit of a point is a fixed point. Consider x which is not a fixed point of x. Consider the largest open interval (a,b) containing x such that $f(y)\neq y$ on (a,b). By continuity of f, we have that $f(a)=a$ and $f(b)=b$. Thus $f\big((a,b)\big)=(a,b)$. By definition of f, $f(y)-y$ is of constant sign on (a,b) and thus the sequence $f^n(x)$ is either strictly increasing or decreasing. It therefore has a limit which is a fixed point, which proves that the ω/α-limit of x must consist of a single fixed point.

In the general case, we apply the above discussion to T^q whose every orbit has for ω/α-limit a single fixed point. This implies that the ω/α-limit of any point for T is a periodic orbit of order at most q. But it cannot be the case that T has a periodic orbit of order less than q, otherwise every single periodic orbit would be of order at most this number (by the same reasoning) and thus negating the existence of a periodic orbit of period q. Finally, the fact that these periodic orbits have same combinatorics (meaning that the cyclic order induced on $\{0,1,\cdots q-1\}$ by that of the circle on the orbit) is the same is easily deduced from the fact that T is order-preserving.

□

4. Rotation number

4.1. Definition. We are now going to define a quantity attached to any (orientation-preserving) circle homeomorphism called the **rotation number**. We consider $T:S^1\longrightarrow S^1$ a circle homeomorphism. For any point x whose orbit under T is not periodic, the cyclic ordering of S^1 induces a cyclic ordering on $\mathbb{Z}$ defined by

$$n_1\leq n_2\leq n_3\leq n_1 \Leftrightarrow T^{n_1}(x)\leq T^{n_2}(x)\leq T^{n_3}(x)\leq T^{n_1}(x).$$

LEMMA 20. *Assume that T does not have a periodic orbit. Then the cyclic ordering on $\mathbb{Z}$ induced by the action of T on orbits does not depend on the choice of an initial point.*

PROOF. Assume it is not the case, *i.e.* there are two points x and y and two different integers k and l such that the triples $(x,T^k(x),T^l(x))$ and $(y,T^k(y),T^l(y))$ do not induce the same cyclic ordering. We claim that this is equivalent to the fact that among the three intervals $[x,y]$, $[T^k(x),T^k(y]$, $[T^l(x),T^l(y)]$, one is contained within one of the two others. But this implies the existence of a periodic orbit for T (because any map of an interval to itself has a fixed point).

□

Case where T has a periodic orbit. We suppose that T has a periodic orbit, of order q. Let x be a periodic point for this periodic point. Define p to be the number of times $x,T(x),T^2(x),\cdots,T^q(x)=x$ goes around S^1. We define the rotation of $\rho(T)$ to be

$$\rho(T):=\frac{p}{q}.$$

Proposition 19 implies that

- $\rho(T)$ is well-defined *i.e.* the ratio $\frac{p}{q}$ does not depend on the choice of a periodic orbit.
- p and q given by this construction are necessarily co-prime.
- $\rho(r_{\frac{p}{q}})=\frac{p}{q}$ (recall that $r_{\frac{p}{q}}$ is the rotation of angle $\frac{p}{q}$).

Case where T does not have a periodic orbit. In that case we define the rotation number using the algorithm that we used to define continued fractions from the dynamical viewpoint. Consider an arbitrary $x \in S^1$. We can run the same algorithm that we used with rotation to define a sequence $a_1, \cdots, a_n, \cdots$. The are two things that we need to check

(1) The algorithm does not stop.
(2) Each step of the algorithm (which consist in iterating an iterate of T until a certain orbit gets past a certain point) can be carried out.

The algorithm stop if and only if there exists $n \in \mathbb{N}^*$ such that $T^{q_n}(x) = x$. We have assumed that T does not have periodic orbits so this case can be ruled out.
We now show that there is no problem carrying out a step of the algorithm. Assume that we have constructed the a_is and q_is up to $i = n$. To construct a_{n+1}, we iterate T^{q_n} starting from $y = T^{q_{n-1}}(x)$ until it gets past x. For a_{n+1} not to be defined, we should have that all the iterates $ty, T^{q_n}(y), T^{2q_n}(y), \cdots, T^{kq_n}(y), \cdots$ remain within the interval $[x, y]$. Because this sequence is increasing (or decreasing depending on the parity of n), it would have an accumulation point z which would satisfy $T^{q_n}(z) = z$ thus contradicting the fact that T does not have a periodic orbit.

We define the rotation number of T to be the sole irrational number with the continued fraction expansion $[a_1, \cdots, a_n, \cdots]$

$$\rho(T) := \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}}$$

Remark 21. We could also have defined rational rotation number using the continued fraction algorithm. As explained above, the only thing that can cause the algorithm to stop is a periodic orbit of combinatorics corresponding to the rational number defined by the finite number of a_is defined up to the moment where the algorithm stops.

The rotation number thus defined is a number in $[0, 1[$. It will also be convenient to think of it as an element of $S^1 = \mathbb{R}/\mathbb{Z}$.

4.2. Semi-conjugacy and conjugacy. We introduce in this paragraph a notion which weakens slightly that of conjugacy.

Definition 7. *Let T_1 and T_2 two circle homeomorphisms. A map $\varphi : S^1 \longrightarrow S^1$ is called a (continuous)* ***semi-conjugacy*** *between T_1 and T_2 if*

(1) any lift of φ to $\mathbb{R}$ is (not necessarily strictly) increasing and continuous;
(2) we have $T_1 \circ \varphi = \varphi \circ T_2$.

We will only consider **continuous** semi-conjugacies and conjugacies; we will therefore omit the adjective in the sequel.
Note that semi-conjugacy does not definition an equivalence relation on circle homeomorphisms for a lack of reflexivity. We therefore do not say that T_1 and T_2 are semi-conjugate but that T_1 is semi-conjugate to T_2.
A instance of non-trivial semi-conjugacy is one between an irrational rotation and a blow-up of it (see 2.3). One can check that the map (which can thought of as an inverse to the blow-up) which maps every blown-up interval to the initial point of S^1 that had been blown-up initially is a semi-conjugacy between the blow-up and the initial rotation. In this case, the semi-conjugacy is locally constant on the union of the interiors of the blown-up intervals; the

only points at which it is strictly increasing is the Cantor set which is the complement of the union of the interiors of the blown-up intervals.

The particular case of semi-conjugation to an irrational rotation. Assume that T is (strictly) semi-conjugate to a rotation. We describe here more precisely the dynamics of such a map T. There exists an irrational α and $\varphi : S^1 \longrightarrow S^1$ increasing such that

$$\varphi \circ T = r_\alpha \circ \varphi.$$

Because φ is a strict semi-conjugation there exists $x \in S^1$ such that $\varphi^{-1}(\{x\})$ is a closed connected interval, we denote by I its interior. α is irrational; this implies that the iterates of x under r_α are disjoint which in turns implies that the iterates of I under T for a collection of disjoint intervals. In other words, I is a *wandering interval.* By definition, a wandering interval I is a connected open interval such that its iterates under T are disjoint and which is maximal for these properties.

PROPOSITION 22. *Let $\alpha \in [0, 1[$ be an irrational number and T be a strictly semi-conjugate to r_α. Let $\mathcal{C} \subset S^1$ be the complement of all wandering intervals. The following hold*

(1) $\mathcal{C}$ is T-invariant;
(2) $\mathcal{C}$ is a Cantor set;
(3) The ω-limit and α-limit of any point under T is equal to $\mathcal{C}$.

PROOF. Exercise. □

This Proposition tells us that a circle homeomorphism semi-conjugate to an irrational rotation is obtained (up to actual conjugation) by blowing-ups orbits of the irrational rotation.

4.3. Poincaré's Theorem.

In this section we prove the following Theorem

THEOREM 23 (Poincaré). *There exists a function $\rho : \mathrm{Homeo}(S^1) \longrightarrow S^1$ satisfying the following properties:*

(1) If $\rho(T) = 0$ then T has a fixed point.
(2) If $\rho(T) = \frac{p}{q}$ with p and q co-prime then T has a periodic of order q.
(3) If $\rho(T) = \alpha$, then T is semi-conjugate to the rotation of angle α.
(4) For any $T \in \mathrm{Homeo}(S^1)$ and $n \in \mathbb{Z}$, $\rho(T^n) = n\rho(T)$.

PROOF. (1) and (2) are given by definition of the rotation number. We prove (3).

Consider any point $x \in S^1$. By definition of the rotation number, the orbit of x under T induces the cyclic ordering on $\mathbb{Z}$ than that induced by the rotation of angle $\alpha = \rho(T)$ on the orbit of $0 \in S^1$. A semi-conjugacy φ mapping x to 0 will have to satisfy

$$\forall k \in \mathbb{Z},\ \varphi(T^k(x)) = r_\alpha^k(x) = \{k \times \alpha\}.$$

A map φ defined using the above formula comes close to defining a semi-conjugation:

- it respects the cyclic ordering *i.e.* it is increasing;
- its image is dense in S^1.

Because φ is increasing, it extends to a continuous increasing map

$$\varphi : \overline{\mathcal{O}_T(x)} \longrightarrow S^1 = \overline{\mathcal{O}_{r_\alpha}(0)}$$

satisfying $\varphi \circ T = r_\alpha \circ \varphi$ on $\overline{\mathcal{O}_T(x)}$. Now, it might happen that $\overline{\mathcal{O}_T(x)}$ is not the entire circle. It is typically the case when one takes x in the invariant Cantor set of a blow-up example.

What we actually expect is that $\overline{\mathcal{O}_T(x)}$ only misses most of the wandering intervals (except maybe countably many points within one, if x had been chosen to belong to one). $S^1 \setminus \overline{\mathcal{O}_T(x)}$ is a union of disjoint open connected intervals. Consider I such an interval. The only way we can extend φ to an increasing map to I is if φ takes the same value on both ends of the intervals. It is indeed the case as φ is increasing and has dense image: if the images at the ends of I were different, the image of φ would miss the entire interval between these two values. φ thus extends to each interval of $S^1 \setminus \overline{\mathcal{O}_T(x)}$ by making it constant equal to its value on the boundary of I.

□

5. The more classical definition of the rotation number

In the literature one will often find a different definition of the rotation number than that that was first given in these notes. This other definition, that we give in this paragraph, is somewhat more compact, does not require the introduction of the continued fraction algorithm or any form of combinatorial analysis of the orbital structure of circle homeomorphism. One the other hand, one could argue that it is more cryptic, and does not give any hint at why the produced number should be a complete combinatorial invariant. It will nonetheless prove to be quite useful to give quick proof of certain statements later on.

Consider T a circle homeomorphism and let $\tilde{T} : \mathbb{R} \longrightarrow \mathbb{R}$ be any lift of T.

Definition 8. *The rotation number of T is by definition the following limit*

$$\lim_{n \to +\infty} \frac{\tilde{T}^n(x)}{n} \mod 1$$

for any $x \in \mathbb{R}$.

This definition hides the following two non-trivial statements:

- the limit in the definition exists;
- it does not depend of the choice of a point x.

