MODULI SPACES OF FLAT TORI AND ELLIPTIC HYPERGEOMETRIC FUNCTIONS

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Abstract: In the genus one case, we make explicit some constructions of Veech [76] on flat surfaces and generalize some geometric results of Thurston [71] about moduli spaces of flat spheres as well as some equivalent ones but of an analytico-cohomological nature of Deligne and Mostow [11], which concern Appell-Lauricella hypergeometric functions.

In the dizygotic twin paper [19], we follow Thurston's approach and study moduli spaces of flat tori with conical singularities and prescribed holonomy by means of geometrical methods relying on surgeries for flat surfaces. In the present paper, we study the same objects making use of analytical and cohomological methods, more in the spirit of Deligne-Mostow's paper.

Our starting point is an explicit formula for flat metrics with conical singularities on elliptic curves, in terms of theta functions. From this, we deduce an explicit description of Veech's foliation: at the level of the Torelli space of n-marked elliptic curves, it is given by an explicit affine first integral. From the preceding result, one determines exactly the leaves of Veech's foliation which are closed subvarieties of the moduli space $\mathcal{M}_{1,n}$ of n-marked elliptic curves. We also give a local explicit expression, in terms of hypergeometric elliptic integrals, for the Veech map which defines the complex hyperbolic structure of a leaf.

Then we focus on the $n = 2$ case: in this situation, Veech's foliation does not depend on the values of the conical angles of the flat tori considered. Moreover, a leaf which is a closed subvariety of $\mathcal{M}_{1,2}$ is actually algebraic and is isomorphic to a modular curve $Y_1(N)$ for a certain integer $N \geq 2$. In the considered situation, the leaves of Veech's foliation are $\mathbb{CH}^1$-curves. By specializing some results of Mano and Watanabe [49], we make explicit the Schwarzian differential equation satisfied by the $\mathbb{CH}^1$-developing map of any leaf and use this to prove that the metric completions of the algebraic ones are complex hyperbolic conifolds which are obtained by adding some of its cusps to $Y_1(N)$. Furthermore, we compute explicitly the conifold angle at any cusp $\mathfrak{c} \in X_1(N)$, the latter being 0 (*i.e.* $\mathfrak{c}$ is an usual cusp) exactly when it does not belong to the metric completion of the considered algebraic leaf.

In the last section of the paper, we discuss various aspects of the objects previously considered, such as: some particular cases that we make explicit, some links with classical hypergeometric functions in the simplest cases. We explain how to compute explicitly the $\mathbb{CH}^1$-holonomy of any given algebraic leaf, which is important in order to determine when the image of such a holonomy is a lattice in $\mathrm{Aut}(\mathbb{CH}^1) \simeq \mathrm{PSL}(2,\mathbb{R})$. Finally, we compute explicitly the hyperbolic volumes of some algebraic leaves of Veech's foliation.

The paper ends with two appendices. The first consists in a short and easy introduction to the notion of $\mathbb{CH}^1$-conifold. The second appendix is devoted to the Gauß-Manin connection associated to our problem: we first give a general and detailed abstract treatment then we consider the specific case of n-punctured elliptic curves which is made completely explicit when $n = 2$.

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1. Introduction

1.1. Previous works.

1.1.1. The classical **hypergeometric series** defined for $|x|< 1$ by

$$F(a,b,c;x) = \sum_{n=0}^{+\infty} \frac{(a)_n(b)_n}{(c)_n(1)_n} x^n \tag{1}$$

together with the **hypergeometric differential equation** it satisfies

$$x(x-1)\cdot F'' + \big(c-(1+a+b)x\big)\cdot F' - ab\cdot F = 0 \tag{2}$$

certainly constitute one of the most beautiful and important parts of the theory of special functions and of complex geometry of 19th century mathematics and has been studied by many generations of mathematicians since its first appearance in the work of Euler (see [28, Chap. I] for an historical account).

The link between the solutions of (2) and complex geometry is particularly well illustrated by the following very famous results obtained by Schwarz in [66]: he proved that when the parameters a, b and c are real and such that the three values $|1-c|$, $|c-a-b|$ and $|a-b|$ all are strictly less than 1, if F_1 and F_2 form a local basis of the space of solutions of (2) at a point distinct from one of the three singularities $0, 1$ and ∞ of the latter, then after analytic continuation, the associated (multivalued) **Schwarz's map**

$$S(a,b,c;\cdot) = \big[F_1 : F_2\big] : \widetilde{\mathbb{P}^1 \setminus \{0,1,\infty\}} \longrightarrow \mathbb{P}^1$$

actually has values into $\mathbb{CH}^1 \subset \mathbb{P}^1$ and induces a conformal isomorphism from the upper half-plane $\mathbb{H} \subset \mathbb{P}^1 \setminus \{0,1,\infty\}$ onto a hyperbolic triangle[1]. Even if it is multivalued, $S(a,b,c;\cdot)$ can be used to pull-back the standard complex hyperbolic structure of $\mathbb{CH}^1$ and to endow $\mathbb{P}^1 \setminus \{0,1,\infty\}$ with a well-defined complete hyperbolic structure with conical singularities of angle $2\pi|1-c|$, $2\pi|c-a-b|$ and $2\pi|a-b|$ at $0, 1$ and ∞ respectively.

It has been known very early[2] that the following **hypergeometric integral**

$$F(x) = \int_0^1 t^{a-1}(1-t)^{c-a-1}(1-xt)^{-b}dt$$

[1] Actually, Schwarz has proved a more general result that covers not only the hyperbolic case but the euclidean and the spherical cases as well. See *e.g.* [28, Chap.III§3.1] for a modern and clear exposition of the results of [66]

[2] It seems that Legendre was the very first to establish that $F(a,b,c;x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 t^{a-1}(1-t)^{c-a-1}(1-xt)^{-b}dt$ holds true when $|x|< 1$ if a and c verify $0 < a < c$, *cf.* [15, p. 26],

.

is a solution of (2). More generally, for any x distinct from $0,1$ and ∞, any 1-cycle γ in $\mathbb{P}^1\setminus\{0,1,x,\infty\}$ and any determination of the multivalued function $t^{a-1}(1-t)^{c-a-1}(1-xt)^{-b}$ along γ, the (locally well-defined) map

$$F_\gamma(x)=\int_\gamma t^{a-1}(1-t)^{c-a-1}(1-xt)^{-b}dt \tag{3}$$

is a solution of (2) and a basis of the space of solutions can be obtained by taking independent integration cycles (*cf.* [85] for a very pleasant modern exposition of these classical results).

1.1.2. Formula (3) leads naturally to a multi-variable generalization, first considered by Pochammer, Appell and Lauricella, then studied by Picard and his student Levavasseur (among others). Let $\alpha=(\alpha_i)_{i=0}^{n+2}$ be a fixed $(n+3)$-uplet of non-integer real parameters strictly bigger than -1 and such that $\sum_{i=0}^{n+2}\alpha_i=-2$.

Given a $(n+3)$-uplet $x=(x_i)_{i=0}^{n+2}$ of distinct points on $\mathbb{P}^1$, one defines a multivalued holomorphic function of the complex variable t by setting

$$T_x^\alpha(t)=\prod_{i=0}^{n+2}(t-x_i)^{\alpha_i}\,.$$

Then, for any 1-cycle γ supported in $\mathbb{P}^1\setminus\{x\}$ with $\{x\}=\{x_0,\ldots,x_{n+2}\}$ and any choice of a determination of $T_x^\alpha(t)$ along γ, one defines a **generalized hypergeometric integral** as

$$F_\gamma^\alpha(x)=\int_\gamma T_x^\alpha(t)dt=\int_\gamma\prod_{i=0}^{n+2}(t-x_i)^{\alpha_i}dt. \tag{4}$$

Since $T_x^\alpha(t)$ depends holomorphically on x and since γ does not meet any of the x_i's, F_γ^α is holomorphic as well. In fact, it is natural to normalize the integrant by considering only $(n+3)$-uplets x's normalized such that $x_0=0, x_1=1$ and $x_{n+2}=\infty$. This amounts to consider (4) as a multivalued function defined on the moduli space $\mathcal{M}_{0,n+3}$ of projective equivalence classes of $n+3$ distinct points on $\mathbb{P}^1$. As in the 1-dimensional case, it can be proved that the generalized hypergeometric integrals (4) are solutions of a linear second-order differential system in n variables which has to be seen as a multi-dimensional generalization of Gauß hypergeometric equation (2). Moreover, one obtains a basis of the space of solutions of this differential system by considering the (germs of) holomorphic functions $F_{\gamma_0}^\alpha,\ldots,F_{\gamma_n}^\alpha$ for some 1-cycles $\gamma_0,\ldots,\gamma_n$ forming a basis of a certain group of twisted homology.

1.1.3. In this multidimensional context, the associated **generalized Schwarz's map** is the multivalued map

$$F^\alpha=\left[F_{\gamma_i}^\alpha\right]_{i=0}^n:\widetilde{\mathcal{M}_{0,n+2}}\longrightarrow\mathbb{P}^n.$$

It can be proved that the monodromy of this multivalued function on $\mathcal{M}_{0,n+3}$ leaves invariant an hermitian form H^α on $\mathbb{C}^{n+1}$ whose signature is $(1,n)$ when all the α_i's are assumed to be negative. In this case:

- F^α is an étale map with values into the image in $\mathbb{P}^n$ of the complex ball $\{H^\alpha < 1\}$ which is a model of the complex hyperbolic space $\mathbb{CH}^n$;
- the monodromy group Γ^α of F^α is the image of the monodromy representation μ^α of the fundamental group of $\mathcal{M}_{0,n+3}$ in

$$\mathrm{PU}\big(\mathbb{C}^{n+1}, H^\alpha\big) \simeq \mathrm{PU}(1,n) = \mathrm{Aut}\big(\mathbb{CH}^n\big).$$

As in the classical 1-dimensional case, these results imply that there is a natural a priori non-complete complex hyperbolic structure on $\mathcal{M}_{0,n+3}$, obtained as the pull-back of the standard one of $\mathbb{CH}^n$ by the Schwarz map. We will denote by $\mathcal{M}_{0,\alpha}$ the moduli space $\mathcal{M}_{0,n+3}$ endowed with this $\mathbb{CH}^n$-structure.

Several authors (Picard, Levavasseur, Terada, Deligne-Mostow) have studied the case when the image of the monodromy $\Gamma^\alpha = \mathrm{Im}(\mu^\alpha)$ is a discrete subgroup of $\mathrm{PU}(1,n)$. In this case, the metric completion of $\mathcal{M}_{0,\alpha}$ is an orbifold isomorphic to a quotient orbifold $\mathbb{CH}^n/\Gamma^\alpha$. Deligne and Mostow have obtained very satisfying results on this problem: in [11, 53] (completed in [54]) they gave an arithmetic criterion on the α_i's, denoted by ΣINT, which is necessary and sufficient (up to a few known cases) to ensure that the hypergeometric monodromy group Γ^α is discrete. Moreover, they have determined all the α's satisfying this criterion and have obtained a list of 94 complex hyperbolic orbifolds of dimension ≥ 2 constructed via the theory of hypergeometric functions. Finally, they obtain that some of these orbifolds are non-arithmetic.

1.1.4. In [71], taking a different approach, Thurston obtains very similar results to Deligne-Mostow's. His approach is more geometric and concerns moduli spaces of flat Euclidean structures on $\mathbb{P}^1$ with $n+3$ conical singularities. For $x \in \mathcal{M}_{0,n+3}$, the metric $m_x^\alpha = |T_x^\alpha(t)dt|^2$ defines a flat structure on $\mathbb{P}^1$ with conical singularities at the x_i's. The bridge between the hypergeometric theory and Thurston' approach is made by the map $x \mapsto m_x^\alpha$.

Using surgeries for flat structures on the sphere as well as Euclidean polygonal representation of such objects, Thurston recovers geometrically Deligne-Mostow's criterion as well as the finite list of 94 complex hyperbolic orbifold quotients. More generally, he proves that for any $\alpha = (\alpha_i)_{i=0}^{n+2} \in]-1,0[^{n+3}$ and not only for the (necessarily rational) ones satisfying ΣINT, the metric completion $\overline{\mathcal{M}}_{0,\alpha}$ carries a complex hyperbolic conifold structure (see [71, 43] or [19] for this notion) which extends the $\mathbb{CH}^n$-structure of the moduli space $\mathcal{M}_{0,\alpha}$.

1.1.5. In the very interesting (but long and hard-reading hence not so well-known) paper [76], Veech generalizes some parts of the preceding constructions to Riemann surfaces of arbitrary genus g. Veech's starting point is a nice result by Troyanov [73] asserting that for any $\alpha = (\alpha_i)_{i=1}^n \in]-1,\infty[^n$ such that

$$\sum_{i=1}^{n} \alpha_i = 2g-2 \tag{5}$$

and any Riemann surface X with a n-uplet $x = (x_i)_{i=1}^n$ of marked distinct points on it, there exists a unique flat metric m_x^α of area 1 on X with conical singularities of angle $\theta_i = 2\pi(1+\alpha_i)$ at x_i for every $i = 1,\ldots,n$ in the conformal class associated to the complex structure of X.

From this, Veech obtains a real analytic isomorphism

$$\begin{aligned} \mathscr{T}eich_{g,n} &\simeq \mathscr{E}_{g,n}^\alpha \\ \big[(X,x)\big] &\mapsto \big[\big(X, m_x^\alpha\big)\big] \end{aligned} \tag{6}$$

between the Teichmüller space $\mathscr{T}eich_{g,n}$ of n-marked Riemann surfaces of genus g and the space $\mathscr{E}_{g,n}^\alpha$ of (isotopy classes of) flat Euclidean structures with n conical points of angles $\theta_1,\ldots,\theta_n$ on the marked surfaces of the same type.

Using (6) to identify the Teichmüller space with $\mathscr{E}_{g,n}^\alpha$, Veech constructs a real-analytic map

$$H_{g,n}^\alpha : \mathscr{T}eich_{g,n} \longrightarrow \mathbb{U}^{2g} \tag{7}$$

which associates to (the isotopy class of) a marked genus g Riemann surface (X,x) the unitary linear holonomy of the flat structure on X induced by m_x^α.

This map is a submersion and even though it is just real-analytic, Veech proves that any level set

$$\mathscr{F}_\rho^\alpha = \big(H_{g,n}^\alpha\big)^{-1}(\rho)$$

is a complex submanifold of $\mathscr{T}eich_{g,n}$ of complex dimension $2g-3+n$ if $\rho \in \mathrm{Im}(H_{g,n}^\alpha)$ is not trivial. For such a unitary character ρ and under the assumption that none of the α_i's is an integer, Veech introduces a certain space of 1-cocycles $\mathcal{H}_\rho^1$. Then considering not only the linear part but the whole Euclidean holonomies of the elements of $\mathscr{F}_\rho^\alpha$ viewed as classes of flat structures, he constructs a 'complete holonomy map'

$$\mathrm{Hol}_\rho^\alpha : \mathscr{F}_\rho^\alpha \longrightarrow \mathbb{P}\mathcal{H}_\rho^1 \simeq \mathbb{P}^{2g-3+n}$$

and proves first that this map is a local biholomorphism, then that there is a hermitian form H_ρ^α on $\mathcal{H}_\rho^1$ and Hol_ρ^α maps $\mathscr{F}_\rho^\alpha$ into the projectivization $X_\rho^\alpha \subset \mathbb{P}^{2g-3+n}$ of the set $\{H_\rho^\alpha < 0\} \subset \mathcal{H}_\rho^1$ (compare with §1.1.3).

By a long calculation, Veech determines explicitly the signature (p,q) of H_ρ^α and shows that it does depend only on α. The most interesting case is when

$(p,q)=(1,2g-3+n)$. Then Hol^α_ρ takes its values into $X^\alpha_\rho \simeq \mathbb{CH}^{2g-3+n}$. By pull-back by Hol^α_ρ, one endows $\mathscr{F}^\alpha_\rho$ with a natural complex hyperbolic structure.

One occurrence of this situation is when $g=0$ and all the α_i's are in $]-1,0[$: in this case there is only one leaf which is the whole Teichmüller space $\mathscr{T}eich_{0,n}$ and one recovers precisely the case studied by Deligne-Mostow and Thurston.

1.1.6. In addition to the genus 0 case, Veech shows that the complex hyperbolic situation also occurs in one other case, namely when

(8) $\quad g=1$ *and all the* α_i*'s are in* $]-1,0[$ *except one which lies in* $]0,1[$.

In this case, the level-sets $\mathscr{F}^\alpha_\rho$'s of the holonomy map $H^\alpha_{g,n}$ form a real-analytic foliation $\mathscr{F}^\alpha$ of $\mathscr{T}eich_{1,n}$ whose leaves carry natural $\mathbb{CH}^{n-1}$-structures.

A remarkable fact established by Veech is that the pure mapping class group $\mathrm{PMCG}_{1,n}$ leaves this foliation invariant and induces biholomorphisms between the leaves which preserve their respective complex hyperbolic structure. Consequently, all the previous constructions pass to the quotient by $\mathrm{PMCG}_{1,n}$. One finally obtains a foliation on the quotient moduli space $\mathcal{M}_{1,n}$, denoted by $\mathcal{F}^\alpha$, by complex leaves carrying a (possibly orbifold) complex hyperbolic structure. Furthermore, it comes from (7) that $\mathcal{F}^\alpha$ is transversally Euclidean, hence one can endow $\mathcal{M}_{1,n}$ with a natural real-analytic volume form Ω^α.

At this point, interesting questions emerge very naturally:

(1) which are the leaves of $\mathcal{F}^\alpha$ that are algebraic submanifolds of $\mathcal{M}_{1,n}$?
(2) given a leaf of $\mathcal{F}^\alpha$ which is an algebraic submanifold of $\mathcal{M}_{1,n}$, what is its topology? Considered with its $\mathbb{CH}^{n-1}$-structure, has it finite volume?
(3) does the $\mathbb{CH}^{n-1}$-structure of an algebraic leaf extend to its metric completion (possibly as a conifold complex hyperbolic structure)?
(4) which are the algebraic leaves of $\mathcal{F}^\alpha$ whose holonomy representation of their $\mathbb{CH}^{n-1}$-structure has a discrete image in $\mathrm{PU}(1,n-1)$?
(5) is it possible to construct new non-arithmetic complex hyperbolic lattices this way?
(6) is the Ω^α-volume of $\mathcal{M}_{1,n}$ finite as conjectured by Veech in [76]?

In view of what has been done in the genus 0 case, one can distinguish two distinct ways to address such questions. The first, à la Thurston, by using geometric arguments relying on surgeries on flat surfaces. The second, à la Deligne-Mostow, through a more analytical and cohomological reasoning.

Our work shows that both approaches are possible, relevant and fruitful. In the twin paper [19], we generalize Thurston's approach whereas in the present text, we generalize the one of Deligne and Mostow to the genus 1 case.

1.2. **Results.** We give below a short review of the results contained in the papers. All of them are new, even if some of them (namely the first ones) are obtained by rather elementary considerations. We present them below in decreasing order of generality, which essentially corresponds to their order of appearance in the text.

Throughout the text, g and n will always refer respectively to the genus of the considered surfaces and to the number of cone points they carry and it will always be assumed that $2g-3+n>0$.

1.2.1. Our first results just consist in a general remark followed by a natural construction concerning Veech's constructions, whichever g and n are.

Let N be a flat surface with conical singularities whose isotopy class belongs to a moduli space $\mathscr{E}^{\alpha}_{g,n} \simeq \mathscr{T}eich_{g,n}$ for some n-uplet α as in §1.1.5. Since the target space of the associated linear holonomy character $\rho : \pi_1(N) \to \mathbb{U}$ is abelian, the latter factors through the abelianization of $\pi_1(N)$, namely the first homology group $H_1(N,\mathbb{Z})$. From this simple remark, one deduces that the linear holonomy map (7) actually factors through the quotient map from $\mathscr{T}eich_{g,n}$ onto the associated Torelli space and consequently, Veech's foliation $\mathscr{F}^{\alpha}$ on $\mathscr{T}or_{g,n}$ admits a global first integral $\mathscr{T}or_{g,n} \to \mathbb{U}^{2g}$, which will be denoted by $h^{\alpha}_{g,n}$.

Let $e : \mathbb{R}^{2g} \to \mathbb{U}^{2g}$ be the group morphism whose components all are the map $s \mapsto \exp(2i\pi s)$. Our second point is that, using classical geometric facts about simple closed curves on surfaces, one can construct a lift $\widetilde{H}^{\alpha}_{g,n} : \mathscr{T}eich_{g,n} \to \mathbb{R}^{2g}$ of Veech's first integral (7). These results can be summarized in the following

Proposition 1.1. *There are canonical real-analytic maps $\widetilde{H}^{\alpha}_{g,n}$ and $h^{\alpha}_{g,n}$ (in blue below) making the following diagram commutative:*

$$\begin{array}{ccc} \mathscr{T}eich_{g,n} & \xrightarrow{\widetilde{H}^{\alpha}_{g,n}} & \mathbb{R}^{2g} \\ \downarrow & \searrow^{H^{\alpha}_{g,n}} & \downarrow e \\ \mathscr{T}or_{g,n} & \xrightarrow[h^{\alpha}_{g,n}]{} & \mathbb{U}^{2g}. \end{array}$$

This result shows that it is more natural to study Veech's foliation on the Torelli space $\mathscr{T}or_{g,n}$. Note that the latter is a nice complex variety without orbifold point. Furthermore, the existence of the lift $\widetilde{H}^{\alpha}_{g,n}$ strongly suggests that the level-subsets of Veech's first integral $H^{\alpha}_{g,n}$ are not connected a priori.

1.2.2. We now consider only the case of elliptic curves and specialize everything to the case when $g = 1$.

First of all, by simple geometric considerations specific to this case, one verifies that the lifted holonomy $\widetilde{H}^{\alpha}_{1,n}$ descends to the corresponding Torelli space. In other terms: there exists a real-analytic map $\widetilde{h}^{\alpha}_{1,n} : \mathscr{T}or_{1,n} \to \mathbb{R}^2$ which fits into the diagram above and makes it commutative.

From now on, we do no longer make abstract considerations but undertake the opposite approach by expliciting everything as much as possible.

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In the genus 0 case, the link between the 'flat surfaces' approach à la Thurston and the 'hypergeometric' one à la Deligne-Mostow comes from the fact that there is an explicit formula for a flat metric with conical singularities on the Riemann sphere (see §1.1.4 above). The crucial point of this paper is that something equivalent can be done in the $g = 1$ case.

Assume that $\alpha_1, \ldots, \alpha_n$ are fixed real numbers bigger than -1 such that $\sum_i \alpha_i = 0$. For $\tau \in \mathbb{H}$, let $E_\tau = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau)$ be the associated elliptic curve. Then, given $z = (z_i)_{i=1}^n \in \mathbb{C}^n$ such that $[z_1], \ldots, [z_n]$ are n distinct points on E_τ, Troyanov's theorem (*cf.* §1.1.5) ensures that, up to normalization, there exists a unique flat metric $m^{\alpha}_{\tau,z}$ on E_τ with a singularity of type $|u^{\alpha_i} du|^2$ at $[z_i]$ for $i = 1, \ldots, n$.

One can give an explicit formula for this metric by means of theta functions:

Proposition 1.2. *Up to normalization, one has*

$$m^{\alpha}_{\tau,z} = \left| T^{\alpha}_{\tau,z}(u) du \right|^2$$

where $T^{\alpha}_{\tau,z}$ is the following multivalued holomorphic function on E_τ:

$$T^{\alpha}_{\tau,z}(u) = \exp\big(2i\pi a_0 u\big) \prod_{i=1}^{n} \theta\big(u - z_i, \tau\big)^{\alpha_i} \tag{9}$$

where θ stands for Jacobi's theta function (16) *and a_0 is given by*

$$a_0 = a_0(\tau, z) = -\frac{\Im\mathrm{m}\big(\sum_{i=1}^n \alpha_i z_i\big)}{\Im\mathrm{m}(\tau)}.$$

While the preceding formula is easy to establish[3], it is the key result on which the rest of the paper relies. Indeed, the 'explicitness' of the above formulae for $T^{\alpha}_{\tau,z}$ and a_0 will propagate and this will allow us to make all Veech's constructions completely explicit in the case of elliptic curves.

[3]It essentially amounts to verify that the monodromy of $T^{\alpha}_{\tau,z}$ on the n-punctured elliptic curve $E_{\tau,z} = E_\tau \setminus \{[z_1], \ldots, [z_n]\}$ is multiplicative and unitary.

1.2.3. Another key ingredient is that there exists a nice and explicit description of the Torelli spaces of marked elliptic curves: this result, due to Nag [55], can be summarized by saying that the parameters $\tau \in \mathbb{H}$ and $z \in \mathbb{C}^n$ as above provide global holomorphic coordinates on $\mathscr{T}or_{1,n}$ for any $n \geq 1$, if we assume the normalization $z_1 = 0$.

Using the coordinates (τ, z) on $\mathscr{T}or_{1,n}$, it is then easy to prove the

Proposition 1.3. *For $(\tau, z) \in \mathscr{T}or_{1,n}$, one sets*

$$a_\infty(\tau, z) = a_0(\tau, z)\tau + \sum_{i=1}^{n} \alpha_i z_i \in \mathbb{R}.$$

(1) *The following map is a primitive first integral of Veech's foliation:*

$$\begin{aligned} \xi^\alpha : \mathscr{T}or_{1,n} &\longrightarrow \mathbb{R}^2 \\ (\tau, z) &\longmapsto \big(a_0(\tau, z), a_\infty(\tau, z)\big). \end{aligned}$$

(2) *One has* $\mathrm{Im}(\xi^\alpha) = \mathbb{R}^2$ *if* $n \geq 3$ *and* $\mathrm{Im}(\xi^\alpha) = \mathbb{R}^2 \setminus \alpha_1 \mathbb{Z}^2$ *if* $n = 2$.

(3) *For $a = (a_0, a_\infty) \in \mathrm{Im}(\xi^\alpha)$, the leaf $\mathscr{F}_a^\alpha = (\xi^\alpha)^{-1}(a)$ in $\mathscr{T}or_{1,n}$ is cut out by*

$$a_0 \tau + \sum_{i=1}^{n} \alpha_i z_i = a_\infty. \tag{10}$$

(4) *Veech's foliation $\mathscr{F}^\alpha$ on $\mathscr{T}or_{1,n}$ only depends on $[\alpha] \in \mathbb{P}(\mathbb{R}^n)$.*

Point (1) above shows that each level-set $\mathscr{F}_\rho^\alpha = (h_{1,n}^\alpha)^{-1}(\rho)$ of the linear holonomy map $h_{1,n}^\alpha : \mathscr{T}or_{1,n} \to \mathbb{U}^2$ is a countable disjoint union of leaves $\mathscr{F}_a^\alpha$'s. Point (2) answers a question of [76]. Finally (3) makes the general and abstract result of Veech mentioned in §1.1.5 completely explicit in the $g = 1$ case.

1.2.4. The pure mapping class group $\mathrm{PMCG}_{1,n}$ does not act effectively on the Torelli space. Indeed, $\mathscr{T}or_{1,n}$ can be seen abstractly as the quotient of $\mathscr{T}eich_{1,n}$ by the normal subgroup of the pure mapping class group formed by mapping classes which act trivially on the homology of the model n-punctured 2-torus. The latter is called the **Torelli group** and is denoted by $\mathrm{Tor}_{1,n}$.

Another key ingredient for what comes next is the fact that

$$\mathrm{Sp}_{1,n}(\mathbb{Z}) := \mathrm{PMCG}_{1,n} \big/ \mathrm{Tor}_{1,n}$$

as well as its action in the coordinates (τ, z) on $\mathscr{T}or_{1,n}$ can be made explicit.

For instance, there is an isomorphism

$$\mathrm{Sp}_{1,n}(\mathbb{Z}) \simeq \mathrm{SL}_2(\mathbb{Z}) \ltimes \left(\mathbb{Z}^2\right)^{n-1}$$

with the $\mathrm{SL}_2(\mathbb{Z})$-part acting as usual on the variable $\tau \in \mathbb{H}$.

It is then straightforward to determine, first which are the lifted holonomies $a \in \mathrm{Im}(\xi^\alpha)$ whose orbits under $\mathrm{Sp}_{1,n}(\mathbb{Z})$ are discrete; then, for such a holonomy a, what is the image $\mathcal{F}_a^\alpha = \pi(\mathscr{F}_a^\alpha) \subset \mathcal{M}_{1,n}$ of the leaf $\mathscr{F}_a^\alpha$ by the quotient map

$$\pi : \mathscr{T}\!or_{1,n} \longrightarrow \mathcal{M}_{1,n} = \mathscr{T}\!or_{1,n}/\mathrm{Sp}_{1,n}(\mathbb{Z}).$$

A m-uplet $\nu = (\nu_i)_{i=1}^n \in \mathbb{R}^m$ is said to be **commensurable** if there exists a constant $\lambda \neq 0$ such that $\lambda\nu = (\lambda\nu_i)_{i=1}^n$ is rational, *i.e.* belongs to $\mathbb{Q}^n$.

Theorem 1.4.
1. *Veech's foliation $\mathcal{F}^\alpha$ on $\mathcal{M}_{1,n}$ admits algebraic leaves if and only if α is commensurable.*
2. *The leaf $\mathcal{F}_a^\alpha$ is an algebraic subvariety of $\mathcal{M}_{1,n}$ if and only if the $(n+2)$-uplet of real numbers (α, a) is commensurable.*

Actually, under the assumption that α is commensurable, one can give an explicit description of the algebraic leaves of $\mathcal{F}^\alpha$. The case when $n = 2$ is particular and will be treated very carefully in §1.2.6 below. But the consideration of the case when $n = 3$ already suggests what happens more generally and we will give a general description of an algebraic leaf $\mathcal{F}_a^\alpha \subset \mathcal{M}_{1,3}$.

First, we remark that the $\mathrm{SL}_2(\mathbb{Z})$-part of $\mathrm{Sp}_{1,3}(\mathbb{Z})$ acts in a natural way on $a \in \mathrm{Im}(\xi^\alpha)$ and that, if the corresponding leaf $\mathcal{F}_a^\alpha$ is algebraic, the latter has a discrete orbit and the image S_a of its stabilizer in $\mathrm{SL}_2(\mathbb{Z})$ is 'big' (*i.e.* of finite index). Second we recall that the linear projection $\mathscr{T}\!or_{1,n} \to \mathbb{H}$, $(\tau, z) \mapsto \tau$ passes to the quotient and induces the map $\mathcal{M}_{1,n} \to \mathcal{M}_{1,1}$, which corresponds to forgetting the last $n-1$ points of a n-marked elliptic curve.

Theorem 1.5. *Assume that $\mathcal{F}_a^\alpha \subset \mathcal{M}_{1,3}$ is an algebraic leaf of $\mathcal{F}^\alpha$.*
1. *There exists an integer N_a such that $S_a \simeq \Gamma_1(N_a)$;*
2. *The leaf $\mathcal{F}_a^\alpha$ is isomorphic to the total space of the elliptic modular surface $\mathcal{E}_1(N_a) \to Y_1(N_a)$ from which the union of a finite number of torsion multi-sections has been removed.*

Actually, the way in which this result is stated here is not completely correct since a finite number of pathological cases do occur. Anyway, it applies to most of the algebraic leaves and can actually be made more precise and explicit: for instance, there is exactly one algebraic leaf for each integer $N > 0$, one can give N_a in terms of a and it is possible to list which are the torsion multisections to be removed from $\mathcal{E}_1(N_a)$ in order to get the leaf $\mathcal{F}_a^\alpha$.

1.2.5. From (10), it comes that $(\tau, z') = (\tau, z_3, \ldots, z_n)$ forms a system of global coordinates on any leaf $\mathscr{F}_a^\alpha$. For $(\tau, z') \in \mathscr{F}_a^\alpha$, we denote by (τ, z) the element of $\mathscr{T}\!or_{1,n}$ where z_2 is obtained from (τ, z') by solving the affine equation (10).

Our next result is about an explicit expression, in these coordinates, of the restriction to $\mathscr{F}_a^\alpha$, denoted by V_a^α, of Veech's full holonomy map Hol_a^α of §1.1.5[4]. From Proposition 1.2, it comes immediately that for any $(\tau, z) \in \mathscr{F}_a^\alpha$ fixed,

$$z \longmapsto \int^z T_{\tau,z}^\alpha(u)du$$

is 'the' developing map of the corresponding flat structure on the punctured elliptic curve $E_{\tau,z} = E_\tau \setminus \{[z_1], \dots, [z_n]\}$. Consequently, there is a local analytic expression for V_a^α whose components are obtained by integrating a fixed determination of the multivalued 1-form $T_{\tau,z}^\alpha(u)du$ along certain 1-cycles in $E_{\tau,z}$.

Using some results of Mano and Watanabe [49], one can extend to our situation the analytico-cohomological approach used in the genus 0 case by Deligne and Mostow in [11]. More precisely, for $(\tau, z) \in \mathscr{T}or_{1,n}$, let $L_{\tau,z}^\vee$ be the local system on $E_{\tau,z}$ whose local sections are given by local determinations of $T_{\tau,z}^\alpha$. Following [49], one defines some $L_{\tau,z}^\vee$-twisted 1-cycles $\boldsymbol{\gamma}_0, \boldsymbol{\gamma}_2, \dots, \boldsymbol{\gamma}_n, \boldsymbol{\gamma}_\infty$ by taking regularizations of the relative twisted 1-simplices obtained by considering certain determinations of $T_{\tau,z}^\alpha$ along the segments $\ell_0, \ell_2, \dots, \ell_n, \ell_\infty$ on $E_{\tau,z}$ represented in Figure 1 below.

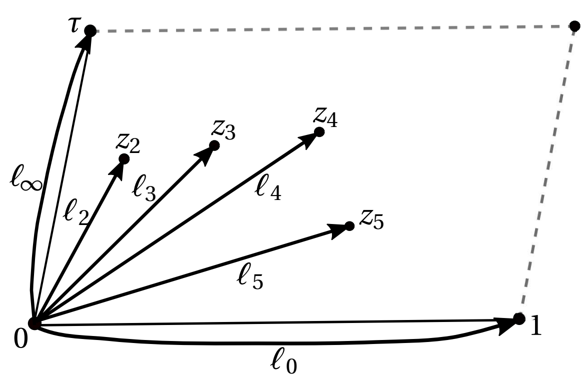

FIGURE 1. For $\bullet = 0, 2, \dots, n, \infty$, $\ell_\bullet$ is the image in the n-punctured elliptic curve $E_{\tau,z}$ of the segment $]0, z_\bullet[$ (with $z_0 = 1$, $z_\infty = \tau$ and assuming the normalization $z_1 = 0$).

By using that the $\boldsymbol{\gamma}_\bullet$'s for $\bullet = 0, 2, \dots, n-1, \infty$ induce a basis of $H_1(E_{\tau,z}, L_{\tau,z}^\vee)$ (*cf.* [49]) and they can be locally continuously extended on the Torelli space, one obtains the following result:

[4]As a global holomorphic map, Veech's map is only defined on the corresponding leaf in $\mathscr{T}eich_{1,n}$. On $\mathscr{F}_a^\alpha \subset \mathscr{T}or_{1,n}$, it has to be considered as a global multivalued holomorphic function, except if this leaf has no topology (as when $n = 2$, a case such that $\mathscr{F}_a^\alpha \simeq \mathbb{H}$ for any a).

Proposition 1.6. (1) *The Veech map of* $\mathscr{F}_a^\alpha$ *has a local analytic expression*

$$V_a^\alpha : (\tau, z') \mapsto \left[F_0(\tau,z) : F_3(\tau,z) : \cdots : F_n(\tau,z) : F_\infty(\tau,z)\right]$$

where for $\bullet = 0, 3, \ldots, n, \infty$*, the component* $F_\bullet$ *is the (locally defined)* **elliptic hypergeomeric integral** *depending on* $(\tau, z) \in \mathscr{T}or_{1,n}$ *defined as*

$$F_\bullet : (\tau, z) \longmapsto \int_{\boldsymbol{\gamma}_\bullet} T_{\tau,z}^\alpha(u)\, du\,.$$

(2) *The matrix of Veech's hermitian form* H_a^α *on* $\mathbb{C}^{n+1}$ *(cf. §1.1.5) in the coordinates associated to the components* $F_\bullet$ *of* V_a^α *considered in* (1) *can be obtained from the twisted intersection products* $\boldsymbol{\gamma}_\bullet \cdot \boldsymbol{\gamma}_\circ^\vee$ *for* $\bullet$ *and* $\circ$ *ranging in* $\{0, 3, \ldots, n, \infty\}$*, all of which can be explicitly computed (see §3.4).*

1.2.6. We now turn to the case of elliptic curves with two conical points. In this case $\alpha = (\alpha_1, \alpha_2)$ is such that $\alpha_2 = -\alpha_1$, so one can take $\alpha_1 \in]0,1[$ as the main parameter and consequently replace α by α_1 in all the notations. For instance, Veech's foliations on $\mathscr{T}or_{1,2}$ and $\mathcal{M}_{1,2}$ will be denoted respectively by $\mathscr{F}^{\alpha_1}$ and $\mathcal{F}^{\alpha_1}$ from now on. It follows from the fourth point of Proposition 1.3 that theses foliations do not depend on α_1. In this case, the leaves of $\mathcal{F}^{\alpha_1}$, and in particular the algebraic ones, can be described very precisely.

Note that for any leaf $\mathscr{F}_a^{\alpha_1}$ of $\mathscr{F}^{\alpha_1}$, the natural projection $\mathscr{T}or_{1,2} \to \mathbb{H}$ induces by restriction an isomorphism $\mathscr{F}_a^{\alpha_1} \simeq \mathbb{H}$. We denote by $\pi_a : \mathscr{F}_a^{\alpha_1} \to \mathcal{F}_a^{\alpha_1}$ the restriction to $\mathscr{F}_a$ of the quotient map $\pi : \mathscr{T}or_{1,2} \to \mathcal{M}_{1,2}$.

Theorem 1.7. *For any leaf* $\mathcal{F}_a^{\alpha_1}$ *in* $\mathcal{M}_{1,2}$*, one of the following situations occurs:*

(1) *either the quotient mapping* π_a *is trivial, hence* $\mathcal{F}_a^{\alpha_1} \simeq \mathbb{H}$*; or*

(2) *the quotient mapping* π_a *is isomorphic to that of* $\mathbb{H}$ *by* $\tau \mapsto \tau + 1$*, hence* $\mathcal{F}_a^{\alpha_1}$ *is conformally isomorphic to an infinite cylinder; or*

(3) *the leaf* $\mathcal{F}_a^{\alpha_1}$ *is algebraic. If* N *stands for the smallest positive integer such that* $Na \in \alpha_1 \mathbb{Z}^2$*, then* $N \geq 2$ *and* $\mathcal{F}_a^{\alpha_1}$ *coincides with the image of*

$$\begin{aligned} \mathbb{H}/_{\Gamma_1(N)} &\longrightarrow \mathcal{M}_{1,2} \\ \left(E_\tau, \left[\frac{1}{N}\right]\right) &\longmapsto \left(E_\tau, [0], \left[\frac{1}{N}\right]\right), \end{aligned} \tag{11}$$

hence is isomorphic to the modular curve $Y_1(N) = \mathbb{H}/\Gamma_1(N)$.