The notion of rotation thus defined agrees with that of paragraph 4. As a historical note, it was this definition of the rotation number that was initially given by Poincaré.

Exercise 11. *Prove that the notions of rotation number defined here and in Section 4 are actually the same.*

6. Consequences for flows on the torus

Recall that we had proven that a flow $\vec{X}$ on the torus $\mathbb{T}^2$ without critical points satisfies the following dichotomy

- either it has a periodic orbit and in which case the ω/α-limit of any point is a periodic orbit;
- or it can be realised as the suspension of a circle diffeomorphism.

We focus on the later case. The ω/α-limit of a point for $\vec{X}$ is exactly the union of the orbits of the corresponding ω/α-limit on the circle. As a consequence of Theorem 23 and Proposition 22 we get the following

Theorem 24. *Let $\vec{X}$ be a $\mathcal{C}^1$ vector field on $\mathbb{T}^2$ without critical point. The one of the following, mutually exclusive, three statements holds true.*

(1) The ω/α-limit of any point is a periodic orbit.
(2) There exists a minimal set $\mathcal{C}$, which is locally the product of a Cantor set with an interval, which is the ω/α-limit of any point.
(3) $\vec{X}$ is minimal.

Exercises

EXERCISE 12. *Prove Proposition 16.*

EXERCISE 13. *Prove Proposition 22.*

CHAPTER 4

Parametrised families

An important question in the theory of dynamical systems is the following:

QUESTION. *Let $\mathcal{M}$ be a family of dynamical systems of the same type (say circle diffeomorphisms of class $\mathcal{C}^{26}$ for instance). By family we mean a topological space $\mathcal{M}$ (possibly with extra structure: it can be a smooth finite dimensional manifold, maybe with a nice measure) and a map*

$$\psi : \mathcal{M} \longrightarrow \mathrm{Diff}^{26}(S^1).$$

$\mathcal{M}$ is the space of parameters. The question we ask is the following: ***what is the "typical" dynamical behaviour of an element in $\mathcal{M}$?*** *Typical in this context can traditionally mean two things:*

- *a property is typical if it holds true for a dense G_δ subset of $\mathcal{M}$;*
- *or a property is typical if it holds true for a full measure set of parameters in $\mathcal{M}$.*

Poincaré-Bendixson theorem is an answer to this meta-question in the case of flows on the sphere. In that case, every single dynamical system of the considered class displays a simple behaviour; every orbit is attracted to a periodic orbit or a closed cycle.

1. The function rotation number

PROPOSITION 25. *The map*

$$\rho : \mathrm{Homeo}(S^1) \longrightarrow S^1$$

which associates to a circle homeomorphism its rotation number is continuous.

To prove this Proposition, we need the following lemma:

LEMMA 26. *Let T be a circle homeomorphism and let $\tilde{T}$ be a lift of T. Assume that $\tilde{T}^n(0) \in [p, p+1]$. Then the rotation number $\rho(T)$ belongs to the interval $[\frac{p}{n}, \frac{p+1}{n}]$ modulo 1.*

This lemma says in substance that the convergence of the limit

$$\lim_{n\to\infty} \frac{\tilde{T}^n(0)}{n}$$

to the rotation number of T is uniform.

PROOF. Assume that $\tilde{T}^n(0) \in [p, p+1]$. Using the fact that $\tilde{T}$ is 1-periodic, we get by induction that for all $k \geq 0$

$$kp \leq \tilde{T}^{kn}(0) \leq k(p+1).$$

Dividing by kn and passing to the limit gives that $\rho(T) \in [\frac{p}{n}, \frac{p+1}{n}]$.

□

We can now move to the proof of Proposition 25

PROOF OF PROPOSITION 25. Consider an open set $U \in S^1$ and T_0 in $\rho^{-1}(U)$. Because U is open there exists n such that $(\rho(T_0) - \frac{4}{n}, \rho(T_0) + \frac{4}{n}) \subset U$. By applying Lemma 1 to T_0 we get that $\tilde{T_0}^n(0)$ belongs to the interval $(\rho(T_0) - \frac{1}{n}, \rho(T_0) + \frac{1}{n})$. Now the map

$$T \longmapsto \frac{\tilde{T}^n(0)}{n}$$

being continuous, we get that there is an open set around T such that $\rho(T) \in [\rho(T_0) - \frac{2}{n}, \rho(T_0) + \frac{2}{n}] \subset (\rho(T_0) - \frac{4}{n}, \rho(T_0) + \frac{4}{n}) \subset U$.

□

2. One-parameter families

Let T_0 be an orientation-preserving circle homeomorphism.

PROPOSITION 27. *The map*

$$\psi := t \in S^1 \longmapsto \rho(r_t \circ T_0) \in S^1$$

is increasing and it is strictly increasing at any t such that $\rho(r_t \circ T_0)$ is irrational.

PROOF. There are two points in this Proposition: proving the monotonicity of ψ and proving the strict monotonicity at irrational points.

The monotonicity comes from the following fact. Let T_0 be a circle homeomoprhism and consider a small ϵ. Let T_ϵ be $x \mapsto T_0(x) + \epsilon$. If $\tilde{T_0}$ is a lift of T_0, then $\tilde{T_\epsilon} = \tilde{T_0} + \epsilon$ is a lift of T_ϵ. Now one easily shows by induction that for all $n \in \mathbb{N}$ we have

$$\tilde{T_\epsilon}^n(0) \geq \tilde{T_0}^n(0) + \epsilon$$

which implies that $\rho(T_\epsilon) \geq \rho(T_0)$.

The second point is more delicate. Assume T_0 has irrational rotation number. We only need to show that $\rho(T_\epsilon) \neq \rho(T_0)$ for any small $\epsilon > 0$.
Consider n such that $T_0^{q_n}(0)$ is at distance less than $\frac{\epsilon}{2}$ and such that $T_0^{q_n}(0)$ is approaching zero "from below". Because we have $\tilde{T_\epsilon}^{q_n}(0) \geq \tilde{T_0}^{q_n}(0) + \epsilon$, $T_\epsilon^{q_n}(0)$ is going to be place on the other side of 0 compared to $T_0^{q_n}(0)$. This implies that the cyclic order on the orbit of 0 under T_0 is different from that of T_ϵ which in turns implies that their rotation numbers differ.

□

3. Morse-Smale diffeomorphisms

DEFINITION 9 (Morse-Smale). *A circle diffeomorphism T (of class at least $\mathcal{C}^1$) is called Morse-Smale if the following hold*

(1) it has finitely many periodic orbits;
(2) all its periodic orbits are either attracting or repelling i.e. for any x and k such that $T^k(x) = x$, $(T^k)'(x) \neq 1$;
(3) for every point $x \in S^1$ whose orbit is not periodic, $\omega(x)$ is an attracting periodic orbit and $\alpha(x)$ is a repelling one.

It turns out that the three conditions in the definition above are redundant; the second one can be shown to imply the two others. We still insist on giving them, as it is the simple dynamical behaviour that they imply altogether which is the motivation for the definition of the concept of *Morse-Smale* diffeomorphism.

THEOREM 28. *The set* $\mathcal{MS} \subset \mathrm{Diff}^1_+(S^1)$ *is open and dense.*

PROOF. **Proof of the openness** Consider a map T that is Morse-Smale. It has k periodic orbits that are either attracting or repelling.

- There exists an open neighbourhood of T in $\mathrm{Diff}^1_+(S^1)$ for which those k attracting/periodic orbit survive. This is an application of Exercise 14 to T^q where q is the order of a periodic orbit.
- The set

$$\mathcal{P} := \{T \in \mathrm{Diff}^1_+(S^1) \mid \exists x \in S^1 \text{ such that } T(x) = x \text{ and } T'x) = 1\}$$

  is closed.
- The intersection of $\mathcal{U}$ with the complement of $\mathcal{P}$ in $\mathrm{Diff}^1_+(S^1)$ is an open neighbourhood consisting of Morse-Smale circle diffeomorphisms.

Proof of the density Consider first a circle diffeomorphism T with rational rotation number. We show that T can be approximated in the $\mathcal{C}^1$-topology by diffeomorphism with finitely many periodic orbits.
Fix $\epsilon > 0$. S^1 can be partitioned into $S^1 = [x_0, , x_1] \cup [x_1, x_2] \cup \cdots [x_n, x_0]$ such that for all k x_k is a periodic point and that we have

- either (x_k, x_{k+1}) does not contain any periodic point;
- or for all $x \in (x_k, x_{k+1})$, $|T'(x_k) - T'x)| < \epsilon$.

(We leave it to the reader as an exercise that such a partition can be constructed) To simplify the discussion we assume that periodic orbits have order 1 *i.e.* are fixed points. Consider an interval $[x_k, x_{k+1}]$ which contains at least on fixed point y in its interior. Because of the choice we have made, the derivative on $[x_k, x_{k+1}]$ is very close to 1. We can therefore make a small perturbation of T by adding a $\mu : [x_k, x_{k+1}] \longrightarrow \mathbb{R}$ for instance such that

- $||\mu'||$ is very small;
- $\mu(x_k) = \mu(x_{k+1}) = 0$;
- $\mu'(x_k) = \mu'(x_{k+1} = 0$;
- $\mu(y) = 0$;
- y is the only fixed point of $T + \mu$ in (x_k, x_{k+1}).

See below the graph of such a perturbation.
By proceeding this way on each interval $[x_k, x_{k+1}]$, we can find an arbitrarily small $\mathcal{C}^1$ perturbation of T which has only finitely many fixed points. Amongst those fixed points, some might have derivative equal to 1. We leave it as an exercise to show that one can find a small perturbation close to such a fixed point which makes the derivative at the fixed point different from and creating at most one other fixed point whose multiplier is different from 1 as well. This proves that a circle diffeomorphism T with rotation number number equal to 0 can be approximated by Morse-Smale circle diffeomorphisms in the $\mathcal{C}^1$-topology. The same proof works for T with rational rotation number, by replacing fixed points with periodic orbits.

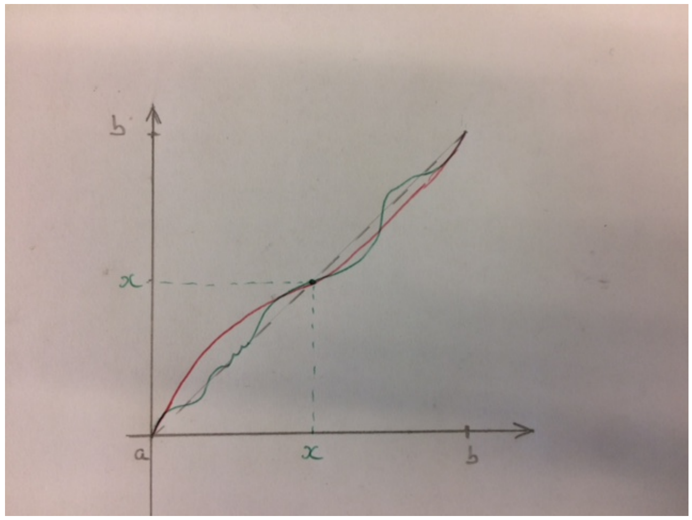

FIGURE 1. In green the graph of the original T and in red its perturbation with only three fixed points.

By Proposition 27, T with irrational rotation number can be approximated by T with rational rotation number (by considering the family $(T+\alpha)_{\alpha\in S^1}$ for instance) thus T can be approximated by Morse-Smale circle diffeomorphisms. This concludes the proof of the theorem.

□

EXERCISE 14. *Consider* $f_0 \in \mathrm{Diff}^1_+([0,1])$ *such that there exists* $x_0 \in (0,1)$ *such that* $f(x_0) = x_0$ *and* $f'(x_0) \neq 1$.

(1) *Show that there exists a continuous function* $x : \mathcal{U} \subset \mathrm{Diff}^1_+([0,1]) \longrightarrow (0,1)$ *where* $\mathcal{U}$ *is a neighbourhood of* $\mathcal{U}$ *such that*

$$\forall f \in \mathcal{U},\ f(x(f)) = x(f) \text{ and } f'(x(f)) \neq 1.$$

(2) *Show that this map is unique.*

EXERCISE 15. *Let* $\mathcal{MS}_k$ *the set of Morse-Smale diffeomorphisms with at most* k *periodic orbits.*

(1) *Show that* $\mathcal{MS}_k$ *is non-empty if and only if* k *is even.*
(2) *Show that for any* k, $\mathcal{MS}_k$ *is not dense.*
(3) *Show that for all* k *even, the closure of* $\mathcal{MS}_k$ *is contains* $\rho^{-1}(S^1 \setminus \mathbb{Q})$.

CHAPTER 5

Ergodic theory and Denjoy theorem

In this chapter we discuss the ergodic theory of circle homeomorphisms. In a nutshell, because circle homeomorphisms preserves the ordering of the circle, they enjoy very strong equirepartition properties.
When working with a relatively regular circle diffeomorphism T, these ergodic-theoretic considerations can be applied to the observable $\log \mathrm{D}T$ to improve on the regularity of semi-conjugation. Precisely, we get Denjoy's theorem that asserts that a class $\mathcal{C}^2$ circle diffeomorphism with irrational rotation number is actually topologically conjugate to the associate rotation (instead of only semi-conjugate in the case of circle homeomorphisms).

1. Invariant measures

PROPOSITION 29. *The rotation of angle α is uniquely ergodic for any α irrational.*

PROOF. R_α is the rotation of angle α. Consider functions of the form

$$f_k := x \mapsto e^{2\pi i k x}.$$

Consider the Birkhoff sums of f_k for an arbitrary point x_0

$$S_n(f_k)(x_0) = \sum_{j=0}^{n-1} e^{2\pi i k(x_0 + j\alpha)} = e^{2\pi i k x_0} \sum_{j=0}^{n-1} (e^{2\pi i k\alpha})^j = e^{2\pi i k x_0} \cdot \frac{(e^{2\pi i k\alpha})^n - e^{2\pi i k\alpha}}{e^{2\pi i k\alpha} - 1}.$$

In particular for any n

$$|S_n(f_k)(x_0)| \leq \frac{2}{|e^{2\pi i k\alpha} - 1|}$$

and consequently for any $x_0 \in S^1$ and any

$$\lim_{n\to+\infty} \frac{S_n(f_k)(x_0)}{n} = 0 = \int_{S^1} f_k d\mathrm{Leb}.$$

The vector space spanned by the $(f_k)_{k\geq 1}$ is dense (with respect to the topology of the uniform convergence) in the subspace of continuous functions which are of mean zero with respect to the Lebesgue measure. We thus get that for any $f \in \mathcal{C}^0(S^1, \mathbb{R})$ such that $\int_{S^1} f d\mathrm{Leb} = 0$ and for any $x_0 \in S^1$

$$\lim_{n\to+\infty} \frac{S_n(f)(x_0)}{n} = 0 = \int_{S^1} f_k d\mathrm{Leb}.$$

This implies that R_α is uniquely ergodic and its unique invariant measure is the Lebesgue measure.

□

PROPOSITION 30. *Let T be a circle homeomorphism with irrational rotation number α. Then it is uniquely ergodic and its unique invariant measure μ supported by its unique invariant minimal Cantor set. Moreover, the image of μ by the semi-conjugacy to R_α is the Lebesgue measure.*

PROOF. We know that T is semi-conjugate to R_α via an increasing map φ. Let $\tilde{\varphi} : \mathbb{R} \longrightarrow \mathbb{R}$ be a lift of φ.

- $\tilde{\varphi}$ is continuous and increasing.
- For all $x \in \mathbb{R}$, $\tilde{\varphi}(x+1) = \tilde{\varphi}(x) + 1$.

For any continuous increasing function, there exists a unique atom-free measure μ such that for all x_0 and x in $\mathbb{R}$

$$\varphi(x) = \varphi(x_0) + \int_{x_0}^{x} d\mu.$$

Because for all $x \in \mathbb{R}$, $\tilde{\varphi}(x+1) = \tilde{\varphi}(x) + 1$, $\mu([x, x+1]) = 1$ for any x.
We claim that μ is invariant under T. This is a consequence of the following facts:

- by definition, $\varphi_* \mu = \text{Leb}$;
- the Lebesgue measure is invariant by $\mathcal{R}_\alpha$.

Let A be any measurable subset of S^1. By definition,

$$\mu(T(A)) = \text{Leb}(\varphi(T(A))) = \text{Leb}(R_\alpha(\varphi(A)) = \text{Leb}(\varphi(A)) = \mu(A)$$

which proves that μ is invariant.
Remains to prove uniqueness. Let μ' be a T-invariant probability measure. Because $\varphi \circ T = R_\alpha \circ \varphi$, $\varphi_* \mu'$ is R_α-invariant and by unique ergodicity of R_α is equal to the Lebesgue measure. For any measurable set A,

$\mu'(A) = \text{Leb}(\varphi(A)) = \mu(A)$ which implies that $\mu' = \mu$; this proves uniqueness of the invariant measure.

□

2. The cohomological equation for rotations

2.1. Ergodic averages. We begin with a general discussion on the ergodic theory of dynamical systems. The Ergodic Theorem states that temporal averages of observables converge to spatial averages for extremal invariant measures. It can be rephrased the following way

THEOREM 31 (Ergodic Theorem). *Let μ be a T-invariant ergodic measure. For μ-almost every x the following holds. For any observable $f \in \mathcal{C}^0(X, \mathbb{R})$, we have*

$$\lim_{n \to +\infty} \frac{1}{n} S_n(f)(x) = \lim_{n \to +\infty} \frac{1}{n} \sum_{0}^{n-1} f \circ T^k(x) = \int_X f d\mu$$

or in other words

$$S_n(f)(x) - n \cdot \int_X f d\mu = o(n).$$

This can be summed up the following way:for an ergodic measure spatial averages are equal to temporal averages. One might wonder whether it is possible to get estimates on the rate of convergence in this Theorem *i.e.* try and find estimates of the $o(n)$ featuring in the second formulation of the Theorem. The smaller it is, the "more ergodic" the transformation is. Of course, we are looking for estimates which are somewhat independent of the choice of the **observable** f. [1]

DEFINITION 10. *Let $T : X \longrightarrow X$ be a continuous transformation of a compact topological space. An observable is called a continuous coboundary if there exists a continuous observable φ such that*

$$f = \varphi \circ T - \varphi.$$

Continuous coboundaries are a remarkable class of observables because their Birkhoff sums $S_n(f)(x)$ are uniformly bounded (independently of the point x). Indeed for *any* $x \in X$ (and NOT almost every with respect to a given measure)

$$S_n(f)(x) = \sum_{i=0}^{n-1} f \circ T^i(x) = \sum_{i=0}^{n-1} \varphi \circ T^{i+1}(x) - \varphi \circ T^i(x) = \varphi \circ T^n(x) - \varphi(x)$$

and thus $|S_n(f)(x)| \leq 2||\varphi||_0$.
This shows that being a coboundary has strong implications: not only does it forces to have mean zero with respect to any invariant measure; it also implies that Birkhoff sums are uniformly bounded. *A priori*, there is no reason why a given continuous function of zero average should be a coboundary.

2.2. The cohomological equation for rotations. In this paragraph we consider the following problem. Let $f : S^1 \longrightarrow \mathbb{R}$ be a smooth observable. What are the conditions on f for it to be a coboundary for the rotation of angle α? We will see that an answer to this question can be given using Fourier analysis.
This is going to the first time where we are going to have to make an assumption of *arithmetic* nature on the angle α.

We consider $f \in \mathcal{C}^r(S^1, \mathbb{R})$ and we look for a continuous φ such that

$$f = \varphi \circ R_\alpha - \varphi.$$

Assume such an observable φ exists and write $f = \sum a_n e^{i2\pi nx}$ and $\varphi = \sum b_n e^{i2\pi nx}$ their Fourier series. We have

$$\varphi \circ R_\alpha = \sum (b_n e^{i2\pi n\alpha}) \cdot e^{i2\pi nx}$$

and we thus get that were such an observable φ to exist, its Fourier coefficients should satisfy

$$\forall n \in \mathbb{Z},\ b_n \cdot (e^{i2\pi n\alpha} - 1) = a_n.$$

First thing we notice (but we alredy knew it) is that it forces $\int_{S^1} f d\text{Leb} = a_0 = 0$.

[1] "Observable" is another way to call a function $X \longrightarrow \mathbb{R}$ in the context of ergodic theory. It underlines the fact that we think of this function as a "test" function for which we compute temporal averages.

THEOREM 32. *Assume that* α *satisfies the following arithmetic assumption: there exists* $C > 0$ *such that for any rational* $\frac{p}{q}$, $|\alpha - \frac{p}{q}| \geq \frac{C}{q^3}$. *Then for any observable* $f \in \mathcal{C}^\infty(S^1, \mathbb{R})$ *there exists a unique* $\varphi \in \mathcal{C}^\infty(S^1, \mathbb{R})$ *such that*

$$f = \int_{S^1} f d\mathrm{Leb} + \varphi \circ R_\alpha - \varphi.$$

Before proving this theorem, we make two important comments.

(1) The hypothesis on α is *generic*, meaning that it is satisfied for a subset of full measure of $\alpha \in [0, 1)$. This hypothesis is one about rational approximations of α and says in substance that α is not too well approximated by rational numbers.
(2) The fact that the observable f has been chosen of class $\mathcal{C}^\infty$ is important, however there exists versions of this theorem in lower/higher regularity. We will not prove that here, but the lowest regularity for which versions of this theorem can hold is $\mathcal{C}^r$ for any $r > 1$.

To prove this Theorem we are going to need the following Lemma:

LEMMA 33. *Let* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *a* 1*-periodic continuous function and let* $(a_n)_{n\in\mathbb{Z}}$ *its Fourier coefficients.* f *is of class* C^∞ *if and only if* $|a_n| = o(\frac{1}{n^k})$ *for all* $k \in \mathbb{N}^*$.
Conversely, if $(a_n)_{n\in\mathbb{Z}}$ *is a sequence of real number such that* $|a_n| = o(\frac{1}{n^k})$ *for all* $k \in \mathbb{N}^*$ *then the series*

$$\sum a_n e^{i2\pi nx}$$

is absolutely convergent and defines a $\mathcal{C}^\infty$, 1*-periodic function.*

PROOF. Exercise.