We thus have described the conformal types of the leaves of $\mathcal{F}^{\alpha_1}$ which are independent from α_1. We now want to go further and describe Veech's complex hyperbolic structures of the leaves and these depend on α_1. Of course, our main interest will be in the algebraic leaves of Veech's foliation.

1.2.7. Let $a=(a_0,a_\infty)\in \mathrm{Im}(\xi^{\alpha_1})$ be fixed. The leaf $\mathscr{F}_a^{\alpha_1}$ in $\mathscr{T}or_{1,2}$ is cut out by the following affine equation in the variables τ and z_2 (*cf.* Proposition 1.3.(3))

$$z_2=t_\tau=\frac{1}{\alpha_1}\big(a_0\tau-a_\infty\big),$$

hence is (conformally) isomorphic to the upper half-plane $\mathbb{H}$.

For $\tau\in\mathbb{H}$, let $T_a^{\alpha_1}(\cdot,\tau)$ be the multivalued holomorphic function defined by

$$T_a^{\alpha_1}(u,\tau)=\exp\big(2i\pi a_0 u\big)\frac{\theta(u,\tau)^{\alpha_1}}{\theta\,(u-t_\tau,\tau)^{\alpha_1}}$$

for $u\in\mathbb{C}$ distinct from 0 and $t(\tau)$ modulo $\mathbb{Z}\oplus\tau\mathbb{Z}$.

One considers the following two holomorphic functions of $\tau\in\mathbb{H}$:

$$F_0(\tau)=\int_{[0,1]}T_a^{\alpha_1}(u,\tau)du \qquad \text{and} \qquad F_\infty(\tau)=\int_{[0,\tau]}T_a^{\alpha_1}(u,\tau)du. \tag{12}$$

Specializing the results of §1.2.5, one obtains the

Proposition 1.8. *There exists a fractional transformation $z\mapsto(Az+B)/(Cz+D)$, and examples of such maps can be given explicitly (see §4.4.5) so that*

$$V_a^{\alpha_1}=\frac{A\cdot F_0+B\cdot F_\infty}{C\cdot F_0+D\cdot F_\infty}:\mathbb{H}\longrightarrow\mathbb{P}^1,$$

is a model of the Veech map of the leaf $\mathscr{F}_a^{\alpha_1}\simeq\mathbb{H}$ which

(1) *takes values into the upper half-plane $\mathbb{H}$;*
(2) *is such that Veech's complex hyperbolic structure of $\mathscr{F}_a^{\alpha_1}$ is the pull-back by V of the standard one of $\mathbb{H}$.*

It follows that the Schwarzian differential equation characterizing Veech's hyperbolic structure of $\mathscr{F}_a^{\alpha_1}$ can be obtained from the second-order differential equation $(E_a^{\alpha_1})$ on $\mathbb{H}$ satisfied by F_0 and F_∞. The definition (12) of these two functions being explicit, one can compute $(E_a^{\alpha_1})$ explicitly.

In order to do so, we specialize some results of [49] and determine explicitly a certain Gauß-Manin connection on $\mathscr{F}_a^{\alpha_1}$. Let $\mathscr{E}_a\to\mathscr{F}_a^{\alpha_1}\simeq\mathbb{H}$ be the universal 2-punctured curve over $\mathscr{F}_a^{\alpha_1}$ whose fiber at $\tau\in\mathbb{H}$ is the punctured elliptic curve $E_{\tau,t_\tau}=E_\tau\setminus\{[0],[t_\tau]\}$. There is a line bundle L_a on $\mathscr{E}_a$ the restriction of which on any fiber E_{τ,t_τ} coincides with the line bundle L_τ on the latter defined by the multivalued function $T_a^{\alpha_1}(\cdot,\tau)$. The push-forward of L_a onto $\mathscr{F}_a^{\alpha_1}$ is a local system of rank 2, denoted by B_a, whose fiber at τ is nothing else but the first group of twisted cohomology $H^1(E_{\tau,t_\tau},L_\tau)$ considered above in §1.2.5.

One sets $\mathscr{B}_a=B_a\otimes\mathscr{O}_{\mathbb{H}}$. We are interested in the Gauß-Manin connection $\nabla_a^{GM}:\mathscr{B}_a\to\mathscr{B}_a\otimes\Omega^1_{\mathbb{H}}$ whose flat sections are the sections of B_a. Following [49], one defines two trivializing explicit global sections $[\varphi_0]$ and $[\varphi_1]$ of $\mathscr{B}_a$.

Proposition 1.9. (1) *In the basis* $([\varphi_0],[\varphi_1])$*, the action of* ∇_a^{GM} *is written*

$$\nabla_a^{GM}\begin{pmatrix}[\varphi_0]\\ [\varphi_1]\end{pmatrix} = M_a\cdot\begin{pmatrix}[\varphi_0]\\ [\varphi_1]\end{pmatrix}$$

for a certain explicit matrix M_a *of holomorphic 1-forms on* $\mathbb{H}$.

(2) *The differential equation on* $\mathbb{H}$ *with* F_0, F_∞ *as a basis of solutions is written*

$$\big(E_a^{\alpha_1}\big) \qquad \ddot{F} - \big(2i\pi a_0^2/\alpha_1\big)\cdot\dot{F} + \varphi_a\cdot F = 0$$

for an explicit global holomorphic function φ_a *on* $\mathbb{H}$.

The interest of this result lies in the fact that everything can be explicited. It will be our main tool to study the $\mathbb{CH}^1$-structures of the algebraic leaves of $\mathcal{F}^{\alpha_1}$ in $\mathcal{M}_{1,2}$.

1.2.8. Let $N\geq 2$ be fixed. For $(k,l)\in\mathbb{Z}^2\setminus N\mathbb{Z}^2$, let $\mathscr{F}_{k,l}^{\alpha_1}(N)$ be the leaf of Veech's foliation on $\mathscr{T}or_{1,2}$ cut out by $z_2=(k/N)\tau+l/N$. It is isomorphic to $\mathbb{H}$ and its image in $\mathcal{M}_{1,2}$ is precisely the image of the embedding (11). When endowed with Veech's $\mathbb{CH}^1$-structure, we denote the latter by $Y_1(N)^{\alpha_1}$ to emphasize the fact that it is $Y_1(N)$ but with a deformation of its usual hyperbolic structure.

Under the assumption that α_1 is rational, it follows from our main result in [19] that for any $N\geq 2$, Veech's hyperbolic structure of $Y_1(N)^{\alpha_1}$ extends as a conifold $\mathbb{CH}^1$-structure to its metric completion $\overline{Y_1(N)}^{\alpha_1}$. The point is that using Proposition 1.9, one can recover this result without the rationality assumption on α_1 and precisely characterize this conifold structure.

Let $X_1(N)$ be the compactification of $Y_1(N)$ obtained by adding to it its set of cusps $C_1(N)$. For $\mathfrak{c}\in C_1(N)$, two situations can occur: either $Y_1(N)^{\alpha_1}$ is metrically complete in the vicinity of $\mathfrak{c}$, either it is not. In the first case, $\mathfrak{c}$ is a cusp in the classical sense[5] and will be called a **conifold point of angle 0**.

To study the geometric structure of $Y_1(N)^{\alpha_1}$ near a cusp $\mathfrak{c}\in C_1(N)$, our approach consists in looking at the Schwarzian differential equation associated to Veech's $\mathbb{CH}^1$-structure on a small punctured neighborhood $U_\mathfrak{c}$ of $\mathfrak{c}$ in $Y_1(N)$.

First, one verifies that there exist k and l such that $\mathscr{F}_{k,l}^{\alpha_1}(N)\to Y_1(N)^{\alpha_1}$ is a uniformization which sends $[i\infty]$ onto $\mathfrak{c}$. Then, since the functions F_0, F_∞ defined in (12) (with the corresponding a, namely $a=\alpha_1(k/N,-l/N)$) are components of the Veech map on $\mathscr{F}_{k,l}^{\alpha_1}(N)$, they can be viewed as the components of the developing map of Veech's hyperbolic structure of $Y_1(N)^{\alpha_1}$. So, looking at the asymptotic behavior of (E_a^α) when τ tends to $i\infty$ while belonging to a

[5] *I.e.* $(Y_1(N)^{\alpha_1},\mathfrak{c})\simeq(\mathbb{H}/(z\mapsto z+1),[i\infty])$ as germs of punctured hyperbolic surfaces.

vertical strip of width equal to that of $\mathfrak{c}$, one obtains that the Schwarzian differential equation of the $\mathbb{CH}^1$-curve $Y_1(N)^{\alpha_1}$ is Fuchsian at $\mathfrak{c}$ and one can compute explicitly the two characteristic exponents at this point.

Then, since a cusp $\mathfrak{c} \in C_1(N)$ is a class modulo $\Gamma_1(N)$ of a rational element of the boundary $\mathbb{P}^1_{\mathbb{R}} \simeq S^1$ of the closure of $\mathbb{H}$ in $\mathbb{P}^1$ hence as such, is written $\mathfrak{c} = [a/c]$ with $a/c \in \mathbb{P}^1_{\mathbb{Q}}$, one eventually gets the following result:

Theorem 1.10. *For any parameter $\alpha_1 \in]0,1[$:*

(1) *Veech's complex hyperbolic structure of $Y_1(N)^{\alpha_1}$ extends as a conifold structure of the same type to the compactification $X_1(N)$. The latter, when endowed with this conifold structure, will be denoted by $X_1(N)^{\alpha_1}$;*

(2) *the conifold angle of $X_1(N)^{\alpha_1}$ at the cusp $\mathfrak{c} = [a/c] \in C_1(N)$ is*

$$\theta_{\mathfrak{c}} = \theta(c) = 2\pi \frac{c'(N-c')}{N \cdot \gcd(c', N)} \cdot \alpha_1$$

where $c' \in \{0, \ldots, N-1\}$ stands for the residue of c modulo N.

According to a classical result going back to Poincaré, a $\mathbb{CH}^1$-conifold structure on a compact Riemann surface is completely characterized by its conifold points and the conifold angles at these points. Thus the preceding theorem completely characterizes $Y_1(N)^{\alpha_1}$ (or rather $X_1(N)^{\alpha_1}$) as a complex hyperbolic conifold. It can be seen as the generalization, to the genus 1 case, of the result by Schwarz on the hypergeometric equation, dating of 1873, evoked in §1.1.1.

Defining N^* as the least common multiple of the integers $c'(N-c')/\gcd(c', N)$ when c' ranges in $\{1, \ldots, N-1\}$, one deduces immediately from above the

Corollary 1.11. *A sufficient condition for $X_1(N)^{\alpha_1}$ to be an orbifold is that*

$$\alpha_1 = \frac{N}{\ell N^*} \qquad \text{for some } \ell \in \mathbb{N}_{>0}.$$

In this case, the image $\mathbf{\Gamma}_1(N)^{\alpha_1}$ of the holonomy representation associated to Veech's $\mathbb{CH}^1$-structure on $Y_1(N)^{\alpha_1}$ is a non-cocompact lattice in $\mathrm{PSL}_2(\mathbb{R})$.

The $\mathbf{\Gamma}_1(N)^{\alpha_1}$'s with $\alpha_1 \in]0,1[$ form a real-analytic deformation of $\mathbf{\Gamma}_1(N) = \mathbf{\Gamma}_1(N)^0$ in $\mathrm{PSL}_2(\mathbb{R})$. The problem of determining which of its elements are lattices (or arithmetic lattices, etc.) is quite interesting but does not seem easy.

An interesting case is when N is equal to a prime number p. It is well-known that $X_1(p)$ is a smooth curve of genus $(p-5)(p-7)/24$ with $p-1$ cusps.

Corollary 1.12. (1) *For $k = 1, \ldots, (p-1)/2$, the conifold angle of $X_1(p)^{\alpha_1}$ at $[-p/k]$ is $2\pi k(1-k/p)\alpha_1$. The $(p-1)/2$ other conical angles are 0.*

(2) *The hyperbolic volume (area) of $Y_1(p)^{\alpha_1}$ is equal to*

$$V_1^{\alpha_1}(p) = \frac{\pi}{6}\big(p^2-1\big)\big(1-\alpha_1\big).$$

If one takes for granted that the measure $p^{-2}\delta_{Y_1(p)^{\alpha_1}}$ tends towards the one associated to Veech's volume form on $\mathcal{M}_{1,2}$ as p tends to infinity among primes, the preceding result gives the following conjectural value for the Ω^{α_1}-volume:

$$\int_{\mathcal{M}_{1,2}} \Omega^{\alpha_1} = \lim_{\substack{p\to+\infty \\ p \text{ prime}}} p^{-2} V_1^{\alpha_1}(p) = \frac{\pi}{6}\big(1-\alpha_1\big).$$

We plan to return to this in some future works.

1.3. **Organization of the paper.** Since this text is quite long, we think that stating what we will do and where in the paper could be helpful to the reader. We then make a few general comments which could also be of help.

1.3.1. In the **first section** of this paper (namely the present one), we first take some time in §1.1 to display some elements about the historical and mathematical background regarding the problem we are interested in. We then present our results in **§1.2**. Finally in **§1.4**, we indicate some of our sources and discuss other works to which the present one is related.

In **Section 2**, we introduce some classical material and fix some notations.

Section 3 is about one of the main tools we use in this paper, namely twisted (co-)homology on Riemann surfaces. After sketching a general theory of what we call '*generalized hypergeometric integrals*', we give a detailed treatment of some results obtained by Mano and Watanabe in [49] concerning the case of punctured elliptic curves. The single novelty here is the explicit computation of the twisted intersection product in **§3.4**. Note also that what is for us the main hero of this text, namely the multivalued function (9), is carefully introduced in **§3.2** where some of its main properties are established.

We begin with two simple general remarks about some constructions of [76] in the first subsection of **Section 4**. One of them leads to the conclusion that Veech's foliation is more naturally defined on the corresponding Torelli space. The relevance of this point of view becomes clear when we start focusing on the genus 1 case in **§4.2**. We then use an explicit description of $\mathcal{T}or_{1,n}$ obtained by Nag, as well as an explicit formula (in terms of the function (9)) for a flat metric with conical singularity on an elliptic curve, to make Veech's foliation $\mathscr{F}^\alpha$ on the Torelli space completely explicit. With that at hand, it is not difficult to obtain some of our main results about $\mathscr{F}^\alpha$, such as Proposition 1.3, Theorem 1.4 or Theorem 1.5. Finally, in **§4.4**, we turn to the study of the Veech map which is used to define the geometric structure (a complex hyperbolic structure under suitable assumptions on the considered conical angles) on the leaves of Veech's foliation. We show that locally, Veech's map admits an analytic description à la Deligne-Mostow in terms of elliptic hypergeometric integrals and we relate Veech's form to the twisted intersection product considered in §3.4.

We specialize and make the previous results more precise in **Section 5** where we restrict ourselves to the case of flat elliptic curves with only two conical points. In this case, one proves that the Torelli space is isomorphic to a product and that, up to this isomorphism, Veech's foliation identifies with the horizontal foliation. It is then not difficult to describe the possible conformal types of the leaves of Veech's foliation (Theorem 1.7 above).

Using some results of Mano and Watanabe [49] and of Mano [46], we use the explicit differential system satisfied by the two elliptic hypergeometric integrals which are the components of Veech's map in this case to look at Veech's $\mathbb{CH}^1$-structure of an algebraic leaf $Y_1(N)^{\alpha_1}$ of Veech's foliation on the moduli space $\mathcal{M}_{1,2}$ in the vicinity of one of its cusps. From an easy analysis, one deduces Theorem 1.10, Corollary 1.11 and Corollary 1.12 stated above.

In **Section 6**, we eventually consider some particular questions or problems to which the results previously obtained naturally lead. In **§6.1**, one uses a result by Mano [47] to give an explicit example of an analytic degeneration of some elliptic hypergeometric integrals towards usual hypergeometric functions. For N small (namely $N \leq 5$), the algebraic leaf $Y_1(N)^{\alpha_1}$ is of genus 0 with 3 or 4 punctures, hence the associated elliptic hypergeometric integrals can be expressed in terms of classical (hypergeometric or Heun's) functions. This is quickly discussed in **§6.2**. The subsection **§6.3**, which constitutes the main part of the sixth section, is devoted to the determination of the hyperbolic holonomy of the algebraic leaves $Y_1(N)$'s. More precisely, after having explained why we consider this problem as particularly relevant, we use some connection formulae in twisted homology (due to Mano and presented at the very end of §3) to describe a general method to construct an explicit representation

$$\pi_1\big(Y_1(N)\big) = \Gamma_1(N) \to \mathrm{PSL}_2(\mathbb{R})$$

corresponding to the $\mathbb{CH}^1$-holonomy of $Y_1(N)^{\alpha_1}$.

Finally in **§6.4**, we say a few words about the volume of the algebraic leaves. We give a simple closed formula for the hyperbolic area of a leaf $Y_1(p)^{\alpha_1}$ when p is prime and explain how this could be used to prove that Veech's Ω^{α_1}-volume of $\mathcal{M}_{1,2}$ is finite and give a conjectural value for $\int_{\mathcal{M}_{1,2}} \Omega^{\alpha_1}$.

Two appendices conclude this paper.

Appendix A introduces the notion of *'complex hyperbolic conifold curve'*. In the 1-dimensional case, everything is quite elementary. Some classical links with the theory of Fuchsian second-order differential equations are recalled as well.

The second appendix, **Appendix B**, is considerably longer. It offers a very detailed treatment of the Gauß-Manin connection which is relevant to construct the differential system satisfied by the elliptic hypergeometric integrals which are the components of Veech's map (see Proposition 1.6). After recalling some general results about the theory in a twisted relative situation of dimension 1, we treat very explicitly the case of 2-punctured elliptic curves over a leaf of Veech's foliation on $\mathscr{T}or_{1,2}$ following [49]. All the results that we present are justified and made explicit. In the end, we use the Gauß-Manin connection to construct the second-order differential equation $(E_a^{\alpha_1})$ of Proposition 1.9.

1.3.2. We think that the length of this text and the originality of the results it offers are worth commenting.

From our point of view, the two crucial technical results of this text on which all the others rely are, first, the explicit global expression (9) in Proposition 1.2 and, secondly, some explicit formulae, first for Veech's map by means of elliptic hypergeometric integrals, then for the differential equation $(E_a^{\alpha_1})$ satisfied by its components F_0 and F_∞ when $n=2$ (*cf.* Proposition 1.9).

If the first aforementioned result follows easily from a constructive proof of Troyanov's theorem (*cf.* the beginning of §1.1.5) described by Kokotov in [33, §2.1], its use to make Veech's constructions of [76] explicit in the genus 1 case is completely original although not difficult. Once one has the explicit formula (9) at hand, it is rather easy to obtain the local expression for the Veech map in terms of elliptic hypergeometric integrals. As for the classical (genus 0) hypergeometric integrals, the relevant technology to study such integrals is that of twisted (co-)homology.

In the case of punctured elliptic curves, this theory has been worked out by Mano and Watanabe in [49] where they also give some explicit formulae for the corresponding Gauß-Manin connection. It follows that, up to a few exceptions, the material we present in Section 3 and in Appendix B is not new and should be attributed to them. So it would have been possible to replace these lengthy parts of the present paper by some references to [49].

The reason why we have chosen to do otherwise is twofold. First, when we began to work on the subject of this paper, we were not very familiar with the modern twisted (co-)homological way to deal with hypergeometric functions. In order to understand this theory better, we began to write down detailed notes. Because these were helpful for our own understanding, we thought that they could be helpful to some readers as well and decided to incorporate them in the text.

The second reason which prompted us to proceed that way is that the context in which the results of [49] lie, namely the context of isomonodromic deformations of linear differential systems with regular singular points on elliptic curves, is more general than ours. More concretely, the authors in [49] deal with a parameter λ which corresponds to a certain line bundle $\mathscr{O}_\lambda$ of degree 0 on the considered elliptic curves. Our case corresponds to the specialization $\lambda = 0$ which corresponds to $\mathscr{O}_\lambda$ being trivial. If the case we are interested in is somehow the simplest one of [49], some of the results of the latter, those about the Gauß-Manin connection in particular, do not apply to the case $\lambda = 0$ in a straightforward manner. In order to fill some details which were not explicitly mentioned in [49], we worked out this case carefully and it lead to Appendix B.

1.4. **Remarks, notes and references.** This text being already very long, we think it is not a problem to add a few lines mentioning other mathematical works to which the present one is or could be linked.

1.4.1. As is well-known (or at least, as it must be clear after reading §1.1), the distinct approaches of Deligne-Mostow [11] on the one hand and of Thurston [71] on the other hand, lead to the same results. As already said before, Thurston's approach is more geometric than the hypergeometric one and basically relies on certain surgeries[6] for flat surfaces (actually flat spheres).

In the present text, we extend the hypergeometric approach of Deligne and Mostow in order to handle the elliptic case. The point is that Thurston's approach, in terms of flat surfaces, can be generalized to the genus 1 case as well.

In the 'non-identical twin' paper [19][7], we introduce several surgeries for flat surfaces (some of which are natural generalizations of the one implicitly used by Thurston) which we use to generalize some statements of [71] to the case of flat tori with conical singularities.

We believe that the important fact highlighted by our work is that both Thurston's geometric approach and Deligne-Mostow's hypergeometric one can be generalized to the genus 1 case. At the moment, we have written two separate

[6] By 'surgery' we mean an operation which transforms a flat surface into a new one which is obtained from the former by cutting along piecewise geodesic segments in it or by removing a part of it with a piecewise geodesic boundary and then identifying certain isometric components of the boundary of the flat surface with geodesic boundary obtained after the cutting operation (see [19, §6] for more formal definitions).

[7] We use this terminology since, if [19] has the same parents and is born at the same time as the present text, both papers clearly do not share the same DNA hence are dizygotic twins.

papers, one for each of these two approaches. In the genus 0 case, any one of these approaches suffices, but we believe that this is specific to this case. Our credo is that the geometric approach (à la Thurston) as well as the hypergeometric one (à la Deligne-Mostow) are truly complementary. Each gives a different light on the objects under study and combining these two approaches should be powerful and even necessary in order to better understand the case $g = 1$ with $n \geq 3$. We plan to illustrate this in forthcoming papers. For the time being, readers are just strongly encouraged to take a look at [19] and compare its methods and results to the ones of the present text.

1.4.2. The main mathematical objects studied in [11] are the monodromy groups attached to the Appell-Lauricella hypergeometric functions which are the ones admitting an Eulerian integral representation of the following form

$$F_{\boldsymbol{\gamma}}(x) = \int_{\boldsymbol{\gamma}} \prod_{i=1}^{n} (t - x_i)^{\alpha_i} \, dt \tag{13}$$

with $x \in \mathbb{C}^n$ and where $\boldsymbol{\gamma}$ is a twisted 1-cycle supported in $\mathbb{P}^1 \setminus \{x\}$ (*cf.* §1.1.2).

In the present text we are interested in the functions which admit an integral representation of the following form (*cf.* §(1.2.5) for some explanations)

$$F_{\boldsymbol{\gamma}}(\tau, z) = \int_{\boldsymbol{\gamma}} \exp\big(2i\pi a_0 u\big) \prod_{i=1}^{n} \theta(u - z_i, \tau)^{\alpha_i} \tag{14}$$

with $(\tau, z) \in \mathbb{H} \times \mathbb{C}^n$ and where $\boldsymbol{\gamma}$ stands for a twisted 1-cycle supported in the punctured elliptic curve $E_{\tau,z}$ (*cf.* §1.1.2). From our point of view, they are the direct generalization to the genus 1 case of the Appell-Lauricella functions (13). For this reason, it seemed to us that the name **elliptic hypergeometric function** was quite adequate to describe them.

Here we have to mention that the Appell-Lauricella hypergeometric functions (13) admit developpements in series similar to (1) (*cf.* [11, (I')] for instance). Taking this as their main feature and motivated by some questions arising in mathematical physics, several people have developed a theory of '*elliptic hypergeometric series*' which have been quickly named '*elliptic hypergeometric functions*' as well (see *e.g.* the survey paper [68]). These share several other similarities with the classical hypergeometric functions such as, for instance, integral representations. We do not know if our elliptic hypergeometric functions are related to the ones considered by these authors, but we doubt it.

Anyway, since it sounds very adequate and because we like it too much, we have decided to use the expression '*elliptic hypergeometric function*' in our paper as well. Note that this terminology has already been used once in a context very similar to the one we are dealing with in this text, see [32].

1.4.3. Note also that in the papers [46, 49], which we use in a crucial way in §6, the authors consider functions defined by integral representations of the form

$$F_{j,\boldsymbol{\gamma}}(\tau,z,\lambda)=\int_{\boldsymbol{\gamma}}\exp\big(2i\pi a_0 u\big)\prod_{i=1}^{n}\theta(u-z_i,\tau)^{\alpha_i}\mathfrak{s}(u-z_j,\lambda) \tag{15}$$

for $(\tau,z)\in\mathscr{T}or_{1,n}$, $\lambda\in\mathbb{C}\setminus\mathbb{Z}_\tau$ and $j=1,\ldots,n$, where $\mathfrak{s}(\cdot,\lambda)$ stands for the function

$$\mathfrak{s}(u,\lambda)=\frac{\theta'(0)\theta(u-\lambda)}{\theta(u)\theta(\lambda)}.$$

Such functions were previously baptized *'Riemann-Wirtinger integrals'* by Mano in [46]. Since $\lambda\mathfrak{s}(u,\lambda)\to -1$ when λ goes to 0, our elliptic hypergeometric functions (14) can be seen as natural limits of renormalized Riemann-Wirtinger integrals. However, if the functions (15) for $j\in\{1,\ldots,n\}$ fixed can be seen as translation periods of a certain flat structure on E_τ (namely the one defined by the square of the modulus of the integrand in (15)), the latter does not have finite volume hence is not of the type which is of interest to us.

1.4.4. One of the origins of the terminology 'Riemann-Wirtinger integrals' (see just above) can be found in the little known paper of Wirtinger [82], dating of 1902, in which he gives an explicit expression for the uniformization of the hypergeometric function (1) to the upper-half plane $\mathbb{H}$. This paper has been followed by a whole series of works by several authors [83, 4, 7, 61, 24, 25, 26, 58, 27] in which they study particular cases of what we call here 'elliptic hypergeometric integrals' (see [35] for an exposition of some of the results obtained by these authors).

The 'uniformized approach' to the study of the hypergeometric functions initiated by Wirtinger does not seem to have generated much interest from 1910 until very recently. Starting from 2007, Watanabe begins to work on this subject again. In the series of papers [77, 78, 79, 80], he applies the modern approach relying on twisted (co-)homology to the Wirtinger integral (see [78] or §3.1.7 below for details) and recovers several classical results about Gauß hypergeometric function. It seems to be some overlap with several results contained in the former papers just aforementioned (see Remark 6.4) but Watanabe was apparently not aware of them since [82] is the only paper of that time he refers to.

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2. **Notations and preliminary material**

We indicate below some notations for the objects considered in this paper as well as a few references. We have chosen to present this material in telegraphic style: we believe that this presentation is the most useful to the reader.

2.1. **Notations for punctured elliptic curves.**

- $\mathbb{H}$ stands for Poincaré's upper half-plane: $\mathbb{H} = \big\{ u \in \mathbb{C} \,\big|\, \Im\mathrm{m}(u) > 0 \big\}$;
- $\mathbb{D}$ denotes the unit disk in the complex plane: $\mathbb{D} = \big\{ u \in \mathbb{C} \,\big|\, |u| < 1 \big\}$;
- τ stands for an a priori arbitrary element of $\mathbb{H}$;
- $A_\tau = A + A\tau$ for any $\tau \in \mathbb{H}$ and any subset $A \subset \mathbb{C}$;
- $E_\tau = \mathbb{C}/\mathbb{Z}_\tau$ is the elliptic curve associated to the lattice $\mathbb{Z}_\tau$ for $\tau \in \mathbb{H}$;
- $[0,1[_\tau$ is the fundamental parallelogram of E_τ;
- $z = (z_1, \ldots, z_n)$ denotes a n-uplet of complex numbers: $(z_i)_{i=1}^n \in \mathbb{C}^n$;
- $[z_i] \in E_\tau$ stands for the class of $z_i \in \mathbb{C}$ modulo $\mathbb{Z}_\tau$ when τ is given;
- Most of the time $z = (z_i)_{i=1}^n \in \mathbb{C}^n$ will be assumed to be
  - such that the $[z_i]$'s are pairwise distinct;
  - normalized up to a translation, that is $z_1 = 0$;
- $E_{\tau,z}$ is the n-punctured elliptic curve $E_\tau \setminus \big\{[z_1], \ldots, [z_n]\big\}$;

2.2. **Notations and formulae for theta functions.**

Our main reference concerning theta functions and related material is Chandrasekharan's book [8].

- $q = \exp(i\pi\tau) \in \mathbb{D}$ for $\tau \in \mathbb{H}$;
- $\theta(\cdot) = \theta(\cdot, \tau)$ for $\tau \in \mathbb{H}$ (viewed as a fixed parameter) stands for Jacobi's theta function defined by, for every $u \in \mathbb{C}$:

$$\theta(u) = \theta(u,\tau) = -i \sum_{n \in \mathbb{Z}} (-1)^n \exp\Big(i\pi\big(n + 1/2\big)^2 \tau + 2i\pi\big(n+1/2\big)u \Big); \tag{16}$$

- For $\tau \in \mathbb{H}$, the following multiplicative functional relations hold true:

$$\theta(u+1) = -\theta(u) \qquad \text{and} \qquad \theta(u+\tau) = -q^{-1}e^{-2i\pi u}\cdot\theta(u)\,; \tag{17}$$

- $\theta'(u)$ and $\dot{\theta}(u)$ stand for the derivative of θ w.r.t u and τ respectively;
- Heat equation: for every $u \in \mathbb{C}$, one has: $\dot{\theta}(u) = (4i\pi)^{-1}\theta''(u)$
- By definition, the four Jabobi's theta functions $\theta_0, \ldots, \theta_3$ are

$$\begin{aligned}
\theta_0(u) &= \theta(u,\tau) & \theta_1(u) &= -\theta\left(u - \frac{1}{2}, \tau\right) \\
\theta_2(u) &= \theta\left(u - \frac{\tau}{2}, \tau\right) i q^{\frac{1}{4}} e^{-i\pi z} & \theta_3(u) &= -\theta\left(u - \frac{1+\tau}{2}, \tau\right) q^{\frac{1}{4}} e^{-i\pi u}\,;
\end{aligned}$$

- Functional equations for the θ_i's: for every $(u,\tau) \in \mathbb{C}\times\mathbb{H}$, one has

$$\begin{aligned}
\theta_1(u+1) &= -\theta_1(u) & \theta_1(u+\tau) &= q^{-1}e^{-2i\pi u}\theta_1(u) \\
\theta_2(u+1) &= \theta_2(u) & \theta_2(u+\tau) &= -q^{-1}e^{-2i\pi u}\theta_2(u) \\
\theta_3(u+1) &= \theta_3(u) & \theta_3(u+\tau) &= q^{-1}e^{-2i\pi u}\theta_3(u)\,;
\end{aligned}$$

- ρ denotes the logarithmic derivative of θ w.r.t. u, *i.e.* $\rho(u) = \theta'(u)/\theta(u)$;
- functional equation for ρ: for every $\tau \in \mathbb{H}$ and every $u \in \mathbb{C}\setminus\mathbb{Z}_\tau$, one has

$$\rho(u+1) = \rho(u) \qquad \text{and} \qquad \rho(u+\tau) = \rho(u) - 2i\pi\,;$$

- $\rho'(\cdot)$ is $\mathbb{Z}_\tau$-invariant hence can be seen as a rational function on E_τ.

2.3. **Modular curves.** A handy reference for the little we use on modular curves is the nice book [13] by Diamond and Shurman.

- N stands for a (fixed) positive integer;
- classical congruence subgroups of level N:

$$\Gamma(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \,\middle|\, a \equiv d \equiv 1 \text{ and } c \equiv d \equiv 0 \bmod N \right\}\,;$$

$$\Gamma_1(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \,\middle|\, a \equiv d \equiv 1 \text{ and } c \equiv 0 \bmod N \right\}\,;$$

- $Y(\Gamma) = \mathbb{H}/\Gamma$ for Γ a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$;
- $Y(N) = Y(\Gamma(N))$ and $Y_1(N) = Y(\Gamma_1(N))$;
- $\mathbb{H}^\star = \mathbb{H} \sqcup \mathbb{P}^1_{\mathbb{Q}} \subset \mathbb{P}^1$ is the extended upper-half plane;
- $X(\Gamma) = \mathbb{H}^\star/\Gamma$ is the compactified modular curve associated to Γ;
- $X(N) = X(\Gamma(N))$ and $X_1(N) = X(\Gamma_1(N))$.

2.4. **Teichmüller material.** There are many good books about Teichmüller theory. A useful one considering what we are doing in this text is [56] by Nag.

- g and n stand for non-negative integers such that $2g-2+n>0$;
- S_g (or just S for short) is a fixed compact orientable surface of genus g;
- $S_{g,n}$ (or S^* for short) denotes either the n-punctured surface $S\setminus\{s_1,\ldots,s_n\}$ or the n-marked surface (S,s) where $s=(s_i)_{i=1}^n$ stands for a fixed n-uplet of pairwise distinct points on S;
- $\mathscr{T}eich_{g,n}$ is a shorthand for $\mathscr{T}eich(S_g,s)$, the Teichmüller space of the n-marked surface of genus g $S_{g,n}$;
- $\mathrm{PMCG}_{g,n}$ denotes the pure mapping class group;
- $\mathrm{Tor}_{g,n}$ is the Torelli group: it is the kernel of the epimorphism of groups $\mathrm{PMCG}_{g,n}\to\mathrm{Aut}\big(H_1(S_{g,n},\mathbb{Z}),\cup\big)$ (where $\cup$ stands for the cup product);
- $\mathscr{T}or_{g,n}=\mathscr{T}eich_{g,n}/\mathrm{Tor}_{g,n}$ is the associated Torelli space;
- $\mathcal{M}_{g,n}=\mathscr{T}eich_{g,n}/\mathrm{PMCG}_{g,n}$ is the associated (Riemann) moduli space.

2.5. **Complex hyperbolic spaces.** We will make practically no use of complex hyperbolic geometry in this text. However, viewed its conceptual importance to understand Veech's constructions, we settle basic definitions and facts below. For a reference, the reader can consult [22].

- $\langle\cdot,\cdot\rangle=\langle\cdot,\cdot\rangle_{1,n}$ is the hermitian form of signature $(1,n)$ on $\mathbb{C}^{n+1}$: one has
$$\langle z,w\rangle=\langle z,w\rangle_{1,n}=z_0\overline{w}_0-\sum_{i=1}^n z_i\overline{w}_i$$
for $z=(z_0,\ldots,z_n)$ and $w=(w_0,\ldots,w_n)$ in $\mathbb{C}^{n+1}$;
- $V_{1,n}^+=\{z\in\mathbb{C}^{n+1}\,|\,\langle z,z\rangle_{1,n}>0\}\subset\mathbb{C}^{n+1}$ is the set of $\langle\cdot,\cdot\rangle$-positive vectors;
- the complex hyperbolic space $\mathbb{CH}^n$ is the projectivization of $V_{1,n}^+$:
$$\mathbb{CH}^n=\mathbf{P}V_{1,n}^+\subset\mathbb{P}^n;$$
- in the affine coordinates $(z_0=1,z_1,\ldots,z_n)$, $\mathbb{CH}^n$ is the complex unit ball:
$$\mathbb{CH}^n=\left\{(z_i)_{i=1}^n\in\mathbb{C}^n\;\Big|\;\sum_{i=1}^n|z_i|^2<1\right\};\tag{18}$$
- the complex hyperbolic metric is the Bergman metric of the unit complex ball (18). For $[z]\in\mathbb{CH}^n$ with $z\in V_+$, it is given explicitly by
$$g_{[z]}^{hyp}=-\frac{4}{\langle z,z\rangle^2}\det\begin{bmatrix}\langle z,z\rangle & \langle dz,z\rangle\\ \langle z,dz\rangle & \langle dz,dz\rangle\end{bmatrix};$$

- $\mathrm{PU}(1,n) = \mathrm{PAut}(\mathbb{C}^{n+1}, \langle\cdot,\cdot\rangle_{1,n}) < \mathrm{PGL}_{n+1}(\mathbb{C})$ acts transitively on $\mathbb{CH}^n$ and coincides with its group of biholomorphisms $\mathrm{Aut}(\mathbb{CH}^n)$;
- being a Bergman metric, g^{hyp} is invariant by $\mathrm{Aut}(\mathbb{CH}^n) = \mathrm{PU}(1,n)$;
- $(\mathbb{CH}^n, g^{hyp})$ is a non-compact complete hermitian symmetric space of rank 1 with constant holomorphic sectional curvature;
- for $n = 1$ and $(1,z) \in V_+$, one has $g^{hyp}_{[z]} = 4\big(1-|z|^2\big)^{-2}|dz|^2$, therefore $\mathbb{CH}^1$ coincides with Poincaré's hyperbolic disk $\mathbb{D}^{hyp}$ hence with the real hyperbolic plane $\mathbb{RH}^2$. In other terms, there are some identifications
$$\mathbb{CH}^1 \simeq \mathbb{D}^{hyp} \simeq \mathbb{H} \simeq \mathbb{RH}^2 \quad \text{and} \quad \mathrm{Aut}\big(\mathbb{CH}^1\big) = \mathrm{PU}(1,1) \simeq \mathrm{PSL}_2(\mathbb{R})\,.$$

3. **Twisted (co-)homology and integrals of hypergeometric type**

It is now well-known that a rigorous and relevant framework to deal with (generalized) hypergeometric functions is the one of twisted (co-)homology.

For the sake of completeness, we give below a short review of this theory in the simplest 1-dimensional case. All this material and its link with the theory of hypergeometric functions is exposed in many modern references, such as [11, 38, 85, 2], to which we refer for proofs and details.

After recalling some generalities, we focus on the case we will be interested in, namely the one of punctured elliptic curves. This case has been studied extensively by Mano and Watanabe. Almost all the material presented below has been taken from [49]. The unique exception is Proposition 3.4 in subsection §3.4, where we compute explicitly the twisted intersection product. If this result relies on simple computations, it is of importance for us since it will allow us to give an explicit expression of the Veech form (*cf.* Proposition 4.20).

3.1. **The case of Riemann surfaces: generalities.**

Interesting general references in arbitrary dimension are [2, 75]. The case corresponding to the classical theory of hypergeometric functions is the one where the ambient variety is a punctured projective line. It is treated in a very nice but informal way in the fourth chapter of Yoshida's love book [85]. A more detailed treatment is given in the second section of Deligne-Mostow's paper [11]. For arbitrary Riemann surfaces, the reader can consult [32].