□

PROOF OF THEOREM 32. Let $f(x) = \sum_{n\in\mathbb{Z}} a_n e^{i2\pi nx}$ be the Fourier expansion of f. We are going to show that the formal solution to the cohomological equation $\varphi(x) = \sum_{n\in\mathbb{Z}^*} \frac{a_n}{e^{i2\pi n\alpha}-1} e^{i2\pi nx}$ actually defines a $\mathcal{C}^\infty$ function. $\mathcal{C}^\infty$, 1-periodic function are characterised by the fact decay of its Fourier coefficients (see Lemma 33). We therefore try to give an estimate of $e^{i2\pi n\alpha} - 1$.

$$|e^{i2\pi n\alpha} - 1|^2 = |\cos(2\pi n\alpha) - 1|^2 + |\sin(2\pi n\alpha)|^2 \geq |\cos(2\pi n\alpha) - 1|^2$$

In a neighbourhood of 0, we have $|\cos(2\pi x) - 1| \geq \frac{2\pi}{3}|x|^2$. In particular there exists a constant $\delta > 0$ such that

$$|e^{i2\pi n\alpha} - 1| \geq \min\left\{\sqrt{\frac{2\pi}{3}}\mathrm{d}(n\alpha, \mathbb{Z}), \delta\right\}.$$

By assumption, we have

$$\mathrm{d}(n\alpha, \mathbb{Z}) \geq \frac{C}{n^2}.$$

We consider the sequence $b_n = \frac{a_n}{e^{i2\pi n\alpha}-1}$. Because of the above estimate for n large enough we have

$$|b_n| \leq \frac{1}{C} \cdot n^2 |a_n|.$$

By assumption, f is $\mathcal{C}^\infty$ and thus by Lemma 33, we have that $|a_n| = o(\frac{1}{n^k})$ for all $k \in \mathbb{N}$. This in turn implies that

$$|b_n| = o(\frac{1}{n^k})$$

for all $k \in \mathbb{N}$. The formal Fourier series $\sum_{n\in\mathbb{Z}} a_n e^{i2\pi nx}$ thus defines a $\mathcal{C}^\infty$ observable φ such that

$$\varphi \circ R_\alpha - \varphi = f.$$

This completes the proof of Theorem 32.

□

This Theorem has an interesting consequence: any smooth observable of mean zero over a rotation of angle satisfying the arithmetic condition of Theorem 32 has its Birkhoff sums uniformly bounded.

2.3. On the arithmetic condition arising in Theorem 32. It is natural (and important) to wonder how restrictive is the assumption that for any rational $\frac{p}{q}$, $|\alpha - \frac{p}{q}| \geq \frac{C}{q^3}$ (main assumption in Theorem 32. We indicate the following Theorem whose proof is beyond the scope of these notes (although it is not too difficult).

Theorem 34. *The set*

$$\{\alpha \in \mathbb{R} \mid \exists C > 0 \text{ such that } \forall \frac{p}{q} \in \mathbb{Q},\ |\alpha - \frac{p}{q}| \geq \frac{C}{q^3}\}$$

has full Lebesgue measure.

3. The Denjoy-Koksma inequality

3.1. Bounded variation functions. We introduce a class of functions that is going to be of importance in the analysis of circle diffeomorphisms.

Definition 11. *A function $f \in \mathcal{C}^0(S^1, \mathbb{R})$ is said to have **bounded variation** if there exists a constant $K > 0$ such that for any ordered partition of S^1, $x_0 \leq x_1 \leq \cdots \leq x_n \leq x_{n+1} = x_0$ we have that*

$$\sum_{i=0}^{n} |f(x_{i+1}) - f(x_i)| < K.$$

*The **variation** of f, which we denote by $\mathrm{Var}(f)$ is defined to be the supremum over all partitions of the sum $\sum_{i=0}^{n} |f(x_{i+1}) - f(x_i)|$.*

One can check that differentiable functions and Lipschitz functions have bounded variation (see Exercises). Another important property is that the property of having bounded variation is invariant under pre-composition by homeomorphisms of S^1.

3.2. Denjoy-Koksma inequality. In Section 2 we showed that for *special rotations* (whose angles are badly approximable by rational numbers) a *smooth* function of mean zero is a coboundary which in particular implies that their Birkhoff sums are bounded. In this paragraph, we show a weaker result (boundedness at special time) but which is valid for

- ANY irrational rotation number;
- ANY circle homeomorphism with irrational number;
- for a class of observables which is much less regular than smooth (bounded variation).

THEOREM 35 (Denjoy-Koksma inequality). *Let T be a circle homeomorphism of irrational rotation number α and let $f : S^1 \longrightarrow \mathbb{R}$ an observable with bounded variation function. Let $(q_n)_{n\in\mathbb{N}^*}$ be the sequence of special times associated with α. Then the following holds:*

$$\forall n \geq 1,\ \forall x \in S^1,\ |\sum_{k=1}^{q_n} f(T^k(x)) - q\int_{S^1} f d\mu| \leq \mathrm{Var}(f)$$

where μ is unique T-invariant probability measure on S^1.

To prove this Theorem we are going to need the following Lemma

LEMMA 36. *Let α be an irrational number, R_α the rotation of angle α and $\frac{p}{q}$ an irrational approximation of α (i.e. $|\alpha - \frac{p}{q}| \leq \frac{1}{q^2}$). Denote by I_k the interval $[\frac{k}{q_n}, \frac{k+1}{q}]$. Then for any k, there is a unique $1 \leq j \leq q$ such that $R^j(0) \in I_k$.*

PROOF. Exercise.

□

PROOF OF THE DENJOY-KOKSMA INEQUALITY. Let μ the invariant measure by T. The map

$$\varphi : y \mapsto \int_x^y d\mu$$

defines a semi-conjugacy from T to R_α which maps x to 0 (see proof of Proposition 30). We therefore have

$$\varphi \circ T = R_\alpha \circ T.$$

Consider any sequence $(y_k)_{0\leq k\leq q-1}$ such that $\varphi(y_k) = \{\frac{k}{q}\}$ and let J_k be the interval $[y_k, y_{k+1}]$. We have the following two properties.

(1) For all k, $\mu(J_k) = \frac{1}{q}$.
(2) For any k, there is a unique $1 \leq j_k \leq q$ such that $T^j(x) \in J_k$. This is an application of Lemma 36 to $\varphi(x) = 0$ and the fact that φ is order-preserving.

For any k, there exists $x_k \in J_k$ such that $f(x_k) = \int_{J_k} f d\mu$ (because f is continuous). This way we get

$$\sum_{j=1}^{n} f(T^j((x) - q\int_{S^1} f d\mu = q\sum_{k=1}^{q} \frac{1}{q}(f(T^{j_k}(x)) - f(x_k)$$

and thus

$$|\sum_{j=1}^{n} f(T^j((x) - q\int_{S^1} f d\mu| \leq |\sum_{k=1}^{q} \mathrm{Var}(f_{|J_k})| \leq \mathrm{Var}(f).$$

□

4. Denjoy theorem

In this Section we prove that a $\mathcal{C}^2$ circle diffeomorphism of irrational rotation number is always conjugate to the associated rotation.

4.1. The logarithm of the derivative. Many proof in smooth one-dimensional dynamics involve the logarithm of the derivative. The main reason for that is that the logarithm of the derivative of T^n can be written as a Birkhoff sum of the logarithm of the derivative of T. Precisely

LEMMA 37. *Let $T : S^1 \longrightarrow S^1$ be a circle diffeomorphism. For any $n \in \mathbb{N}$ and $x \in S^1$ we have the following equality*

$$\log \mathrm{D}(T^n)(x) = \sum_{i=0}^{n-1} (\log \mathrm{D}T(T^i(x))$$

PROOF. Exercise.

□

We give an immediate application of Lemma 37.

PROPOSITION 38. *Let T be a circle diffeomorphism whose derivative is continuous. Assume the rotation number of T is irrational. Denote by μ the unique invariant measure of T. Then*

$$\int_{S^1} \log \mathrm{D}T d\mu = 0.$$

PROOF. Assume by contradiction that $\int_{S^1} \log \mathrm{D}T d\mu \neq 0$. Up to considering T^{-1}) we can assume that $\int_{S^1} \log \mathrm{D}T d\mu = a > 0$. The ergodic theorem implies that the sequence of functions

$$\sum_{i=0}^{n-1} (\log \mathrm{D}T(T^i(x))$$

converges uniformly to the constant function a. In particular there exists n_0 such that for all $n \geq n_0$, $\sum_{i=0}^{n-1} (\log \mathrm{D}T(T^i(x)) > 0$. By Lemma 37, this implies that for all $x \in \mathrm{S}^1$

$$\mathrm{D}(T^n)(x) > 1.$$

But T^n is a diffeomorphism so $\mathrm{Leb}(S^1) = \mathrm{Leb}(T^n(S^1)) = \int_{S^1} \mathrm{D}(T^n)(x)dx = 1$ which contradicts $\mathrm{D}(T^n)(x) > 1$ for all $x \in S^1$.

□

4.2. Absence of wandering intervals in $\mathcal{C}^2$-regularity. In this paragraph we prove the following celebrated theorem due to Denjoy.

THEOREM 39. *Let T be a $\mathcal{C}^2$ circle diffeomorphism with irrational rotation number α. Then*

(1) T has no wandering intervals.
(2) Equivalently, T is conjugate to R_α.

We have now developed all the tools to make an efficient proof of this theorem: the Denjoy-Koksma inequality (Theorem 35) and the chain rule for the logarithm of the derivative (Lemma 37).

PROOF. The idea that we follow is the following: because the derivative is sufficiently regular, at special times q_ns the logarithm of $\mathrm{D}(T^{q_n}$ is uniformly bounded and in turns it implies that $\mathrm{D}(T^{q_n}$ is bounded from below away from 0. But it there were a wandering interval, the derivative at a point of the wandering interval should tend to zero to allow the lengths of the images of the wandering interval to tend to zero. We make this rigorous hereafter.

Assume by contradiction that an interval I is a wandering interval. Because all the $T^n(I)$ of I are disjoint we must have

$$\sum_{n\in\mathbb{N}} \mathrm{Leb}(T^n(I)) \leq \mathrm{Leb}(S^1) = 1.$$

T is of class $\mathcal{C}^2$ which implies that $\mathrm{D}T$ and thus $\log \mathrm{D}T$ are of class $\mathcal{C}^1$ and in particular hae bounded variation. Applying the Denjoy-Koksma inequality to $\log \mathrm{D}T$, which we know by Proposition 38 to have mean zero with respect to the unique invariant measure of T we get that for all $n \geq 1$ and for all $x \in S^1$

$$|\sum_{i=1}^{q_n} (\log \mathrm{D}T(T^i(x))| \leq V = \mathrm{Var}(\log \mathrm{D}T).$$

By Lemma 37, this implies that for all $x \in S^1$ we have

$$e^-V \leq \mathrm{D}(T^{q_n})(x) \leq e^V.$$

Because

$$\mathrm{Leb}(T^{q_n}(I)) = \int_I \mathrm{D}(T^{q_n})(x)dx$$

we obtain that for all $n \geq 1$, $\mathrm{Leb}(T^{q_n}(I)) \geq e^{-V}\mathrm{Leb}(I)$. There are infinitely many such q_n, this thus contradicts the fact that

$$\sum_{n\in\mathbb{N}} \mathrm{Leb}(T^n(I))$$

is finite. This terminates the proof of Theorem 39.

□

5. Denjoy counterexamples

We close this chapter with a construction showing that Theorem 39 is essentially optimal.

THEOREM 40. *For any irrational α, there exists a $\mathcal{C}^1$-circle diffeomorphism of rotation number α which has a wandering interval.*

The construction of such "Denjoy counter-examples" is achieved by going through the blow-up construction (paragraph 2.3) with a little extra care. We recall the notation we used

- I_0 is a wandering interval;
- for all $n \in \mathbb{Z}$, $I_n = T^n(I_0)$;
- $\mathcal{C} = S^1 \setminus \bigcup_n I_n$.

The construction we carried out then was purely topological. We can attempt at making it more smooth, this requires

(1) deciding of lengths (l_n) for the intervals (I_n) such that $\sum_{n\in\mathbb{Z}} l_n < \infty$;
(2) a specification of the map T on each of the branches $T : I_n \to I_{n+1}$.

Once such a specification is given, the resulting map is $\mathcal{C}^1$ on the whole of S^1 if and only if for any $x \in \mathcal{C}$, the limit

$$\lim_{y\to x} T'(y)$$

exists and defines a continuous function on S^1. This is by virtue of the following fact

PROPOSITION 41. *Let $f : (a, b) \longrightarrow \mathbb{R}$ a continuous function which is differentiable on a dense subset A. Assume that $f' : A \longrightarrow \mathbb{R}$ extends to a continuous function g on the whole of (a, b). Then f is of class $\mathcal{C}^1$ and for all $x \in (a, b)$, $f'(x) = g(x)$.*

PROOF. Exercise. □

We therefore need to find a slick way of achieving this. The idea is that we started from a map whose derivative was one everywhere and since blown-up intervals accumulate to the Cantor set that is the remnant of the initial $\mathcal{C}^1$-structure of the circle, we should try to get the derivative of the branches $T : I_n \longrightarrow I_{n+1}$ tend to 1 as $|n| \to +\infty$.
This will be impossible to achieve for an arbitrary choice of the lengths (l_n). For instance, the choice $l_n = \frac{1}{S^{|n|}}$ will force $T'(x_n) = 2$ or $\frac{1}{2}$ for at least one $x_n \in I_n$. We make the better choice

$$l_n = \frac{1}{n^2}.$$

We have the following

PROPOSITION 42. *Let $I_n = [a_n, b_n]$ and assume that $|I_n| = l_n = \frac{1}{n^2}$ for all $n \in \mathbb{Z}^*$. Then there exists a $\mathcal{C}^1$ diffeomorphism $T_n : I_n \longrightarrow I_{n+1}$ such that*

(1) $T'(a_n) = T'(b_n) = 1$;
(2) $||T' - 1||_0 \longrightarrow 0$.

PROOF. For any $l > 0$, consider a tent function of magnitude a

$$\begin{array}{rccc} g_l^a & := & [0,l] & \longrightarrow & \mathbb{R} \\ & & x & \longmapsto & 1 - a(1 - \frac{2}{l}|x - \frac{l}{2}|) \end{array}$$

We define T_l^a to be $x \mapsto \int_0^x g_l^a(t)dt$. The map T_l^a is such that

- T_l^a is strictly increasing if $a < 1$;

- $(T_l^a)'(0) = (T_l^a)'(l) = 1$;
- $T_l^a([0, l]) = l - a\frac{l}{2}$.

If we take $l_n = \frac{1}{n^2}$ and $a_n = 2(1 - \frac{n^2}{(n+1)^2})$ we obtain that T maps an interval of length $\frac{1}{n^2}$ to an interval of length $\frac{1}{(n+1)^2}$. Furthermore, a_n tends to 0 when $|n|$ tends to ∞, thus $T_{l_n}^{a_n}$ satisfies the conclusions of the Proposition.

□

One can now use the diffeomorphisms $T_n : I_n \longrightarrow I_{n+1}$ of Proposition 42 when doing the blow-up construction to obtain a $\mathcal{C}^1$ circle diffeomorphism of S^1 with a wandering interval. This completes the proof of Theorem 40.

6. Exercises

EXERCISE 16. *Show that $f_k := x \mapsto e^{2\pi ikx}$ is always a coboundary for R_α, α irrational, irrespective of α's arithmetic properties.*

EXERCISE 17.

1. *Assume that $f : S^1 \longrightarrow \mathbb{R}$ is a μ-Lipschitz function. Show that f has bounded variation and that $\mathrm{Var}(f) \leq \mu$.*
2. *Assume furthermore that f is of class $\mathcal{C}^1$. Show that in this case $\mathrm{Var}(f) = \int_{S^1} |f'| d\mathrm{Leb}$.*

EXERCISE 18. *Assume that α is an irrational number of bounded type (i.e. there exists K_1 such that for all $n \in \mathbb{N}^*$, $a_n \leq K_1$). Show that for any bounded variation observable f of mean zero, there exists $K_2 > 0$ such that for all $k \in \mathbb{N}^*$*

$$|\sum_{i=0}^{k-1} f(R_\alpha^i(x))| \leq K_2 \cdot \log n.$$

CHAPTER 6

The local rigidity theorem and KAM theory

In this chapter we discuss a proof of the celebrated local rigidity theorem of Arnol'd. We discuss briefly the context of this result. It is known, by Denjoy's theorem, that a $\mathcal{C}^2$ circle diffeomorphism of irrational rotation number is topologically conjugate to the associated rotation. The question we are concerned with here is the following

QUESTION 43. *Could it be that this conjugacy is more regular?*

This question is considerably more difficult than that of the topological conjugacy as it is essentially a question of analysis. Because we expect this problem to be difficult in general, we are going to consider a simplified version of it, hoping that it will allow us to isolate the difficulty without being bothered by too much technical complication. We are going to try and solve the following question

QUESTION 44. *Let R_α be the rotation of irrational angle α. Consider a map of the form $T : x \mapsto x + \alpha + \eta(x)$ where η is a very small **analytic** 1-periodic function such that T also has rotation number α. Is it true the map conjugating T to R_α is also analytic?*

The main reason why we make this particular simplification is because we want to study a linearised problem and use some form of implicit function theorem.

1. The linearised problem

1.1. Analytic maps. We first define the class of maps we are going to be working with. This choice my seem somewhat arbitrary, but it should become apparent later on that it minimises technical issues.

DEFINITION 12. *A $\mathcal{C}^\infty$-map $f : (a, b) \longrightarrow \mathbb{R}$ is analytic if at every point $x \in (a, b)$ there is a neighbourhood $\mathcal{U}_x$ of x in (a, b) such that*

(1) its Taylor expansion at x has a positive radius of convergence and is defined on $\mathcal{U}_x$;
(2) f is equal to its Taylor expansion on $\mathcal{U}_x$.

Now, we say that a circle diffeomorphism T is analytic if and only if any of its lift to $\mathbb{R}$ is analytic. Consider a lift $\tilde{T}$ of T, it is an increasing map $\mathbb{R} \longrightarrow \mathbb{R}$ which satisfies $\tilde{T}(x + 1) = \tilde{T}(x) + 1$. By definition, $\tilde{T}$ extends to a holomorphic function that is defined on a strip

$$S_\delta = \big\{ z \in \mathbb{C} \mid \mathrm{Im}(z) < \delta \big\}.$$

We will make use of this fact later on in the discussion.

1.2. The linearised equation. For the remainder of the chapter, T is an analytic diffeomorphism close to R_α a lift of which we write

$$\tilde{T} := x \mapsto x + \alpha + \eta(x)$$

where η is a 1-periodic analytic function. Our ultimate goal is to show that T is conjugate to R_α via an analytic map close to the identity. We can therefore try an write the equation that such an analytic map φ should satisfy. We write $\varphi = \mathrm{Id} + h$ where h is to be thought of as a small analytic 1-periodic function. $\varphi, \mathrm{R}_\alpha$ and T must satisfy

$$\varphi \circ \mathrm{R}_\alpha = T \circ \varphi.$$

Writing $\varphi(x) = x + h(x)$ and $T(x) = x + \alpha + \eta(x)$ for all $x \in \mathbb{R}$ we get

$$\varphi(x + \alpha) = \varphi(x) + \alpha + \eta(\varphi(x))$$

$$x + \alpha + h(x + \alpha) = x + h(x) + \alpha + \eta(x + h(x))$$

and then

$$h(x + \alpha) - h(x) = \eta(x + h(x)).$$

The map $x \mapsto \eta(x+h(x))$ can be approximated at first order by $x \mapsto \eta(x)$. Thus, the equation $\varphi \circ \mathrm{R}_\alpha = T \circ \varphi$ can be linearly approximated by the **linear** equation

$$h \circ R_\alpha - h = \eta.$$

This is nothing but the cohomological equation that we have already partially studied in 2.

Remark 45. At a very heuristic level, what we are trying to do is to invert the map

$$\varphi \longmapsto \varphi \circ \mathrm{R}_\alpha \circ \varphi^{-1}$$

in a neighbourhood of Id in the Fréchet manifold $\mathrm{Diff}^\omega_+(S^1)$ where the map

$$h \longmapsto h \circ R_\alpha - h$$

from the space of 1-periodic analytic functions (which is the tangent space at Id of $\mathrm{Diff}^\omega_+(S^1)$) is its derivative. If we manage to put ourselves in a context where we can apply a local inversion theorem (or implicit function theorem), we should be able to solve our problem.

2. Inverse function theorem

Motivated by the discussion of the preceding section, we state and prove a somewhat general version of the inverse function theorem and try to see if it (or elements of its proof) apply to our context.

Theorem 46 (Inverse function). *Let $\mathcal{E}$ and $\mathcal{F}$ be two Banach spaces and let $f : \mathcal{E} \longrightarrow \mathcal{F}$ be a continuously differentiable function such that at a certain point $x \in \mathcal{E}$, the differential of f*

$$\mathrm{D}_x(f) : \mathcal{E} \longrightarrow \mathcal{F}$$

is a continuous linear isomorphism whose inverse is also continuous. Then there exists a neighbourhood $\mathcal{U}$ of x and a neighbourhood $\mathcal{V}$ of $y = f(x)$ such that $f(\mathcal{U}) = \mathcal{V}$ and f is a bijection restricted to $\mathcal{U}$.

If we wanted to apply this theorem to the map $\varphi \longmapsto \varphi \circ \mathrm{R}_\alpha \circ \varphi^{-1}$ in some reasonable space, as suggested in the previous section, we shall fail for the following two reasons.