3.1.1. Let μ be a **multiplicative complex character** on the fundamental group of a (possibly non-closed) Riemann surface X, *i.e.* a group homomorphism $\mu : \pi_1(X) \to \mathbb{C}^*$. Since the target group is abelian, it factorizes through a homomorphism of abelian groups $\pi_1(X)^{\mathrm{ab}} = H_1(X,\mathbb{Z}) \to \mathbb{C}^*$, denoted abusively by the same notation μ. We will denote by L_μ or just L 'the' **local system associated to $\boldsymbol{\mu}$**. We use the notation $L_\mu^\vee$, $L^\vee$ for short, to designate the dual local system which is the local system $L_{\mu^{-1}}$ associated to the dual character μ^{-1}.

Assume that T is a multivalued holomorphic function on X whose monodromy is multiplicative and equal to μ^{-1}: the analytic continuation along any loop $\gamma : S^1 \to X$ of a determination of T at $\gamma(1)$ is $\mu(\gamma)^{-1}$ times this initial determination. Then T can be seen as a global section of $L^\vee$. Conversely, assuming that T does not vanish on X, one can define L as the line bundle, the stalk of which at any point x of X is the 1-dimensional complex vector space spanned by a (or equivalently by all the) determination(s) of T^{-1} at x.

Assuming that T satisfies all the preceding assumptions, the logarithmic derivative $\omega = (d\log T) = T^{-1}dT$ of T is a global (univalued) holomorphic 1-form on X. Then one can define L more formally as the local system formed by the solutions of the global differential equation $dh + \omega h = 0$ on X.

3.1.2. For $k = 0,1,2$, a **($L^\vee$-)twisted k-simplex** is the data of a k-simplex α in X together with a determination T_α of T along α. We will denote this object by $\alpha \otimes T_\alpha$ or, more succinctly, by $\boldsymbol{\alpha}$. A **twisted k-chain** is a finite linear combination with complex coefficients of twisted k-simplices. By taking the restriction of T_α to the corresponding facet of $\partial\alpha$, one defines a boundary operator ∂ on twisted k-simplices which extends to twisted k-chains by linearity. It satisfies $\partial^2 = 0$, which allows to define the **twisted homology group** $H_k(X, L^\vee)$.

More generally, one defines a **locally finite twisted k-chain** as a possibly infinite linear combination $\sum_{i\in I} c_i \cdot \boldsymbol{\alpha}_i$ with complex coefficients of $L^\vee$-twisted k-simplices $\boldsymbol{\alpha}_i$, but such that there are only finitely many indices $i \in I$ such that α_i intersects non-trivially any previously given compact subset of X. The boundary operator previously defined extends to such chains and allows to define the **groups of locally finite twisted homology** $H_k^{\rm lf}(X, L^\vee)$ for $k = 0,1,2$.

3.1.3. A **(L-)twisted k-cochain** is a map which associates a section of L over α to any k-simplex α (or equivalently, it is a map which associates a complex number to any $L^\vee$-twisted k-simplex $\alpha \otimes T_\alpha$). The fact that such a section extends in a unique way to any $(k+1)$-simplex admitting α as a face allows to define a coboundary operator. The latter will satisfy all the expected properties in order to construct the **twisted cohomology groups** $H^k(X, L)$ for $k = 0,1,2$. Similarly, by considering twisted k-cochains with compact support, one constructs the groups of **twisted cohomology with compact support** $H_c^k(X, L)$.

The (co)homology spaces considered above are dual to each other: for any $k = 0,1,2$, there are natural dualities

$$H_k\big(X, L^\vee\big)^\vee \simeq H^k\big(X, L\big) \qquad \text{and} \qquad H_k^{\rm lf}\big(X, L^\vee\big)^\vee \simeq H_c^k\big(X, L\big). \tag{19}$$

3.1.4. A twisted k-chain being locally finite, there are natural maps $H_k(X,L^\vee) \to H_k^{\mathrm{lf}}(X,L^\vee)$. We focus on the case when $k=1$ which is the only one to be of interest for our purpose. In many situations, when μ is sufficiently generic, it turns out that the natural map $H_1(X,L^\vee) \to H_1^{\mathrm{lf}}(X,L^\vee)$ actually is an isomorphism. In this case, one denotes the inverse map by

$$\mathrm{Reg} : H_1^{\mathrm{lf}}\big(X,L^\vee\big) \longrightarrow H_1\big(X,L^\vee\big) \tag{20}$$

and call it the **regularization morphism**. Note that it is canonical.

Assume that $\boldsymbol{\alpha}_1,\ldots,\boldsymbol{\alpha}_N$ are locally finite twisted 1-chains (or even better, twisted 1-simplices) in X whose homology classes generate $H_1^{\mathrm{lf}}(X,L^\vee)$.

A **regularization map** is a map $\mathrm{reg} : \boldsymbol{\alpha}_i \mapsto \mathrm{reg}(\boldsymbol{\alpha}_i)$ such that:

(1) for every $i=1,\ldots,n$, $\mathrm{reg}(\boldsymbol{\alpha}_i)$ is a $L^\vee$-twisted 1-chain which is no longer locally finite but finite on X;
(2) reg factors through the quotient and the induced map $H_1^{\mathrm{lf}}(X,L^\vee) \to H_1(X,L^\vee)$ coincides with the regularization morphism (20).

3.1.5. **Poincaré duality** holds true for twisted (co)homology: for $i=0,1,2$, there are natural isomorphisms $H^i(X,L) \simeq H_{2-i}^{\mathrm{lf}}(X,L^\vee)$ (*cf.* [75, Theorem 1.1 p. 218] or [2, §2.2.11] for instance). Combining the latter isomorphism with (19), one obtains a non-degenerate bilinear pairing $H_1(X,L^\vee) \otimes H_1^{\mathrm{lf}}(X,L) \to \mathbb{C}$. When the regularization morphism (20) exists, it matches the induced pairing

$$H_1\big(X,L^\vee\big) \otimes H_1\big(X,L\big) \longrightarrow \mathbb{C} \tag{21}$$

which, in particular, is non-degenerate.

3.1.6. Assume now that μ is unitary, *i.e.* $\mathrm{Im}(\mu) \subset \mathbb{U}$. Then μ^{-1} coincides with the conjugate morphism $\overline{\mu}$, thus the twisted homology groups $H_1(X,L)$ and $H_1(X,L_{\overline{\mu}})$ are equal. On the other hand, the map $\alpha \otimes T_\alpha \to \alpha \otimes \overline{T_\alpha}$ defined on $L^\vee$-twisted 1-simplices induces a complex conjugation $\boldsymbol{\alpha} \mapsto \overline{\boldsymbol{\alpha}}$ from $H_1(X,L^\vee)$ onto $H_1(X,L_{\overline{\mu}})$.

Using §3.1.5, one gets the following non-degenerate hermitian pairing

$$\begin{aligned} H_1\big(X,L^\vee\big)^2 &\longrightarrow \mathbb{C} \\ \big(\boldsymbol{\alpha},\boldsymbol{\beta}\big) &\longmapsto \boldsymbol{\alpha}\cdot\overline{\boldsymbol{\beta}} \end{aligned} \tag{22}$$

which in this situation is called the **twisted intersection product**.

3.1.7. Let η be a holomorphic 1-form on X. By setting

$$\int_{\boldsymbol{\alpha}} T\cdot\eta = \int_\alpha T_\alpha \cdot \eta \tag{23}$$

for any twisted 1-simplex $\boldsymbol{\alpha} = \alpha \otimes T_\alpha$, and by extending this map by linearity, one defines a complex linear map $\int T\cdot\eta$ on the spaces of twisted 1-cycles. The value (23) does not depend on $\boldsymbol{\alpha}$ but only on its (twisted) homology class. Consequently, the preceding map factorizes and gives rise to a linear map

$$\int T\cdot\eta \,:\, H_1\big(X,L^\vee\big)\longrightarrow\mathbb{C}$$
$$[\boldsymbol{\alpha}]\longmapsto\int_{\boldsymbol{\alpha}} T\cdot\eta.$$

On the other hand, there is an exact sequence of sheaves $0\to L\to\Omega^0_X(L)\stackrel{d}{\to}\Omega^1_X(L)\to 0$ on X whose hypercohomology groups are proved to be isomorphic to the simplicial ones $H^k(X,L)$. Then for any η as above, it can be verified that (23) depends only on the associated class $[T\cdot\eta]$ in $H^1(X,L)$ and its value on $\boldsymbol{\alpha}$ is given by means of the pairing (21): for η and $\boldsymbol{\alpha}$ as above, one has

$$\int_{\boldsymbol{\alpha}} T\cdot\eta=\big\langle\big[T\cdot\eta\big],[\boldsymbol{\alpha}]\big\rangle\,.$$

From this, one deduces a precise cohomological definition for what we call a **generalized elliptic integral**, that is an integral of the form

$$\int_{\boldsymbol{\gamma}} T\cdot\eta \tag{24}$$

where η is a 1-form on X and $\boldsymbol{\gamma}$ a twisted 1-cycle (or a twisted homology class).

3.1.8. Assume that T is a non-vanishing function as in §3.1.1. Then using $\omega=d\log(T)$, one defines a twisted covariant differential operator ∇_ω on $\Omega^\bullet_X$ by setting $\nabla_\omega(\eta)=d\eta+\omega\wedge\eta$ for any holomorphic form η on X. In this way one gets a complex $(\Omega^\bullet_X,\nabla_\omega)$ called **the twisted De Rham complex** of X.

There is an exact sequence of sheaves on X

$$0\longrightarrow L\longrightarrow\Omega^0_X\stackrel{\nabla_\omega}{\longrightarrow}\Omega^1_X\longrightarrow 0\,,$$

from which it comes that $(\Omega^\bullet_X,\nabla_\omega)$ is a resolution of L. Consequently (see *e.g.* [2, §2.4.3 and §2.4.6]), the twisted simplicial cohomology groups of X are naturally isomorphic to the twisted hypercohomology groups $H^k(\Omega^\bullet_X,\nabla_\omega)$ for $k=0,1,2$. The main conceptual interest of using this twisted de Rham formalism is that it allows to construct what is called the associated **Gauß-Manin connection** which in turn can be used to construct (and actually is essentially equivalent to) the linear differential system satisfied by the hypergeometric integrals (24). We will return to this in Appendix B, where we will treat the case of 2-punctured elliptic curves very explicitly.

When X is affine (a punctured compact Riemann surface for instance), the hypercohomology groups $H^k(\Omega^\bullet_X,\nabla_\omega)$ can be shown to be isomorphic to some particular cohomology groups built from global holomorphic objects on X.

For instance, in the affine case, there are natural isomorphisms

$$H^1\big(X,L\big) \simeq H^1\big(\Omega_X^\bullet,\nabla_\omega\big) \simeq \frac{H^0\big(X,\Omega_X^1\big)}{\nabla_\omega\big(H^0(X,\mathscr{O}_X)\big)}\,. \tag{25}$$

3.1.9. Assume that X is a punctured compact Riemann surface, *i.e.* $X = \overline{X}\setminus\Sigma$ where Σ is a non-empty finite subset of a compact Riemann surface $\overline{X}$. If ω extends to a rational 1-form on $\overline{X}$ (with poles on Σ), then one can consider the **algebraic twisted de Rham complex** $(\Omega_X^\bullet(*\Sigma),\nabla_\omega)$. It is the subcomplex of $(\Omega_X^\bullet,\nabla_\omega)$ formed by the restrictions to X of the rational forms on $\overline{X}$ with poles supported exclusively on Σ. The **(twisted) algebraic de Rham comparison theorem** (*cf.* [2, §2.4.7]) asserts that these two resolutions of L are quasi-isomorphic, *i.e.* their associated hypercohomology groups $H^k(\Omega_X^\bullet(*\Sigma),\nabla_\omega)$ and $H^k(\Omega_X^\bullet,\nabla_\omega)$ are isomorphic.

Taking one step further, one gets that the singular L-twisted cohomology of X can be described by means of global holomorphic objects on X which actually are restrictions to X of some rational forms on the compact Riemann surface $\overline{X}$. More precisely, there is a isomorphism

$$H^1\big(X,L\big) \simeq \frac{H^0\big(X,\Omega_X^1(*\Sigma)\big)}{\nabla_\omega\big(H^0\big(X,\mathscr{O}_X(*\Sigma)\big)\big)}\,. \tag{26}$$

The interest of this isomorphism lies in the fact that it allows to describe the twisted cohomology group $H^1\big(X,L\big)$ by means of rational 1-forms on $\overline{X}$. For instance, this is quite useful to simplify the computations involved in making the Gauß-Manin connection mentioned in 3.1.8 explicit. Usually (for instance for classical hypergeometric functions, see [2, §2.5]), one even uses a (stricly proper) subcomplex of $(\Omega_X^\bullet(*\Sigma),\nabla_\omega)$ by considering rational forms on $\overline{X}$ with logarithmic poles on Σ. However, such a simplification is not always possible. An example is precisely the case of punctured elliptic curves we are interested in, for which it is necessary to consider rational 1-forms with poles of order 2 at (at least one of) the punctures in order to get an isomorphism similar to (26), see §3.3.2 below.

3.2. **On punctured elliptic curves.** We now specialize and make the theory described above explicit in the case of punctured elliptic curves. This case has been treated very carefully in [49] to which we refer for proofs and details. For some particular cases with few punctures, the interested reader can consult [78, 31, 47, 48].

In this subsection, $n \geq 2$ is an integer and $\alpha_0, \alpha_1, \alpha_2, \dots, \alpha_n$ are fixed real parameters such that

$$\alpha_i \in]-1, \infty[\quad \text{for } i = 1, \dots, n \qquad \text{and} \qquad \sum_{i=1}^{n} \alpha_i = 0\,. \tag{27}$$

Note that unlike the others α_i's, α_0 is arbitrary.

3.2.1. For $\tau \in \mathbb{H}$ and $z = (z_1, z_2, \dots, z_n) \in \mathbb{C}^n$, one denotes by $E_{\tau,z}$ the punctured elliptic curve $E_\tau \setminus \{[z_i] \,|\, i = 1, \dots, n\}$, where $[z_i]$ stands for the class of z_i in E_τ. We will always assume that the $[z_i]$'s are pairwise distinct and that z has been normalized, meaning that $z_1 = 0$.

For τ and z as above, one considers the holomorphic multivalued function

$$T^\alpha(\cdot, \tau, z) \,:\, u \longmapsto T^\alpha(u; \tau, z) = \exp\big(2i\pi\alpha_0 u\big) \prod_{k=1}^{n} \theta\big(u - z_k\big)^{\alpha_k}, \tag{28}$$

of a complex variable u, where θ stands for the theta function $\theta(\cdot, \tau)$, *cf.* (16).

Since τ, z and the α_i's will stay fixed in this section, we will write $T(\cdot)$ instead of $T^\alpha(\cdot, \tau, z)$ to make the notations simpler.

A straightforward computation gives

$$\omega := d \log T = \big(\partial \log T / \partial u\big) du = 2i\pi\alpha_0 du + \sum_{k=1}^{n} \alpha_k \rho(u - z_k) du$$

where $\rho(\cdot)$ stands for the logarithmic derivative of $\theta(\cdot)$, see again (16). Using (27), this can be rewritten as

$$\omega = 2i\pi\alpha_0 du + \sum_{k=2}^{n} \alpha_k \big(\rho(u - z_k) - \rho(u)\big) du\,. \tag{29}$$

Starting from 2 instead of 1 in the summation above forces to subtract $\rho(u)$ at each step. The advantage is that the functions $(\rho(u - z_k) - \rho(u))$, $k = 2, \dots, n$ which appear in (29) are all rational on $E_{\tau,z}$. This shows that ω is a logarithmic rational 1-form on E_τ with poles exactly at the $[z_i]$'s.

Clearly, T is nothing else but the pull back by the universal covering map $\mathbb{C} \to E_\tau$ of a solution of the differential operator

$$\begin{aligned} \nabla_{-\omega} : \mathcal{O}_{E_{\tau,z}} &\longrightarrow \Omega^1_{E_{\tau,z}} \\ h &\longmapsto dh - \omega \cdot h\,, \end{aligned} \tag{30}$$

hence can be considered as a multivalued holomorphic function on $E_{\tau,z}$.

Since $\omega = d \log T$ is a rational 1-form, the monodromy of $\log T$ is additive, hence that of T is multiplicative. For this reason, it is not necessary to refer to a base point to specify the monodromy of T. Thus the latter can be encoded by means of a morphism

$$\rho : H_1(E_{\tau,z}, \mathbb{Z}) \longrightarrow \mathbb{C}^* \tag{31}$$

that we are going to give explicitely below.

Let us define $L^\vee$ as the kernel of the differential operator (30). It is the local system of $E_{\tau,z}$ the local sections of which are local determinations of T.

3.2.2. Let $\epsilon > 0$ be very small and set $\star = -\epsilon(1+i) \in \mathbb{C}$. Denote by β_0 (resp. by β_∞) the image in $E_{\tau,z}$ of the rectilinear segment in $\mathbb{C} \setminus \cup_{i=1}^n (z_i + \mathbb{Z}_\tau)$ linking $\star$ to $\star + 1$ (resp. $\star + \tau$). For $i = 1, \ldots, n$, let β_i stand for the image in $E_{\tau,z}$ of a circle centered at z_i of radius $\epsilon/2$ and positively oriented (see Figure 2).

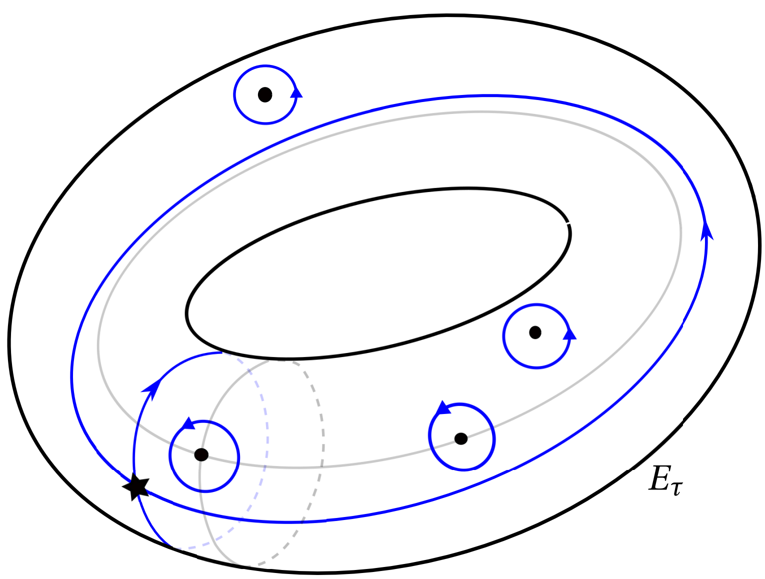


FIGURE 2. In blue, the 1-cycles $\beta_\bullet$, $\bullet = 0, 1, \ldots, n, \infty$ (the two cycles in grey are the images in E_τ of the segments $[0,1]$ and $[0,\tau]$).

For $\bullet \in \{0, \infty, 1, \ldots, n\}$, the analytic continuation of any determination $T_\star$ of T at $\star$ along $\beta_\bullet$ is equal to $T_\star$ times a complex number $\rho_\bullet = \rho(\beta_\bullet)$ which does not depend on ϵ or on the initially chosen determination $T_\star$. Moreover, since $H_1(E_{\tau,z}, \mathbb{Z})$ is spanned by the homology classes of the 1-cycles $\beta_\bullet$ (which do not depend on ϵ if the latter is sufficiently small), the $n+2$ values $\rho_\bullet$ completely characterize the monodromy morphism (31).

For any $k=1,\dots,n$, up to multiplication by a non-vanishing constant, one has $T(u)\sim(u-z_k)^{\alpha_k}$ for u in the vicinity of z_k. It follows immediately that

$$\rho_k=\exp\big(2i\pi\alpha_k\big).$$

It is necessary to use different kinds of arguments in order to determine the values ρ_0 and ρ_∞ which account for the monodromy coming from the global topology of E_τ. We will deal only with the monodromy along β_∞ since the determination of the monodromy along β_0 relies on similar (and actually simpler) computations. For u close to $\star$, using the functional equation (17) satisfied by θ and because $\sum_{i=1}^n\alpha_i=0$, the following equalities hold true:

$$\begin{aligned}T(u+\tau)=&e^{2i\pi\alpha_0(u+\tau)}\prod_{k=1}^n\Big(-q^{\frac14}e^{-2i\pi(u-z_k)}\theta(u-z_k,\tau)\Big)^{\alpha_k}\\ &=\big(-q^{\frac14}\big)^{\sum_k\alpha_k}e^{2i\pi(\alpha_0\tau-\sum_k\alpha_k(u-z_k))}\,T(u)\\ &=e^{2i\pi(\alpha_0\tau+\sum_k\alpha_kz_k)}\,T(u)\,.\end{aligned}$$

Setting

$$\alpha_\infty=\alpha_0\tau+\sum_{k=1}^n\alpha_kz_k\,, \tag{32}$$

the preceding computation shows that

$$\rho_\infty=\exp\big(2i\pi\alpha_\infty\big)\,.$$

By similar computations, one proves that $\rho_0=\exp\big(2i\pi\alpha_0\big)$.

All the above computations can be summed up in the following

Lemma 3.1. *The monodromy of T is multiplicative and the values $\rho_\bullet$ characterizing the monodromy morphism* (31) *are given by*

$$\rho_\bullet=\exp\big(2i\pi\alpha_\bullet\big)$$

for $\bullet\in\{0,1,\dots,n,\infty\}$*, where* α_∞ *is given by* (32).

3.2.3. Let $\tau\in\mathbb{H}$ and $z\in\mathbb{C}^n$ be as above. For $i=2,\dots,n$, let $\tilde z_i$ be the element of $z_i+\mathbb{Z}_\tau$ lying in the fundamental parallelogram $[0,1[_\tau\subset\mathbb{C}$ and denote by $\tilde\ell_i$ the image of $]0,\tilde z_i[$ in $E_{\tau,z}$. Then let us modify the $\tilde\ell_i$'s, each in its respective relative homotopy class, in order to get locally finite 1-cycles ℓ_i in $E_{\tau,z}$ which do not intersect nor have non-trivial self-intersection (*cf.* Figure 3 below where, to simplify the notation, we have assumed that $z_i=\tilde z_i$ for $i=2,\dots,n$).[8]

[8] If the ℓ_i's are not formally defined as a locally finite linear combinations of twisted 1-simplices, a natural way to see them like this is by subdividing each segment $]0,\tilde z_i[$ into a countable union of 1-simplices overlapping only at their extremities. There is no canonical way to do this, but two locally finite twisted 1-chains obtained by this construction are clearly homotopically equivalent.

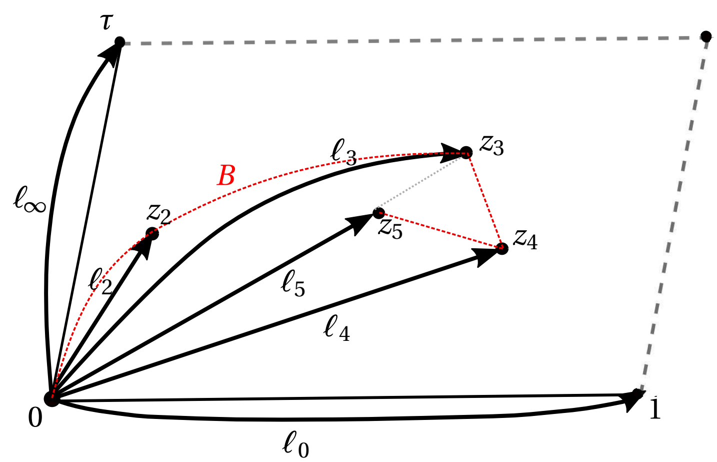


FIGURE 3. The locally finite 1-cycles $\ell_0, \ell_2, \ldots, \ell_n$ and ℓ_∞.

Let $\varphi : [0,1] \to \mathbb{R}$ be a non-negative smooth function, such that $\varphi(0) = \varphi(1) = 0$ and such that $\varphi > 0$ on $]0,1[$. Define $\tilde{\ell}_0$ as the image of $]0,1[$ in E_τ. Let ℓ_0 be the image in the latter tori of $f : t \mapsto t - i\epsilon\varphi(t)$ with ϵ positive and sufficiently small so that the bounded area delimited by the segment $[0,1]$ from above and by the image of f from below does not contain any element $\mathbb{Z}_\tau$-congruent to one of the z_i's. By a similar construction but starting from the segment $]0,\tau[$, one constructs a locally finite 1-chain ℓ_∞ in $E_{\tau,z}$ (see Figure 3 above). We prefer to consider small deformations of the segments $]0,1[$ and $]0,\tau[$ to define ℓ_0 and ℓ_∞ in order to avoid any ambiguity if some of the $\tilde{z}_i$'s happen to be located on one (or on both) of these segments.

Let B be the branch cut in E_τ defined as the image of an embedding $[0,1] \to [0,1[_\tau$ sending 0 to 0 and 1 to $\tilde{z}_n$ which does not meet the ℓ_i's except at their extremities $\tilde{z}_i$'s which all belong to B (*cf.* the curve in red in Figure 3).

Denote by U the complement in E_τ of the topological closure of the union of ℓ_0, ℓ_∞ and B. Then U is a simply connected open set which is naturally identified to the bounded open subset of $\mathbb{C}$, denoted by V, the boundary of which is the union of B with the 1-chains ℓ_0, ℓ_∞ and their respective horizontal and 'vertical' translations $1 + \ell_\infty$ and $\tau + \ell_0$.

Thus it makes sense to speak of a (global) determination of the function T defined in (28) on U. Let T_U stand for such a global determination on U. It extends continuously to the topological boundary ∂U of U in $\mathbb{C}$ minus the $n-1+4$ points of $\cup_{i=1}^n (z_i + \mathbb{Z}_\tau)$ lying on ∂U. Then for any $\bullet \in \{0, 2, \ldots, n, \infty\}$, the restriction $T_\bullet$ of this continuous extension to $\ell_\bullet$ is well defined and one defines a locally finite $L^\vee$-twisted 1-chain (*cf.* the footnote of the preceding page) by setting

$$\boldsymbol{\ell}_\bullet = \ell_\bullet \otimes T_\bullet\,.$$

The continuous extension of T_U to $\overline{U}\setminus\cup_{i=1}^n(z_i+\mathbb{Z}_\tau)$ does not vanish. Hence for any $\bullet$ as above, one defines a locally L-twisted 1-chain by setting:

$$\boldsymbol{l}_\bullet = \ell_\bullet \otimes (T_\bullet)^{-1}.$$

We let the reader verify that the $\boldsymbol{\ell}_\bullet$'s as well as the $\boldsymbol{l}_\bullet$'s actually are (twisted) 1-cycles. Therefore they induce twisted locally finite homology classes, respectively in $H_1^{\rm lf}(E_{\tau,z},L^\vee)$ and $H_1^{\rm lf}(E_{\tau,z},L)$. A bit abusively, we will denote these homology classes by the same notation $\boldsymbol{\ell}_\bullet$ and $\boldsymbol{l}_\bullet$. This will not cause any problem whatsoever.

3.2.4. Such as they are defined above, the locally finite twisted 1-cycles $\boldsymbol{\ell}_0,\boldsymbol{\ell}_2,\ldots,\boldsymbol{\ell}_n$ and $\boldsymbol{\ell}_\infty$ depend on some choices. Indeed, except for $\boldsymbol{\ell}_0$ and $\boldsymbol{\ell}_\infty$, the way the supports ℓ_i's are chosen is anything but constructive. Less important issues are the choices of a branch cut B and of a determination of T on U, which are not specified.

There is a way to remedy to this lack of determination by considering specific z_i's. Let us say that these are in **(very) nice position** if

> *for every $i=1,\ldots,n-1$, the principal argument of $\tilde z_i$ is (stricly) bigger than that of $\tilde z_{i+1}$.*

Remark that when $n=2$, the z_i's are always in very nice position.

When the z_i's are in very nice position, there is no need to modify the $\tilde\ell_i$'s considered above since they already satisfy all the required properties. For the branch cut B, we take the union of a small deformation of $[0,\tilde z_1]$ with the segments $[\tilde z_i,\tilde z_{i+1}]$ for $i=2,\ldots,n-1$ (see Figure 4 just below).

As to the choice of a determination of T on U, let us remark that $\theta(\cdot,\tau)$ takes positive real values on $]0,1[$ for any purely imaginary modular parameter τ. If Log stands for the principal determination of the logarithm, one can define $\theta(u-z_i,\tau)^{\alpha_i}$ as $\exp(\alpha_i{\rm Log}\theta(u-z_i,\tau))$ on the intersection of the suitable translate of V with a disk centered at z_i and of very small radius, for any $i=1,\ldots,n$ (remember the normalization $z_1=0$). By analytic continuation, one gets a global determination of this function on V. Now, since τ varies in the upper half-plane which is simply connected, there is no problem to perform analytic continuation with respect to this parameter in order to obtain a determination of the $\theta(u-z_i,\tau)^{\alpha_i}$'s, hence of T on U for any τ in $\mathbb{H}$.

The $\ell_\bullet$'s as well as the chosen determination T_U on U being perfectly well determined, the same holds true for the twisted 1-cycles $\boldsymbol{\ell}_\bullet$'s and, by extension, for the $\boldsymbol{l}_\bullet$'s (hence for the corresponding twisted homology classes as well).

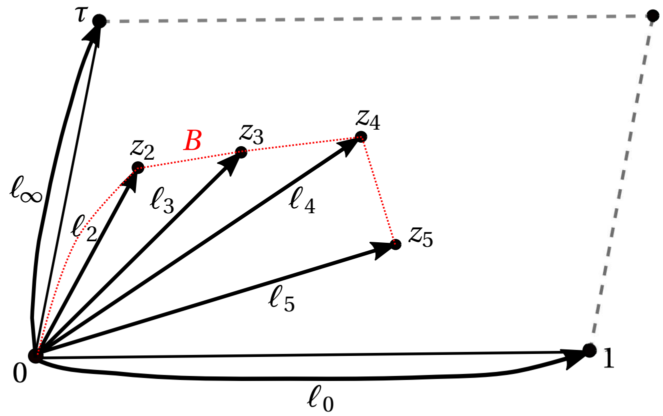
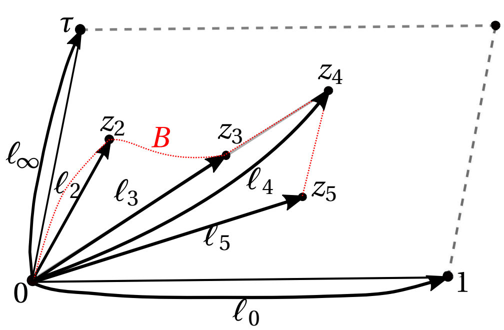
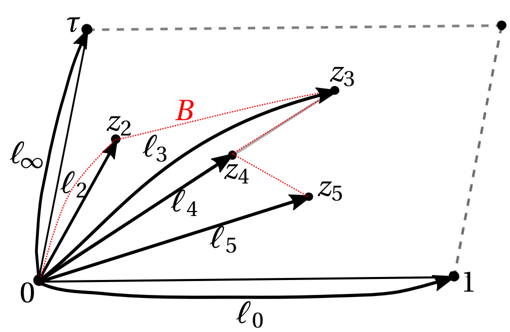


FIGURE 4. The 1-cycles $\ell_\bullet$ for $\bullet = 0, 2, \ldots, n, \infty$ and the branch cut B, for points z_i's in very nice position.

Finally, by continuous deformation of the ℓ_i's (*cf.* [11, Remark (**3.6**)]), one constructs canonical twisted 1-cycles (and associated homology classes) $\boldsymbol{\ell}_\bullet$ and $\boldsymbol{l}_\bullet$ for points z_i's only supposed to be in nice position (see the two pictures below).





3.3. **Description of the first twisted (co)homology groups.** In this subsection, we follow [49] very closely and give explicit descriptions of the (co)homology groups $H_1(E_{\tau,z}, L_\rho)$ and $H^1(E_{\tau,z}, L_\rho)$.

In what follows, we assume that the points z_i's are in nice position.

Recall that $\rho_\bullet = \exp(2i\pi\alpha_\bullet)$ for any $\bullet \in \{0, 2, \ldots, n, \infty\}$, with α_∞ given by (32).[9] Given m elements $\bullet_1, \ldots, \bullet_m$ of the set of indices $\{0, 2, \ldots, n, \infty\}$, one sets:

$$\rho_{\bullet_1 \ldots \bullet_m} = \rho_{\bullet_1} \cdots \rho_{\bullet_m} \qquad \text{and} \qquad d_{\bullet_1 \ldots \bullet_m} = \rho_{\bullet_1 \ldots \bullet_m} - 1\,.$$

3.3.1. **The first twisted homology group $\boldsymbol{H_1(E_{\tau,z}, L_\rho)}$.**

[9]It could be useful for the reader to indicate here the relation between our $\alpha_\bullet$'s and the corresponding notations $c_\bullet$ used in [46, 49]: one has $\alpha_j = c_j$ for $j = 0, 1, \ldots, n$ but $\alpha_\infty = -c_\infty$.

3.3.1.1. Denote by V the bounded simply connected open subset of $\mathbb{C}$ whose boundary is the topological closure of the union of the $\ell_\bullet$'s for $\bullet$ in $\{0,2,\ldots,n,\infty\}$ with the two translated cycles $1+\ell_0$ and $\tau+\ell_\infty$. By analytic extension of the restriction of the determination T_U of T on U in the vicinity of $1+\tau$, one gets a determination T_V of T on V. Considering now V as an open subset of $E_{\tau,z}$, one defines a locally-finite L_ρ-twisted 2-chain[10] by setting

$$\overline{\boldsymbol{V}} = \overline{V} \otimes T_V .$$

This is not a cycle: one verifies easily that the following relation holds true:

$$\begin{aligned}\partial\overline{\boldsymbol{V}} = &\,\boldsymbol{\ell}_0 + \rho_0\boldsymbol{\ell}_\infty - \rho_\infty\boldsymbol{\ell}_0 - \boldsymbol{\ell}_\infty \\ &+ (\rho_n-1)\boldsymbol{\ell}_n + \rho_n(\rho_{n-1}-1)\boldsymbol{\ell}_{n-1} + \cdots + \rho_{3\cdots n}(\rho_2-1)\boldsymbol{\ell}_2 .\end{aligned}$$

It follows that, in $H_1^{\rm lf}(E_{\tau,z},L_\rho)$, one has

$$-d_\infty\cdot\boldsymbol{\ell}_0 + d_0\cdot\boldsymbol{\ell}_\infty + \sum_{k=2}^{n}\frac{d_k}{\rho_{1\cdots k}}\cdot\boldsymbol{\ell}_k = 0 .$$

3.3.1.2. In order to construct a regularization map, we fix $\epsilon>0$. The constructions given below are all independent of ϵ (at the level of homology classes) if the latter is supposed sufficiently small. Of course, we assume that it is the case in what follows.

For any $k=2,\ldots,n$, let $\sigma_k : S^1 \to \mathbb{C}$ be a positively oriented parametrization of the circle centered at $\tilde{z}_k$ and of radius ϵ such that the point $p_k=\sigma_k(1)$ is on the branch locus B. The image of $]0,1[$ by $s_k=\sigma_k(\exp(2i\pi\cdot)) : [0,1]\to\mathbb{C}$ is included in U, hence $s_k^*(T_U)$ is well defined and extends continuously to the closure $[0,1]$. Denoting this extension by T_k, one defines a twisted 1-simplex in $E_{\tau,z}$ by setting

$$\boldsymbol{s}_k = [0,1]\otimes T_k .$$

3.3.1.3. Let $\varphi\in]0,\pi[$ be the principal argument of τ, set $I^0=[0,\varphi]$, $I^1=[\varphi,\pi]$, $I^2=[\pi,\pi+\varphi]$ and $I^3=[\varphi+\pi,2\pi]$ and denote by σ_0^ν the restriction of $[0,2\pi]\to S^1$, $t\mapsto\epsilon\exp(it)$ to I^ν for $\nu=0,1,2,3$. We denote by m^ν the image of σ_0^ν viewed as a subset of $E_{\tau,z}$. These are circular arcs the union of which is a circle of radius ϵ centered at 0 in $E_{\tau,z}$.

In order to specify a determination of T on each of the m^ν, we are going to use the fact that each of them is also the image in E_τ of a suitable translation of σ_0^ν, the (interior of) the image of which is included in U. More precisely, one sets $\widehat{\sigma}_0^\nu(\cdot)=\sigma_0^\nu(\cdot)+x^\nu$ for $\nu=1,\ldots,3$, with $x^1=1$, $x^2=1+\tau$ and $x^3=\tau$.

[10] Strictly speaking, we do not define $\overline{\boldsymbol{V}}$ as a locally-finite 2-chain but there is a natural way to see it as such (by using similar arguments to the ones in footnote 8 above).

For $\nu = 1,2,3$, the image of the interior of I^ν by $\widehat{\sigma}_0^\nu$ is included in U. The restriction of T_U to this image extends continuously to I^ν. Denoting these extensions by T_U^ν, one defines twisted 1-simplices in $E_{\tau,z}$ by setting

$$\boldsymbol{m}^1 = I^1 \otimes \left(\rho_0^{-1} T_U^1\right), \qquad \boldsymbol{m}^2 = I^2 \otimes \left(\rho_{0\infty}^{-1} T_U^2\right) \qquad \text{and} \qquad \boldsymbol{m}^3 = I^3 \otimes \left(\rho_\infty^{-1} T_U^3\right).$$

Since the image of σ_0^0 meets the branch cut B, one cannot proceed as above in this particular case. We use the fact that $p_0 = \sigma_0^0(0) = \epsilon$ belongs to U. Since $m^0 \subset E_{\tau,z}$, the germ of $(\sigma_0^0)^* T_U$ at $0 \in I^0$ extends to the whole simplex I^0. Denoting this extension by T_U^0, one defines a twisted 1-simplex in $E_{\tau,z}$ by setting

$$\boldsymbol{m}^0 = I^0 \otimes T_U^0.$$

3.3.1.4. For $k = 2, \ldots, n$, let ℓ_k^ϵ be the rectilinear segment linking p_0 to p_k in $\mathbb{C}$: $\ell_k^\epsilon = [p_0, p_k]$. Setting $p_\infty = \sigma_0^0(\varphi) = \epsilon\tau$ and deforming the two segments $[p_0, 1 - p_0] = [\epsilon, 1-\epsilon]$ and $[p_\infty, \tau - p_\infty] = [\epsilon\tau, (1-\epsilon)\tau]$ by means of a function φ as in 3.2.3, one constructs two 1-simplices in U, denoted by ℓ_0^ϵ and ℓ_∞^ϵ respectively.