- Either we decide to work in a Banach space of circle diffeomorphisms of class $\mathcal{C}^r$ for a certain r large enough. In that case, the "differential" of our map $h \longmapsto h \circ R_\alpha - h$ cannot really be inverted. Indeed, in the process of solving the cohomological equation there is a loss of regularity (a solution h is twice fewer differentiable than the datum η).
- Or we decide to work in a space of $\mathcal{C}^\infty$ or analytic circle diffeomorphism, then we lose the local structure of Banach space. The differential $h \longmapsto h \circ R_\alpha - h$ can be inverted but the $\mathcal{C}^\infty$ or analytic topology is defined by a family of semi-norms which endows it with the structure of a Fréchet space which is weaker than a Banach structure.

Nonetheless, in the latter setting we are dealing with a very particular case and the differential is completely explicit, so we might hope that the proof of Theorem 46 adapts. Let us go through this proof.

Proof of the Inverse Function Theorem. The proof starts with reducing the problem to a formally simpler setting. Up to pre and post-composing f by two translations, and post-composing by $\big(D_x(f)\big)^{-1}$, we can assume that

- $\mathcal{E} = \mathcal{F}$;
- $x = f(x) = 0$;
- $D_0(f) = \mathrm{Id}$.

The map $f - \mathrm{Id}$ is continuously differentiable and its derivative vanishes at 0. Hence there exists $r > 0$ such that for any $x \in B(0, r)$,

$||D(f - \mathrm{Id})(x)|| \leq \frac{1}{2}$. Which means that on $B(0, r)$, $f - \mathrm{Id}$ is $\frac{1}{2}$-Lipschitz. In particular

$$(f - \mathrm{Id})(B(0, r) \subset B(0, \frac{r}{2}).$$

Now for any $y \in B(0, \frac{r}{2})$ we consider the following map

$$\varphi_y := x \mapsto f(x) - x + y.$$

Notice that $f(x) = y$ if and only $\varphi_y(x) = x$. By the triangle inequality, we have

$$\varphi_y(B(0, r) \subset B(0, r)$$

. Moreover, φ_y is $\frac{1}{2}$-Lipschitz. By the contraction mapping theorem, φ_y has a unique fixed point in $B(0, r)$ and thus there exists a unique $x \in B(0, r)$ such that $f(x) = y$.

We have obtained the existence of a pre-image via f for any point in a neighbourhood of 0. The rest of the proof is just checking that the local inverse to f thus defined enjoys all the nice properties that one would expect. We do not carry out this discussion as we have explained the bit we need for our problem.

□

3. The cohomological equation II

We have seen that the differential of the map

$$\varphi \longmapsto \varphi \circ R_\alpha \circ \varphi^{-1}$$

is more or less given by

$$h \longmapsto h \circ R_\alpha \circ h^{-1}.$$

This is not completely rigorous; the correct statement is that $h \circ R_\alpha \circ h^{-1} = \eta$ is the linearised version of the equation $\varphi \circ R_\alpha = T \circ \varphi$. The inverse function theorem discussed in last section (Theorem 46)

(1) requires the structure of a Banach space which we do not have on $\mathrm{Diff}^\omega(S^1)$;
(2) requires that the differential of the function that we are trying to invert be a continuous (equivalently) bounded operator.

The first thing that we suggest we do is to try and improve the Fréchet structure on $\mathrm{Diff}^\omega(S^1)$ and hope that the cohomological equation can be solved within this space, and that the operator which associate to an observable the associated solution to the cohomological equation be continuous. A natural thing to do is consider the space

$$\mathcal{E}_\delta := \{1-\text{periodic } \eta \text{ analytic on } S_\delta \mid \exists C_T > 0 \ |\eta(x)| \leq C_T \text{ for } x \in S_\delta\}.$$

For any $\eta \in \mathcal{E}_\delta$, $T = \mathrm{Id}+\eta$ passes to the quotient $S^1 = \mathbb{R}/\mathbb{Z}$ to an analytic circle diffeomorphism provided η is small enough. Note the following things

- Not every (lift of an) analytic circle diffeomorphism can be extended to the whole strip S_δ. δ is somehow a measure of how analytic T is. The bigger the δ the more "analytic" the circle diffeomorphism.
- $\mathcal{E}_\delta$ can be endowed with the structure of a Banach space by simply considering the norm

$$||\eta||_\delta = \sup_{x \in S_\delta} |\eta(x)|.$$

We now do the analysis of the cohomological equation for observables $\eta \in \mathcal{E}_\delta$.

THEOREM 47. *Let α be an irrational number such that there exists $K > 0$ such that for any irrational number $\frac{p}{q} \in \mathbb{Q}$, we have $|\alpha - \frac{p}{q}| \geq \frac{K}{q^3}$. Let η be an element of $\mathcal{E}_\delta$. Then*

(1) there exists an analytic h such that

$$h \circ R_\alpha - h = \eta;$$

(2) for any $\mu > 0$, h belongs to $\mathcal{E}_{\delta-\mu}$;
(3) there exists a constant $C > 0$ such that for any $\mu > 0$,

$$||h||_{\delta-\mu} \leq \frac{C}{\mu^3} ||\eta||_\delta.$$

In order to prove this theorem we will need the following lemma.

LEMMA 48. *(1) Let $\eta \in \mathcal{E}_\delta$ and consider its Fourier development*

$$\eta(x) = \sum_{n\in\mathbb{Z}} a_n e^{inx}.$$

Its Fourier coefficients satisfy

$$|a_n| \leq ||\eta||_\delta e^{-|n|2\pi\delta}.$$

(2) Conversely, for any sequence$(a_n)_{n\in\mathbb{Z}}$ satisfying $|a_n| \leq Ce^{-|n|2\pi\delta}$ the series

$$\sum_{n\in\mathbb{Z}} a_n e^{inx}$$

defines an element of $\mathcal{E}_\delta$ whose norm is less than C.

PROOF. We make use of the fact that

$$a_n = \int_0^{2\pi} e^{-n2x\pi i}\eta(e^{2\pi ix})dx = \int_0^{2\pi} z^{-n}\eta(z)dz.$$

Integrating $z^{-n}\eta(z)dz$ along the rectangle defined by vertices $0, 2\pi, 2\pi + i\epsilon, i + \epsilon$, we obtain

$$a_n = -\int_{i\epsilon}^{2\pi+i\epsilon} z^{-n}\eta(z)dz = -\int_{i\epsilon}^{2\pi+i\epsilon} e^{-n\epsilon 2\pi}e^{-n2x\pi i}\eta(e^{2\pi ix})dx$$

from which we get

$$|a_n| \leq ||\eta||_\delta e^{-2\pi|n|\epsilon}$$

and that for any $\epsilon < \delta$ which gives the result. The converse is obtained by similar calculations to establish the convergence of the formal series $\sum a_n e^{2\pi in}$.

□

PROOF OF THEOREM 47. We have seen in Section 2 that if $\eta = \sum a_n e^{2\pi in}$ the Fourier coefficients of h are $b_n = \frac{a_n}{e^{2\pi in\alpha}-1}$. Thus

$$|b_n| \leq |a_n|\frac{1}{|e^{2\pi in\alpha} - 1|}.$$

$|e^{2\pi in\alpha} - 1| \geq |\sin(2\pi in\alpha)|$ and when $\{2\pi in\alpha\}$ is sufficiently small (say less than $\frac{1}{2}$) we have

$$|e^{2\pi in\alpha} - 1| \geq |\sin(2\pi n\alpha)| \geq \frac{1}{2}\{2\pi n\alpha\}.$$

But for all $n \in \mathbb{N}$, $|\{n\alpha\}| \geq \frac{K}{n^2}$ which yields for $z = e^{2\pi i(x+iy)}$ for small y

$$|h(z)| \leq \sum_{n\in\mathbb{Z}} \frac{|a_n|n^2}{2\pi K}|z|^n.$$

By Lemma 48, we have $|a_n| \leq ||\eta||_\delta e^{-|n|2\pi\delta}$. Assume $|y| \leq \epsilon < \delta$ we get $|z|^n \leq e^{2\pi|ny|}$ and thus

$$|h(z)| \leq \sum_{n\in\mathbb{Z}} \frac{1}{2\pi K}||\eta||_\delta e^{-|n|2\pi(\delta-\epsilon)}n^2 e^{-|n|2\pi\epsilon}$$

$$|h(z)| \leq \frac{1}{2\pi K}||\eta||_\delta \sum_{n\in\mathbb{Z}} e^{-|n|2\pi(\delta-\epsilon)}n^2.$$

Set $\Gamma = \int_0^{+\infty} x^2e^{-x}dx$. We have $\int_0^{+\infty} t^2e^{-2\pi(\delta-\epsilon)t}dt = \frac{1}{8\pi^3(\delta-\epsilon)^3}\int_0^{+\infty} x^2e^{-x}dx$. Ultimately this yields

$$|h(z)| \leq \frac{||\eta||_\delta\Gamma}{16\pi^4(\delta-\epsilon)^3}.$$

If we write $\epsilon = \delta - \mu$ and $C = \frac{\Gamma}{16\pi^4}$ we get the expect result.

□

Unfortunately we notice that Theorem 47 predicts a loss of regularity when solving the cohomological equation. This loss of regularity is responsible for the impossibility of applying the Inverse Function Theorem at our problem, as to invert the differential of the map defining the conjugacy equation (which is equivalent to solving the cohomological equation), one has to place oneself in a bigger Banach space.

4. Newton's method

In this paragraph we recall Newton's method for finding roots of an equation. This algorithm will be the base idea behind an iterative scheme to solve the local conjugacy problem for circle diffeomorphisms.

We consider a $\mathcal{C}^2$ map $F : \mathbb{R} \longrightarrow \mathbb{R}$ and assume that it vanishes at a point z, but that we do not know the exact value of z. We explain here Newton's method which is a way to construct approximations of z converging extremely fast to z.
The idea is to start from any point $x_0 \in \mathbb{R}$ (not too far from z) and replace F by its linear approximation at x_0. One constructs this way x_1 to be the point at which this linear approximation vanishes. x_2 is built in a similar fashion by replacing F at x_1 by its linear approximation. Iterating the process one constructs a sequence (x_n) which is going to converge to z. We make this formal.

4.1. Formal definition of the approximating sequence. Following what is suggested in the above paragraph, we recursively define given any x_0 the sequence (x_n) by

$$x_{n+1} = x_n - \frac{F(x_n)}{F'(x_n)}.$$

For this to work, we must place ourselves in a neighbourhood of z on which F' does not vanish.

4.2. Proof of the convergence. At any point y, the linear approximation of F is the map

$$L_y := x \mapsto F(y) + (x-y)F'(y).$$

Applying Taylor's theorem, we get that for any y

$$L_y(z) - F(z) = \frac{1}{2}F''(u)(z-y)^2$$

for a certain $u \in [z, y]$. Set $\epsilon_n = |z - x_n|$. Applying the formula above we get the existence of $u_n \in [z, x_n]$ such that

$$F(x_n) - (z - x_n)F'(x_n) = \frac{1}{2}F''(u_n)(z - x_n)^2$$

One can divide this equality by $F'(x_n)$ and use the fact that $x_{n+1} = x_n - \frac{F(x_n)}{F'(x_n)}$ to find

$$(x_{n+1} - z) = \frac{F''(u_n)}{2F'(x_n)}(z - x_n)^2$$

from which we derive

$$\epsilon_{n+1} \leq \frac{||F''||}{2||F'||}\epsilon_n^2.$$

If we start with an x_0 such that $(x_0 - z)^2$ is sufficiently small compared to $\frac{||F''||}{2||F'||}$, we get the existence of a constant C such that $\epsilon_n \leq C\epsilon_0^{2^n}$ hence the super-exponential convergence of the sequence x_n to z.

5. Arnold's theorem

In this last Section, we prove the following theorem

THEOREM 49 (Arnol'd, 1961). *Let α be a real number such that there exists $K > 0f$ such that for any rational $\frac{p}{q}$, $|\alpha - \frac{p}{q}| \geq \frac{K}{q^3}$. Then there exists $\epsilon(\alpha, \delta) > 0$ such that for any $\eta \in \mathcal{E}_\delta$, if*

- $||\eta||_\delta \leq \epsilon(\alpha, \delta)$;
- *the rotation number of $T = R_\alpha + \eta$ is α;*

then the map conjugating T to R_α is analytic.

We explain the strategy, that builds on Newton's method, that we implement to prove Theorem 49.

Step 1: Linearisation We have seen in paragraph 1.2 that the equation $\varphi \circ R_\alpha = T \circ \varphi$ is equivalent to

$$h \circ R_\alpha - h = \eta \circ \varphi$$

if we write $T = R_\alpha + \eta$ and $\varphi = \mathrm{Id} + h$. Because φ is assumed to be close to the identity, this equation can be linearised to

$$h \circ R_\alpha - h = \eta.$$

Step 2: Solving the linear equation In paragraph 2.2 we have solved the linearised equation. We have noticed that in the process of solving this equation there is a loss of regularity. Namely, if the input η is analytic and defined on a strip of a certain size about the real axis, its solution h is still analytic but defined on a strip marginally thinner. This can refined to a parametrised control of the $||\cdot||_{\delta-\epsilon}$-norm of h in function of the $||\cdot||_\delta$-norm of η. This is given by Theorem 47.

Step 3: Error term in the linearised solution We can use the linearised solution to defined a conjugating map $\varphi_1 = \mathrm{Id} + h$. Newton's method suggest that

$$\varphi_1 \circ T \circ \varphi_1^{-1}$$

will miss $\mathcal{R}_\alpha$ by a quantity whose norm is of the order of $||\eta||_\delta^2$. In the process there is a loss of regularity, and the smaller the loss we consider (*i.e.* the smaller the ϵ is the measurement of the $||\cdot||_{\delta-\epsilon}$ of the norm of the solution h) the bigger the constant in front of the $||\eta||_\delta^2$ error term. This constant is made illicit.

Step 4: Fast convergence in the Newton scheme We define inductively $T_{n+1} = \varphi_n \circ T_n \circ \varphi_n^{-1}$ and φ_{n+1} the linearised solution to the equation $\varphi \circ R_\alpha = T \circ \varphi_n$. The fast incontinence of the scheme allows to show that the maps

$$\varphi_n \circ \cdots \circ \varphi_2 \circ \varphi_1$$

actually converges to an analytic circle diffeomorphisms conjugating T to R_α.

6. Estimates on the linearised solution

In this paragraph we essentially implement Step 3 of the outline of the proof of Theorem 49.

6.1. Estimates for φ_1 and φ_1^{-1}. Recall that $h \in \mathcal{E}_{\delta-\mu}$ is the solution of the equation

$$h \circ R_\alpha - h = \eta - \int_{S^1} \eta d\mathrm{Leb}$$

and that we have set $\varphi_1 = \mathrm{Id} + h$. We first need the following Lemma

Lemma 50. *Let f be an element of $\mathcal{E}_\nu$ for some $\nu > 0$. Then there exists a constant $C_1 > 0$ such that*

$$||f'||_{\nu-\epsilon} \leq \frac{C_1}{\epsilon} ||f||_\nu.$$

Proof. This is a standard lemma of complex analysis. By Cauchy's integral formula we have that for every r such $B(z,r) \subset S_\nu$

$$f'(z) = \int_{\partial B(z,r)} \frac{f(w)}{(z-w)^2} dw.$$

From which we deduces

$$|f'(z)| \leq ||f||_\nu \int_{\partial B(z,r)} \frac{dw}{r^2} dw = \frac{2\pi}{r} ||f||_\nu.$$

If $z \in S_{\nu-\epsilon}$, r can be made arbitrarily close to ϵ hence the result.

□

From this Lemma we can deduce

Proposition 51. *If $||\eta||_\delta < \frac{\mu^4}{CC_1}$ then*

(1) $\varphi_1 = \mathrm{Id} + h$ is a diffeomorphism from $S_{\delta-2\mu}$ onto its image;
(2) The image of $S_{\delta-\mu}$ contains $S_{\delta-3\mu}$ and thus φ_1^{-1} is well-defined on $S_{\delta-3\mu}$.

PROOF. (1) By Lemma 50, $||h'||_{\delta-2\mu} \leq \frac{C_1}{\mu}||h||_{\delta-\mu}$ and by Theorem 47

$$||h'||_{\delta-2\mu} \leq \frac{CC_1}{\mu^4}||\eta||_\delta.$$

Hence if $||\eta||_\delta < \frac{\mu^4}{CC_1}$ then $\varphi_1'(z) \neq 0$ on $S_{\delta-2\mu}$; which implies that φ is a diffeomorphism on its image when restricted to $S_{\delta-2\mu}$.

(2) Since $||\eta||_\delta \leq \frac{\mu^4}{CC_1}$ by Theorem 47 $||h||_{\delta-\mu} \leq \frac{\mu}{C_1}$. Since $C_1 = 2\pi > 1$, $||h||_{\delta-\mu} \leq \mu$. Thus the image of $S_{\delta-2\mu}$ by $\varphi_1 = \mathrm{Id} + h$ contains $S_{\delta-3\mu}$.

□

We now need an estimate on φ_1^{-1}. We write $\varphi^{-1} = \mathrm{Id} - h + g$ where g is an analytic function in $\mathcal{E}_{\delta-3\mu}$.

PROPOSITION 52. *With the notation above, there exists a universal constant $C_2 > 0$ such that*

$$||g||_{\delta-4\mu} \leq \frac{C_2}{\mu^7}||\eta||_\delta^2.$$

PROOF. We just write that for all $z \in S_{\delta-3\mu}$, $\varphi_1^{-1} \circ \varphi_1(z) = z$, which rewrites

$$z + h(z) - h(z + h(z)) + g(z + h(z)) = z$$

from which we get

$$g(z + h(z)) = h(z + h(z)) - h(z).$$

We write $g(z + h(z)) = |h(z)| \int_0^1 h'(z + th(z))dz$ for any z such that the segment $[z, z + h(z)]$ belongs to $S_{\delta-\mu}$. This is in particular the case when $z \in S_{\delta-2\mu}$. Restrict z further so that $[z, z + h(z)] \subset S_{\delta-2\mu}$; this is achieved if $z \in S_{\delta-3\mu}$. We can use the estimates of $||h'||_{\delta-2\mu}$ given by Lemma 50. We get that for any $z \in S_{\delta-3\mu}$

$$|g(z + h(z))| \leq ||h||_{\delta-\nu}||h'||_{\delta-2\mu} \leq \frac{C_1C^2}{\mu^7}||\eta||_\delta^2.$$

Now any $y \in S_{\delta-4\mu}$ can be written as $z + h(z)$ with $z \in S_{\delta-3\mu}$ hence the conclusion of the Proposition.

□

6.2. Main estimate. In this paragraph we prove the main estimate that we will need to implement Newton's scheme. Consider $T = \mathrm{R}_\alpha + \eta$ with $\eta \in \mathcal{E}_\delta$ and such that T is an analytic circle diffeomorphism. We set $\varphi_1 = \mathrm{Id} + h$ where h is the solution to the equation

$$h \circ R_\alpha - h = \eta - \int_{S^1} \eta d\mathrm{Leb}.$$

By Proposition 51, we know that if η is chosen small enough, φ_1 is an analytic circle diffeomorphism which extends to a strip of width $S_{\delta-2\mu}$ where μ depends on how small η is.

THEOREM 53. *Consider $T = \mathrm{R}_\alpha + \eta$, h and φ_1 as above and assume that $\rho(T) = \alpha$. Consider $\varphi_1^{-1} \circ T \circ \varphi_1$ which we write $\mathrm{R}_\alpha + \tilde{\eta}$. There exists a constant $C_3 > 0$ such that if $||\eta||_\delta \leq \frac{\mu^4}{CC_1}$ then*

$$||\tilde{\eta}||_{\delta-6\mu} \leq \frac{C_3}{\mu^7}||\eta||_\delta^2.$$

PROOF. The strategy of the proof is to compute $\varphi_1^{-1} \circ T \circ \varphi_1$ in term of h, g and η and to use Theorem 47 and Proposition 51 to get the estimate. We get

$$T \circ \varphi_1(z) = z + \alpha + h(z) + \eta \circ \varphi_1(z)$$

and

$$\varphi_1^{-1} \circ T \circ \varphi_1(z) = z + \alpha + h(z) + \eta \circ \varphi_1(z) - h \circ T \circ \varphi_1(z) + g \circ T \circ \varphi_1(z).$$

We now group terms together in order to use the cohomological equation estimate

$$\begin{aligned}\varphi_1^{-1} \circ T \circ \varphi_1(z) =& z + \alpha + h(z) - h \circ \mathrm{R}_\alpha(z) + \eta(z) \\ &+ \eta \circ \varphi_1(z) - \eta(z) \\ &+ h \circ \mathrm{R}_\alpha(z) - h \circ T \circ \varphi_1(z) \\ &+ g \circ T \circ \varphi_1(z).\end{aligned}$$

We see that

$\tilde{\eta}(z) = h(z) - h \circ \mathrm{R}_\alpha(z) + \eta(z) + \eta \circ \varphi_1(z) - \eta(z) + h \circ \mathrm{R}_\alpha(z) - h \circ T \circ \varphi_1(z) + g \circ T \circ \varphi_1(z)$

and we now give estimates for all the coloured terms.