For ϵ small enough, the $\ell_\bullet$'s, $\bullet = 0,2,\ldots,n,\infty$, are pairwise disjoint and included in U, hence one defines twisted 1-simplices in $E_{\tau,z}$ by setting

$$\boldsymbol{\ell}_\bullet^\epsilon = \ell_\bullet^\epsilon \otimes \left(T_U|_{\ell_\bullet^\epsilon}\right).$$

The 1-simplices $\ell_\bullet^\epsilon$ for $\bullet = 0,2,\ldots,n,\infty$, s_k for $k = 2,\ldots,n$ and m^ν for $\nu = 0,1,2,3$ are pictured in blue in Figure 5 below (in the case when $n = 3$).

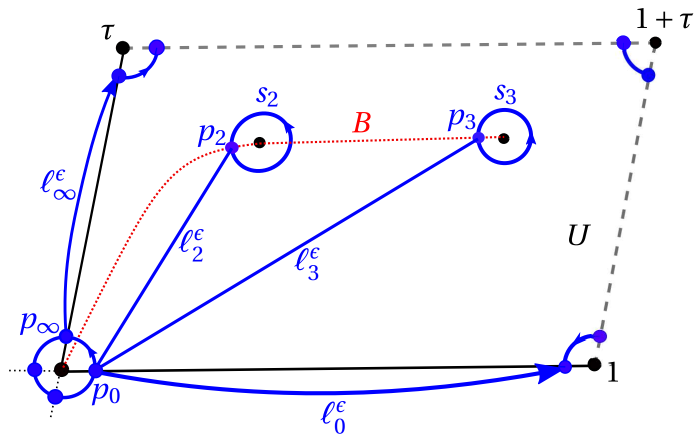


FIGURE 5.

3.3.1.5. Using the twisted 1-simplices constructed above, one defines L_ρ-twisted 1-chains in $E_{\tau,z}$ by setting

$$
\begin{aligned}
\boldsymbol{\gamma}_\infty &= \frac{1}{d_1}\Big[\boldsymbol{m}^0+\boldsymbol{m}^1+\boldsymbol{m}^2+\boldsymbol{m}^3\Big] && +\boldsymbol{\ell}_\infty^\epsilon && -\frac{\rho_\infty}{d_1}\Big[\boldsymbol{m}^3+\boldsymbol{m}^0+\rho_1\big(\boldsymbol{m}^1+\boldsymbol{m}^2\big)\Big],\\
\boldsymbol{\gamma}_0 &= \frac{1}{d_1}\Big[\boldsymbol{m}^0+\rho_1\big(\boldsymbol{m}^1+\boldsymbol{m}^2+\boldsymbol{m}^3\big)\Big] && +\boldsymbol{\ell}_0^\epsilon && -\frac{\rho_0}{d_1}\Big[\boldsymbol{m}^2+\boldsymbol{m}^3+\boldsymbol{m}^0+\rho_1\boldsymbol{m}^1\Big]
\end{aligned}
\tag{33}
$$

[11]

$$
\text{and}\quad \boldsymbol{\gamma}_k = \frac{1}{d_1}\Big[\boldsymbol{m}^0+\rho_1\big(\boldsymbol{m}^1+\boldsymbol{m}^2+\boldsymbol{m}^3\big)\Big] +\boldsymbol{\ell}_k^\epsilon -\frac{1}{d_k}\boldsymbol{s}_k \qquad \text{for } k=2,\ldots,n.
$$

By straightforward computations (similar to the one in [2, Example 2.1] for instance), one verifies that the $\boldsymbol{\gamma}_\bullet$'s actually are 1-cycles hence define twisted homology classes in $H_1(E_{\tau,z},L_\rho)$. We will again use $\boldsymbol{\gamma}_\bullet$ to designate the corresponding twisted homology classes. It is quite clear that these do not depend on ϵ. In particular, this justifies that ϵ does not appear in the notation $\boldsymbol{\gamma}_\bullet$.

When the z_i's are in nice position, the following proposition holds true:

Proposition 3.2 (Mano-Watanabe [49])**.**

(1) *The map* $\mathrm{reg}:\boldsymbol{\ell}_\bullet\mapsto \mathrm{reg}(\boldsymbol{\ell}_\bullet)=\boldsymbol{\gamma}_\bullet$ *is a regularization map: at the homological level, it induces the isomorphism* $H_1^{\mathrm{lf}}(E_{\tau,z},L_\rho)\simeq H_1(E_{\tau,z},L_\rho)$.

(2) *The homology classes* $\boldsymbol{\gamma}_\infty,\boldsymbol{\gamma}_0,\boldsymbol{\gamma}_2,\ldots,\boldsymbol{\gamma}_n$ *satisfy the following relation:*

$$
-d_\infty\boldsymbol{\gamma}_0+d_0\boldsymbol{\gamma}_\infty+\sum_{k=2}^{n}\frac{d_k}{\rho_{1\ldots k}}\boldsymbol{\gamma}_k=0\,. \tag{34}
$$

(3) *The twisted homology group* $H_1(E_{\tau,z},L_\rho)$ *is of dimension n and admits*

$$
\boldsymbol{\gamma}=\big(\boldsymbol{\gamma}_\infty,\boldsymbol{\gamma}_0,\boldsymbol{\gamma}_3,\ldots,\boldsymbol{\gamma}_n\big)
$$

as a basis.

Since T does not vanish on $E_{\tau,z}$, all the preceding constructions can be done with replacing T by its inverse T^{-1}. The regularizations $\boldsymbol{\gamma}_\bullet^\vee=\mathrm{reg}(\boldsymbol{l}_\bullet)$ of the $L_\rho^\vee$-twisted 1-cycles $\boldsymbol{l}_\bullet$ defined at the end of §3.2.3 are defined by the same formulae than (33) but with replacing $\rho_\bullet$ by $\rho_\bullet^{-1}$ for $\bullet=0,2,\ldots,n,\infty$. Then

$$
\boldsymbol{\gamma}^\vee=\big(\boldsymbol{\gamma}_\infty^\vee,\boldsymbol{\gamma}_0^\vee,\boldsymbol{\gamma}_3^\vee,\ldots,\boldsymbol{\gamma}_n^\vee\big)
$$

is a basis of $H_1(E_{\tau,z},L_\rho^\vee)$.

[11] Note that here is a typo in the formula for $\boldsymbol{\gamma}_0$ in [49]. With the notation of the latter, the numerator of the coefficient of the term $(m_0+e^{2\pi\sqrt{-1}c_1}m_1)$ appearing in the definition of γ_0 page 3877 should be $1-e^{2\pi\sqrt{-1}c_0}$ and not $1-e^{-2\pi\sqrt{-1}c_0}$.

3.3.2. **The first cohomology group $H^1(E_{\tau,z}, L_\rho)$.** In [49], the authors give a very detailed treatment of the material described above in §3.1.9 in the case of a punctured elliptic curve. In particular, they show that in this case, it is not possible to use only logarithmic differential forms to describe $H^1(E_{\tau,z}, L_\rho)$.

We continue to use the previous notation. In [49, Proposition 2.4], the authors prove that $H^1(E_{\tau,z}, L_\rho)$ is isomorphic to the quotient of $H^0(E_{\tau,z}, \Omega^1_{E_{\tau,z}})$ by the image by ∇_ω of the space of holomorphic functions on $E_{\tau,z}$ (*cf.* (25)). Then they give a direct proof of the twisted algebraic de Rham comparison theorem (*cf.* [49, Proposition 2.5], see also §3.1.9 above) which asserts that one can consider only rational objects on E_τ (but with poles at the $[z_i]$'s).

Viewing $Z = \sum_{i=1}^n [z_i]$ as a divisor on E_τ, one has

$$H^1\big(E_{\tau,z}, L_\rho\big) \simeq \frac{H^0\big(E_\tau, \Omega^1_{E_\tau}(*Z)\big)}{\nabla_\omega\big(H^0\big(E_\tau, \mathscr{O}_{E_\tau}(*Z)\big)\big)} \tag{35}$$

(recall that, with our notations, $H^0(E_\tau, \mathscr{O}_{E_\tau}(*Z))$ (resp. $H^0(E_\tau, \Omega^1_{E_\tau}(*Z))$) stands for the space of rational functions (resp. 1-forms) on E_τ with poles only at the z_i's.

We now consider the non-reduced divisor $Z' = Z + [0] = 2[0] + \sum_{k=2}^n [z_k]$. There is a natural map from the space $H^0(E_\tau, \Omega^1_{E_\tau}(Z'))$ of rational 1-forms on E_τ with poles at most Z' to the right hand quotient space of (35):

$$H^0\big(E_\tau, \Omega^1_{E_\tau}\big(Z'\big)\big) \longrightarrow \frac{H^0\big(E_\tau, \Omega^1_{E_\tau}(*Z)\big)}{\nabla_\omega\big(H^0\big(E_\tau, \mathscr{O}_{E_\tau}(*Z)\big)\big)}\,. \tag{36}$$

One of the main results of [49] is Theorem 2.7 which says that the preceding map is surjective with a kernel of dimension 1.

It is not difficult to see that, as a vector space, $H^0\big(E_\tau, \Omega^1_{E_\tau}\big(Z'\big)\big)$ is spanned by

$$\begin{aligned} &\varphi_0 = du\,, \\ &\varphi_1 = \rho'(u)du \\ \text{and}\quad &\varphi_j = \big(\rho(u - z_j) - \rho(u)\big)\,du \qquad \text{for } j = 2, \ldots, n. \end{aligned}$$

Remark that all these forms are logarithmic on E_τ (*i.e.* have at most poles of the first order), at the exception of φ_1 which has a pole of order 2 at the origin.

On the other hand, 1 is holomorphic on E_τ and, according to (29), one has:

$$\nabla_\omega(1) = \omega = 2i\pi\alpha_0 \cdot \varphi_0 + \sum_{j=2}^n \alpha_j \cdot \varphi_j\,. \tag{37}$$ [12]

One then deduces the following description of the cohomology group we are interested in:

Theorem 3.3. (1) *Up to the isomorphisms* (35) *and* (36), *the space* $H^1(E_{\tau,z}, L_\rho)$ *is identified with the space spanned by the rational 1-forms* φ_m *for* $m = 0,1,2,\ldots,n$, *modulo the relation* $0 = 2i\pi\alpha_0\varphi_0 + \sum_{k=2}\alpha_k\varphi_k$, *i.e.:*

$$H^1\big(E_{\tau,z}, L_\rho\big) \simeq \frac{\bigoplus_{m=0}^n \mathbb{C}\cdot\varphi_m}{\big\langle 2i\pi\alpha_0\varphi_0 + \sum_{k=2}\alpha_k\varphi_k\big\rangle}.$$

(2) *In particular when* $n = 2$, *up to the preceding isomorphism, the respective classes* $[\varphi_0]$ *and* $[\varphi_1]$ *of* du *and* $\rho'(u)du$ *form a basis of* $H^1(E_{\tau,z}, L_\rho)$.

3.4. **The twisted intersection product.** It follows from Lemma 3.1 that the monodromy character ρ is unitary if and only if the quantity α_∞ defined in (32) is real. Starting from now on, we assume that it is indeed the case.

Since ρ is unitary, the constructions of §3.1.6 apply. We want to make them completely explicit. More precisely, we want to express the intersection product (22) in the basis $\boldsymbol{\gamma}$, *i.e.* we want to compute the coefficients of the following intersection matrix:

$$I\!I_\rho = \big(\boldsymbol{\gamma}_\circ\cdot\overline{\boldsymbol{\gamma}}_\bullet\big)_{\circ,\bullet=\infty,0,3,\ldots,n}.$$

Since ρ is unitary, $\rho^{-1} = \overline{\rho}$, hence for any $\bullet$, $\overline{\boldsymbol{\gamma}}_\bullet$ is the regularization of the locally finite L-twisted 1-cycle $\boldsymbol{l}_\bullet$ and consequently $\boldsymbol{\gamma}_\circ\cdot\overline{\boldsymbol{\gamma}}_\bullet = \boldsymbol{\gamma}_\circ\cdot\boldsymbol{l}_\bullet$ for every $\circ,\bullet$ in $\{\infty,0,2,3,\ldots,n\}$. Using the method explained in [38], it is just a computational task to determine these twisted intersection numbers.

Assuming that the z_i's are in nice position, one has the

[12] Even if we are interested only in the case when $\lambda = 0$, we mention here that the general formula given in [49, Remark 2.8] is not correct. Setting, as in [49], $\mathfrak{s}(u) = \mathfrak{s}(u;\lambda) = \theta'\theta(u-\lambda)/(\theta(u)\theta(\lambda))$ for $u,\lambda\in\mathbb{C}\setminus\mathbb{Z}_\tau$, the correct formula when $\lambda\neq 0$ is

$$\nabla_\omega(1) = \Big[2i\pi\alpha_0 - \alpha_1\rho(\lambda) + \sum_{j=2}^n \alpha_j\big(\mathfrak{s}(z_j) - \rho(z_j)\big)\Big]\cdot\varphi_0 + \lambda(\alpha_1 - 1)\cdot\varphi_1 - \lambda\sum_{j=2}^n \alpha_j\,\mathfrak{s}(z_j)\cdot\varphi_j\,.$$

Proposition 3.4. *For $i = 2,\dots,n$, $j = 2,\dots,j-1$ and $k = i+1,\dots,n$, one has:*

$$\boldsymbol{\gamma}_\infty \cdot \boldsymbol{l}_\infty = \frac{d_\infty d_{1\infty}}{d_1 \rho_\infty} \qquad \boldsymbol{\gamma}_i \cdot \boldsymbol{l}_\infty = -\frac{\rho_1 d_\infty}{\rho_\infty d_1}$$

$$\boldsymbol{\gamma}_\infty \cdot \boldsymbol{l}_0 = \frac{1-\rho_0+\rho_{0\infty}-\rho_{1\infty}}{\rho_0 d_1} \qquad \boldsymbol{\gamma}_i \cdot \boldsymbol{l}_0 = -\frac{\rho_1 d_0}{\rho_0 d_1}$$

$$\boldsymbol{\gamma}_\infty \cdot \boldsymbol{l}_i = \frac{d_\infty}{d_1} \qquad \boldsymbol{\gamma}_j \cdot \boldsymbol{l}_i = -\frac{\rho_1}{d_1}$$

$$\boldsymbol{\gamma}_0 \cdot \boldsymbol{l}_\infty = \frac{\rho_1-\rho_{1\infty}-\rho_0+\rho_{01\infty}}{\rho_\infty d_1} \qquad \boldsymbol{\gamma}_i \cdot \boldsymbol{l}_i = -\frac{d_{1i}}{d_1 d_i}$$

$$\boldsymbol{\gamma}_0 \cdot \boldsymbol{l}_0 = \frac{d_0}{d_1}\left(1-\frac{\rho_1}{\rho_0}\right) \qquad \boldsymbol{\gamma}_k \cdot \boldsymbol{l}_i = -\frac{1}{d_1}$$

$$\boldsymbol{\gamma}_0 \cdot \boldsymbol{l}_i = \frac{d_0}{d_1}.$$

Proof. Let $\boldsymbol{\alpha}$ and $\boldsymbol{a}$ stand for the classes, in $H_1(E_{\tau,z}, L^\vee)$ and $H_1^{\mathrm{lf}}(E_{\tau,z}, L)$ respectively, of two twisted 1-simplices denoted somewhat abusively by the same notation. Denote respectively by α and a the supports of these twisted cycles and let T_α and T_a be the two determinations of T along α and a respectively, such that

$$\boldsymbol{\alpha} = \alpha \otimes T_\alpha \qquad \text{and} \qquad \boldsymbol{a} = a \otimes T_a^{-1}.$$

Since the intersection number $\boldsymbol{\alpha}\cdot\boldsymbol{a}$ depends only on the associated twisted homotopy classes and because α is a compact subset of $E_{\tau,z}$, one can assume that the topological 1-cycles α and a are smooth and intersect transversally in a finite number of points. As explained in [38], $\boldsymbol{\alpha}\cdot\boldsymbol{a}$ is equal to the sum of the local intersection numbers at the intersection points of the supports α and a of the two considered twisted 1-simplices. In other terms, one has

$$\boldsymbol{\alpha}\cdot\boldsymbol{a} = \sum_{x\in\alpha\cap a} \langle \boldsymbol{\alpha}\cdot\boldsymbol{a}\rangle_x$$

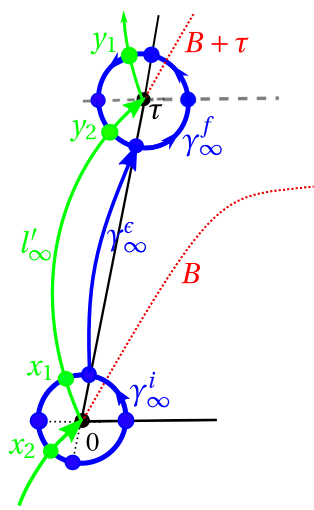
where for any intersection point x of α and a, the twisted local intersection number $\langle \boldsymbol{\alpha}, \boldsymbol{a}\rangle_x$ is defined as the product of the usual topological local intersection number $\langle \alpha, a\rangle_x \in \mathbb{Z}$ with the complex ratio $T_\alpha(x)/T_a(x)$, *i.e.*

$$\langle \boldsymbol{\alpha}, \boldsymbol{a}\rangle_x = \left\langle \alpha\otimes T_\alpha, a\otimes T_a^{-1}\right\rangle_x = \left\langle \alpha, a\right\rangle_x \cdot T_\alpha(x) T_a(x)^{-1} \in \mathbb{C}.$$

With the preceding result at hand, determining all the intersection numbers of the proposition is just a computational task. We will detail only one case below. The others can be computed in a similar way.[13].

[13]Some of these computations can be considered as classical since they already appear in the existing literature, such as the one of $\boldsymbol{\gamma}_i \cdot \boldsymbol{l}_i$ for $i = 2,\dots,n$, which follows immediately from the computations given p. 294 of [38] (see also [2, §2.3.3]).

As an example, let us detail the computation of $\boldsymbol{\gamma}_\infty \cdot \boldsymbol{l}_\infty$. The picture below is helpful for this. On it, the 1-cycle $\boldsymbol{\gamma}_\infty$ has been drawn in blue whereas the locally finite 1-cycle $\boldsymbol{l}_\infty$ is pictured in green.



The picture shows that l_∞ does not meet γ^ϵ_∞ and intersects γ^i_∞ and γ^i_∞ at the points x_1, x_2 and y_1, y_2 respectively.

It follows that

$$\begin{aligned}\boldsymbol{\gamma}_\infty \cdot \boldsymbol{l}_\infty =& \boldsymbol{\gamma}^i_\infty \cdot \boldsymbol{l}_\infty + \boldsymbol{\gamma}^f_\infty \cdot \boldsymbol{l}_\infty \\ =& \sum_{k=1}^2 \big\langle \boldsymbol{\gamma}^i_\infty \cdot \boldsymbol{l}_\infty \big\rangle_{x_k} + \sum_{k=1}^2 \big\langle \boldsymbol{\gamma}^f_\infty \cdot \boldsymbol{l}_\infty \big\rangle_{y_k} \\ =& \frac{1}{d_1} \sum_{k=1}^2 \big\langle \boldsymbol{m}^k \cdot \boldsymbol{l}_\infty \big\rangle_{x_k} - \frac{\rho_{1\infty}}{d_1} \sum_{k=1}^2 \big\langle \boldsymbol{m}^k \cdot \boldsymbol{l}_\infty \big\rangle_{y_k}, \qquad (38)\end{aligned}$$

the last equality coming from the formula for $\boldsymbol{\gamma}^i_\infty$ and $\boldsymbol{\gamma}^f_\infty$ and from the fact that $x_k, y_k \in m^k$ for $k = 1, 2$.

The topological intersection numbers are the following:

$$\big\langle m^1 \cdot l_\infty \big\rangle_{x_1} = \big\langle m^1 \cdot l_\infty \big\rangle_{y_1} = -1 \quad \text{and} \quad \big\langle m^2 \cdot l_\infty \big\rangle_{x_2} = \big\langle m^2 \cdot l_\infty \big\rangle_{y_2} = 1\,.$$

It is then easy to compute the four intersection numbers appearing in (3.4):

- let ζ_1 stand for x_1 or y_1. The determination of T associated to $\boldsymbol{l}_\infty$ at ζ_1 is the same as the one associated to $\boldsymbol{m}^1$ at this point. It follows that

$$\langle \boldsymbol{m}^1 \cdot \boldsymbol{l}_\infty \rangle_{\zeta_1} = \langle m^1 \cdot l_\infty \rangle_{\zeta_1} = -1\,;$$

- let ζ_2 stand for x_2 or y_2. The determination of T associated to $\boldsymbol{l}_\infty$ at ζ_2 is ρ_∞ times the one associated to $\boldsymbol{m}^2$ at this point. It follows that

$$\langle \boldsymbol{m}^2 \cdot \boldsymbol{l}_\infty \rangle_{\zeta_2} = \langle m^2 \cdot l_\infty \rangle_{\zeta_2} \cdot \rho_\infty^{-1} = \rho_\infty^{-1} .$$

From all the preceding considerations, it comes that

$$\boldsymbol{\gamma}_\infty \cdot \boldsymbol{l}_\infty = \frac{1}{d_1}\Big[-1+\rho_\infty^{-1}\Big] - \frac{\rho_{1\infty}}{d_1}\Big[-1+\rho_\infty^{-1}\Big] = \frac{d_\infty d_{1\infty}}{\rho_\infty d_1}.$$

□

3.5. **The particular case $n=2$.** In this case, the complete intersection matrix is

$$I_\rho = \big(\boldsymbol{\gamma}_\circ \cdot \boldsymbol{l}_\bullet\big)_{\circ,\bullet=0,\infty,2} = \begin{bmatrix} \frac{d_\infty d_{1\infty}}{d_1\rho_\infty} & \boldsymbol{\gamma}_\infty \cdot \boldsymbol{l}_0 & \frac{d_\infty}{d_1} \\ \boldsymbol{\gamma}_0 \cdot \boldsymbol{l}_\infty & \frac{d_0}{d_1}\big(1-\frac{\rho_1}{\rho_0}\big) & \frac{d_0}{d_1} \\ -\frac{\rho_1 d_\infty}{\rho_\infty d_1} & -\frac{\rho_1 d_0}{\rho_0 d_1} & 0 \end{bmatrix}$$

with

$$\boldsymbol{\gamma}_\infty \cdot \boldsymbol{l}_0 = -\frac{1}{d_1} + \frac{\rho_0^{-1}}{d_1} + \frac{\rho_\infty}{d_1} - \frac{\rho_1\rho_\infty\rho_0^{-1}}{d_1}$$

$$\text{and} \quad \boldsymbol{\gamma}_0 \cdot \boldsymbol{l}_\infty = -\frac{\rho_1}{d_1} + \frac{\rho_0\rho_1}{d_1} + \frac{\rho_1\rho_\infty^{-1}}{d_1} - \frac{\rho_0\rho_\infty^{-1}}{d_1} .$$

The linear relation between the twisted 1-cycles $\boldsymbol{\gamma}_0, \boldsymbol{\gamma}_\infty$ and $\boldsymbol{\gamma}_2$ is

$$(1-\rho_\infty)\boldsymbol{\gamma}_0 - (1-\rho_0)\boldsymbol{\gamma}_\infty = (1-\rho_2)\boldsymbol{\gamma}_2 .$$

Thus, since $\rho_2 \neq 1$, one can express $\boldsymbol{\gamma}_2$ in function of $\boldsymbol{\gamma}_0$ and $\boldsymbol{\gamma}_\infty$ as follows:

$$\boldsymbol{\gamma}_2 = -\rho_1\frac{d_\infty}{d_1}\boldsymbol{\gamma}_0 + \rho_1\frac{d_0}{d_1}\boldsymbol{\gamma}_\infty. \tag{39}$$

The intersection matrix relative to the basis $(\boldsymbol{\gamma}_0, \boldsymbol{\gamma}_\infty)$ and $(\boldsymbol{l}_0, \boldsymbol{l}_\infty)$ is

$$I\!I_\rho = \begin{bmatrix} \boldsymbol{\gamma}_\infty \\ \boldsymbol{\gamma}_0 \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{l}_\infty & \boldsymbol{l}_0 \end{bmatrix} = \begin{bmatrix} \frac{d_\infty d_{1\infty}}{d_1\rho_\infty} & \frac{-1+\rho_0^{-1}+\rho_\infty-\rho_1\rho_\infty\rho_0^{-1}}{d_1} \\ \frac{-\rho_1+\rho_0\rho_1+\rho_1\rho_\infty^{-1}-\rho_0\rho_\infty^{-1}}{d_1} & \frac{d_0}{d_1}\Big(1-\frac{\rho_1}{\rho_0}\Big) \end{bmatrix}.$$

By a direct computation, one verifies that the determinant of this anti-hermitian matrix is always equal to 1, hence this matrix is invertible. Then one can consider

$$I\!H_\rho = \big(2i\,I\!I_\rho\big)^{-1} = \frac{1}{2i}\begin{bmatrix} \frac{d_0}{d_1}\Big(1-\frac{\rho_1}{\rho_0}\Big) & \frac{\rho_0-1-\rho_0\rho_\infty+\rho_1\rho_\infty}{\rho_0 d_1} \\ \frac{\rho_0-\rho_1-\rho_0\rho_1\rho_\infty+\rho_1\rho_\infty}{\rho_\infty d_1} & \frac{d_\infty d_{1\infty}}{d_1\rho_\infty} \end{bmatrix}. \tag{40}$$

This matrix is hermitian and its determinant is $-1/4<0$. It follows that the signature of the hermitian form associated to $I\!H_\rho$ is $(1,1)$, as expected.

3.5.1. **Some connection formulae.** We now let the parameters τ and z vary. More precisely, let $f : [0,1] \to \mathbb{H} \times \mathbb{C}^n$, $s \mapsto (\tau(s), z(s))$ be a smooth path in the corresponding parameter space: for every $s \in [0,1]$, $z_1(s) = 0$ and $z_1(s), \ldots, z_n(s)$ are pairwise distinct modulo $\mathbb{Z}_{\tau(s)}$. For $s \in [0,1]$, let ρ_s be the corresponding monodromy morphism (namely, the one corresponding to the monodromy of $T(\cdot, \tau(s), z(s))$) and denote by $L_s = L_{\rho_s}$ the associated local system on $E_{\tau(s),z(s)}$.

Since the $E_{\tau(s),z(s)}$'s form a topologically trivial family of n-punctured elliptic curves over $[0,1]$, the corresponding twisted homology groups $H_1(E_{\tau(s),z(s)}, L_s)$ organize themselves into a local system over $[0,1]$, which is necessarily trivial. If in addition the $z_i(0)$'s are in very nice position, then the twisted 1-cycles $\boldsymbol{\gamma}_\bullet^0$ (for $\bullet = \infty, 0, 3, \ldots, n$) are well-defined and can be smoothly deformed along f. One obtains a deformation parametrized by $s \in [0,1]$

$$\boldsymbol{\gamma}^s = \big(\boldsymbol{\gamma}_\bullet^s\big)_{\bullet=\infty,0,3,\ldots,n}$$

of the initial $\boldsymbol{\gamma}^0$'s, such that the map $\boldsymbol{\gamma}_\bullet^0 \mapsto \boldsymbol{\gamma}_\bullet^1$ induces an isomorphism denoted by f_* between the corresponding twisted homology spaces $H_1(E_{\tau(0),z(0)}, L_0)$ and $H_1(E_{\tau(1),z(1)}, L_1)$. It only depends on the homotopy class of f. Similarly, one constructs an analytic deformation $\boldsymbol{l}^s = (\boldsymbol{l}_\infty^s, \ldots, \boldsymbol{l}_n^s)$, $s \in [0,1]$.

Let us suppose furthermore that the $z_i(1)$'s also are in very nice position. Then let $\boldsymbol{\gamma}' = (\boldsymbol{\gamma}_\bullet')_{\bullet=\infty,0,3,\ldots,n}$ be the nice basis of $H_1(E_{\tau(1),z(1)}, L_1)$ constructed in §3.3.1.2. The matrix of $f_* : H_1(E_{\tau(0),z(0)}, L_0) \simeq H_1(E_{\tau(1),z(1)}, L_1)$ expressed in the nice bases $\boldsymbol{\gamma}^0$ and $\boldsymbol{\gamma}'$ is nothing else but the $n \times n$ matrix M_f such that

$$ {}^t\boldsymbol{\gamma}' = M_f \cdot {}^t\boldsymbol{\gamma}^1 . \tag{41}$$

Such a relation is called a **connection formula**. In the particular case when f is a loop, one has $(\tau', z') = (\tau, z)$ and such a formula appears as nothing else but a monodromy formula.

One verifies easily that the twisted intersection product is constant up to small deformations. In particular, for any $\bullet, \circ$ in $\{\infty, 0, 2, \ldots, n\}$, the twisted intersection number $\boldsymbol{\gamma}_\bullet^s \cdot \boldsymbol{l}_\circ^s$ does not depend on $s \in [0,1]$, thus ${}^t\boldsymbol{\gamma}^1 \cdot \boldsymbol{l}^1 = I\!I_{\rho_0}$. Combining this with (41), it comes that the following matricial relation holds true:

$$I\!I_{\rho_1} = M_f \cdot I\!I_{\rho_0} \cdot {}^t\overline{M}_f .$$

In what follows, we give several natural connection formulae in the case when $n = 2$. All these are particular cases of the formulae given in [46, §6] for $n \geq 2$ arbitrary. Note that the reader will not find rigorous proofs of these formulae in [46] but rather some pictures explaining what is going on. However, with the help of these pictures and using similar arguments than those of [11, Proposition (9.2)], it is not too difficult to give rigorous proofs of the formulae below. Since it is rather long, it is left to the motivated reader.

In what follows, the modular parameter $\tau \in \mathbb{H}$ is fixed as well as $z = (z_1, z_2)$ which is a pair of points of $\mathbb{C}$ which are not congruent modulo $\mathbb{Z}_\tau$. As remarked before, z_1, z_2 are in very nice position, hence the twisted 1-cycles $\boldsymbol{\gamma}_\bullet$, $\bullet = \infty, 0, 2$ are well-defined. To remain close to [46], we will not write that z_1 is 0 in the lines below, even if one can suppose that z_1 as been normalized in this way.

3.5.1.1. **Half-Twist formula "$z_1 \longleftrightarrow z_2$".** We first deal with the connection formula associated to the (homotopy class of a) half-twist exchanging z_1 and z_2 with z_2 passing above z_1 as pictured in red in Figure 6 below. This case is the one treated at the bottom of p. 15 of [46].[14]

Setting $z' = (z_2, z_1)$, there is a linear isomorphism HTwist_ρ from $H_1(E_{\tau,z}, L_\rho)$ onto $H_1(E_{\tau,z'}, L_{\rho'})$ with $\rho' = R_{\mathrm{HTwist}}(\rho)$ where

$$R_{\mathrm{HTwist}} : (\rho_\infty, \rho_0, \rho_1) \longmapsto \left(\rho_\infty, \rho_0, \rho_1^{-1}\right).$$

Setting ${}^t\boldsymbol{\gamma} = {}^t(\boldsymbol{\gamma}_\infty, \boldsymbol{\gamma}_0, \boldsymbol{\gamma}_2)$ and $\boldsymbol{l} = (\boldsymbol{l}_\infty, \boldsymbol{l}_0, \boldsymbol{l}_2)$ with analogous notations for $\boldsymbol{\gamma}'$ and $\boldsymbol{l}'$ one has ${}^t\boldsymbol{\gamma}' = \mathrm{HTwist}_\rho \cdot {}^t\boldsymbol{\gamma}$ and $\boldsymbol{l}' = \boldsymbol{l} \cdot {}^t\overline{\mathrm{HTwist}_\rho}$ with

$$\mathrm{HTwist}_\rho = \begin{bmatrix} 1 & 0 & \frac{d_\infty}{\rho_1} \\ 0 & 1 & \frac{d_0}{\rho_1} \\ 0 & 0 & -\frac{1}{\rho_1} \end{bmatrix}. \tag{42}$$

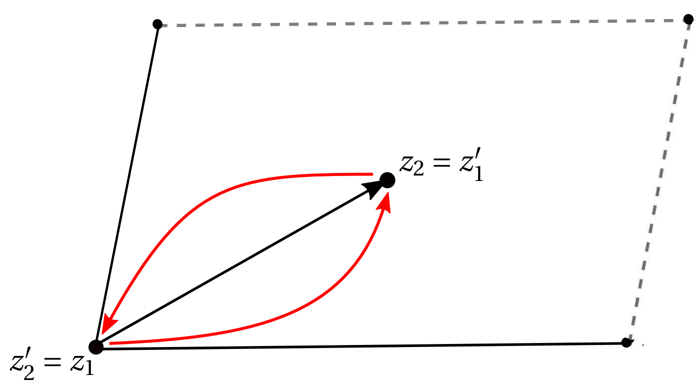


FIGURE 6. Half-twist in the direct sense exchanging z_1 and z_2.

[14] Note that there is a typo in the formula for the half-twist in page 15 of [46]. With the formula given there, relation (43) does not hold true.

Verification: one should have $I_{\rho'} = \boldsymbol{\gamma}' \cdot \boldsymbol{l}' = \mathrm{HTwist}_\rho \cdot \boldsymbol{\gamma} \cdot \boldsymbol{l} \cdot {}^t\overline{\mathrm{HTwist}_\rho} = \mathrm{HTwist}_\rho \cdot I_\rho \cdot {}^t\overline{\mathrm{HTwist}_\rho}$ and, indeed, one verifies that the following relation holds true:

$$I_{\rho'} = \mathrm{HTwist}_\rho \cdot I_\rho \cdot {}^t\overline{\mathrm{HTwist}_\rho}\,. \tag{43}$$

3.5.1.2. **First horizontal translation formula “$z_1 \longrightarrow z_1+1$”.** We now consider the connection formula associated to the path

$$f_{\mathrm{HTrans1}} : [0,1] \longrightarrow \mathbb{H}\times\mathbb{C}^2,\ s \longmapsto \big(\tau, z_1+s, z_2\big)\,.$$

We define $\widetilde{\rho} = R_{\mathrm{HTrans1}}(\rho)$ with

$$R_{\mathrm{HTrans1}} : (\rho_\infty, \rho_0, \rho_1) \longmapsto (\rho_\infty \rho_1^{-1}, \rho_0, \rho_1)\,.$$

We set $\widetilde{z} = (z_1+1, z_2) = f_{\mathrm{HTrans1}}(1)$. The path f_{HTrans1} gives us a linear isomorphism from $H_1(E_{\tau,z}, L_\rho)$ onto $H_1(E_{\tau,\widetilde{z}}, L_{\widetilde{\rho}})$ which will be denoted by $\mathrm{HTrans1}_\rho$.

The corresponding connection matrix is

$$\mathrm{HTrans1}_\rho = \begin{bmatrix} \frac{1}{\rho_1} & -\frac{d_\infty}{\rho_0\rho_1} & 0 \\ 0 & \frac{1}{\rho_0} & 0 \\ 0 & \frac{1}{\rho_0} & \frac{1}{\rho_1} \end{bmatrix}.$$

One verifies that the following relation is satisfied:

$$I_{\widetilde{\rho}} = \mathrm{HTrans1}_\rho \cdot I_\rho \cdot {}^t\overline{\mathrm{HTrans1}_\rho}. \tag{44}$$

3.5.1.3. **Second horizontal translation formula “$z_2 \longrightarrow z_2+1$”.** We now consider the connection formula associated to the path

$$f_{\mathrm{HTrans2}} : [0,1] \longrightarrow \mathbb{H}\times\mathbb{C}^2,\ s \longmapsto \big(\tau, z_1, z_2+s\big)\,.$$

We define $\rho'' = R_{\mathrm{HTwist}} \circ R_{\mathrm{HTrans}} \circ R_{\mathrm{HTwist}}(\rho)$, that is

$$\rho'' = \big(\rho''_\infty, \rho''_0, \rho''_1\big) = \Big(\rho_\infty\rho_1, \rho_0, \rho_1\Big)\,.$$

We set $z'' = (z_1, z_2+1) = f_{\mathrm{HTrans2}}(1)$. The map f_{HTrans2} gives us a linear isomorphism from $H_1(E_{\tau,z}, L_\rho)$ onto $H_1(E_{\tau,z''}, L_{\rho''})$, which will be denoted by $\mathrm{HTrans2}_\rho$.

The corresponding connection matrix is

$$\mathrm{HTrans2}_\rho = \mathrm{HTwist}_{\tilde{\rho}'} \cdot \mathrm{HTrans1}_{\rho'} \cdot \mathrm{HTwist}_\rho$$

with $\tilde{\rho}' = R_{\mathrm{HTrans}} \circ R_{\mathrm{HTwist}}(\rho)$. Explicitly, one has

$$\mathrm{HTrans2}_\rho = \begin{bmatrix} \rho_1 & \frac{\rho_{1\infty} d_1}{\rho_0} & -\frac{d_1(\rho_{01\infty}+\rho_\infty-\rho_0)}{\rho_0} \\ 0 & \frac{1+d_0\rho_1}{} & -\frac{d_0 d_1(\rho_{01}+1)}{\rho_0\rho_1} \\ 0 & -\frac{\rho_1}{\rho_0} & \frac{\rho_0 d_1+1}{\rho_0} \end{bmatrix}.$$

We verify that the following relation is satisfied:

$$I_{\rho''} = \mathrm{HTrans2}_\rho \cdot I_\rho \cdot {}^t\overline{\mathrm{HTrans2}_\rho}.$$

The matrix HT2 of the isomorphism $\mathrm{HTrans2}_\rho$ expressed in the basis $(\boldsymbol{\gamma}_\infty, \boldsymbol{\gamma}_0)$ and $(\boldsymbol{\gamma}''_\infty, \boldsymbol{\gamma}''_0)$ is more involved. But since this formula will be used later (in Lemma 6.3), we give it explicitly below:

(45)
$$\mathrm{HT2}_\rho = \begin{bmatrix} \frac{(\rho_{01\infty}-\rho_0\rho_{01\infty}+\rho_\infty-\rho_{0\infty}+\rho_0{}^2)\rho_1}{\rho_0} & \frac{(\rho_{10\infty}\rho_\infty-\rho_{01\infty}+\rho_{1\infty}-2\rho_\infty+\rho_0+\rho_\infty{}^2-\rho_{0\infty})\rho_1}{\rho_0} \\ -\frac{(\rho_0-1)^2(\rho_{01}+1)}{\rho_0} & \frac{-\rho_{01\infty}+2\rho_{01}-\rho_1+\rho_0\rho_{01\infty}-\rho_0\rho_{01}-\rho_\infty+\rho_{0\infty}+2-\rho_0}{\rho_0} \end{bmatrix}.$$

This matrix satisfies the following relation: $I\!I_{\rho''} = \mathrm{HT2}_\rho \cdot I\!I_\rho \cdot {}^t\overline{\mathrm{HT2}}_\rho$.

3.5.1.4. **First vertical translation formula "$z_1 \longrightarrow z_1 + \tau$"**. We now consider the connection formula associated to the path

$$f_{\mathrm{VTrans1}} : [0,1] \longrightarrow \mathbb{H} \times \mathbb{C}^2, \; s \longmapsto \big(\tau, z_1 + s\tau, z_2\big).$$

We define $\widehat{\rho} = R_{\mathrm{VTrans1}}(\rho)$ where R_{VTrans1} stands for the following map:

$$R_{\mathrm{VTrans1}} : (\rho_\infty, \rho_0, \rho_1) \longmapsto (\rho_\infty, \rho_0\rho_1, \rho_1)\,.$$

We set $\widehat{z} = (z_1 + \tau, z_2) = f_{\mathrm{VTrans1}}(1)$. The map f_{VTrans1} gives us a linear isomorphism from $H_1(E_{\tau,z}, L_\rho)$ onto $H_1(E_{\tau,\widehat{z}}, L_{\widehat{\rho}})$ which will be denoted by $\mathrm{VTrans1}_\rho$.

The corresponding connection matrix is

$$\mathrm{VTrans1}_\rho = \begin{bmatrix} \frac{1}{\rho_\infty} & 0 & 0 \\ -\frac{d_0\rho_1}{\rho_\infty} & \rho_1 & 0 \\ \frac{\rho_1}{\rho_\infty} & 0 & \rho_1 \end{bmatrix}.$$

One verifies that the following relation is satisfied:

$$I_{\widehat{\rho}} = \mathrm{VTrans1}_\rho \cdot I_\rho \cdot {}^t\overline{\mathrm{VTrans1}_\rho}\,. \tag{46}$$

3.5.1.5. **Second vertical translation formula "$z_2 \to z_2 + \tau$".** We finally consider the connection formula associated to the path

$$f_{\mathrm{VTrans2}} : [0,1] \longrightarrow \mathbb{H} \times \mathbb{C}^2, \; s \longmapsto \big(\tau, z_1, z_2 + s\tau\big).$$

We define $\rho^* = R_{\mathrm{HTwist}} \circ R_{\mathrm{VTrans}} \circ R_{\mathrm{HTwist}} \circ (\rho)$, that is

$$\rho^* = \big(\rho^*_\infty, \rho^*_0, \rho^*_1\big) = \Big(\rho_\infty, \rho_0\rho_1^{-1}, \rho_1\Big).$$

We set $z^* = (z_1, z_2 + \tau) = f_{\mathrm{VTrans2}}(1)$. The map f_{VTrans2} gives us a linear isomorphism $\mathrm{VTrans2}_\rho$ from $H_1(E_{\tau,z}, L_\rho)$ onto $H_1(E_{\tau,z^*}, L_{\rho^*})$.

The corresponding connection matrix is

$$\mathrm{VTrans2}_\rho = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{d_1}{\rho_1\rho_\infty} & \frac{1}{\rho_1} & \frac{d_1}{\rho_1^2\rho_\infty} \\ -\frac{1}{\rho_\infty} & 0 & \frac{1}{\rho_1\rho_\infty} \end{bmatrix}.$$

One verifies that the following relation is satisfied:

$$I_{\rho^*} = \mathrm{VTrans2}_\rho \cdot I_\rho \cdot {}^t\overline{\mathrm{VTrans2}_\rho}. \tag{47}$$

The matrix $\mathrm{VT2}_\rho$ of the isomorphism $\mathrm{VTrans2}_\rho$ expressed in the basis $(\boldsymbol{\gamma}_\infty, \boldsymbol{\gamma}_0)$ and $(\boldsymbol{\gamma}_\infty^*, \boldsymbol{\gamma}_0^*)$ is quite simple compared to (45). It is

$$\mathrm{VT2}_\rho = \begin{bmatrix} 1 & 0 \\ \frac{\rho_0-\rho_1}{\rho_1\rho_\infty} & \frac{1}{\rho_1\rho_\infty} \end{bmatrix}. \tag{48}$$

This matrix satisfies the following relation: $I\!I_{\rho^*} = \mathrm{VT2}_\rho \cdot I\!I_\rho \cdot {}^t\overline{\mathrm{VT2}_\rho}$.

3.5.2. **Normalization in the case when $\boldsymbol{\rho_0 = 1}$.** If we assume that $\rho_0 = 1$, then (40) simplifies and one has:

$$I\!H_\rho = (2i)^{-1} \begin{bmatrix} 0 & \rho_\infty \\ -\frac{1}{\rho_\infty} & \frac{d_\infty d_{1\infty}}{d_1\rho_\infty} \end{bmatrix}.$$

Then setting

$$Z_\rho = \sqrt{2} \begin{bmatrix} \rho_\infty & -\frac{d_{1\infty}}{d_1} \\ 0 & 1 \end{bmatrix},$$

one verifies that

$$\boldsymbol{H} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = {}^t\overline{Z_\rho} \cdot I\!H_\rho \cdot Z_\rho\,.$$

The interest of considering $\boldsymbol{H}$ instead of $I\!H_\rho$ is clear: the automorphism group of the former is $\mathrm{SL}_2(\mathbb{R})$, hence, in particular, does not depend on ρ.

This normalization will be used later in §6.3.6.

4. An explicit expression for Veech's map and some consequences

4.1. **Some general considerations about Veech's foliation.** In this subsection, we make general remarks about Veech's foliation in the general non-resonant case. Hence g and n are arbitrary positive integers such that $2g-2+n>0$ and $\alpha = (\alpha_k)_{k=1}^n \in]-1,\infty[^n$ is supposed to be non-resonant, *i.e.* none of the α_k's is an integer.

In [76], Veech defines the isoholonomic foliation $\mathscr{F}^\alpha$ on the Teichmüller space by means of a real analytic map $H^\alpha_{g,n} : \mathscr{T}eich_{g,n} \to \mathbb{U}^{2g}$. The point is that this map descends to the Torelli space $\mathscr{T}or_{g,n}$ and even on this quotient, it is probably not a primitive integral of the foliations formed by its level-sets (this is proved below only when $g=1$). It is what we explain below.

4.1.1. Let $(S,(s_k)_{k=1}^n)$ be a reference model for n-marked surfaces of genus g. We fix a base point $s_0 \in S^* = S\setminus\{s_k\}_{k=1}^n$. One can find a natural 'symplectic basis' (A_i,B_i,C_k), $i=1,\ldots,g$, $k=1,\ldots,n$ of $\pi_1(S^*,s_0)$ such that the latter group is isomorphic to

$$\pi_1(g,n)=\left\langle A_1,\ldots,A_g,B_1,\ldots,B_g,C_1,\ldots,C_n \,\middle|\, \prod_{i=1}^{g}[A_i,B_i]=C_n\cdots C_1\right\rangle,$$

see the picture just below (case $g=n=2$):

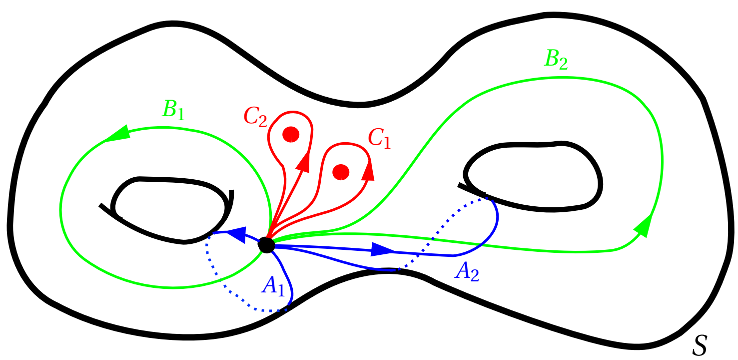


FIGURE 7. The base-point s_0 is the black dot, s_1,s_2 are the red ones.

4.1.2.We recall Veech's definition of the space $\mathscr{E}^{\alpha}_{g,n}$: it is the space of isotopy classes of flat structures on S with conical singularity of type $|u^{\alpha_k}du|^2$ (or equivalently, with cone angle $2\pi(1+\alpha_k)$) at s_k for $k=1,\ldots,n$. Since a flat structure of this type induces a natural conformal structure on S, there is a natural map

$$\mathscr{E}^{\alpha}_{g,n} \longrightarrow \mathscr{T}eich_{g,n} \tag{49}$$

which turns out to be a real analytic[15] diffeomorphism. We need to describe the inverse map of (49). To this end we are going to use a somehow old-fashioned definition of the Teichmüller space that will be useful for our purpose.

Let $(X,x)=(X,(x_1,\ldots,x_n))$ be a n-marked Riemann surface of genus g. Considering a point over it in $\mathscr{T}eich_{g,n}$ amounts to specify a marking of its fundamental group, that is a class, up to inner automorphisms, of isomorphisms ψ :

[15]The space $\mathscr{E}^{\alpha}_{g,n}$ carries a natural intrinsic real analytic structure, *cf.* [76].

$\pi_1(g,n) \simeq \pi_1(X^*,x_0)$ for any $x_0 \in X^* = X \setminus \{x_k\}_{k=1}^n$ (see *e.g.* [70, §2] or [1, 81][16]). Finally, we denote by $m^\alpha_{X,x}$ the unique flat metric of area 1 in the conformal class corresponding to the complex structure of X, with a conical singularity of exponent α_k at x_k for every k (*cf.* Troyanov's theorem mentioned in §1.1.5).

With these notations, the inverse of (49) is written

$$\big(X,x,\psi\big) \longmapsto \big(X,x,\psi,m^\alpha_{X,x}\big).$$

4.1.3. Since $m^\alpha_{X,x}$ induces a smooth flat structure on X^*, its linear holonomy along any loop γ in X^* is a complex number of modulus 1, noted by $\mathrm{hol}^\alpha_{X,x}(\gamma)$. Of course, this number actually only depends on the homotopy class of γ in X^*. With this formalism at hand, it is easy to describe the map constructed in [76] to define the foliation $\mathscr{F}^\alpha$ on the Teichmüller space: it is the map which associates to (X,x,ψ) the holonomy character induced by $m^\alpha_{X,x}$.

Note that, since the conical angles are fixed, for every $k = 1,\ldots,n$, one has

$$\mathrm{hol}^\alpha_{X,x}\big(\psi(C_k)\big) = \exp\big(2i\pi\alpha_k\big) \in \mathbb{U}.$$

Consequently, there is a well-defined map

$$\begin{aligned} \chi^\alpha_{g,n} : \mathscr{T}eich_{g,n} &\longrightarrow \mathrm{Hom}^\alpha\big(\pi_1(g,n),\mathbb{U}\big) \\ \big(X,x,\psi\big) &\longmapsto \mathrm{hol}^\alpha_{X,x}\circ\psi, \end{aligned} \tag{50}$$

the exponent α in the formula of the target space meaning that one considers only unitary characters on $\pi_1(g,n)$ which map C_k to $\exp(2i\pi\alpha_k)$ for every k.

4.1.4. Let $H_1(g,n)$ be the abelianization of $\pi_1(g,n)$: it is the $\mathbb{Z}$-module generated by the A_i's, the B_j's and the C_k's up to the relation $\sum_k C_k = 0$. We denote by a_i, b_i and c_k the corresponding homology classes. We take $H_1(g,n)$ as a model for the first homology group of n-punctured genus g Riemann surfaces.

The Torelli space $\mathscr{T}or_{g,n}$ can be defined as the set of triples (X,x,ϕ) where (X,x) is a marked Riemann surface as above and ϕ an isomorphism from $H_1(g,n)$ onto $H_1(X^*,\mathbb{Z})$. Moreover, the projection from the Teichmüller space onto the Torelli space is given by

$$\begin{aligned} p_{g,n} : \mathscr{T}eich_{g,n} &\longrightarrow \mathscr{T}or_{g,n} \\ \big(X,x,\psi\big) &\longmapsto \big(X,x,[\psi]\big) \end{aligned}$$

where $[\psi]$ stands for the isomorphism in homology induced by ψ.

[16] The definition of a point of the Teichmüller space (of a closed surface and without marked points) by means of a marking of the fundamental group follows from a result attributed to Dehn by Weil in [81], whereas in [70], Teichmüller refers for this to the paper [45] by Mangler.

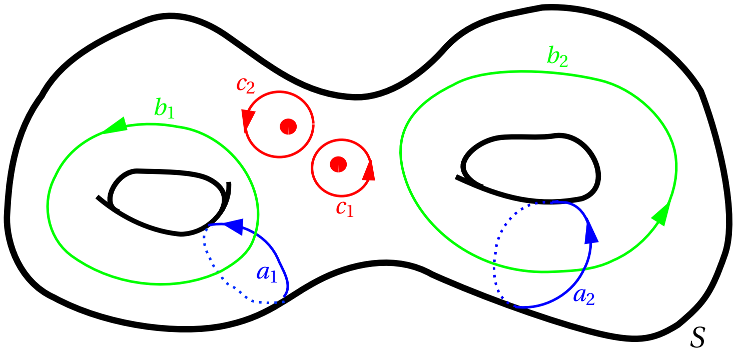


FIGURE 8. A model for the homology of the punctured surface S^*.

4.1.5. Now, the key (but obvious) point is that the holonomy $\mathrm{hol}^{\alpha}_{(X,x)}(\gamma)$ for $\gamma \in \pi_1(X^*)$ does not depend on the base point but only on the (base-point) free homology class $[\gamma] \in H_1(X^*, \mathbb{Z})$. Since $\mathbb{U}$ is commutative, any unitary representation of $\pi_1(g,n)$ factors trough $\pi_1(g,n)^{\mathrm{ab}} = H_1(g,n)$, thus there is a natural map $\mathrm{Hom}^{\alpha}\big(\pi_1(g,n), \mathbb{U}\big) \to \mathrm{Hom}^{\alpha}\big(H_1(g,n), \mathbb{U}\big)$. Since $\mu \in \mathrm{Hom}^{\alpha}(\pi_1(g,n), \mathbb{U})$ is completely determined by its values on the A_i's and the B_i's (it verifies $\mu(C_k) = \exp(2i\pi\alpha_k)$ for $k = 1, \ldots, n$), the space $\mathrm{Hom}^{\alpha}(\pi_1(g,n), \mathbb{U})$ is naturally isomorphic to $\mathbb{U}^{2g}$. This applies verbatim to $\mathrm{Hom}^{\alpha}(H_1(g,n), \mathbb{U})$ as well. It follows that these two spaces of unitary characters are both naturally isomorphic to $\mathbb{U}^{2g}$.

From the preceding discussion, it comes that one can define a map $\mathscr{T}or_{g,n} \to \mathrm{Hom}^{\alpha}(H_1(g,n), \mathbb{U})$ which makes the following square diagram commutative:

$$
\begin{array}{ccccc}
\mathscr{T}eich_{g,n} & \xrightarrow{\chi^{\alpha}_{g,n}} & \mathrm{Hom}^{\alpha}\big(\pi_1(g,n), \mathbb{U}\big) & \simeq & \mathbb{U}^{2g} \\
\downarrow{\scriptstyle p_{g,n}} & & \downarrow & & \| \\
\mathscr{T}or_{g,n} & \longrightarrow & \mathrm{Hom}^{\alpha}\big(H_1(g,n), \mathbb{U}\big) & \simeq & \mathbb{U}^{2g}.
\end{array}
\tag{51}
$$

Both maps with values into $\mathbb{U}^{2g}$ given by the two lines of the preceding diagram will be called the **linear holonomy maps**. We will use the (a bit abusive) notation $H^{\alpha}_{g,n} : \mathscr{T}eich_{g,n} \to \mathbb{U}^{2g}$ for the first and the second will be denoted by

$$
h^{\alpha}_{g,n} : \mathscr{T}or_{g,n} \longrightarrow \mathbb{U}^{2g}. \tag{52}
$$

Since $p_{g,n}$ is the universal covering map of the Torelli space which is a complex manifold, the maps $H^{\alpha}_{g,n}$ and $h^{\alpha}_{g,n}$ enjoy the same local analytic properties. Then from [76, Theorem 0.3], one deduces immediately the

Corollary 4.1. *If α is non-resonant, the linear holonomy map $h^\alpha_{g,n}$ is a real analytic submersion. Its level sets are complex submanifolds of the Torelli space $\mathscr{T}or_{g,n}$, of complex dimension $2g-3+n$.*[17]

This implies that the foliation constructed by Veech on $\mathscr{T}eich_{g,n}$ in [76] actually is the pull-back of a foliation defined on $\mathscr{T}or_{g,n}$. We will also call the latter Veech's foliation and will denote it the same way, that is by $\mathscr{F}^\alpha$.

4.1.6. We now explain that Veech's linear holonomy map $\chi^\alpha_{g,n} : \mathscr{T}eich_{g,n} \to \mathbb{U}^{2g}$ actually admits a canonical lift to $\mathbb{R}^{2g}$. To this end, we use elementary arguments (which can be found in [51], p. 488-489).

Let (X,x) be as above and consider γ, a smooth simple curve in X^*. If ℓ stands for its length for the flat metric $m^\alpha_{X,x}$, there exists a ℓ-periodic smooth map $g : \mathbb{R} \to X^*$ which induces a isomorphism of flat circles $\mathbb{R}/\ell\mathbb{Z} \simeq \gamma$ (*i.e.* the pull-back of $m^\alpha_{X,x}$ by g coincides with the Euclidean metric on $\mathbb{R}$). For any t, $g'(t)$ is a unit tangent vector at $g(t) \in \gamma$, thus there exists a unique other tangent vector at this point, noted by $g(t)^\perp$, such that $(g'(t), g(t)^\perp)$ form a direct orthonormal basis of $T_{g(t)}X^*$. Then there exists a smooth function $w : [0,\ell] \to \mathbb{R}$ such that $g''(t) = w(t)\cdot g(t)^\perp$ for any $t \in [0,\ell]$ and one defines the **total angular curvature of the loop γ in the flat surface** $(X, m^\alpha_{X,x})$ as the real number

$$\kappa(\gamma) = \kappa^\alpha_{X,x}(\gamma) = \int_0^\ell w(t)dt\,.$$

There is a nice geometric interpretation of this number as a sum of the oriented interior angles of the triangles of a given Delaunay triangulation of X which meet γ (see [51, §6]). In particular, one obtains that $\kappa(\gamma)$ only depends on the free isotopy class of γ and that $\exp(2i\pi\kappa(\gamma))$ is nothing else but the linear holonomy of $(X, m^\alpha_{X,x})$ along γ, that is:

$$\exp\big(2i\pi\kappa(\gamma)\big) = \mathrm{hol}^\alpha_{X,x}(\gamma)\,. \tag{53}$$

Let $\widetilde{\gamma}$ be another simple curve in the free homotopy class $\langle\gamma\rangle$ of γ. According to a classical result of the theory of surfaces (see [16]), $\widetilde{\gamma}$ and γ actually are isotopic, hence $\kappa(\gamma) = \kappa(\widetilde{\gamma})$. Consequently, the following definition makes sense:

$$\kappa^\alpha_{X,x}\big(\langle\gamma\rangle\big) = \kappa^\alpha_{X,x}(\gamma)\,. \tag{54}$$

[17] Actually, the statement is valid for any α but on the complement of the preimage by the linear holonomy map of the trivial character on $H_1(g,n)$. Note that the latter does not belong to $\mathrm{Im}(h^\alpha_{g,n})$ (so its preimage is empty) as soon as at least one of the α_k's is not an integer.

4.1.7. Now assume that a base point $x_0 \in X^*$ has been fixed. By the preceding construction, one can attached a real number $\kappa_{X,x}(\langle\gamma\rangle)$ to each element $[\gamma] \in \pi_1(X^*, x_0)$ which is representable by a simple loop γ. If η stands for an inner automorphism of $\pi_1(X^*, x_0)$, a classical result of the topology of surfaces ensures that γ and $\eta(\gamma)$ are freely homotopic, *i.e.* $\langle\gamma\rangle = \langle\eta(\gamma)\rangle$.

We now have explicited everything needed to construct a lift of Veech's linear holonomy map. Let $(X, x, \psi) \simeq (X, x, m^{\alpha}_{X,x}, \psi)$ be a point of $\mathscr{T}eich_{g,n} \simeq \mathscr{E}^{\alpha}_{g,n}$. Then for any element D of $\{A_k, B_k \,|\, k = 1, \ldots, g\} \subset \pi_1(g, n)$, its 'image' D^{ψ} by ψ can be seen as the conjugacy class of the homotopy class of a simple curve in X^*. By the preceding discussion, it comes that the map

$$
\begin{aligned}
\widetilde{H}^{\alpha}_{g,n} : \mathscr{T}eich_{g,n} &\longrightarrow \mathbb{R}^{2g} \\
\big(X, x, \psi\big) &\longmapsto \Big(\kappa\big(A_1^{\psi}\big), \ldots, \kappa\big(A_g^{\psi}\big), \kappa\big(B_1^{\psi}\big), \ldots, \kappa\big(B_g^{\psi}\big)\Big)
\end{aligned}
\tag{55}
$$

is well-defined. This map is named the **lifted holonomy map**.

It is easy (left to the reader) to verify that it enjoys the following properties:

(1) it is a lift of $H^{\alpha}_{g,n}$ to $\mathbb{R}^{2g}$: if $e : \mathbb{R}^{2g} \to \mathbb{U}^{2g}$ is the universal covering, then
$$H^{\alpha}_{g,n} = e \circ \widetilde{H}^{\alpha}_{g,n};$$

(2) it is real analytic.

The first point follows at once from (53) and the second is an immediate consequence of the first combined with the obvious fact that $\widetilde{H}^{\alpha}_{g,n}$ is continuous.

4.1.8. From the lines above, it comes that $\widetilde{H}^{\alpha}_{g,n}$ is a real analytic first integral for Veech's foliation on $\mathscr{T}or_{g,n}$ which could enjoy better properties than $H^{\alpha}_{g,n}$. Note that, among the lifts of the latter which are continuous, it is unique up to translation by an element of $2\pi\mathbb{Z}^{2g}$.

For $a \in \mathbb{R}^{2g}$, one defines $\mathscr{F}^{\alpha}_{a}$ as the inverse image of a by the lifted holonomy map in the Teichmüller space. In particular, for $\rho = e(a) \in \mathbb{U}^{2g}$, one has

$$\mathscr{F}^{\alpha}_{\rho} = \big(H^{\alpha}_{g,n}\big)^{-1}(\rho) = \bigcup_{\boldsymbol{m} \in \mathbb{Z}^{2g}} \mathscr{F}^{\alpha}_{a+2\pi\boldsymbol{m}} \tag{56}$$

hence it is natural to expect that any level-set $\mathscr{F}^{\alpha}_{\rho}$ has a countable set of connected components. We will prove below that it is indeed the case when $g = 1$, by completely expliciting the lifted holonomy map (see Remark 4.6.(2)). We conjecture that it is also true when $g \geq 2$ but it is not proved yet.

To conclude this subsection, we would like to warn the reader that the terminology '*lifted holonomy*' that we use to designate $\widetilde{H}^{\alpha}_{g,n}$ can be misleading. Indeed, the latter map is not a holonomy in a natural sense. This will make be clear below when considering the genus 1 case, see Example 4.3 below.

4.2. **An explicit description of Veech's foliation when $g = 1$.** We now focus on the case when $g = 1$, with $n \geq 2$ arbitrary.

The first particular and interesting feature of this special case is that it is possible to define a 'lifted holonomy map' on the Torelli space $\mathscr{T}or_{1,n}$.

Then, when dealing with elliptic curves, all the rather abstract considerations of the preceding subsection can be made completely explicit. The reason for this is twofold: first, there is a nice explicit description of the Torelli space $\mathscr{T}or_{1,n}$ due to Nag; second, on tori, one can give an explicit formula for a metric inducing a flat structure with conical singularities in terms of theta functions.

4.2.1. Our goal here is to construct abstractly a lift to $\mathbb{R}^2$ of the map $h^{\alpha}_{1,n}$: $\mathscr{T}or_{1,n} \to \mathbb{U}^2$. Our construction is based on the following crucial result:

Lemma 4.2. *On a punctured torus, two simple closed curves which are homologous are actually isotopic.*

Proof. Let Σ stand for a finite subset of $T = \mathbb{R}^2/\mathbb{Z}^2$. We consider two simple closed curves a and b in $T^* = T \setminus \Sigma$, assumed to be homologous.

We need first to treat the case when Σ is empty. Since $\pi_1(T) = \mathbb{Z}^2$ is abelian, it coincides with its abelianization, namely $H_1(T,\mathbb{Z})$. From this, it follows that a and b are homotopic. Then a classical result from the theory of surfaces [16] allows to conclude that these two curves are isotopic.

We now consider the case when Σ is not empty which is the one of interest for us. The hypothesis implies that a and b are *a fortiori* homologous in T. From above, it follows that they are isotopic through an isotopy $I : [0,1] \times S^1 \to T$. We can assume that this isotopy is minimal in the sense that the number m of couples $(t,\theta) \in [0,1] \times S^1$ such that $I(t,\theta) \in \Sigma$ is minimal.

We denote by $(t_1,\theta_1),\ldots,(t_m,\theta_m)$ the elements of $I^{-1}(\Sigma)$. For any $i = 1,\ldots,m$, we set $s_i = I(t_i,\theta_i) \in \Sigma$ and define $\epsilon(i) \in \{\pm 1\}$ as follows: $\epsilon(i) = 1$ if $(\partial I/\partial\theta, \partial I/\partial t)$ form an direct basis of the tangent space of T at s_i; otherwise, we set $\epsilon(i) = -1$.

Therefore we have

$$[b] = [a] + \sum_{i=1}^{m} \epsilon(i)[\delta_{s_i}]$$

in $H_1(T^*,\mathbb{Z})$, where δ_s stands for a small circle turning around s counterclockwise for any $s \in \Sigma$. If i and i' are two indices such that $s_i = s_{i'}$ then $\epsilon(i)$ and $\epsilon(i')$ must be equal. Indeed otherwise these two crossings could be cancelled what would contradict the minimality of m. Since $[b] = [a]$ by assumption, we have $\sum_{i=1}^m \epsilon(i)[\delta_{s_i}] = 0$. This relation is necessarily an integer multiple of $\sum_{s\in\Sigma}[\delta_s]$. Remark that the latter can be be cancelled by modifying I: a simple closed curve on T^* cuts T into a cylinder and we can find an isotopy consisting of going along this cylinder crossing every puncture once in the same direction.

Post-composing I by such an isotopy the appropriate number of time allows to cancel all the remaining crossings. The lemma follows. □

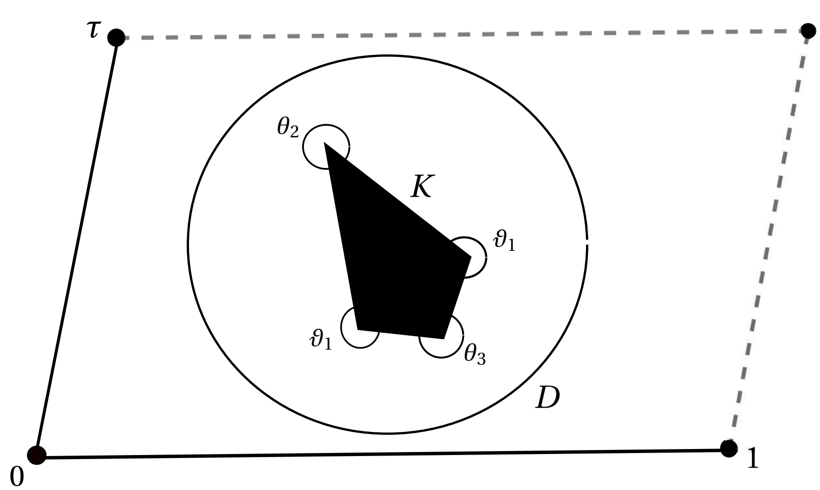
With this lemma at hand, one can proceed as in §4.1.7 and construct a real analytic map $\widetilde{h}^{\alpha}_{1,n} : \mathscr{T}or_{1,n} \to \mathbb{R}^2$ making the following diagram commutative:

$$\begin{array}{ccc} \mathscr{T}or_{1,n} & \xrightarrow{\widetilde{h}^{\alpha}_{1,n}} & \mathbb{R}^2 \\ & \searrow^{h^{\alpha}_{1,n}} & \downarrow e(\cdot) \\ & & \mathbb{U}^2 . \end{array}$$

Note that the lift of $h^{\alpha}_{1,n}$ to $\mathbb{R}^2$ is unique, up translation by an element of $2\pi\mathbb{Z}^2$, as soon as one demands that it is continuous. We have proved above that such a *'lifted holonomy'* exists on the Torelli space of punctured elliptic curves. We will give an explicit and particularly simple expression for it in §4.2.3 below.

To conclude these generalities, we would like to warn the reader that the terminology *'lifted holonomy'* we use to designate $\widetilde{h}^{\alpha}_{1,n}$ is misleading. Let E^*_τ be a n-punctured elliptic curve. With the notations of §4.1.4, the homology classes a_1, b_1 and $c_1, \ldots, c_{n-1}$ are representable by closed simple curves and span freely $H_1(E^*_\tau, \mathbb{Z})$. Using (54), one can construct a lift $\mathscr{T}or_{1,n} \to \mathrm{Hom}(H_1(1,n), \mathbb{R})$ of the map $\mathscr{T}or_{1,n} \to \mathrm{Hom}^{\alpha}(H_1(1,n), \mathbb{U})$ in (51). If the latter is a genuine (linear) holonomy map, this is not the case for the former. Indeed, this additive character on $H_1(1,n)$ is not geometric in a meaningful sense: as the example below shows, its value on $c_1 + \cdots + c_{n-1}$ a priori differs from $-\kappa(c_n)$.

Example 4.3. Let $\tau \in \mathbb{H}$ be arbitrary. Consider a disk D in $]0,1[_\tau \subset E_\tau$ and a kite $K \subset D$, whose exterior angles are $\vartheta_1, \theta_2, \theta_3 \in]0, 2\pi[$ (see the picture below).



Consider E_τ with its non-singular canonical flat structure. Removing the interior of K and gluing pairwise the edges of its boundary which are of the same length, one ends up with a flat tori $E_{\tau,K}$ with three conical singularities, of cone angles $\theta_1 = 2\vartheta_1 \in]0,4\pi[$ and $\theta_2, \theta_3 \in]0,2\pi[$ (in the language of [19, §6], we have performed a 'Kite surgery' on the flat torus E_τ in order to construct $E_{\tau,K}$).

Let a_1 and b_1 be the loops in $E_{\tau,K}$ which correspond to the images in E_τ of the two segments $[0,1]$ and $[0,\tau]$ respectively. With the same meaning for c_1, c_2 and c_3 as above, one can see $(E_{\tau,K}, a_1, b_1, c_1, c_2, c_3)$ as a point in $\mathscr{T}or_{1,3}$. If c stands for the loop given by the boundary of D oriented in the direct order, then $c = c_1 + c_2 + c_3$, hence $c = 0$ in $H_1(E^*_{\tau,K}, \mathbb{Z})$. But clearly, computing the total angular curvature of c depends only on the flat geometry along ∂D, hence can be performed in the flat tori E_τ. One gets $\kappa(c) = 2\pi \neq 0$ although c is trivial in homology. This shows that κ does not induce a real character on $H_1(1,3) = \pi_1(1,3)^{ab}$ in a natural way.

To summarize the discussion above: what we have constructed is a natural lift to $\mathbb{R}^2$ of the map $h^\alpha_{1,n} : \mathscr{T}or_{1,n} \to \mathbb{U}^2$ but not a lift to $\mathrm{Hom}(H_1(1,n), \mathbb{R})$ of the genuine holonomy map $\mathscr{T}or_{1,n} \to \mathrm{Hom}^\alpha(H_1(1,n), \mathbb{U})$ in diagram (51).

4.2.2. **The Torelli space of punctured elliptic curves.** For (g,n) arbitrary, the Torelli group $\mathrm{Tor}_{g,n}$ is defined as the subgroup of the pure mapping class group $\mathrm{PMCG}_{g,n}$ which acts trivially on the first homology group of fixed n-punctured model surface $S_{g,n}$.[18] It is known that it acts holomorphically, properly, discontinuously and without any fixed point on the Teichmüller space (*cf.* §2.8.3 in [56]). Consequently, the Torelli space $\mathscr{T}or_{1,n} = \mathscr{T}eich_{1,n}/\mathrm{Tor}_{1,n}$ is a smooth complex variety (in particular, it has no orbifold point).

In [55], the author shows that, setting $z_1 = 0$, one has an identification

$$\mathscr{T}or_{1,n} = \Big\{ (\tau, z_2, \ldots, z_n) \in \mathbb{H} \times \mathbb{C}^{n-1} \,\Big|\, z_i - z_j \notin \mathbb{Z}_\tau \text{ for } i,j = 1, \ldots, n,\ i \neq j \Big\}.$$

Moreover there is a universal curve

$$\mathscr{E}_{1,n} \longrightarrow \mathscr{T}or_{1,n}$$

whose fiber over $(\tau, z) = (\tau, (z_2, \ldots, z_n))$ is the elliptic curve $E_\tau = \mathbb{C}/\mathbb{Z}_\tau$. It comes with n global sections σ_i defined by $\sigma_i(\tau, z) = [z_i] \in E_\tau$ for $i = 1, \ldots, n$.

The action of the pure mapping class group $\mathrm{PMCG}_{1,n}$ on the Torelli space is not effective and its kernel is precisely the Torelli group. We denote by $\mathrm{Sp}_{1,n}(\mathbb{Z})$

[18]Beware that several kinds of Torelli groups have been considered in the literature, especially in geometric topology (see *e.g.* [63] where this is carefully explained). The Torelli group we are considering in this text is known as the *'small Torelli group'*.

the quotient $\mathrm{PMCG}_{1,n}/\mathrm{Tor}_{1,n}$. It is isomorphic to the group of automorphisms of the first homology group of a n-punctured genus g surface which leaves the cup-product invariant.[19]

From [55], one deduces that there is an isomorphism

$$\mathrm{Sp}_{1,n}(\mathbb{Z}) \simeq \mathrm{SL}_2(\mathbb{Z}) \ltimes \left(\mathbb{Z}^2\right)^{n-1}$$

where the semi-direct product is given by

$$\Big(M', \big(\boldsymbol{k}', \boldsymbol{l}'\big)\Big) \cdot \Big(M, \big(\boldsymbol{k}, \boldsymbol{l}\big)\Big) = \Big(M'M, \rho(M) \cdot \big(\boldsymbol{k}', \boldsymbol{l}'\big) + \big(\boldsymbol{k}, \boldsymbol{l}\big)\Big)$$

for $M, M' \in \mathrm{SL}_2(\mathbb{Z})$ and $(\boldsymbol{k}, \boldsymbol{l}) = \big((k_i, l_i)\big)_{i=2}^n, (\boldsymbol{k}', \boldsymbol{l}') = \big((k_i', l_i')\big)_{i=2}^n \in \left(\mathbb{Z}^2\right)^{n-1}$, with

$$\rho(M) = \begin{bmatrix} d & b \\ c & a \end{bmatrix} \quad \text{and} \quad M \cdot \big(\boldsymbol{k}, \boldsymbol{l}\big) = \Big(\big(ak_i + bl_i, ck_i + dl_i\big)\Big)_{i=2}^n \tag{57}$$

if $M = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{SL}_2(\mathbb{Z})$.

Moreover, one has

$$\big(M, (\boldsymbol{k}, \boldsymbol{l})\big)^{-1} = \Big(M^{-1}, \Big(\big(ak_i - bl_i, -ck_i + dl_i\big)_{i=2}^n\Big)\Big).$$

The action of $\big(M, (\boldsymbol{k}, \boldsymbol{l})\big) \in \mathrm{SL}_2(\mathbb{Z}) \ltimes \left(\mathbb{Z}^2\right)^{n-1}$ on the Torelli space is given by

$$\Big(M, \big(\boldsymbol{k}, \boldsymbol{l}\big)\Big) \cdot (\tau, z) = \left(\frac{a\tau + b}{c\tau + d}, \frac{z_2 + k_2 + l_2\tau}{c\tau + d}, \ldots, \frac{z_n + k_n + l_n\tau}{c\tau + d}\right) \tag{58}$$

for $(\tau, z) = (\tau, z_2, \ldots, z_n) \in \mathcal{T}or_{1,n}$.

The epimorphism of groups $\mathrm{SL}_2(\mathbb{Z}) \ltimes \left(\mathbb{Z}^2\right)^{n-1} \to \mathrm{SL}_2(\mathbb{Z})$ is compatible with the natural projection

$$\mu = \mu_{1,n} : \mathcal{T}or_{1,n} \longrightarrow \mathcal{T}or_{1,1} = \mathbb{H}.$$

In other terms: there is a surjective morphism in the category of analytic G-spaces:

$$\begin{array}{ccc} \mathcal{T}or_{1,n} & \xrightarrow{\quad\mu\quad} & \mathbb{H} \\ \circlearrowleft & & \circlearrowleft \\ \mathrm{SL}_2(\mathbb{Z}) \ltimes \left(\mathbb{Z}^2\right)^{n-1} & \longrightarrow & \mathrm{SL}_2(\mathbb{Z}) . \end{array} \tag{59}$$

[19] We are not aware of any other proof of this result than the one given in the unpublished thesis [74].

4.2.3. **Flat metrics with conical singularities on elliptic curves.** As in the genus 0 case, there is a general explicit formula for the flat metrics on elliptic curves we are interested in. We fix $n>1$ and $\alpha=(\alpha_i)_{i=1}^n \in]-1,\infty[$ such that $\sum_i \alpha_i=0$.

For any elliptic curve $E_\tau=\mathbb{C}/\mathbb{Z}_\tau$, we will denote by u the usual complex coordinates on $\mathbb{C}=\widetilde{E_\tau}$. Let τ be fixed in $\mathbb{H}$ and assume that $z=(z_1,\dots z_n)$ is a n-uplet of pairwise distinct points on $\mathbb{C}$. If one defines a_0 as the real number

$$a_0=-\frac{\Im\mathrm{m}\big(\sum_{i=1}^n \alpha_i z_i\big)}{\Im\mathrm{m}(\tau)}, \tag{60}$$

then $\Im\mathrm{m}(a_0\tau+\sum_i \alpha_i z_i)=0$, hence the constant

$$a_\infty=a_0\tau+\sum_{i=1}^n \alpha_i z_i \tag{61}$$

is a real number as well.

Considering $(\tau,z)\in\mathscr{T}or_{1,n}$ as a fixed parameter, we recall the definition of the function T^α introduced in §3.2 above: it is the function of the variable u defined by

$$T^\alpha_{\tau,z}(u)=T^\alpha(u,\tau,z)=e^{2i\pi a_0 u}\prod_{i=1}^n \theta(u-z_i,\tau)^{\alpha_i}.$$

We see this function as a holomorphic multivalued function on the n-punctured elliptic curves $E_{\tau,z}$. From Lemma 3.1, we know that the monodromy of $T^\alpha_{\tau,z}$ is multiplicative and is given by the character ρ whose characteristic values are

$$\rho_0=e^{2i\pi a_0},\qquad \rho_k=e^{2i\pi\alpha_k}\quad \text{for } k=1,\dots,n \qquad \text{and} \qquad \rho_\infty=e^{2i\pi a_\infty}.$$

Since a_0, a_∞ and the α_k's are real, ρ is unitary. Thus for any $(\tau,z)\in\mathscr{T}or_{1,n}$,

$$m^\alpha_{\tau,z}=\big|T^\alpha_{\tau,z}(u)du\big|^2$$

defines a flat metric on $E_{\tau,z}$. Moreover, the theta function $\theta(\cdot)$, viewed as a section of a line bundle on E_τ, has a single zero at the origin, which is simple.