- Because h solves the cohomological equation, we get that

$$h(z) - h \circ \mathrm{R}_\alpha(z) + \eta(z) = -\int_{S^1} \eta.$$

- We write

$$\eta \circ \varphi_1(z) - \eta(z) = |h(z)| \int_0^1 \eta'(z + th(z))dt$$

for any z such that $[z, h(z)] \subset S_{\delta-2\mu}$. This is the case for $z \in S_{\delta-3\mu}$. There we can use the estimate that $|\eta'(z)| \leq \frac{C_1}{\mu}||\eta||_\delta$ (Proposition 50) and by Theorem 47 we get

$$|\eta \circ \varphi_1(z) - \eta(z)| \leq \frac{C_1 C}{\mu^4}||\eta||_\delta^2.$$

- We have $h \circ \mathrm{R}_\alpha(z) - h \circ T \circ \varphi_1(z) = h(z + \alpha) - h(z + h(z) + \alpha + \eta(z + h(z))$ and thus

$$h \circ \mathrm{R}_\alpha(z) - h \circ T \circ \varphi_1(z) = |h(z) + \eta(z + h(z))| \cdot \int_0^1 h'(z + \alpha + t[h(z) + \eta(z + h(z))])dt.$$

Provided that $[z + \alpha, z + \alpha + h(z) + \eta(z + h(z))] \subset S_{\delta-2\mu}$ we get the following estimates

$$|h \circ \mathrm{R}_\alpha(z) - h \circ T \circ \varphi_1(z)| \leq [||h(z)||_{\delta-\mu} + ||\eta||_\mu] \cdot ||h'||_{\delta-2\mu}.$$

The condition $[z + \alpha, z + \alpha + h(z) + \eta(z + h(z))] \subset S_{\delta-2\mu}$ is satisfied if $z \in S_{\delta-4\mu}$.

- Finally we have the estimate

$$|g(y)| \leq \frac{C_2}{\mu^7} ||\eta||_\delta^2$$

for $y \in S_{\delta-4\mu}$. Applying this to $y = T \circ \varphi_1(z) = z + \alpha + h(z) + \eta(z + h(z)$ we get that provided that $z \in S_{\delta-6\mu}$

$$|g \circ T \circ \varphi_1(z)| \leq \frac{C_2}{\mu^7} ||\eta||_\delta^2.$$

Putting all this information together we get that for any $z \in S_{\delta-6\mu}$ we have

$$|\tilde{\eta}(z)| \leq |\int_{S^1} \eta| + \frac{C_4}{\mu^7} ||\eta||_\delta^2$$

which is not quite what we wanted, because of this extra term $\int_{S^1} \eta$ for which we have *a priori* no control. To get around this problem, we use the fact that because $x \mapsto x + \alpha + \tilde{\eta}(x)$ defines a circle homeomorphism of rotation number α, there must be an $x_0 \in [0, 1)$ such that $\tilde{\eta}(x_0) = 0$ (this is essentially contained in the proof of Proposition 27). At such a point we have

$$\int_{S^1} \eta = \eta \circ \varphi_1(x_0) - \eta(x_0) + h \circ \mathrm{R}_\alpha(x_0) - h \circ T \circ \varphi_1(x_0) + g \circ T \circ \varphi_1(x_0)$$

and using the upper bounds that we have just obtained on the green, blue and purple terms, we obtain that for a certain universal constant D,

$$|\int_{S^1} \eta| \leq \frac{D}{\mu^7} ||\eta||_\delta^2.$$

We can now re-inject in the formula of $\tilde{\eta}$ to obtain the conclusion of the Theorem.

□

7. Convergence in the inductive scheme

We have now set the stage for the implementation of Newton's scheme. Start with $T = T_0 = \mathrm{R}_\alpha + \eta_0$ and define the following inductively.

(1) h_{n+1} is the solution to

$$h_{n+1} \circ R_\alpha - h_{n+1} = -\eta_n + \int_{S^1} \eta_n d\mathrm{Leb}.$$

(2) $\varphi_{n+1} = \mathrm{Id} + h_{n+1}$, $\varphi_{n+1} : S^1 \longrightarrow S^1$ is an analytic diffeomorphism provided h_{n+1} is small enough.

(3) $T_{n+1} := \varphi_{n+1}^{-1} \circ T_n \circ \varphi_{n+1}$.

(4) η_{n+1} is the 1-periodic function such that $T_{n+1} = R_\alpha + \eta_{n+1}$.

We also define $\psi_n = \varphi_1 \circ \varphi_2 \circ \cdots \circ \varphi_n$. Note that with this notation we have

$$T_n = \psi_n^{-1} \circ T_0 \circ \psi_n.$$

Our ultimate goal is to show the following Proposition

PROPOSITION 54. *There exists $\delta' > 0$ such that*

- *η_n is analytic, belongs to $\mathcal{E}_{\delta'}$ and $||\eta_n||_{\delta'} \longrightarrow 0$.*

- *h_n is analytic, belongs to $\mathcal{E}_{\delta'}$ and there exists an analytic ψ_∞ which extends analytically to $S_{\delta'}$ and such that*

$$\psi_n \longrightarrow \psi_\infty$$

in the topology defined by the norm $||\cdot||_{\delta'}$.

The path to follow to prove the first point is quite clear : apply the main estimate given by Theorem 53 to get an inequality of the form

$$||\eta_{n+1}||_{\delta_n-\mu_n} \leq \frac{C_3}{\mu^7}||\eta_n||^2_{\delta_n}$$

and adjust μ_n so that

(1) on the one hand, the cumulated loss of regularity $\sum \mu_n$ is finite (and less than the an initial δ_0;
(2) on the other hand, the term $\frac{C_3}{\mu_n^7}$ tends to infinity slowly enough not to break the fast convergence in Newton's scheme allowed by the quadratic error in $||\eta_n||_{\delta_n}$.

Note that given any analytic $T_0 = \mathrm{R}_\alpha + \eta_0$, there exists a δ_0 such that $\eta_0 \in \mathcal{E}_{\delta_0}$. We make the following assumption

$$||\eta_0||_{\delta_0} \leq \epsilon_0 = \max\{\frac{\delta_0}{CC_1}, a\}$$

where $a > 0$ is a quantity whose value is to be decided later. We now set

(1) $\epsilon_n = \epsilon_0^{(\frac{3}{2})^n}$;
(2) $\mu_n = \frac{\delta_0}{6(n+1)^2}$ (so that $\sum_{n\in\mathbb{N}} \mu_n < \frac{\delta_0}{2}$).

We claim that with these assumption and notation we have

Proposition 55. *$\forall n \in \mathbb{N}$ we have*

$$||\eta_n||_{\delta_0-\sum_0^n \mu_i} \leq \epsilon_n.$$

Proof. The proof goes by induction. Assume that the Proposition holds true for a given n *i.e.* $||\eta_n||_{\delta_0-\sum_0^n \mu_i} \leq \epsilon_n$. By the main estimate Theorem 53 we get

$$||\eta_n||_{\delta_0-\sum_0^{n+1} \mu_i} \leq \frac{6^7 C_3}{\delta_0^7} \cdot (n+1)^{14} \epsilon_n^{\frac{1}{2}} \epsilon_n^{\frac{3}{2}}.$$

What we have just done is split the ϵ_n^2 into $\epsilon_n^{\frac{1}{2}}\epsilon_n^{\frac{3}{2}}$ to get ϵ_{n+1} and an extra term which we hope is going to be small enough to counter-balance the term $\frac{6^7C_3}{\delta_0^7} \cdot (n+1)^{14}$. What we want precisely for the induction to go through is

$$\frac{6^7 C_3}{\delta_0^7} \cdot (n+1)^{14} \cdot \epsilon_n^{\frac{1}{2}} \leq 1$$

which is equivalent to

$$\epsilon_n \leq (\frac{6^7 C_3}{\delta_0^7} \cdot (n+1)^{14})^{-2}.$$

The quantity on the right hand side can be written under the form $(A(n+1))^{-14}$. Recall that $\epsilon_n = \epsilon_0^{(\frac{3}{2})^n} \leq a^{(\frac{3}{2})^n}$. If a is initially chosen small enough, the inequality above obviously holds true. Thus, up to choosing a carefully, the induction can be carried out.

□

This Proposition has the following consequence. Because for any $n \in \mathbb{N}$, $\delta - \sum_{i=0}^n \mu_i \geq \frac{\delta}{2}$, we get

$$||\eta_n||_{\frac{\delta}{2}} \leq \epsilon_n$$

and thus the sequence (T_n) converges to R_α in the topology induced by the norm $||\cdot||_{\frac{\delta_0}{2}}$ (we can take $\delta' = \frac{\delta_0}{2}$ for the conclusion of Proposition 54).

Convergence of (ψ_n). The one thing left to establish is the convergence of the sequence (ψ_n) to an analytic map ψ_∞ such that

$$\psi_\infty^{-1} \circ T_0 \circ \psi_\infty = R_\alpha.$$

Recall that by definition

$$\psi_n = \psi_1 \circ \cdots \circ \varphi_n$$

and since $\varphi_n = \mathrm{Id} + h_n$ with h_n very small for the $||\cdot||_{\delta'}$-norm, it is reasonable to expect the convergence (ψ_n). We prove it formally now. Consider lifts

$$\tilde{\varphi_n} := z \mapsto z + h_n(z)$$

defined on $S_{\delta - \sum_0^n \mu_i}$. Note that the μ_i have been chosen so that

$$\tilde{\varphi_{n+1}}(S_{\delta - \sum_0^{n+1} \mu_i}) \subset S_{\delta - \sum_0^n \mu_i}$$

in application of Proposition 51 and thus $\tilde{\psi_n} := \tilde{\psi_1} \circ \cdots \circ \tilde{\varphi_n}$ is well-defined on $S_{\delta'}$ for all $n \in \mathbb{N}^*$. Furthermore we can write

$$\tilde{\psi_n}(z) = z + h_1(\psi_{n-1}(z)) + h_2(\psi_{n-2}(z)) + \cdots + h_n(z)$$

and therefore

$$||\tilde{\psi_n}(z) - z||_{\delta'} \leq \sum_{i=1}^n ||h_i||_{\delta'}.$$

By Theorem 47

$$||h_i||_{\delta'} \leq \frac{C}{\mu_{i-1}^3} \epsilon_{i-1}^2$$

from which we easily find that $||h_i||_{\delta'} \leq \epsilon_{i-1}^{3/2}$ from which we get $||\tilde{\psi_n}(z) - z||_{\delta'}$ is uniformly bounded. We now conclude by showing that $(\tilde{\psi_n})$ is Cauchy sequence

$$\tilde{\psi_{n+1}}(z) - \tilde{\psi_n}(z) = \psi_n(z + h_{n+1}(z)) - \psi_n(z) = |h_{n+1}(z)| \int_0^1 \tilde{\psi_n}'(z + th_{n+1}(z))dt.$$

Using Lemma 50, we get control for any $\epsilon > 0$

$$||\tilde{\psi_{n+1}} - \tilde{\psi_n}||_{\delta'-\epsilon} \leq \frac{2\pi}{\epsilon}||h_{n+1}||_\delta||\tilde{\psi_n}||_\delta.$$

Since $||h_{n+1}||_{\delta'} \leq \epsilon_0^{(\frac{3}{2})^n}$ we obtain that (ψ_n) is a Cauchy sequence with respect to the $||\cdot||_{\delta-\epsilon}$-norm for any $\epsilon > 0$. $\mathcal{E}_\delta$ is a Banach space for any δ, hence we obtain that the sequence (ψ_n) converges to a analytic circle diffeomorphism which extends to $S_{\delta-\epsilon}$ for any $\epsilon > 0$. This completes the proof of Arnold's Theorem (Theorem 49).