This implies that up to multiplication by a positive constant, one has

$$m^\alpha_{\tau,z}\sim\big|(u-z_k)^{\alpha_k}du\big|^2$$

on a neighborhood of z_k, for $k=1,\dots,n$. This shows that the flat structure induced by $m^\alpha_{\tau,z}$ on $E_{\tau,z}$ has conical singularities with exponent α_k at $[z_k]$ for every $k=1,\dots,n$. We recall the reader that assuming that $(\tau,z)\in\mathscr{T}or$ implies that $z=(z_1,\dots,z_n)\in\mathbb{C}^n$ has been normalized such that $z_1=0$.

It follows from Troyanov's theorem that considering $\mathscr{T}or_{1,n}$ as the quotient of the space $\mathscr{E}^\alpha_{1,n}$ of isotopy classes of flat tori by the Torelli group $\mathrm{Tor}_{1,n}$ amounts to associate the triplet $(\tau,z,m^\alpha_{\tau,z})$ to any $(\tau,z)\in\mathscr{T}or_{1,n}$.

Any element of $(\tau,z)\in\mathscr{T}or_{1,n}$ comes with a well-defined system of generators of $H_1(E_{\tau,z},\mathbb{Z})$. Let $\epsilon>0$ be very small. For $k=1,\ldots,n$, let γ_k^ϵ be a positively oriented small circle centered at $[z_k]$ in E_τ, of radius ϵ. Let γ_0^ϵ (resp. γ_∞^ϵ) be the loop in $E_{\tau,z}$ defined as the image of $[0,1]\ni t\mapsto \epsilon(1+i)+t$ (resp. of $[0,1]\ni t\mapsto\epsilon(1+i)+t\cdot\tau$) by the canonical projection. For ϵ sufficiently small, the homology classes of the $\gamma_\bullet^\epsilon$'s for $\bullet=0,1,\ldots,n,\infty$ do not depend on ϵ. We just denote by $\gamma_\bullet$ the associated homology classes. These generate $H_1(E_{\tau,z},\mathbb{Z})$ and $\sum_{k=1}^n\gamma_n=0$ is the unique linear relation they satisfy (see Figure 9 below).

At this point, it is quite obvious that the linear holonomies of the flat surface $(E_{\tau,z},m_{\tau,z}^\alpha)$ along γ_0 and γ_∞ are respectively

$$\rho_0=\rho_0(\tau,z)=\exp\big(2i\pi a_0\big)\qquad\text{and}\qquad\rho_\infty=\rho_\infty(\tau,z)=\exp\big(2i\pi a_\infty\big).$$

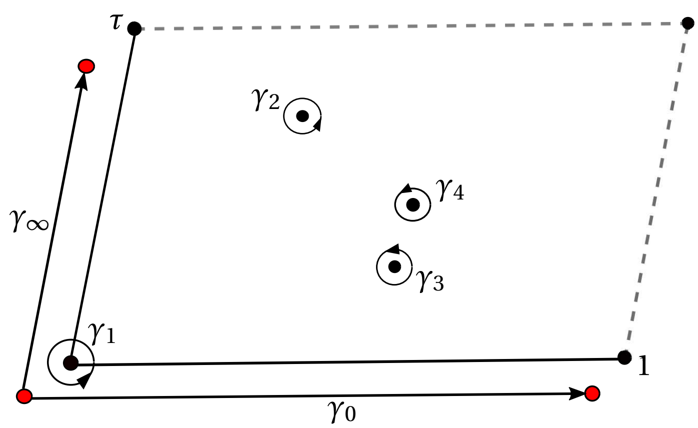


FIGURE 9.

It follows that one has the following explicit expression for the linear holonomy map (52) (to make the notations simpler, we do not specify the subscripts $1,n$ starting from now):

$$\begin{aligned} h^\alpha:\mathscr{T}or_{1,n}&\longrightarrow\mathbb{U}^2\\ (\tau,z)&\longmapsto\big(\rho_0(\tau,z),\rho_\infty(\tau,z)\big).\end{aligned}$$

This map is the composition with the exponential map $e(\cdot)=\exp(2i\pi\cdot)$ of

$$\begin{aligned} \xi^\alpha:\mathscr{T}or_{1,n}&\longrightarrow\mathbb{R}^2\\ (\tau,z)&\longmapsto\big(a_0(\tau,z),a_\infty(\tau,z)\big).\end{aligned}\tag{62}$$

where $a_0(\tau,z)$ and $a_\infty(\tau,z)$ are defined in (60) and (61) respectively.

Remark 4.4. The map ξ^α defined above is then a real-analytic lift of h^α to $\mathbb{R}^2$. We do not know if it coincides with the lifted holonomy map $\widetilde{h}^\alpha$ constructed in §4.1.7. But since the former differs from the latter up to translation by an element of $2\pi\mathbb{Z}^2$, this will be irrelevant for our purpose.

With the material introduced so far, one can make some of the general results obtained by Veech in [76] in the case when $g=1$ completely explicit.

Since we are mainly interested in the case when the leaves of Veech's foliation carry a complex hyperbolic structure, we will assume from now on that (8) holds true. Moreover, there is no loss of generality by assuming that theα_i's are presented in decreasing order. Hence, from now on, one assumes that

$$-1<\alpha_n\leq \alpha_{n-1}\leq\cdots\leq\alpha_2<0<\alpha_1<1. \tag{63}$$

Proposition 4.5 (Explicit description of $\mathscr{F}^\alpha$ on the Torelli space)**.**

(*i*) *The map ξ^α is a primitive first integral of Veech's foliation on $\mathscr{T}or_{1,n}$.*

(*ii*) *For any $a=(a_0,a_\infty)\in \mathrm{Im}(\xi^\alpha)$, the leaf $\mathscr{F}^\alpha_a=(\xi^\alpha)^{-1}(a)$ is the complex subvariety of $\mathscr{T}or_{1,n}$ cut out by the affine equation*

$$a_0\tau+\sum_{k=2}^n \alpha_k z_k=a_\infty\,. \tag{64}$$

(*iii*) *The image of ξ^α is $\mathbb{R}^2$ if $n\geq 3$ and $\mathbb{R}^2\setminus\alpha_1\mathbb{Z}^2$ if $n=2$:*

$$\mathrm{Im}\big(\xi^\alpha\big)=\begin{cases}\mathbb{R}^2\setminus\alpha_1\mathbb{Z}^2 & \text{if } n=2;\\ \mathbb{R}^2 & \text{if } n\geq 3\,.\end{cases}$$

(*iv*) *Veech's foliation $\mathscr{F}^\alpha$ is invariant by $\mathrm{Sp}_{1,n}(\mathbb{Z})\simeq \mathrm{SL}_2(\mathbb{Z})\ltimes\big(\mathbb{Z}^2\big)^{n-1}$.*

More precisely, one has

$$g^{-1}\big(\mathscr{F}^\alpha_a\big)=\mathscr{F}^\alpha_{g\bullet a}$$

for any $a=(a_0,a_\infty)\in\mathrm{Im}(\xi^\alpha)$ and any $g=\big(M,(\boldsymbol{k},\boldsymbol{l})\big)\in\mathrm{SL}_2(\mathbb{Z})\ltimes(\mathbb{Z}^2)^{n-1}$, for a certain action $\bullet$ of this group on $\mathbb{R}^2$ given explicitly by

$$\Big(M,\big(\boldsymbol{k},\boldsymbol{l}\big)\Big)\bullet\big(a_0,a_\infty\big)=\left(a_0a-a_\infty c+\sum_{i=2}^n\alpha_i l_i\,,\,-a_0b+a_\infty d-\sum_{i=2}^n\alpha_i k_i\right) \tag{65}$$

if $M=\left[\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right]\in\mathrm{SL}_2(\mathbb{Z})$ and $(\boldsymbol{k},\boldsymbol{l})=\big((k_i,l_i)\big)_{i=2}^n\in\big(\mathbb{Z}^2\big)^{n-2}$.

Proof. The fact that ξ^α is a first integral for $\mathscr{F}^\alpha$ has been established in the discussion preceding the proposition. The fact that it is primitive follows from (*ii*) since any equation of the form (64) cuts out a connected subset of $\mathscr{T}or_{1,n}$.

Considering the formulae (60) and (61), the proof of (*ii*) is straightforward.

To prove (*iii*), remark that $\mathscr{T}or_{1,n}$ is nothing else but $\mathbb{H}\times\mathbb{C}^{n-1}$ minus the union of the complex hypersurfaces $\Sigma_{i,j}^{p,q}$ cut out by

$$z_i - z_j + p + q\tau = 0\,, \tag{66}$$

for $(p,q)\in\mathbb{Z}^2$ and i,j such that $1\leq i<j\leq n$ (remember that $z_1=0$ according to our normalization). For $(a_0,a_\infty)\in\mathbb{R}^2$, (64) has no solution in $\mathscr{T}or_{1,n}$ if and only if it cuts out one of the hypersurfaces $\Sigma_{i,j}^{p,q}$. As a consequence, the linear parts (in (τ,z)) of the affine equations (64) and (66) should be proportional. Since all the α_i's are negative for $i\geq 2$ according to our assumption (63), this is clearly impossible if $n\geq 3$. Consequently, one has $\mathrm{Im}(\xi^\alpha)=\mathbb{R}^2$ when $n\geq 3$.

When $n=2$, the equations (63) reduce to the following one: $q\tau+z_2+p=0$ with $(p,q)\in\mathbb{Z}^2$. Such an equation is proportional to an equation of the form $a_0\tau+\alpha_2 z_2-a_\infty=0$ if and only if $(a_0,a_\infty)\in\alpha_2\mathbb{Z}^2$. Since $\alpha_1=-\alpha_2$ when $n=2$, one obtains that $\mathrm{Im}(\xi^\alpha)=\mathbb{R}^2\setminus\alpha_1\mathbb{Z}^2$ in this case.

Finally, the fact that $\mathscr{F}^\alpha$ is invariant by the suitable quotient of the pure mapping class group has been proved in greater generality by Veech. In the particular case we are considering, this can be verified by direct and explicit computations by using the material of §4.2.2. In particular, formula (65) for the action of $\mathrm{SL}_2(\mathbb{Z})\ltimes(\mathbb{Z}^2)^{n-1}$ on $\mathbb{R}^2$ follows easily from (58). □

Remark 4.6. (1). The description of the image of the map ξ^α given in (*iii*) allows to answer (in the particular case when $g=1$) a question raised implicitly by Veech (see the sentence just after Proposition 7.10 in [76]).

(2). From Remark 4.4 and from the proposition above, it follows that the phenomenon evoked at the end of §4.1.8 occurs indeed when $g=1$: in this case, any level subset $\mathscr{F}_\rho^\alpha$ of the linear holonomy map (52), which is called a 'leaf' by Veech in [76], actually has a countable set of connected components, *cf.* (56).

From the explicit and elementary description of Veech's foliation on $\mathscr{T}or_{1,n}$ given above, one deduces easily the following results.

Corollary 4.7. *Veech's foliation $\mathscr{F}^\alpha$ depends only on $[\alpha_1:\alpha_2:\cdots:\alpha_n]\in\mathbb{P}(\mathbb{R}^n)$.*

In particular, when $n=2$, Veech's foliation does not depend on α.

The preceding statement concerns only $\mathscr{F}^\alpha$ viewed as a real-analytic foliation of $\mathscr{T}or_{1,n}$. If its leaves do depend only on α up to a scaling factor, it does not apply to the complex hyperbolic structures they carry: they do not depend only on $[\alpha]$ but on α as well (see Theorem 1.10 when $n=2$ for instance).

Proposition 4.8. *For any leaf $\mathscr{F}_a^\alpha$ of Veech's foliation on the Torelli space:*

1. *the inclusion $\mathscr{F}_a^\alpha\subset\mathscr{T}or_{1,n}$ induces an injective morphism of the corresponding fundamental groups;*

(2) *any connected component of the preimage of* $\mathscr{F}_a^\alpha$ *in* $\mathscr{T}eich_{1,n}$ *is simply connected.*

Proof. In the case when $n=2$, any leaf $\mathscr{F}_r^\alpha$ is isomorphic to $\mathbb{H}$ (see §4.3 below for some details) thus is simply connected hence there is nothing to prove.

We sketch a proof of the proposition in the case when $n=3$. The proof in the general case is similar and left to the reader. Let $r\in\mathbb{R}^2$ be fixed. One verifies that the linear projection $\mathscr{T}or_{1,n}\to\mathbb{H}$ induces a trivial topological bundle $\mathscr{F}_r^\alpha\to\mathbb{H}$ by restriction. Let $\tau\in\mathbb{H}$ be fixed and denote by $\mathscr{F}_r^\alpha(\tau)$ the fiber over this point. Since $\mathbb{H}$ is simply connected, the inclusion of $\mathscr{F}_r^\alpha(\tau)$ into $\mathscr{F}_r^\alpha$ induces an identification of the corresponding fundamental groups.

Let $\eta=(\tau,z_2^*,z_3^*)$ be an arbitrary element of $\mathscr{F}_r^\alpha(\tau)$ and consider the map from $\mathbb{C}$ into $\{\tau\}\times\mathbb{C}^2$ that associates the 3-uplet $(\tau,z_2^*+\alpha_3\xi,z_3^*-\alpha_2\xi)$ to any $\xi\in\mathbb{C}$. One promptly verifies that there exists a discrete countable subset $C_\eta=C_{r,\eta}^\alpha\subset\mathbb{C}$ such that, by restriction, the preceding injective affine map induces an isomorphism: $i:\mathbb{C}\setminus C_\eta\simeq\mathscr{F}_r^\alpha(\tau)$. Moreover, for η sufficiently generic in $\mathscr{F}_r^\alpha(\tau)$, the segment $]0,c[$ does not meet C_η, this for any $c\in C_\eta$. Then for any such c, let γ_c be the homotopy class of a path in $(\mathbb{C}\setminus C_\eta,0)$ consisting of the concatenation of $]0,(1-\epsilon_c)c[$ with $0<\epsilon_c<<1$, then of a circular loop in the direct send, of center c, with radius ϵ_c starting and finishing at $(1-\epsilon_c)c$ then going back to 0 along the segment $](1-\epsilon_c)c,0[$. The classes γ_c for $c\in C_\eta$ freely generate $\pi_1(\mathbb{C}\setminus C_\eta,0)\simeq\pi_1(\mathscr{F}_r^\alpha,\eta)$, hence to prove the proposition, it suffices to prove that the class of $i_*(\gamma_c)$ is not trivial in $\pi_1(\mathscr{T}or_{1,n},\eta)$ for every $c\in C_\eta$.

For $c\in C_\eta$ arbitrary, there is a divisor $D_c(p,q)$ cut out by an equation of the form $z_2-(p+q\tau)=0$, $z_3-(p+q\tau)=0$ or $z_2-z_3-(p+q\tau)=0$ for some integers $p,q\in\mathbb{Z}$ such that c is the intersection point of $\mathscr{F}_r^\alpha$ with $D_c(p,q)$.

Fact: for any $c\in C_\eta$, the homology class of γ_c in $\mathscr{T}or_{1,n}$ is not trivial. In particular, this implies that γ_c, viewed as an element of $\pi_1(\mathscr{T}or_{1,n})$, is not trivial. Then the natural map $\pi_1(\mathscr{F}_r^\alpha)\to\pi_1(\mathscr{T}or_{1,n})$ is injective which proves (1).

Finally, the second point of the proposition follows at once from the first since the projection $\mathscr{T}eich_{1,n}\to\mathscr{T}or_{1,n}$ is nothing else but the universal covering map of the Torelli space. □

4.2.4. **Algebraic leaves of Veech's foliation.** Using Proposition 4.5, it is easy to determine and to describe the algebraic leaves of Veech's foliation on $\mathcal{M}_{1,n}$.

Since the case $n=2$ is particular and because we are going to focus on it in the sequel, we left it aside until §4.3 and assume $n\geq 3$ in the lines below.

4.2.4.1. For $a=(a_0,a_\infty)\in\mathbb{R}^2$, let $\mathcal{F}_a^\alpha$ be the corresponding leaf of Veech's foliation $\mathcal{F}^\alpha$ on $\mathcal{M}_{1,n}$: it is the image of $\mathscr{F}_a^\alpha\subset\mathscr{T}or_{1,n}$ by the action of $\mathrm{SL}_2(\mathbb{Z})\ltimes$

$(\mathbb{Z}^2)^{n-1}$. The question we are interested in it twofold: first, we want to determine the lifted holonomies a's such that $\mathcal{F}_a^\alpha$ is an algebraic subvariety of the moduli space; secondly, we would like to give a description of such leaves.

A preliminary remark is in order: on the moduli space $\mathcal{M}_{1,n}$, Veech's foliation is not truly a foliation but an orbi-foliation. Consequently, from a rigorous point of view, the algebraic leaves of $\mathcal{F}^\alpha$, if any, are a priori algebraic suborbifolds of $\mathcal{M}_{1,n}$. However, this subtlety, if important for what concerns the complex hyperbolic structure on the algebraic leaves, is not really relevant for what interests us here, namely their topological/geometric description. For this reason, we will not consider this point further and will abusively speak only of subvarieties and not of sub-orbifolds in the lines below.

4.2.4.2. For $a = (a_0, a_\infty) \in \mathbb{R}^2$, we denote its orbit under $\mathrm{SL}_2(\mathbb{Z}) \ltimes (\mathbb{Z}^2)^{n-1}$ by:

$$[a] = \big[a_0, a_\infty\big] = \Big(\mathrm{SL}_2(\mathbb{Z}) \ltimes \big(\mathbb{Z}^2\big)^{n-1}\Big) \bullet a \subset \mathbb{R}^2 .$$

According to a classical result of the theory of foliations (see [6, p.51] for instance) a necessary and sufficient condition for the leaf $\mathcal{F}_a^\alpha$ to be an (analytic) subvariety of $\mathcal{M}_{1,n}$ is that $[a]$ be a discrete subset of $\mathbb{R}^2$.

From (65), one gets easily $\mathrm{Id} \ltimes (\mathbb{Z}^2)^{n-1} \bullet a = a + \mathbb{Z}(\alpha)^2$ where $\mathbb{Z}(\alpha)$ stands for the $\mathbb{Z}$-submodule of $\mathbb{R}$ spanned by the α_i's, *i.e.* $\mathbb{Z}(\alpha) = \sum_{i=1}^n \alpha_i \mathbb{Z}$. Thus a necessary condition for $[a]$ to be discrete is that the α_i's all are commensurable, *i.e.* there exists a non-zero real constant λ such that $\lambda\alpha_i \in \mathbb{Q}$ for $i = 1, \ldots, n$.

Assuming that α is commensurable, let λ be the positive real number such that $\mathbb{Z}(\alpha) = \lambda\mathbb{Z}$. Thus one has

$$\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \ltimes \big(\mathbb{Z}^2\big)^{n-1}\right) \bullet a = a + \lambda\mathbb{Z}^2 \tag{67}$$

We denote by $\mathbb{Z}_{\boldsymbol{l}}^{n-1}$ the subgroup of $(\mathbb{Z}^{n-1})^2$ formed by pairs $(\boldsymbol{k}, \boldsymbol{l}) \in (\mathbb{Z}^{n-1})^2$ with $\boldsymbol{k} = 0$. Setting $a = (a_0, a_\infty)$, it follows immediately from (65) that

$$\left(\begin{bmatrix} 1 & 1 \\ \mathbb{Z} & 0 \end{bmatrix}, \mathbb{Z}_{\boldsymbol{l}}^{n-1}\right) \bullet a = \Big(a_0 + a_\infty\mathbb{Z} + \mathbb{Z}(\alpha), a_\infty\Big).$$

It comes that if $[a]$ is discrete then $a_\infty\mathbb{Z} + \mathbb{Z}(\alpha) = a_\infty\mathbb{Z} + \lambda\mathbb{Z}$ is discrete in $\mathbb{R}$ which implies that $a_\infty \in \lambda\mathbb{Q}$. Using a similar argument, one obtains that $a_0 \in \lambda\mathbb{Q}$ is also a necessary condition for the orbit $[a]$ to be discrete in $\mathbb{R}^2$.

At this point, we have proved that in order for the leaf $\mathcal{F}_a^\alpha$ to be a closed analytic subvariety of $\mathcal{M}_{1,n}$, it is necessary that $(\alpha, a) = (\alpha_1, \ldots, \alpha_n, a_0, a_\infty)$ be commensurable. We are going to see that this condition is also sufficient and actually implies the algebraicity of the considered leaf.

4.2.4.3. We assume that (α, a) is commensurable. Our goal now is to prove that the leaf $\mathcal{F}_a^\alpha$ is an algebraic subvariety of $\mathcal{M}_{1,n}$. We will give a detailed proof of this fact only in the case when $n = 3$. We claim that the general case when $n \geq 3$ can be treated in the exact same way but let the verification of that to the reader.

As above, let $\lambda > 0$ be such that $\lambda\mathbb{Z} = \mathbb{Z}(\alpha)$ (note that λ is uniquely characterized by this equality). Since the two foliations $\mathscr{F}^\alpha$ and $\mathscr{F}^{\alpha/\lambda}$ coincide (more precisely, from (64), it comes that $\mathscr{F}_b^\alpha = \mathscr{F}_{b/\lambda}^{\alpha/\lambda}$ for every $b \in \mathbb{R}^2$), there is no loss in generality by assuming that $\lambda = 1$ or equivalently, that

(68)
- one has $\alpha_i = -p_i$ for $i = 1, \ldots, n$, for some positive integers $p_1, \ldots, p_n$ such that $p_1 + \cdots + p_n = 0$ and $\gcd(p_2, \ldots, p_n) = 1$;
- a is rational, *i.e.* $a \in \mathbb{Q}^2$.

In what follows, we assume that these assumptions hold true.

4.2.4.4. To show that $[a]$ is discrete when a is rational, we first determine a normal form for a representative of such an orbit.

Proposition 4.9.

(1) *For $a \in \mathbb{Q}^2$, let N be the smallest positive integer such that $Na \in \mathbb{Z}^2$.*
   (a) *One has $[a] = \big[0, -1/N\big]$.*
   (b) *If $N = 1$ (that is, if $a \in \mathbb{Z}^2$), then $[a] = \big[0, 0\big]$.*

(2) *The orbit $[a]$ is discrete in $\mathbb{R}^2$ if and only if (α, a) is commensurable.*

Proof. For $a \in \mathbb{Q}^2$, one can write $a_0 = p_0/q$ and $a_\infty = p_\infty/q$ for some integers p_0, p_∞ and $q > 0$ such that $\gcd(p_0, p_\infty, q) = 1$. Let p be the greatest common divisor of p_0 and p_∞: $p = \gcd(p_0, p_\infty)$.

From the proof of Lemma 3 in [52], it comes that $\Gamma(2) \bullet a$, hence $[a]$ contains one of the three following elements: $(p/q, 0), (p/q, p/q)$ or $(0, p/q)$.

Since
$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \bullet \left(\frac{p}{q}, 0\right) = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \bullet \left(\frac{p}{q}, \frac{p}{q}\right) = \left(0, \frac{p}{q}\right),$$
it comes that $(0, p/q) \in \mathrm{SL}_2(\mathbb{Z}) \bullet a \subset [a]$.

Because $\gcd(p_0, p_\infty, q) = \gcd(p, q) = 1$, there exist two integers d and k such that $dp - kq = 1$. From the relation
$$\left(\begin{bmatrix} 1 & 1-d \\ -1 & d \end{bmatrix}, (k, 0)\right) \bullet \left(0, \frac{p}{q}\right) = \left(\frac{p}{q}, \frac{dp - kq}{q}\right) = \left(\frac{p}{q}, \frac{1}{q}\right),$$
one deduces that $(p/q, 1/q) \in [a]$. Since $(p, q, 1) = 1$, it follows from the arguments above that $(0, 1/q) \in \mathrm{SL}_2(\mathbb{Z}) \bullet (p/q, 1/q)$. This implies that $(0, 1/q)$ belongs to $[a]$, hence the same holds true for $(0, -1/q) = (-\mathrm{Id}) \bullet (0, 1/q)$.

Since $qa = (p_0, p_\infty) \in \mathbb{Z}^2$ one has $N \leq q$ where N stands for the integer defined in the statement of the lemma. On the other hand, since $\gcd(p_0, p_\infty, q) = 1$, there exists $u_0, u_\infty, v \in \mathbb{Z}$ such that $u_0 p_0 + u_\infty p_\infty + vq = 1$. Since $Na \in \mathbb{Z}^2$, one has $u_0 N a_0 + u_\infty N a_\infty = N(1 - vq)/q \in \mathbb{Z}$, which implies that q divides N. This shows that $q = N$, thus that the first point of (1) holds true.

When $a \in \mathbb{Z}^2$, the fact that $(0,0) \in [a]$ follows immediately from (67) (recall that we have assumed that $\lambda = 1$), which proves (b).

Finally, using (65), it is easy to verify that all the orbits $[0,0]$ and $[0,-1/N]$ with $N \geq 2$ are discrete subsets of $\mathbb{R}^2$. Assertion (2) follows immediately. □

From the preceding proposition, it follows that the leaves of Veech's foliation $\mathcal{F}^\alpha$ which are closed analytic subvarieties of $\mathcal{M}_{1,3}$ are exactly the one associated to the following 'lifted holonomies'

$$(0,0) \qquad \text{and} \qquad (0,-1/N) \quad \text{with } N \geq 2\,. \tag{69}$$

We will use the following notations for the corresponding leaves:

$$\mathcal{F}^\alpha_0 = \mathcal{F}^\alpha_{(0,0)} \qquad \text{and} \qquad \mathcal{F}^\alpha_N = \mathcal{F}^\alpha_{(0,-1/N)}\,. \tag{70}$$

Let p_1, p_2 and p_3 be the positive integers such that $\alpha_i = -p_i$ for $i = 1,2,3$ (remember our simplifying assumption (68)). The leaves in the Torelli space which correspond to the 'lifted holonomies' (69) are the following:

$$\mathscr{F}^\alpha_0 = \mathscr{F}^\alpha_{(0,0)} = \Big\{ (\tau, z_2, z_3) \in \mathscr{T}or_{1,3} \,\big|\, p_2 z_2 + p_3 z_3 = 0 \Big\}$$

$$\text{and} \quad \mathscr{F}^\alpha_N = \mathscr{F}^\alpha_{(0,1/N)} = \left\{ (\tau, z_2, z_3) \in \mathscr{T}or_{1,3} \,\Big|\, p_2 z_2 + p_3 z_3 = \frac{1}{N} \right\}. \tag{71}$$

Note that the preceding leaves are (possibly orbifold) coverings of the leaves (70): for any $N \neq 1$, the image of $\mathscr{F}^\alpha_N$ by the quotient map $\mathscr{T}or_{1,3} \to \mathcal{M}_{1,3}$ is $\mathcal{F}^\alpha_N$.

4.2.4.5. We are going to consider carefully the case of the leaf $\mathcal{F}^\alpha_0$. We will deal with the case of $\mathcal{F}^\alpha_N$ with $N \geq 2$ more succinctly in the next subsection.

In what follows, we set $p = (p_1, p_2, p_3)$. Remember that p_2 and p_3 determine p_1 since the latter is the sum of the two former: $p_1 = p_2 + p_3$. Note that according to (68), one has $\gcd(p_1, p_2, p_3) = \gcd(p_2, p_3) = 1$.

Consider the affine map from $\mathbb{H} \times \mathbb{C}$ to $\mathbb{H} \times \mathbb{C}^2$ defined for any $(\tau, \xi) \in \mathbb{H} \times \mathbb{C}$ by

$$U_0(\tau, \xi) = \big(\tau, p_3 \xi, -p_2 \xi\big)\,.$$

Let $\mathscr{U}_p$ be the inverse image of $\mathscr{T}or_{1,3} \subset \mathbb{H} \times \mathbb{C}^2$ by U_0. One verifies easily that

$$\mathscr{U}_p = \left\{ (\tau, \xi) \in \mathbb{H} \times \mathbb{C} \,\Big|\, \xi \notin \Big(\frac{1}{p_1}\mathbb{Z}_\tau \cup \frac{1}{p_2}\mathbb{Z}_\tau \cup \frac{1}{p_3}\mathbb{Z}_\tau \Big) \right\} \tag{72}$$

and, by restriction, U_0 induces a global holomorphic isomorphism

$$U_0 : \mathscr{U}_p \simeq \mathscr{F}_0^\alpha \subset \mathscr{T}or_{1,3}\,. \tag{73}$$

Let Fix(0) be the subgroup of $\mathrm{Sp}_{1,3}(\mathbb{Z})$ which leaves $\mathscr{F}_0^\alpha$ globally invariant. It is nothing else but the subgroup of $g \in \mathrm{SL}_2(\mathbb{Z}) \ltimes (\mathbb{Z}^2)^2$ such that $g \bullet (0,0) = (0,0)$. From (65), it is clear that $g = (M, (k_2, l_2), (k_3, l_3))$ is of this kind if and only if $p_2 l_2 + p_3 l_3 = p_2 k_2 + p_3 k_3 = 0$. It follows that

$$\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2 \simeq \mathrm{Fix}(0)\,,$$

where the injection $\mathbb{Z}^2 \hookrightarrow \big(\mathbb{Z}^2\big)^2$ is given by $(k,l) \mapsto \big(p_3(k,l), -p_2(k,l)\big)$.

By pull-back by U_0, one obtains immediately that the corresponding action of $\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ on $\mathscr{U}_p$ is given by

$$\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}, (k,l) \right) \cdot (\tau, \xi) = \left(\frac{a\tau + b}{c\tau + d}, \frac{\xi + k + l\tau}{c\tau + d} \right). \tag{74}$$

For any subgroup Γ in $\mathrm{SL}_2(\mathbb{Z})$, one sets

$$\mathcal{M}_{1,3}(\Gamma) := \mathscr{T}or_{1,3} \big/ \Gamma \ltimes \big(\mathbb{Z}^2\big)^2\,.$$

It is an orbifold covering of $\mathcal{M}_{1,3}$ which is finite hence algebraic if Γ has finite index in $\mathrm{SL}_2(\mathbb{Z})$. In this case, the image $\mathcal{F}_0^\alpha(\Gamma)$ of $\mathscr{F}_0^\alpha$ in $\mathcal{M}_{1,3}(\Gamma)$ is algebraic if and only if $\mathcal{F}_0^\alpha$ is an algebraic subvariety of $\mathcal{M}_{1,3}$. We are going to use this equivalence for a group Γ_p which satisfies the following properties:

(P1). it is a subgroup of finite index of $\Gamma\big(\mathrm{lcm}(p_1, p_2, p_3)\big)$; and

(P2). it acts without fixed point on $\mathbb{H}$.

For instance, setting

$$M_p = \begin{cases} 4 & \text{if } p_2 = p_3; \\ \mathrm{lcm}(p_1, p_2, p_3) & \text{otherwise,} \end{cases}$$

one can take for Γ_p the congruence subgroup of level M_p:

$$\Gamma_p = \Gamma\big(M_p\big)$$

(the case when $p_2 = p_3$ is particular: this equality implies that $p_2 = p_3 = 1$ since $\gcd(p_2, p_3) = 1$. Consequently $\Gamma(\mathrm{lcm}(p_1, p_2, p_3)) = \Gamma(2)$ and this group contains $-\mathrm{Id}$ hence does not act effectively on $\mathbb{H}$).

Since $M = M_p \geq 3$ in every case, Γ_p satisfies the properties (P1) and (P2) stated above. Consequently, the quotient of $\mathbb{H} \times \mathbb{C}$ by the action (74) is the total space of the (non-compact) modular elliptic surface of level M:[20]

$$\mathscr{E}_p := \mathscr{E}(M_p) \longrightarrow Y(M_p)\,. \tag{75}$$

According to [67, §5], $\mathscr{E}_p$ comes with M^2 sections of M-torsion forming an abelian group $S(\mathscr{E}_p)$ isomorphic to $(\mathbb{Z}/M\mathbb{Z})^2$. For any divisor m of M, one denotes by $\mathscr{E}_p[m]$ the union of the images of the elements of order m of $S(\mathscr{E}_p)$:

$$\mathscr{E}_p[m] = \bigcup_{\substack{\sigma\in S(\mathscr{E}_p)\\ m\cdot\sigma=0}} \sigma\big(Y(M_p)\big) \subset \mathscr{E}_p\,.$$

We are almost ready to state our result about the leaf $\mathcal{F}_0^\alpha$. To simplify the notations, we denote respectively by $\mathcal{M}_{1,3}(p)$ and $\mathcal{F}_0^\alpha(p)$ the intermediary moduli space $\mathcal{M}_{1,3}(\Gamma_p)$ and the image of the leaf $\mathscr{F}_0^\alpha$ in it.

The map U_0 induces an isomorphism

$$\mathscr{U}_p\big/\big(\Gamma_p \ltimes \mathbb{Z}^2\big) \simeq \mathcal{F}_0^\alpha(p)\,.$$

Using (72) and (74), it is then easy to deduce the

Proposition 4.10. *The map* (73) *induces an embedding*

$$\mathscr{E}_p \setminus \Big(\mathscr{E}_p[p_1] \cup \mathscr{E}_p[p_2] \cup \mathscr{E}_p[p_3]\Big) \longhookrightarrow \mathcal{M}_{1,3}(p)$$

which is algebraic and whose image is the leaf $\mathcal{F}_0^\alpha(p)$.

In short, this result says that the inverse image of $\mathcal{F}_0^\alpha$ in a certain finite covering of $\mathcal{M}_{1,3}$ is an elliptic modular surface from which the images of some torsion sections have been removed. There is no difficulty to deduce from it a description of $\mathcal{F}_0^\alpha$ itself. But since $\mathrm{SL}_2(\mathbb{Z})$ has elliptic points and contains $-\mathrm{Id}$, the latter is not as nice as the description of $\mathcal{F}_0^\alpha(p)$ given above.

Corollary 4.11. (1) *The projection* $\mathscr{T}or_{1,3} \to \mathbb{H}$ *induces a dominant rational map* $\mathcal{F}_0^\alpha \to Y(1) \simeq \mathbb{C}$ *whose fibers are punctured projective lines.*

(2) *The fiber over any* $j(\tau)$ *distinct from* 0 *and* 1728 *(the two elliptic points of* $Y(1)=\mathbb{C}$*) is the quotient of the punctured elliptic curve* $E_\tau \setminus \big(E_\tau[p_1] \cup E_\tau[p_2] \cup E_\tau[p_3]\big)$ *by the elliptic involution.*

(3) *Both the description of the fibers over 0 and* 1728 *are similar but more involved.*

To end this subsection, we would like to make the particular case when $p_2 = p_3 = 1$ more explicit (note that this condition is equivalent to $\alpha_2 = \alpha_3$). It is more convenient to describe the inverse image $\mathcal{F}_0^\alpha(\Gamma(2))$ of $\mathcal{F}_0^\alpha$ in $\mathcal{M}_{1,3}(\Gamma(2))$: it

[20] See [67] for a reference. Note that we do not use the most basic construction of the theory of modular elliptic surfaces, namely that (75) can be compactified over $X(M_p)$ by adding as fibers over the cusps some generalized elliptic curves (*cf.* also [40, §8]).

is the bundle over $Y(2)=\mathbb{P}^1\setminus\{0,1,\infty\}$, the fiber of which over $\lambda\in\mathbb{C}\setminus\{0,1\}$ is the 4-punctured projective line $\mathbb{P}^1\setminus\{0,1,\lambda,\infty\}$. As an algebraic variety, $\mathcal{F}_0^\alpha(\Gamma(2))$ is then isomorphic to the moduli space $\mathcal{M}_{0,5}$. Actually, there is more: in §6.2.1, we will see that, endowed with Veech's complex hyperbolic structure, $\mathcal{F}_0^\alpha(\Gamma(2))$ can be identified with a Picard/Deligne-Mostow/Thurston moduli space.

4.2.4.6. We now consider succinctly the case of the leaf $\mathcal{F}_N^\alpha$ when N is a fixed integer bigger than or equal to 2. One proceeds as for $\mathscr{F}_0^\alpha$.

Since $\mathcal{F}_N^\alpha$ is cut out by $p_2z_2+p_3z_3=1/N$ in the Torelli space (see (71)), one gets that, by restriction, the affine map

$$\xi\longmapsto\left(p_3\xi+\frac{1}{Np_2},-p_2\xi\right)$$

induces a global holomorphic parametrization of $\mathcal{F}_N^\alpha$ by an open subset $\mathscr{U}_{p,N}$ of $\mathbb{H}\times\mathbb{C}$ which is not difficult to describe explicitly.

The stabilizer $\mathrm{Fix}(N)$ of $(0,-1/N)$ for the action $\bullet$ is easily seen to be the image of the injective morphism of groups

$$\begin{aligned}\Gamma_1(N)\ltimes\mathbb{Z}^2&\longmapsto\mathrm{SL}_2(\mathbb{Z})\ltimes\left(\mathbb{Z}^2\right)^2\\ \left(\begin{bmatrix}a&b\\c&d\end{bmatrix},(k,l)\right)&\longmapsto\left(\begin{bmatrix}a&b\\c&d\end{bmatrix},(k_2,l_2),(k_3,l_3)\right)\end{aligned}$$

with

$$(k_2,l_2)=q_2\left(\frac{d-1}{N},\frac{c}{N}\right)+p_3(k,l)\quad\text{and}\quad(k_3,l_3)=q_3\left(\frac{d-1}{N},\frac{c}{N}\right)-p_2(k,l)$$

where (q_2,q_3) stands for a (fixed) pair of integers such that $q_2p_2+q_3p_3=1$.

Embedding $\Gamma_1(Np_2)\ltimes\mathbb{Z}^2$ into $\mathrm{Fix}(N)$ by setting

$$(k_2,l_2)=\left(\frac{d-1}{Np_2},\frac{c}{Np_2}\right)+p_3(k,l)\quad\text{and}\quad(k_3,l_3)=-p_2(k,l)$$

one verifies that the induced action of $\Gamma_1(Np_2)\ltimes\mathbb{Z}^2$ on $(\tau,\xi)\in\mathscr{U}_{p,N}$ is the usual one (*i.e.* is given by (74)). Consequently, when $Np_2\geq 3$[21], the inclusion $\mathscr{U}_{p,N}\subset\mathbb{H}\times\mathbb{C}$ induces an algebraic embedding

$$\mathcal{F}_N^\alpha\big(\Gamma_1(Np_2)\big)\simeq\mathscr{U}_{p,N}\big/_{\Gamma_1(Np_2)\ltimes\mathbb{Z}^2}\subset\mathscr{E}_1(Np_2)\to Y_1(Np_2)$$

where $\mathscr{E}_1(Np_2)$ stands for the total space of the elliptic modular surface associated to $\Gamma_1(Np_2)$. Moreover, it can be easily seen that the complement of $\mathcal{F}_N^\alpha\big(\Gamma_1(Np_2)\big)$ in $\mathscr{E}_1(Np_2)$ is the union of certain torsion sections.

[21]The case when $p_2=p_3=1$ and $N=2$ is particular and must be treated separately.

Proposition 4.12. (1) *For a certain congruence group $\Gamma_{p,N}$ (which can be explicited), the inverse image of $\mathcal{F}_N^\alpha$ in the intermediary moduli space $\mathcal{M}_{1,3}(\Gamma_{p,N})$ is algebraic and isomorphic to the total space of the modular elliptic surface $\mathcal{E}(\Gamma_{p,N})$ from which the union of some torsion sections have been removed.*

(2) *For $N \geq 3$, the leaf $\mathcal{F}_N^\alpha$ is an algebraic subvariety of $\mathcal{M}_{1,3}$ isomorphic to the total space of the modular elliptic surface $\mathcal{E}_1(N) \to Y_1(N)$ from which the union of certain torsion multi-sections have been removed.*

(3) *The leaf $\mathcal{F}_2^\alpha$ is a bundle in punctured projective lines over $Y_1(2)$.*

Here again, the dichotomy between the cases when $N = 2$ and $N \geq 3$ comes from the fact that $-\mathrm{Id}$ belongs to $\Gamma_1(2)$ whereas it is not the case for $N \geq 3$.

4.2.4.7. We think that considering an explicit example will be quite enlightening.

We assume that $p_2 = p_3 = 1$ (which is equivalent to $\alpha_2 = \alpha_3$) and we fix $N \geq 2$. The preimage $\mathcal{F}_N(2N)$ of $\mathcal{F}_N$ in $\mathcal{M}_{1,2}(\Gamma(2N))$ admits a nice description.

Let $\mathcal{E}(2N) \to Y(2N)$ be the modular elliptic curve associated to $\Gamma(2N)$. We fix a 'base point' $\tau_0 \in \mathbb{H}$. Then for any integers k, l, $(k + l\tau_0)/2N$ defines a point of $2N$-torsion of E_{τ_0} which belongs to exactly one of the $(2N)^2$ $2N$-torsion sections of $\mathcal{E}(2N) \to Y(2N)$. We denote the latter section by $[(k + l\tau)/2N]$.

Then $\mathcal{F}_N(2N)$ is isomorphic to the complement in $\mathcal{E}(2N)$ of the union of $[0]$ and $[1/N]$ with the translation by $[1/2N]$ of the four sections of 2-torsion:

$$\mathcal{F}_N(2N) \simeq \mathcal{E}(2N) \setminus \left([0] \cup \left[\frac{1}{N}\right] \cup \left(\bigcup_{k,l=0,1} \left[\frac{1}{2N} + \frac{k + l\tau}{2} \right] \right) \right).$$

(See also Figure 10 below).

4.2.4.8. To finish our uncomplete study of the algebraic leaves of Veech's foliation on $\mathcal{M}_{1,n}$ when $n \geq 3$, we state the following result which follows easily from all what has been said before (we do not assume that (68) holds true anymore):

Corollary 4.13. *Let $a \in \mathbb{R}^2$. The three following assertions are equivalent:*

(1) *(α, a) is commensurable, i.e. $[\alpha_1 : \cdots : \alpha_n : a_0 : a_\infty] \in \mathbb{P}(\mathbb{Q}^{n+2})$;*

(2) *the leaf $\mathcal{F}_a^\alpha$ is a closed analytic subvariety of $\mathcal{M}_{1,n}$;*

(3) *the leaf $\mathcal{F}_a^\alpha$ is a closed algebraic subvariety of $\mathcal{M}_{1,n}$.*

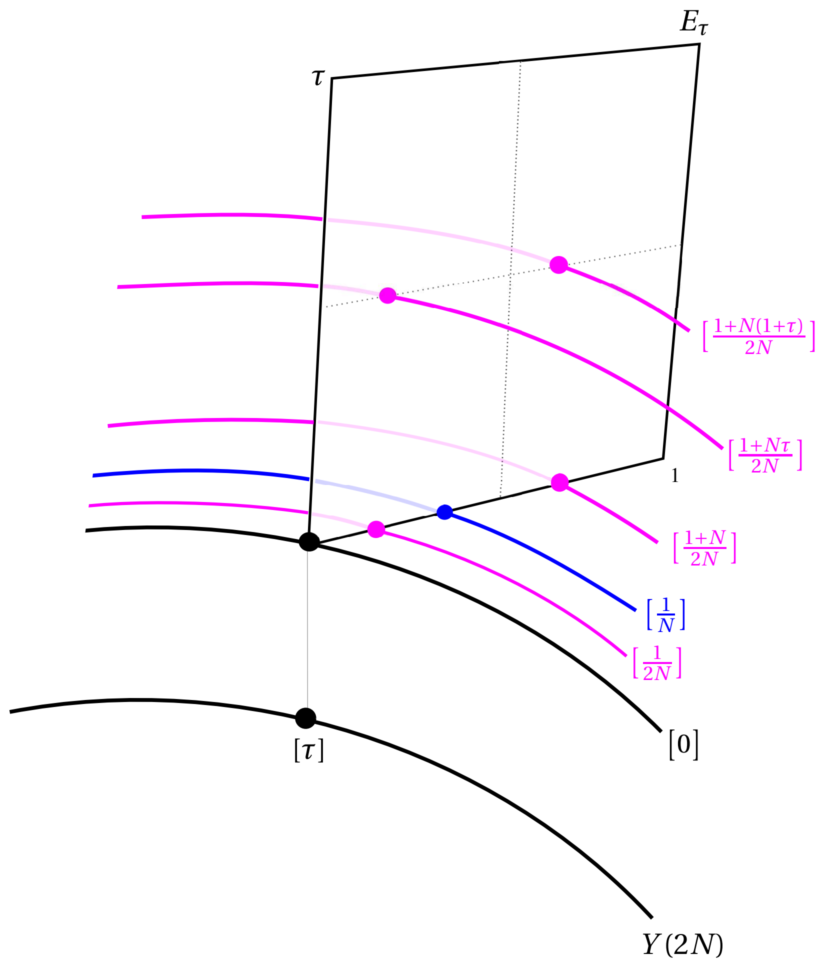


FIGURE 10. The covering $\mathcal{F}_N(2N)$ of the leaf $\mathcal{F}_N$ is the total space of the modular surface $\mathscr{E}(2N) \to Y(2N)$ with the six sections $[0]$, $[1/N]$ and $[(1+N(k+l\tau))/2N]$ for $k,l=0,1$ removed.

4.2.5. **Some algebraic leaves in $\mathcal{M}_{1,3}$ related with some Picard/Deligne-Mostow's orbifolds.** In the $n=3$ case, assume that $\alpha=(\alpha_1,\alpha_2,\alpha_3)$ is such that $\alpha_2 = \alpha_3 = -\alpha_1/2$. Then the leaf $\mathcal{F}_0^\alpha$ formed of flat tori with 3 conical singularities is related to a moduli space of flat spheres with five cone points.

Indeed, the equation of $\mathscr{F}_0^\alpha$ in the Torelli space is $z_2+z_3=0$. It follows that an element $E_{\tau,z}$ of this leaf is a flat structure on E_τ with a conical point of angle $\theta_1 = 2\pi(\alpha_1+1)$ at the origin and two equal conical angles $\theta_2=\theta_3=\pi(2-\alpha_1)$ at $[z_2]$ and $[z_3]=[-z_2]$. This flat structure is invariant by the elliptic involution $i:[z]\mapsto[-z]=-[z]$ on E_τ hence can be pushed-forward by $\wp: E_\tau \to E_\tau/\langle i\rangle \simeq \mathbb{P}^1$. The flat structure one obtains on $\mathbb{P}^1$ has three cone points of angle π at the image of the three 2-torsion points of E_τ by $\wp$, one cone point of angle $\theta_1/2 = \pi(\alpha_1+1)$ at $\wp(0)=\infty$ and one cone point of angle $\pi(2-\alpha_1)$ at $\wp(z_2)=\wp(z_3)$.

At a more global level, this shows that when $\alpha_2=\alpha_3$, the leaf $\mathcal{F}_0^\alpha \subset \mathcal{M}_{1,3}$ admits a special automorphism of order 2 which induces a biholomorphism

$$\mathcal{F}_0^\alpha \longrightarrow \mathcal{M}_{0,\theta(\alpha)}$$

onto the moduli space of flat spheres with five conical points $\mathcal{M}_{0,\theta(\alpha)}$ with associated angle datum

$$\theta(\alpha) = \big(\pi,\pi,\pi,\pi(1+\alpha_1),\pi(2-\alpha_1)\big).$$

Moreover, it is easy to verify that the preceding map is compatible with the $\mathbb{CH}^2$-structures carried by each of these two moduli spaces of flat surfaces.

The 5-uplet $\mu(\alpha)=(\mu_1(\alpha),\ldots,\mu_5(\alpha))\in]0,1[^5$ corresponding to the angle datum $\theta(\alpha)$ in Deligne-Mostow's notation of [11] is given by

$$\mu(\alpha) = \Big(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1-\alpha_1}{2},\frac{\alpha_1}{2}\Big).$$

Then looking at the table page 86 in [11], it comes easily that the metric completion of $\mathcal{M}_{0,\theta(\alpha)}$ is a complex hyperbolic orbifold exactly for two values of α_1, namely $\alpha_1=1/3$ and $\alpha_1=2/3$. It follows that the image of the holonomy of Veech's $\mathbb{CH}^2$-structure of the leaf $\mathcal{F}_0^\alpha$ is a lattice in PU(1,2) exactly when α_1 is equal to one of these two values. Note that the two corresponding lattices are isomorphic, arithmetic and not cocompact.

4.2.6. **Towards a description of the metric completion of an algebraic leaf of Veech's foliation.** We consider here how to describe the metric completion of an algebraic leaf of Veech's foliation when it is endowed with the metric associated to the complex hyperbolic structure it carries. This is a natural question in view of Thurston's results [71] in the genus 0 case.

4.2.6.1. In [19], we have generalized the approach initiated by Thurston which relies on surgeries on flat surfaces to the genus 1 case. From our main result in this paper, it comes that, when α is assumed to be rational, then the metric completion $\overline{\mathcal{F}_N}$ of an algebraic leaf $\mathcal{F}_N$ of Veech's foliation on $\mathcal{M}_{1,n}$:

(1) carries a complex hyperbolic conifold structure of finite volume which extends Veech's hyperbolic structure of $\mathcal{F}_N$;
(2) $\overline{\mathcal{F}_N}$ is obtained by adding to $\mathcal{F}_N$ some (covering of some) strata of flat tori and of flat spheres obtained as particular degenerations of flat tori whose moduli space is $\mathcal{F}_N$.

The strata mentioned in (2) which parametrize flat tori with $n'<n$ cone points are obtained by making several cone points collide hence are called ***C*-strata** (C stands here for *'colliding'*). The others parametrizing flat spheres with $n''\geq$

$n+1$ cone points are obtained by pinching an essential simple curve on some element of $\mathcal{F}_N$ hence are called ***P*-strata** (P stands here for *'pinching'*).[22]

Actually, the main result of [19] concerns the algebraic 'leaves' in $\mathcal{M}_{1,n}$ in the sense of Veech [76]. In our notation, such a leaf $\mathcal{F}_\rho^\alpha$ is the image in $\mathcal{M}_{1,n}$ of a level-subset $\mathscr{F}_\rho^\alpha = (\chi_{1,n}^\alpha)^{-1}(\rho) \subset \mathscr{T}eich_{1,n}$ by Veech's linear holonomy map $\chi_{1,n}^\alpha$ of some element ρ of $\mathrm{Hom}^\alpha(\pi_1(1,n),\mathbb{U}) \simeq \mathbb{U}^2$ (see §4.1.8). The point is that it is not completely clear yet which are the connected components of such a 'leaf' $\mathcal{F}_\rho^\alpha$ in terms of the irreducible leaves $\mathcal{F}_N^\alpha$ considered in the present paper (for instance, the answer depends on α already in the $n=2$ case, see §4.3.1 below).

It follows that the methods developed in [19] to list the strata which must be added to $\mathcal{F}_\rho^\alpha$ in order to get its metric completion do not apply in an effective way to any of the irreducible leaves $\mathcal{F}_N$'s considered here. An interesting feature of the analytic approach to Veech's foliation developed above is that it suggests an explicit and effective way to describe $\overline{\mathcal{F}_N}$ for any N given.

4.2.6.2. In the $n=2$ case, one can give a complete and explicit description of the metric completion of any leaf $\mathcal{F}_N \subset \mathcal{M}_{1,2}$, see §5.3.4 further. In particular, using the results of [19], it comes that the metric completion of a leaf $\mathcal{F}_N$ is obtained by adjoining to it a finite number of P-strata which, in this case, are moduli spaces of flat spheres with three cone points $\mathcal{M}_{0,\theta}$ for some angle data $\theta = (\theta_1,\theta_2,\theta_3) \in]0,2\pi[^3$ which can be explicitly given.

4.2.6.3. We now say a few words about the $n=3$ case. We take $N \geq 4$ in order to avoid any pathological case. Let $\mathscr{E}_1(N) \to Y_1(N)$ be the modular elliptic surface associated to $\Gamma_1(N)$. Then, as proven above in §4.2.4.6, there exists a finite number of torsion multi-sections $\Sigma(1),\ldots,\Sigma(s) \subset \mathscr{E}_1(N)$ such that $\mathcal{F}_N$ is isomorphic to $\mathscr{E}_1(N) \setminus \Sigma$ with $\Sigma = \Sigma(1) \cup \ldots \cup \Sigma(s)$. By restriction, one gets a surjective map

$$\nu_N : \mathcal{F}_N = \mathscr{E}_1(N) \setminus \Sigma \to Y_1(N)$$

which is relevant to describe the first strata (namely the ones of complex codimension 1) which must be attached to $\mathcal{F}_N$ along the inductive process described in [19, §7.1] giving $\overline{\mathcal{F}_N}$ at the end.

Indeed, by elementary analytic considerations, it is not difficult to see that the C-strata of codimension 1 which must be added to $\mathcal{F}_N$ are precisely the multi-sections $\Sigma(i)$ for $i = 1,\ldots,s$, which then appear as being horizontal for ν_N. It is then rather easy to see that each $\Sigma(i)$ is a non-ramified cover of a certain leaf algebraic leaf $\mathcal{F}(i) = \mathcal{F}_{N(i)}^{\alpha(i)}$ of Veech's foliation on $\mathcal{M}_{1,2}$, for a certain integer $N(i) \geq 0$ and a certain 2-uplet $\alpha(i) = (\alpha_1(i), -\alpha_1(i))$ with $\alpha_1(i) \in]0,1[$.

[22]See [19, §10.1] where this terminology is introduced.

Moreover, all these objects (namely $N(i)$, $\alpha(i)$ as well as the cover $\Sigma(i) \to \mathcal{F}(i)$) can be determined explicitly.

At the moment, we do not have as nice and precise descriptions of the P-strata of codimension 1 which must be added to $\mathcal{F}_N$ as the one we have for the C-strata. What seems likely is that these P-strata, which are (coverings of) moduli spaces of flat spheres with 4 cone points, are vertical with respect to v_N. More precisely, we believe that they are vertical fibers at some cusps of a certain extension of v_N over a partial completion of $Y_1(N)$ contained in $X_1(N)$.

Thanks to some classical works by Kodaira and Shioda [40, 67], it is known that $v_N : \mathscr{E}_1(N) \to Y_1(N)$ admits a compactification $\overline{v}_N : \overline{\mathscr{E}_1(N)} \to X_1(N)$ obtained by gluing some generalized elliptic curves as vertical fibers over the cusps of $Y_1(N)$. Note that such compactified modular surfaces (but for the level N congruence group $\Gamma(N)$) have been used by Livné in his thesis [41] (see also [12, §16]) to construct some non-arithmetic lattices in $\mathrm{PU}(1,2)$. This fact prompts us to believe that it might be possible to construct the metric completion of $\mathcal{F}_N$ from $\overline{\mathscr{E}_1(N)}$ by means of geometric operations similar to the ones used by Livné. This could provide a nice way to investigate further the topology and the complex analytic geometry of the $\mathbb{C}\mathbb{H}^2$-conifold $\overline{\mathcal{F}_N}$.

We hope to return on this in some future works.

4.3. **Veech's foliation for flat tori with two conical singularities.** We now specialize in the special case when $g = 1$ and $n = 2$.

In this case, the 2-uplet $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$ is necessarily such that

$$\alpha_1 = -\alpha_2 \in]0,1[. \tag{76}$$

Since Veech's foliation does depend only on $[\alpha_1 : \alpha_2]$ and in view of our hypothesis (76), one obtains that $\mathscr{F}^\alpha$ does not depend on α (Corollary 4.7).

4.3.0.4. In the case under study, it is relevant to consider the rescaled first integral

$$\Xi = \big(\alpha_1\big)^{-1}\xi^\alpha : \mathscr{T}or_{1,2} \longrightarrow \mathbb{R}^2$$

which is independent of α. Indeed, for $(\tau, z_2) \in \mathscr{T}or_{1,2}$, one has

$$\Xi(\tau, z_2) = \left(\frac{\Im\mathrm{m}(z_2)}{\Im\mathrm{m}(\tau)}, \frac{\Im\mathrm{m}(z_2)}{\Im\mathrm{m}(\tau)} \cdot \tau - z_2\right).$$

Moreover, it follows immediately from the third point of Proposition 4.5 that the image of Ξ is exactly $\mathbb{R}^2 \setminus \mathbb{Z}^2$. We denote by Π_1 the restriction to $\mathscr{T}or_{1,2}$ of the linear projection $\mathbb{H} \times \mathbb{C} \to \mathbb{H}$ onto the first factor.

Proposition 4.14. (1) *The following map is a real analytic diffeomorphism*

$$\Pi_1 \times \Xi : \mathscr{T}or_{1,2} \longrightarrow \mathbb{H} \times \big(\mathbb{R}^2 \setminus \mathbb{Z}^2\big). \tag{77}$$

(2) *The push-forward of Veech's foliation $\mathscr{F}^{\alpha}$ by this map is the horizontal foliation on the product $\mathbb{H} \times (\mathbb{R}^2 \setminus \mathbb{Z}^2)$.*

(3) *By restriction, the projection Π_1 induces a biholomorphism between any leaf of $\mathscr{F}^{\alpha}$ and Poincaré upper half-plane $\mathbb{H}$. In particular, the leaves of Veech's foliation on $\mathscr{T}or_{1,2}$ are topologically trivial.*

Since Veech's foliation does not depend on α, we will drop the subscript α in the notations of the rest of this section. We will also identify $\mathscr{T}or_{1,2}$ with $\mathbb{H} \times (\mathbb{R}^2 \setminus \mathbb{Z}^2)$. The corresponding action of $\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ is given by

$$\Big(M, \big(k, l\big)\Big) \circ \Big(\tau, \big(r_0, r_\infty\big)\Big) = \Big(M \cdot \tau, \big(r_0 + l, r_\infty - k\big) \cdot \rho(M)\Big)$$

for any $\big(M, (k, l)\big) \in \mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ and any $\big(\tau, (r_0, r_\infty)\big) \in \mathbb{H} \times (\mathbb{R}^2 \setminus \mathbb{Z}^2)$.[23]

The preceding proposition shows that, on the Torelli space, Veech's foliation is trivial from a topological point of view. It is really more interesting to look at $\mathcal{F}$, which is by definition the push-forward of $\mathscr{F}$ by the natural quotient map

$$\mu : \mathscr{T}or_{1,2} \longrightarrow \mathcal{M}_{1,2}. \tag{78}$$

For a 'rescaled lifted holonomy' $r = (r_0, r_\infty) \in \mathbb{R}^2 \setminus \mathbb{Z}^2$, one sets

$$\begin{aligned} [r] &= \big(\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2\big) \bullet r \subset \mathbb{R}^2\,, \\ \mathscr{F}_r &= \Xi^{-1}(r) = \Big\{\big(\tau, z_2\big) \in \mathscr{T}or_{1,2} \,\big|\, r_0\tau - z_2 = r_\infty\Big\} \subset \mathscr{T}or_{1,2} \\ \text{and} \quad \mathcal{F}_r &= \mu\big(\mathscr{F}_r\big) \subset \mathcal{M}_{1,2}\,. \end{aligned}$$

(Note that the correspondence with the notations of §4.2.4 are obtained via $r \leftrightarrow a$ with $a = \alpha_1 r$, *i.e.* $a_0 = \alpha_1 r_0$ and $a_\infty = \alpha_1 r_\infty$).

4.3.0.5. It turns out that when $g = 1$ and $n = 2$, one can give a complete and explicit description of all the leaves of Veech's foliation on $\mathcal{M}_{1,2}$ and in particular of the algebraic ones.

For $r \in \mathbb{R}^2$, one denotes by $\mathrm{Stab}(r)$ its stabilizer for the action $\bullet$:

$$\mathrm{Stab}(r) = \Big\{g \in \mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2 \,\Big|\, g \bullet r = r\Big\}$$

and one sets:

$$\delta(r) = \dim_{\mathbb{Q}} \big\langle r_0\,,\, r_\infty\,,\, 1\big\rangle \in \big\{1, 2, 3\big\}.$$

The following facts are easy to establish:

---

[23]We remind the reader that $\rho(M)$ is the matrix obtained from M by exchanging the antidiagonal coefficients, see (57).

- $\delta(r)$ does depend only on $[r]$, *i.e.* $\delta(r)=\delta(r')$ if $r'\in[r]$; in fact
$$\delta(r)=\dim_{\mathbb{Q}}\big\langle [r]\big\rangle;$$
- one has $\delta(r)=1$ if and only if $r\in\mathbb{Q}^2$;
- one has $\delta(r)=2$ if and only if there exists a triplet $(u_0,u_\infty,u_1)\in\mathbb{Z}^3$, unique up to multiplication by -1, such that

$$u_0r_0+u_\infty r_\infty=u_1 \qquad \text{and} \qquad \gcd(u_0,u_\infty,u_1)=1. \tag{79}$$

Proposition 4.15. *Let r be an element of $\mathbb{R}^2$.*

(1) *If $\delta(r)=3$ then* $\mathrm{Stab}(r)$ *is trivial.*
(2) *If $\delta(r)=2$ then* $\mathrm{Stab}(r)$ *is isomorphic to $\mathbb{Z}$.*
(3) *If $\delta(r)=1$ then $r\in\mathbb{Q}^2$ and* $\mathrm{Stab}(r)$ *is isomorphic to the congruence subgroup $\Gamma_1(N)$ where N is the smallest integer such that $Nr\in\mathbb{Z}^2$.*
(4) *For any positive integer N, the stabilizer of $(0,1/N)$ in $\mathrm{SL}_2(\mathbb{Z})\ltimes\mathbb{Z}^2$ is the image of the following injective morphism of groups*
$$\begin{aligned}\Gamma_1(N)&\longrightarrow \mathrm{SL}_2(\mathbb{Z})\ltimes\mathbb{Z}^2\\ \begin{bmatrix} a & b\\ c & d\end{bmatrix}&\longmapsto\left(\begin{bmatrix} a & b\\ c & d\end{bmatrix},\Big(\frac{d-1}{N},\frac{c}{N}\Big)\right).\end{aligned}$$

Proof. For $r=(r_0,r_\infty)\in\mathbb{R}^2$ and $g=\big(\big[\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\big],(k,l)\big)\in\mathrm{SL}_2(\mathbb{Z})\ltimes\mathbb{Z}^2$, one has

$$g\bullet r=r \quad\Longleftrightarrow\quad \begin{bmatrix} a-1 & -c\\ -b & d-1\end{bmatrix}\cdot\begin{bmatrix} r_0\\ r_\infty\end{bmatrix}=\begin{bmatrix} -l\\ k\end{bmatrix}. \tag{80}$$

We first consider the case when $r\notin\mathbb{Q}^2$. If $\delta(r)=3$ it is easy to deduce from the preceding equivalence that $\mathrm{Stab}(r)$ is trivial. Assume that $\delta(r)=2$ and let $u(r)=(u_0,u_\infty,u_1)\in\mathbb{Z}^3$ be such that (79) holds true. We denote by u the greatest common divisor of u_0 and u_∞: $u=\gcd(u_0,u_\infty)\in\mathbb{N}_{>0}$.

The equality $g\bullet r=r$ is equivalent to the fact that $(a-1,-c,-l)$ and $(-b,d-1,k)$ are integer multiples of $u(r)$. From this remark, one deduces easily that in this case, any $g\in\mathrm{Stab}(r)$ is written $g=\mathbf{1}+\frac{\lambda}{u}X(u)$ for some $\lambda\in\mathbb{Z}$, where $X(u)$ stands for the following element of $\mathrm{SL}_2(\mathbb{Z})\ltimes\mathbb{Z}^2$:

$$X(u)=\left(\begin{bmatrix} u_0u_\infty & u_0^2\\ -u_\infty^2 & -u_0u_\infty\end{bmatrix},\big(u_0u_1,u_\infty u_1\big)\right). \tag{81}$$

Then a short (but a bit laborious hence left to the reader) computation shows that, if $\mathbf{1}$ stands for the unity $(\mathrm{Id},(0,0))$ of $\mathrm{SL}_2(\mathbb{Z})\ltimes\mathbb{Z}^2$, then the map

$$\begin{aligned}\mathbb{Z}&\longrightarrow \mathrm{Stab}(r)\\ \lambda&\longmapsto \mathbf{1}+\frac{\lambda}{u}X(u).\end{aligned}$$

is an isomorphism of group.

Finally, we consider the case when $r \in \mathbb{Q}^2$. Let N be the integer as in the statement of the proposition. Then $[r] = [0, 1/N]$ according to Proposition 4.9, hence (3) follows from (4). For $r = (0, 1/N)$, (80) is equivalent to the fact that the integers c, d, k and l verify $c = lN$ and $d - 1 = kN$. The fourth point of the proposition follows then easily. □

4.3.0.6. With the preceding proposition at hand, it is then not difficult to determine the conformal types of the leaves of Veech's foliation $\mathcal{F}$ on $\mathcal{M}_{1,2}$ (as abstract complex orbifolds of dimension 1, not as embedded subsets of $\mathcal{M}_{1,2}$).

Corollary 4.16. *Let r be an element of* $\mathrm{Im}(\Xi) = \mathbb{R}^2 \setminus \mathbb{Z}^2$.

1. *If $\delta(r) = 3$ then μ induces an isomorphism $\mathscr{F}_r = \mathbb{H} \simeq \mathcal{F}_r$.*
2. *If $\delta(r) = 2$ then the leaf $\mathcal{F}_r$ is isomorphic to the infinite cylinder $\mathbb{C}/\mathbb{Z}$.*
3. *If $\delta(r) = 1$ then the leaf $\mathcal{F}_r$ is isomorphic (as a complex orbicurve) to the modular curve $Y_1(N) = \mathbb{H}/\Gamma_1(N)$ where N is the smallest positive integer such that $Nr \in \mathbb{Z}^2$.*

Proof. Since any leaf $\mathscr{F}_r$ is simply connected, one has $\mathcal{F}_r = \mu(\mathscr{F}_r) \simeq \mathscr{F}_r/_{\mathrm{Stab}(r)}$. (the latter being a isomorphism of orbifolds if $\mathrm{Stab}(r)$ has fixed points on $\mathscr{F}_r$).

Let $S(r)$ be the image of $\mathrm{Stab}(r)$ by the epimorphism $\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2 \to \mathrm{SL}_2(\mathbb{Z})$. The restriction of (59) to $\mathscr{F}_r$ gives an isomorphism of G-analytic spaces

$$
\begin{array}{ccc}
\mathscr{F}_r & \longrightarrow & \mathbb{H} \\
\circlearrowleft & & \circlearrowleft \\
\mathrm{Stab}(r) & \longrightarrow & S(r)\,.
\end{array}
\tag{82}
$$

This implies that (as complex orbifolds if $S(r)$ has fixed points on $\mathbb{H}$) one has

$$\mathcal{F}_r \simeq \mathbb{H}/S(r).$$

If $\delta(r) = 3$ then $S(r)$ is trivial by Proposition 4.15, hence (1) follows.

If $\delta(r) = 2$, it comes from the second point of Proposition 4.15 that $S(r)$ coincides with the infinite cyclic group spanned by $\mathrm{Id} + M(u) \in \mathrm{SL}_2(\mathbb{Z})$ where $M(u)$ stands for the matrix component of the element $X(u)$ defined in (81). Since $\mathrm{Tr}(\mathrm{Id} + M(u)) = 2 + \mathrm{Tr}(M(u)) = 2$, this generator is parabolic, hence the action of $S(r)$ on the upper half-plane is conjugated with the one spanned by the translation $\tau \mapsto \tau + 1$. It follows that $\mathcal{F}_r$ is isomorphic to the infinite cylinder.

The fact that $\delta(r) = 1$ means that $r \in \mathbb{Q}^2 \setminus \mathbb{Z}^2$. In this case, let N be the integer defined in the third point of the proposition. Then $(0, 1/N) \in [r]$ according to Proposition 4.9, hence $\mathcal{F}_r = \mathcal{F}_{(0,1/N)}$. From the fourth point of Proposition 4.15, it follows that $S(0, 1/N) = \Gamma_1(N)$. Consequently, one has $\mathcal{F}_r \simeq \mathbb{H}/\Gamma_1(N) = Y_1(N)$. □

For any integer N greater than or equal to 2, one sets

$$\mathcal{F}_N = \mathcal{F}_{(0,1/N)} \subset \mathcal{M}_{1,2}\,. \tag{83}$$

From the preceding results, we deduce the following very precise description of the algebraic leaves of Veech's foliation on $\mathcal{M}_{1,2}$:

Corollary 4.17.

(1) *The leaves of $\mathcal{F}$ which are closed analytic sub-orbifolds of $\mathcal{M}_{1,2}$ are exactly the $\mathcal{F}_N$'s for $N \in \mathbb{N}_{\geq 2}$;*

(2) *For any integer $N \geq 2$, the leaf $\mathcal{F}_N$ is algebraic, isomorphic to the modular curve $Y_1(N)$ and is the image of the following embedding:*

$$\begin{aligned} Y_1(N) &\longhookrightarrow \mathcal{M}_{1,2} \\ \big[\big(E_\tau,[1/N]\big)\big] &\longmapsto \big[\big(E_\tau,[0],[1/N]\big)\big]\,. \end{aligned}$$

Note that the modular curve $Y_1(N)$ (hence the leaf $\mathcal{F}_N$) has orbifold points only for $N = 2,3$ (*cf.* [13, Figure 3.3]).

The (orbi-)leaves $\mathcal{F}_N$ of Veech's foliation on $\mathcal{M}_{1,2}$ are clearly the most interesting ones. Thus is true at the topological level already. In the next sections we will go further by taking into account the parameter α. Our goal will be to study Veech's hyperbolic structures of the algebraic leaf $\mathcal{F}_N$ for any $N \geq 2$ as far as possible.

4.3.1. **A remark about the 'leaves' $\mathcal{F}_\theta(M)$ considered in** [19]**.** From the description of the leaves of Veech's foliation given above, it is possible to give an explicit example of the non-connectedness phenomenon mentioned in [19].

4.3.1.1. We first recall some notations of [19] for ($g = 1$ and) $n \geq 1$ arbitrary. We consider a fixed $\alpha = (\alpha_1, \ldots, \alpha_n)$ satisfying (63) and we denote by $\theta = (\theta_1, \ldots, \theta_n)$ the associated angles datum. For $\rho \in \mathrm{Hom}(\pi_1(1,n), \mathbb{U})$, let $\mathscr{F}_\rho \subset \mathscr{E}^\alpha_{1,n} \simeq \mathscr{T}eich_{1,n}$ be the preimage of ρ by Veech's linear holonomy map $\chi^\alpha_{1,n}$ (see §4.1.3 above).

If $[\rho]$ stands for the orbit of ρ under the action of $\mathrm{PMCG}_{1,n}$, then $\mathcal{F}_{[\rho]}$ is the notation used in [19] for the image of $\mathscr{F}_\rho$ into $\mathcal{M}_{1,n}$.

We now assume that the image $\mathrm{Im}(\rho)$ of ρ in $\mathbb{U}$ is finite. From our results above, it comes that $\mathcal{F}_{[\rho]}$ is an algebraic subvariety of $\mathcal{M}_{1,n}$. Moreover, α is necessarily rational, hence $G_\theta = \langle e^{i\theta_1}, \ldots, e^{i\theta_n}\rangle$ is a finite subgroup of the group $\mathbb{U}_\infty$ of roots of unity. If ω_ρ stands for a generator of $\mathrm{Im}(\rho) \subset \mathbb{U}_\infty$, one denotes by $M = M_\theta(\rho)$ the smallest positive integer such that $G_\theta = \langle \omega_\rho^M \rangle$. Now in [19, §10], it is proved that the leaf $\mathcal{F}_{[\rho]}$ is uniquely determined by this integer M (remember that θ is fixed) and the corresponding notation for it is $\mathcal{F}_\theta(M)$.

In [19], using certain surgery for flat surfaces, one proves several results about the geometric structure of an arbitrary algebraic leaf $\mathcal{F}_\theta(M)$. However, the geometrical methods used to establish these results, if quite relevant to answer some questions, do not allow to answer some basic other questions such as the connectedness of a given leaf $\mathcal{F}_\theta(M)$. Using our results presented in the preceding subsections, it is easy to give explicit examples showing that a 'leaf' $\mathcal{F}_\theta(M)$ can have several distinct connected components.

Sticking to the case when $n=2$ we are now going to consider two examples.

4.3.1.2. We first deal with the case when $\alpha_1 = 4/5$. In this case, one has $\theta = 2\pi(9/5, 1/5)$, hence $G_\theta = \langle e^{2i\pi/5}\rangle$. We consider the leaves $\mathcal{F}_2$ and $\mathcal{F}_4$ (*cf.* (83)). These are the images in $\mathcal{M}_{1,2}$ of the two subsets of the Torelli space $\mathcal{T}or_{1,2}$ cut out by the equations $z_2 = 1/2$ and $z_2 = 1/4$ respectively. Then, the two associated linear holonomies ρ_2 and ρ_4 are characterized by the following relations:

$$\rho_2(a_1) = \rho_4(a_1) = 1\,, \quad \rho_2(b_1) = e^{-\frac{2i\pi\alpha_1}{2}} \qquad \text{and} \qquad \rho_4(b_1) = e^{-\frac{2i\pi\alpha_1}{4}}$$

(we use here the notations of §4.1). It comes that

$$\mathrm{Im}\big(\rho_2\big) = \mathrm{Im}\big(\rho_4\big) = \big\langle e^{\frac{2i\pi}{5}}\big\rangle = G_\theta\,,$$

which shows that $\mathcal{F}_2$ and $\mathcal{F}_4$ are two distinct connected components of $\mathcal{F}_\theta(1)$.

4.3.1.3. We now consider the case when $\alpha_1 = 1/2$. In this case, one has $\theta = (3\pi, \pi)$, hence $G_\theta = \langle e^{i\pi}\rangle = \{\pm 1\}$. For any $M \geq 2$, the leaf $\mathcal{F}_M$ is the image in $\mathcal{M}_{1,2}$ of the subset cut out by $z_2 = 1/M$ in the Torelli space. The associated linear holonomy $\rho_M \in \mathrm{Hom}^\alpha(\pi_1(1,2), \mathbb{U})$ is characterized by $\rho_M(a_1) = 1$ and $\rho_M(b_1) = e^{-i\pi/M}$. It is then easy to verify that $\mathrm{Im}(\rho) = \langle e^{i\pi/M}\rangle$. It follows that $\mathcal{F}_\theta(M) = \mathcal{F}_M \simeq Y_1(M)$ for any $M \geq 2$, in sharp contrast with the preceding case.

We find the question of how to determine the $\mathcal{F}_N$'s which are the connected components of a given 'leaf' $\mathcal{F}_\theta(M)$ quite interesting. Note that the two cases considered above show that the answer depends on θ.

4.3.2. **An aside: a connection with Painlevé theory.** The leaves of Veech's foliation on the Torelli space are cut out by the affine equations (64). The fact that the latter have real coefficients reflects the fact that Veech's foliation is transversely real analytic (but not holomorphic) on $\mathcal{T}or_{1,n}$. A natural way to get a holomorphic object is by allowing the coefficients a_0 and a_∞ to take any complex value (the α_i's staying fixed). Performing this complexification, one obtains a 2-dimensional complex family of hypersurfaces in $\mathcal{T}or_{1,n}$ that are nothing else but the solutions of the second-order linear differential system

$$\text{(84)} \qquad \frac{\partial^2 \tau}{\partial z_i^2} = 0 \quad \text{and} \quad \alpha_i \frac{\partial \tau}{\partial z_j} - \alpha_j \frac{\partial \tau}{\partial z_i} = 0 \qquad \text{for } i, j = 2, \ldots, n,\ i \neq j.$$

Using the explicit formula given above, it is not difficult to verify that (84) is invariant by the action of the mapping class group. This implies that it can be pushed-down onto $\mathcal{M}_{1,n}$ and give rise to a (no more linear) second-order holomorphic differential system on this moduli space, denoted by $\mathcal{D}^\alpha$. The integral varieties of $\mathcal{D}^\alpha$ form a complex 2-dimensional family of 1-codimensional locally analytic subsets of $\mathcal{M}_{1,n}$ which can be seen as a kind of complexification of Veech's foliation.

If the preceding construction seems natural, one could have some doubt concerning its interest. What shows that it is actually relevant is the consideration of the simplest case when $n=2$. In this situation, (84) reduces to the second order differential equation $d^2\tau/dz_2^2=0$. In order to avoid considering orbifold points, it is more convenient to look at the push-forward modulo the action of $\Gamma(2)\ltimes\mathbb{Z}^2<\mathrm{Sp}_{1,n}(\mathbb{Z})$. It is known that

$$\mathscr{T}or_{1,2}/_{\Gamma(2)\ltimes\mathbb{Z}^2}\simeq\left(\mathbb{P}^1\setminus\{0,1,\infty\}\right)\times\mathbb{C}$$

and that the quotient map is given by

$$\begin{aligned}\nu:\mathscr{T}or_{1,2}&\longrightarrow\left(\mathbb{P}^1\setminus\{0,1,\infty\}\right)\times\mathbb{C}\\(\tau,z_2)&\longmapsto\left(\lambda(\tau),\frac{\wp(z_2)-e_1}{e_2-e_1}\right).\end{aligned}$$

Here $\wp:E_\tau\to\mathbb{P}^1$ is the Weirerstrass $\wp$-function associated to the lattice $\mathbb{Z}_\tau$, one has $e_i=\wp(\omega_i)$ for $i=1,2,3$ where $\omega_1=1/2$, $\omega_2=\tau/2$ and $\omega_3=\omega_1+\omega_2=(1+\tau)/2$ and where $\lambda:\tau\mapsto(e_3-e_1)/(e_2-e_1)$ stands for the classical elliptic modular lambda function (the usual Hauptmodul[24] for $\Gamma(2)$).

As already known by Picard [59, Chap. V, §17] (see [44] for a recent proof), the push-forward of $d^2\tau/dz_2^2=0$ by ν is the following non-linear second order differential equation

$$\text{(PPVI)}\qquad\begin{aligned}\frac{d^2X}{d\lambda^2}=&\frac{1}{2}\left(\frac{1}{X}+\frac{1}{X-1}+\frac{1}{X-\lambda}\right)\left(\frac{dX}{d\lambda}\right)^2\\&-\left(\frac{1}{\lambda}+\frac{1}{\lambda-1}+\frac{1}{X-\lambda}\right)\cdot\frac{dX}{d\lambda}+\frac{1}{2}\frac{X(X-1)}{\lambda(\lambda-1)(X-\lambda)}.\end{aligned}$$

This equation, now known as **Painlevé-Picard equation**, was first considered by Picard in [59]. The name of Painlevé is associated to it since it is a particular case (actually the simplest case) of the sixth-Painlevé equation.[25]

For $r=(r_0,r_\infty)\in\mathbb{R}^2\setminus\mathbb{Z}^2$, the image of the leaf $\mathscr{F}_r$ in $(\mathbb{P}^1\setminus\{0,1,\infty\})\times\mathbb{C}$ is parametrized by $\tau\mapsto(\lambda(\tau),(\wp(r_0\tau-r_\infty)-e_1)/(e_2-e_1))$, hence the leaves of

[24]We recall here the definition of what is a **Hauptmodul** for a genus 0 congruence group $\Gamma\subset\mathrm{SL}_2(\mathbb{Z})$: it is a Γ-modular function on $\mathbb{H}$ which induces a generator of the field of rational functions on $X_\Gamma=\overline{\mathbb{H}/\Gamma}\simeq\mathbb{P}^1$ with a pole of order 1 with residue 1 at the cusp $[i\infty]\in X_\Gamma$.

[25]Actually, the sixth Painlevé equation has not been discovered by Painlevé himself (due to some mistakes in some of its computations) but by his student R. Fuchs in 1905.

Veech's foliation can be considered as particular solutions of (PPVI). The leaves $\mathcal{F}_r$ with $r \in \mathbb{Q}^2$ are precisely the algebraic solutions of (PPVI), a fact already known to Picard (see [59, p. 300]).

The existence of this link between the theory of Veech's foliations and the theory of Painlevé equations is not so surprising. Indeed, both domains are related to the notion of isomonodromic (or isoholonomic) deformation. But this leads to interesting questions such as the following ones:

(1) is there a geometric characterization of the leaves $\mathcal{F}_N$, $N \in \mathbb{N}_{\geq 2}$ of Veech's foliation among the solutions of (PPVI)?

(2) given $\alpha = (\alpha_1, \alpha_2)$ as in (76), is it possible to obtain the hyperbolic structures constructed by Veech on the leaves of $\mathcal{F}^\alpha$ within the framework of (PPVI)? Moreover, does the general solution of (PPVI) carry a hyperbolic structure which specializes to Veech' one on a leaf $\mathcal{F}_N^\alpha$?

(3) for $n \geq 2$ arbitrary, is there a nice formula for the push-forward $\mathcal{D}^\alpha$ of the differential system (84) onto a suitable quotient of $\mathscr{T}or_{1,n}$? If yes, does such a push-forward enjoy a generalization for differential systems in several variables of the Painlevé property?

4.4. **An analytic expression for the Veech map when $g = 1$.** In this section, we begin with dealing with the general case when ($g = 1$ and) $n \geq 2$. Our goal here is to get an explicit local analytic expression for the Veech map.

After having recalled the definition of this map, we define another map by adapting/generalizing to our context the approach developed in the genus 0 case by Deligne and Mostow. We show that, after some identifications, these two maps are identical. Although all this is not really necessary to our purpose (which is to study Veech's hyperbolic structure on a leaf $\mathscr{F}_a^\alpha \subset \mathcal{M}_{1,n}$), we believe that it is worth considering, since it shows how the constructions of the famous papers [76] and [11] are related in the genus 1 case.

Our aim here is to study Veech's hyperbolic structure on a leaf $\mathscr{F}_a^\alpha$ of Veech's foliation in the Torelli space $\mathscr{T}or_{1,n}$ as extensively as possible. We denote by $\widetilde{\mathscr{F}_a^\alpha}$ its preimage in the Teichmüller space $\mathscr{T}eich_{1,n}$. Then Veech constructs a holomorphic map

$$\widetilde{V}_r^\alpha : \widetilde{\mathscr{F}_a^\alpha} \longrightarrow \mathbb{C}\mathbb{H}^{n-1} \tag{85}$$

and the hyperbolic structure he considers on $\widetilde{\mathscr{F}_a^\alpha}$ is just the one obtained by pull-back by this map (which is étale according to [76, Section 10]). To study this hyperbolic structure, we are going to give an explicit analytic expression for $\widetilde{V}_a^\alpha$, or more precisely, for its push-forward by the (restriction to $\widetilde{\mathscr{F}_a^\alpha}$ of the) projection $\mathscr{T}eich_{1,n} \rightarrow \mathscr{T}or_{1,n}$ which is a multivalued holomorphic function on $\mathscr{F}_a^\alpha$ that will be denoted by V_a^α.

We recall and fix some notations that will be used in the sequel.

In what follows, $a = (a_0, a_\infty)$ stands for a fixed element of $\mathbb{R}^2 \setminus \alpha_1\mathbb{Z}^2$ if $n = 2$ or of $\mathbb{R}^2$ if $n \geq 3$. We denote by ρ_a the linear holonomy map shared by the elements of $\mathscr{F}_a^\alpha \subset \mathscr{T}or_{1,n}$: we recall that it is the element of $H^1(E_{\tau,z}, \mathbb{U})$ defined for any n-punctured torus $E_{\tau,z}$ by the following relations:

$$\rho_{a,0} = e^{2i\pi a_0}, \quad \rho_{a,k} = e^{2i\pi \alpha_k} \text{ for } k = 1, \ldots, n \quad \text{and} \quad \rho_{a,\infty} = e^{2i\pi a_\infty} \tag{86}$$

(see §3.2.2 where all the corresponding notations are fixed).

Note that ρ_a can also be seen as an element of $\mathrm{Hom}^\alpha(\pi_1(E_{\tau,z}), \mathbb{U})$. Considering the latter as a $\mathbb{C}$-valued morphism, one gets a 1-dimensional $\pi_1(E_{\tau,z})$-module that will be denoted by $\mathbb{C}_a^\alpha$. Finally, for any $(\tau, z) \in \mathscr{F}_a^\alpha$, the function $T_{\tau,z}^\alpha(\cdot) = T^\alpha(\cdot, \tau, z)$ defined in (28) is a multivalued function on $E_{\tau,z}$ whose monodromy is multiplicative and given by (86). One denotes by $L_{\tau,z}^\alpha$ the local system on $E_{\tau,z}$ defined as the kernel of the connexion (30) on $\mathscr{O}_{E_{\tau,z}}$. Note that the representation of $\pi_1(E_{\tau,z})$ associated to $L_{\tau,z}^\alpha$ is precisely $\mathbb{C}_a^\alpha$.

4.4.1. **The original Veech's map.** We first recall Veech's abstract definition of (85). We refer to the ninth and tenth sections of [76] for proofs and details.

4.4.1.1. Let $(E_{\tau,z}, m_{\tau,z}^\alpha, \psi)$ be a point of $\mathscr{E}_{1,n}^\alpha \simeq \mathscr{T}eich_{1,n}$ (see §4.1). We fix $x \in E_{\tau,z}$ as well as a determination D at x of the developing map on $E_{\tau,z}$ associated with the flat structure induced by $m_{\tau,z}^\alpha$. For any loop c centered at x in $E_{\tau,z}$, one denotes by $M^c(D)$ the germ at x obtained after the analytic continuation of D along c. Then one has $M^c(D) = \rho(c)\, D + \mu_D(c)$ with $\rho(c) \in \mathbb{U}$ and $\mu_D(c) \in \mathbb{C}$.

One verifies that the affine map $m^c : z \mapsto \rho(c)z + \mu_D(c)$ only depends on the pointed homotopy class of c and one gets this way the **(complete) holonomy representation** of the flat surface $(E_{\tau,z}, m_{\tau,z}^\alpha)$:

$$\begin{aligned} \pi_1\big(E_{\tau,z}, x\big) &\longrightarrow \mathrm{Isom}^+(\mathbb{C}) \simeq \mathbb{U} \ltimes \mathbb{C} \\ [c] &\longmapsto m^c : z \mapsto \rho(c)z + \mu_D(c)\,. \end{aligned}$$

Now assume that $(E_{\tau,z}, m_{\tau,z}^\alpha, \psi)$ belongs to $\widetilde{\mathscr{F}}_a^\alpha$. Then the map $[c] \mapsto \rho(c)$ is nothing else than the linear holonomy map $\chi_{1,n}^\alpha(E_{\tau,z}, m_{\tau,z}^\alpha, \psi)$ (*cf.* (50)) and will be denoted by ρ_a in what follows. Let μ_a be the complex-valued map on $\pi_1(E_{\tau,z}, x)$ which associates $1 - \rho_a(c)$ to any homology class $[c]$. The translation part of the holonomy $[c] \mapsto \mu_D(c)$ can be seen as a complex map on the fundamental group of the punctured torus $E_{\tau,z}$ at x which satisfies the following properties:

(1) for any two homotopy classes $[c_1], [c_2] \in \pi_1(E_{\tau,z}, x)$, one has:

$$\begin{aligned} \mu_D(c_1 c_2) &= \rho(c_1)\mu_D(c_2) + \mu_D(c_1) \\ \text{and} \quad \mu_D(c_1 c_2 c_1^{-1}) &= \rho_a(c_1)\mu_D(c_2) + \mu_D(c_1)\mu_a(c_2)\,; \end{aligned}$$

(2) if $\widetilde{D} = \kappa D + \ell$ is another germ of the developing map at x, with $u \in \mathbb{C}^*$ and $v \in \mathbb{C}$, then one has the following relation between μ_D and $\mu_{\widetilde{D}}$:

$$\mu_{\widetilde{D}} = \kappa \mu_D + \ell \mu_a \, .$$

Now remember that considering $(E_{\tau,z}, m^{\alpha}_{\tau,z}, \psi)$ as a point of $\mathscr{T}eich_{1,n}$ means that the third component ψ stands for an isomorphism $\pi_1(1,n) \simeq \pi_1(E_{\tau,z}, x)$ well defined up to inner automorphisms. Consider then the composition $\mu_{\tau,z} = \mu_D \circ \psi : \pi_1(1,n) \to \mathbb{C}$. It is an element of the following space of 1-cocycles for the $\pi_1(1,n)$-module, denoted by $\mathbb{C}^{\alpha}_a$, associated to the unitary character ρ_a:

$$Z^1\big(\pi_1(1,n), \mathbb{C}_{\rho_a}\big) = \Big\{ \mu : \pi_1(1,n) \to \mathbb{C} \,\big|\, \mu(\gamma\gamma') = \rho_a(\gamma)\mu(\gamma') + \mu(\gamma) \ \forall \gamma, \gamma' \Big\} \, .$$

One denotes again by μ_a the map $\gamma \mapsto 1 - \rho_a(\gamma)$ on $\pi_1(1,n)$. From the properties (1) and (2) satisfied by μ_D and from the fact that ψ is canonically defined up to inner isomorphisms, it follows that the class of $\mu_{\tau,z}$ in the projectivization of the first group of group cohomology

$$H^1\big(\pi_1(1,n), \mathbb{C}^{\alpha}_a\big) = Z^1\big(\pi_1(1,n), \mathbb{C}^{\alpha}_a\big) / \mathbb{C}\mu_a \, ,$$

is well defined. Then one defines the **Veech map $\widetilde{V}^{\alpha}_a$** as the map

$$\begin{aligned} \widetilde{\mathscr{F}}^{\alpha}_a &\longrightarrow \mathbf{P}H^1\big(\pi_1(1,n), \mathbb{C}^{\alpha}_a\big) \\ \big(E_{\tau,z}, m^{\alpha}_{\tau,z}, \psi\big) &\longmapsto \big[\mu_{\tau,z}\big] \, . \end{aligned} \tag{87}$$

In [76, §10], Veech proves the following result:

Theorem 4.18. *The map $\widetilde{V}^{\alpha}_a$ is a local biholomorphism.*

Actually, Veech proves more. Under the assumption that at least one of the α_i's is not an integer, there is a projective bundle $\mathbf{P}\mathcal{H}^1$ over $\mathrm{Hom}^{\alpha}(\pi_1(1,n), \mathbb{U})$, the fiber of which at ρ is $\mathbf{P}H^1(\pi_1(1,n), \mathbb{C}_\rho)$. Then Veech proves that the $\widetilde{V}^{\alpha}_a$'s considered above are just the restrictions of a global real-analytic immersion

$$\widetilde{V}^{\alpha} \, : \, \mathscr{T}eich_{1,n} \longrightarrow \mathbf{P}\mathcal{H}^1 \, .$$

An algebraic-geometry inclined reader may see the preceding result as a kind of local Torelli theorem for flat surfaces: *once the conical angles have been fixed, a flat surfaces is locally determined by its complete holonomy representation.*

An differential geometer may rather this preceding result as a particular occurrence of the Ehresmann-Thurston's theorem which asserts essentially the same thing but in the more general context of geometric structures on manifolds (see [23] for a nice general account of this point of view).

4.4.2. **Deligne-Mostow's version.** We now adapt and apply the constructions and results of the third section of [11] to the genus 1 case we are considering here.

Let

$$\pi : \mathscr{E}_{1,n} \longrightarrow \mathscr{T}or_{1,n}$$

be the universal elliptic curve: for $(\tau, z) = (\tau, z_2, \ldots, z_n) \in \mathscr{T}or_{1,n}$, the fiber of π over (τ, z) is nothing else but the n-punctured elliptic curve $E_{\tau,z}$.

For any $(\tau, z) \in \mathscr{F}_a^\alpha$, recall the local system $L_{\tau,z}^\alpha$ on $E_{\tau,z}$ which admits the multivalued holomorphic function $T_{\tau,z}^\alpha(u) = T^\alpha(u, \tau, z)$ defined in (28) as a section. All these local systems can be glued together over the leaf $\mathscr{F}_a^\alpha$: there exists a local system L_a^α over $\mathscr{E}_a^\alpha = \pi^{-1}(\mathscr{F}_a^\alpha) \subset \mathscr{E}_{1,n}$ whose restriction along $E_{\tau,z}$ is $L_{\tau,z}^\alpha$ for any $(\tau, z) \in \mathscr{F}_a^\alpha$ (see §B.2 in Appendix B for a detailed proof).

Since the restriction of π to $\mathscr{E}_a^\alpha$ is a topologically locally trivial fibration, the spaces of twisted cohomology $H^1(E_{\tau,z}, L_{\tau,z}^\alpha)$'s organize themselves into a local system $R^1\pi_*(L_a^\alpha)$ on $\mathscr{F}_a^\alpha$. We will be interested in its projectivization:

$$B_a^\alpha = \mathbf{P}R^1\pi_*\big(L_a^\alpha\big).$$

It is a flat projective bundle whose fiber $B_{\tau,z}^\alpha$ at any point (τ, z) of $\mathscr{F}_a^\alpha$ is just $\mathbf{P}H^1(E_{\tau,z}, L_{\tau,z}^\alpha)$. Its flat structure is of course the one induced by the local system $R^1\pi_*(L_a^\alpha)$. For $(\tau, z) \in \mathscr{F}_a^\alpha$, let $\omega_{\tau,z}^\alpha$ be the (projectivization of the) twisted cohomology class defined by $T_{\tau,z}^\alpha(u)du$ in cohomology:

$$\omega_{\tau,z}^\alpha = \big[T_{\tau,z}^\alpha(u)du\big] \in B_{\tau,z}^\alpha.$$

As in the genus 0 case (see [11, Lemma **(3.5)**], it can be proved that the class $\omega_{\tau,z}^\alpha$ is never trivial hence induces a global holomorphic section ω_a^α of B_a^α over the leaf $\mathscr{F}_a^\alpha$. Since the inclusion of the latter into $\mathscr{T}or_{1,n}$ induces an injection of the corresponding fundamental groups (see Proposition 4.8), any connected component of $\widetilde{\mathscr{F}_a^\alpha}$ (quite abusively denoted by the same notation in what follows) is simply connected hence can be seen as the universal cover of the leaf $\mathscr{F}_a^\alpha$. It follows that the pull-back $\widetilde{B}_a^\alpha$ of B_a^α by $\widetilde{\mathscr{F}_a^\alpha} \to \mathscr{F}_a^\alpha$ can be trivialized: the choice of any base-point in $\widetilde{\mathscr{F}_a^\alpha}$ over $(\tau_0, z_0) \in \mathscr{F}_a^\alpha$ gives rise to an isomorphism

$$\widetilde{B}_a^\alpha \simeq \widetilde{\mathscr{F}_a^\alpha} \times B_{\tau_0,z_0}^\alpha. \tag{88}$$

It follows that the section ω_a^α of B_a^α on $\mathscr{F}_a^\alpha$ gives rise to a holomorphic map

$$\widetilde{V}_a^{\alpha,DM} : \widetilde{\mathscr{F}_a^\alpha} \longrightarrow B_{\tau_0,z_0}^\alpha = \mathbf{P}H^1\big(E_{\tau_0,z_0}, L_{\tau_0,z_0}^\alpha\big). \tag{89}$$

We remark now that the results of [11, p. 23] generalize verbatim to the genus 1 case which we are considering here. In particular, for any (local) horizontal basis $(\boldsymbol{C}_i)_{i=1}^n$ of the twisted homology with coefficients in L_a^α on $\mathscr{F}_a^\alpha$, $(\int_{\boldsymbol{C}_i} \cdot)_{i=1}^n$

forms a local horizontal system of projective coordinates on B_a^α. Generalizing the first paragraph of the proof of [11, Lemma (3.5)] to our case, it comes that

$$\big(\boldsymbol{\gamma}_0, \boldsymbol{\gamma}_3, \ldots, \boldsymbol{\gamma}_n, \boldsymbol{\gamma}_\infty\big)$$

is such a basis, where the $\boldsymbol{\gamma}_\bullet$'s are the twisted cycles defined in (33).

As a direct consequence, it follows that the push-forward $V_a^{\alpha,DM}$ of $\widetilde{V}_a^{\alpha,DM}$ onto the leaf $\mathscr{F}_a^\alpha$ in the Torelli space $\mathscr{T}or_{1,n}$ (which is a multivalued holomorphic function) admits the following local analytic expression whose components are expressed in terms of elliptic hypergeometric integrals:

$$\begin{aligned} V_r^{\alpha,DM} : \mathscr{F}_a^\alpha &\longrightarrow \mathbb{P}^{n-1} \\ (\tau, z) &\longmapsto \left[\int_{\boldsymbol{\gamma}_\bullet} T^\alpha(u,\tau,z)\,du\right]_{\bullet=0,3,\ldots,n,\infty}. \end{aligned} \tag{90}$$

4.4.3. **Comparison of $\widetilde{V}_a^\alpha$ and $\widetilde{V}_a^{\alpha,DM}$.** We intend here to prove that Veech's and Deligne-Mostow's maps coincide, up to some natural identifications.

4.4.3.1. At first sight, the two abstractly defined maps (87) and (89) do not seem to have the same target space. It turns out that they actually do, but up to some natural isomorphisms.

Indeed, for any $(\tau, z) \in \mathscr{F}_a^\alpha$, since $L_{\tau,z}^\alpha$ is 'the' local system on $E_{\tau,z}$ associated to the $\pi_1(E_{\tau,z})$-module $\mathbb{C}_a^\alpha$, there is a natural morphism

$$H^1\big(\pi_1(E_{\tau,z}), \mathbb{C}_a^\alpha\big) \longrightarrow H^1\big(E_{\tau,z}, L_{\tau,z}^\alpha\big). \tag{91}$$

Since $E_{\tau,z}$ is uniformized by the unit disk $\mathbb{D}$ which is contractible, it follows that the preceding map is an isomorphism (see [22, §2.1] for instance).

On the other hand, for any $(E_{\tau,z}, m_{\tau,z}^\alpha, \psi) \in \widetilde{\mathscr{F}_a^\alpha}$, the (class of) map(s) $\psi : \pi_1(1,n) \simeq \pi_1(E_{\tau,z})$ induces a well defined isomorphism $[\psi^*] : H^1(\pi_1(E_{\tau,z}), \mathbb{C}_a^\alpha) \simeq H^1(\pi_1(1,n), \mathbb{C}_a^\alpha)$. Then, up to the isomorphism (91), one can see the lift of $(\tau, z) \mapsto [\mu_{\tau,z}] \circ [\psi^*]$ as a global section of $\widetilde{B}_a^\alpha$ over $\widetilde{\mathscr{F}_a^\alpha}$. Then using (88), one eventually obtains that Veech's map $\widetilde{V}_a^\alpha$ can be seen as a map with the same source and target spaces than Deligne-Mostow's map $\widetilde{V}_a^{\alpha,DM}$.

4.4.3.2. Comparing the two maps (87) and (89) is not difficult and relies on some arguments elaborated by Veech. In [76], to prove that (87) is indeed a holomorphic immersion, he explains how to get a local analytic expression for this map. It is then easy to relate this expression to (90) and eventually get the

Proposition 4.19. *The two maps $\widetilde{V}_a^\alpha$ and $\widetilde{V}_a^{\alpha,DM}$ coincide. In particular,* (90) *is also a local analytic expression for the push-forward V_a^α of Veech's map on $\mathscr{F}_a^\alpha$.*

Proof. We first review some material from the tenth and eleventh sections of [76] to which the reader may refer for some details and proofs.

Remember that for (τ,z) in $\mathscr{F}_a^\alpha$, one sees $E_{\tau,z}$ as a flat torus with n conical singularities, the flat structure being the one induced by the singular metric $m_a^\alpha(\tau,z)=|T^\alpha(u,\tau,z)du|^2$. Given such an element (τ',z'), there exists a geodesic polygonation $\mathscr{T}=\mathscr{T}_{\tau',z'}$ of $E_{\tau',z'}$ whose set of vertices is exactly the set of conical singularities of this flat surface (for instance, one can consider its Delaunay decomposition[26]). Moreover, the set of points $(\tau,z)\in\mathscr{F}_a^\alpha$ such that the associated flat surface admits a geodesic triangulation $\mathscr{T}_{\tau,z}$ combinatorially equivalent to $\mathscr{T}$ is open (according to [76, §5]), hence contains an open ball $U_{\mathscr{T}}\subset\mathscr{F}_a^\alpha$ to which the considered base-point (τ',z') belongs.

As explained in [76, §10], by removing the interior of some edges (the same edges for every point (τ,z) in $U_{\mathscr{T}}$), one obtains a piecewise geodesic graph $\Gamma_{\tau,z}\subset E_{\tau,z}$ formed by $n+1$ edges of $\mathscr{T}_{\tau,z}$ such that $Q_{\tau,z}=E_\tau\setminus\Gamma_{\tau,z}$ is homeomorphic to the open disk $\mathbb{D}\subset\mathbb{C}$. Then one considers the length metric on $Q_{\tau,z}$ associated to the restriction of the flat structure of $E_{\tau,z}$. The metric completion $\overline{Q}_{\tau,z}$ for this intrinsic metric is isomorphic to the closed disk $\overline{\mathbb{D}}$. Moreover, the latter carries a flat structure with (geodesic) boundary, whose singularities are $2n+2$ conical points $v_1,\ldots,v_{2n+2}$ located on the boundary circle $\partial\mathbb{D}$. One can and will assume that the v_i's are cyclically enumerated in the trigonometric order, v_1 being chosen arbitrarily. For $i=1,\ldots,2n+2$, let I_i be the circular arc on $\partial\mathbb{D}$ whose endpoints are v_i and v_{i+1} (with $v_{2n+3}=v_1$ by convention).

The developing map $D_{\tau,z}$ of the flat structure on $Q_{\tau,z}\simeq\mathbb{D}$ extends continuously to $\overline{Q}_{\tau,z}\simeq\overline{\mathbb{D}}$. For every i, this extension maps I_i onto the segment $[\zeta_i,\zeta_{i+1}]$ in the Euclidean plane $\mathbb{E}^2\simeq\mathbb{C}$, where we have set for $i=1,\ldots,2n+1$:

$$\zeta_i=\zeta_i(\tau,z)=D_{\tau,z}(v_i)\,.$$

There exists an involution θ without fixed point on the set $\{1,\ldots,2n+2\}$ such that the flat torus $E_{\tau,z}$ is obtained from the flat closed disk $\overline{\mathbb{D}}\simeq\overline{Q}_{\tau,z}$ by gluing isometrically the 'flat arcs' $I_i\simeq[\zeta_i,\zeta_{i+1}]$ and $I_{\theta i}\simeq[\zeta_{\theta i},\zeta_{\theta(i+1)}]$. Let J be a subset of $\{1,\ldots,2n+2\}$ such that $J\cap\theta J=\emptyset$. Then J has cardinality $n+1$ and if one sets

$$\xi_j=\xi_j(\tau,z)=\zeta_{j+1}-\zeta_j$$

for every $j\in J$, then these complex numbers satisfy a linear relation which depends only on $\mathscr{T},\theta$ and on the linear holonomy ρ_a (*cf.* [76, §11]).

Consequently the ξ_j's are the components of a map

$$U_{\mathscr{T}}\to\mathbb{P}^{n-1}:(\tau,z)\mapsto\big[\xi_j(\tau,z)\big]_{j\in J} \tag{92}$$

which it is nothing else but a local holomorphic expression of V_a^α on $U_{\mathscr{T}}$ (see [76, §10]).

[26]The *'Delaunay decomposition'* of a compact flat surface is a canonical polygonation of it by Euclidean polygons inscribed in circles (see [51, §4] or [5] for some details).

It it then easy to relate (92) to (90). Indeed $T^{\alpha}_{\tau,z}$ admits a global determination on the complement of $\Gamma_{\tau,z}$ since the latter is simply connected. The crucial but easy point is that the developing map $D_{\tau,z}$ considered above is a primitive of the global holomorphic 1-form $T^{\alpha}_{\tau,z}(u)du$ on $Q_{\tau,z}$. Once one is aware of this, it comes at once that for every $j \in J$, $\xi_j(\tau,z)$ can be written as $\int_{e_j} T^{\alpha}(u,\tau,z)du$ where e_j stands for the edge of $\mathscr{T}_{\tau,z}$ in E_τ which corresponds to the 'flat circular arc' I_j. In other terms: $\xi_j(\tau,z)$ is equal to the integral along e_j of a determination of the multivalued 1-form $T^{\alpha}(u,\tau,z)du$.

Then for every $j \in J$, setting $\boldsymbol{e}_j = \mathrm{reg}(e_j) \in H_1(E_{\tau,z}, L^{\alpha}_{\tau,z})$ where reg is the regularization map considered in §3.1.7, one obtains:

$$\xi_j(\tau,z) = \int_{\boldsymbol{e}_j} T^{\alpha}(u,\tau,z)\,du.$$

It is not difficult to see that $(\boldsymbol{e}_j)_{j\in J}$ is a basis of $H_1(E_{\tau,z}, L^{\alpha}_{\tau,z})$ for every $(\tau,z) \in U_{\mathscr{T}}$. Even better, it follows from [11, *Remark* **(3.6)**] that $(\int_{\boldsymbol{e}_j}\cdot)_{j\in J}$ constitutes a horizontal system of projective coordinates on B^{α}_{a} over $U_{\mathscr{T}}$. Since two such systems of projective coordinates are related by a constant projective transformation when both are horizontal, it follows that (92) coincides with (90) up to a constant projective transformation. The proposition is proved. □

By definition, Veech's hyperbolic structure on $\mathscr{F}^{\alpha}_{a}$ is obtained by pull-back by V^{α}_{a} of the natural one on the target space $\mathbb{CH}^{n-1} \subset \mathbb{P}^{n-1}$. What makes the elliptic-hypergeometric definition of V^{α}_{a} à la Deligne-Mostow interesting is that it allows to make everything explicit. Indeed, in addition to the local explicit expression (90) obtained above, the use of twisted-(co)homology also allows to give an explicit expression for Veech's hyperbolic hermitian form on the target space.

4.4.4. **An explicit expression for the Veech form.** Since we are going to focus only on the $n=2$ case in the sequel, we only consider this case in the next result. However, the proof given hereafter generalizes in a straightforward way to the general case when $n \geq 2$.

Proposition 4.20. *The hermitian form of signature* $(1,1)$ *on the target space of* (90) *which corresponds to Veech's form is the one given by*

$$Z \longmapsto \overline{Z} \cdot I\!H_{\rho_a} \cdot {}^tZ$$

for $Z = (z_\infty, z_0) \in \mathbb{C}^2$, *where* $I\!H_{\rho_a}$ *stands for the matrix defined in* (40).

Note that the arguments of [85, Chap. IV, §7] apply to our situation. Consequently, the hermitian form associated to $I\!H_{\rho_a}$ is invariant by the hyperbolic holonomy of the corresponding leaf $\mathcal{F}^{\alpha}_{a}$ of Veech's foliation. In the classical hypergeometric case (*i.e.* when $g=0$), this is sufficient to characterize the Veech form and get the corresponding result. However, in the genus 1 case, since some leaves of Veech's foliation $\mathcal{F}^{\alpha}$ on the moduli space $\mathcal{M}_{1,2}$ (such as the generic ones, see Corollary 4.16) are simply connected, there is no holonomy whatsoever to consider hence such a proof is not possible.

The proof of Proposition 4.20 which we give below is a direct generalization of the one of Proposition 1.11 in [42] to our case. Remark that although elementary, this proof is long and computational. It would be interesting to give a more conceptual proof of this result.

Proof. We continue to use the notations introduced in the proof of Proposition 4.19. Let v be the hermitian form on the target space $\mathbb{P}^{n-1}$ of (90) which corresponds to the one considered by Veech in [76].

For $(\tau, z) \in \mathscr{F}_a^{\alpha}$, the wedge-product

$$\eta_{\tau,z} = \omega^{\alpha}_{\tau,z} \wedge \overline{\omega^{\alpha}_{\tau,z}} = \left| T^{\alpha}_{\tau,z}(u) \right|^2 du \wedge d\overline{u}$$

does not depend on the determination of $T^{\alpha}_{\tau,z}(u)$ and extends to an integrable positive 2-form on $E_{\tau,z}$. Moreover, according to [76, §12], one has (up to multiplication by -1):

$$v\big(V^{\alpha}_r(\tau, z)\big) = -\frac{i}{2}\int_{E_\tau} \eta_{\tau,z} < 0.$$

The complementary set $Q_{\tau,z} = E_{\tau,z} \setminus \gamma_{\tau,z}$ of the union of the supports of the three 1-cycles γ_0, γ_2 and γ_∞ in E_τ is homeomorphic to a disk. Its boundary in the metric completion $\overline{Q}_{\tau,z}$ (defined as in the proof of Proposition 4.19) is

$$\partial \overline{Q}_{\tau,z} = \overline{\gamma}_0 + \overline{\gamma}'_\infty - \overline{\gamma}'_0 - \overline{\gamma}_\infty + \overline{\gamma}'_2 - \overline{\gamma}_2$$

where the six elements in this sum are the boundary segments defined in the figure below.

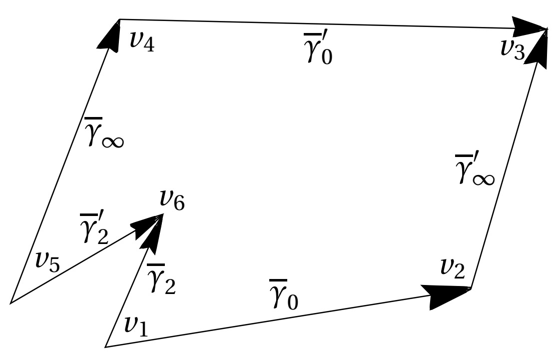


FIGURE 11. The closed disk $\overline{Q}_{\tau,z}$ and its boundary

Let $\Phi=\Phi_{\tau,z}$ be a primitive of $\omega=\omega^{\alpha}_{\tau,z}$ on $\overline{Q}_{\tau,z}$. For any symbol $\bullet\in\{0,2,\infty\}$ we will denote by $\omega_\bullet$ and $\omega'_\bullet$ the restriction of ω on $\overline{\gamma}_\bullet$ and $\overline{\gamma}'_\bullet$ respectively. We will use similar notations for $\overline{\omega}$ and for Φ as well, being aware that $\Phi'_\bullet$ has nothing to do with a derivative but refers to the restriction of Φ on $\overline{\gamma}'_\bullet$.

Since $d(\Phi\cdot\overline{\omega})=\eta_{\tau,z}$, it follows from Stokes theorem that

$$
\begin{aligned}
-2i\nu\big(V_r^{\alpha}(\tau,z)\big)=\int_{\partial\overline{\gamma}^c_{\tau,z}}\Phi\cdot\overline{\omega}=\int_{\gamma_0}&\Phi_0\cdot\overline{\omega}_0-\int_{\gamma'_0}\Phi'_0\cdot\overline{\omega}'_0\\
&+\int_{\gamma'_\infty}\Phi'_\infty\cdot\overline{\omega}'_\infty-\int_{\gamma_\infty}\Phi_\infty\cdot\overline{\omega}_\infty\\
&+\int_{\gamma'_2}\Phi'_2\cdot\overline{\omega}'_2-\int_{\gamma_2}\Phi_2\cdot\overline{\omega}_2\,.
\end{aligned}\tag{93}
$$

For any $\bullet\in\{0,2,\infty\}$, both segments $\overline{\gamma}_\bullet$ and $\overline{\gamma}'_\bullet$ identify to $\gamma_\bullet\subset E_{\tau,z}$, hence there is a natural identification between them. For $\zeta\in\overline{\gamma}_\bullet$, we will denote by ζ' the corresponding point on $\overline{\gamma}'_\bullet$. Up to these correspondences, one has

$$\omega'_0=\rho_\infty\omega_0,\qquad \omega'_2=\rho_2\omega_2\qquad\text{and}\qquad \omega'_\infty=\rho_0\omega_\infty.$$

It follows that for for every ζ' in $\overline{\gamma}'_0$, in $\overline{\gamma}'_2$ and in $\overline{\gamma}'_\infty$, one has respectively

$$
\begin{aligned}
&\Phi'_0(\zeta')=F^0+\rho_0F^\infty-\rho_\infty F^0+\int_{v_4}^{\zeta'}\omega'_0=(1-\rho_\infty)F^0+\rho_0F^\infty+\rho_\infty\int_{v_1}^{\zeta}\omega_0\,,\\
&\Phi'_2(\zeta')=F^2-\rho_2F^2+\int_{v_5}^{\zeta'}\omega'_2=(1-\rho_2)F^2+\rho_2\int_{v_1}^{\zeta}\omega_2\\
\text{and}\quad&\Phi'_\infty(\zeta')=F^0+\int_{v_2}^{\zeta'}\omega'_\infty=F^0+\rho_0\int_{v_5}^{\zeta}\omega_\infty
\end{aligned}\tag{94}
$$

with

$$
\begin{aligned}
&F^0=\Phi(v_2)-\Phi(v_1)=\int_{v_1}^{v_2}\omega=\int_{\boldsymbol{\gamma}_0}T^{\alpha}(u,\tau,z)du\,;\\
&F^2=\Phi(v_6)-\Phi(v_1)=\int_{v_1}^{v_6}\omega=\int_{\boldsymbol{\gamma}_2}T^{\alpha}(u,\tau,z)du\\
\text{and}\quad&F^\infty=\Phi(v_4)-\Phi(v_5)=\int_{v_4}^{v_5}\omega=\int_{\boldsymbol{\gamma}_\infty}T^{\alpha}(u,\tau,z)du\,.
\end{aligned}\tag{95}
$$

Since $\overline{\rho_0}=\rho_0^{-1}$, it follows from (94) that for every $\zeta\in\overline{\gamma}_0$, one has

$$
\begin{aligned}
\big(\Phi_0\cdot\overline{\omega}_0\big)(\zeta)-\big(\Phi'_0\cdot\overline{\omega}'_0\big)(\zeta')&=\Phi(\zeta)\cdot\overline{\omega}_0-\Big[(1-\rho_\infty)F_0+\rho_0F_\infty+\rho_\infty\Phi(\zeta)\Big]\cdot\rho_\infty^{-1}\overline{\omega}_0\\
&=\left[\frac{d_\infty}{\rho_\infty}F_0-\frac{\rho_0}{\rho_\infty}F_\infty\right]\cdot\overline{\omega}_0(\zeta)\,.
\end{aligned}
$$

Similarly, since $\overline{\rho_2} = \rho_2^{-1}$, it follows from (94) that for every $\zeta \in \overline{\gamma}_2$, one has

$$\big(\Phi_2 \cdot \overline{\omega}_2\big)(\zeta) - \big(\Phi_2' \cdot \overline{\omega}_2'\big)(\zeta') = \Phi(\zeta) \cdot \overline{\omega}_2 - \Big[(1-\rho_2)F_2 + \rho_2 \Phi(\zeta)\Big] \cdot \rho_2^{-1}\overline{\omega}_2$$
$$= \left[\frac{d_2}{\rho_2} F_2\right] \cdot \overline{\omega}_2(\zeta)\,.$$

Finally, since $\overline{\rho_\infty} = \rho_\infty^{-1}$, it follows from (94) that for every $\zeta \in \overline{\gamma}_\infty$, one has

$$\begin{aligned}\big(\Phi_\infty \cdot \overline{\omega}_\infty\big)(\zeta) - \big(\Phi_\infty' \cdot \overline{\omega}_\infty'\big)(\zeta') &= \Phi(\zeta) \cdot \overline{\omega}_\infty - \left[F_0 + \rho_0 \int_{v_5}^{\zeta} \omega_\infty\right] \cdot \rho_0^{-1}\overline{\omega}_\infty \\ &= \left[(1-\rho_2)F_2 + \int_{v_5}^{\zeta} \omega_\infty\right] \cdot \overline{\omega}_\infty - \left[F_0 + \rho_0 \int_{v_5}^{\zeta} \omega_\infty\right] \cdot \rho_0^{-1}\overline{\omega}_\infty \\ &= \left[(1-\rho_2)F_2 - \frac{1}{\rho_0} F_0\right] \cdot \overline{\omega}_\infty(\zeta).\end{aligned}$$

From the three relations above, one deduces that

$$\int_{\overline{\gamma}_\bullet} \Phi \cdot \overline{\omega} - \int_{\overline{\gamma}_\bullet'} \Phi \cdot \overline{\omega} = \begin{cases} d_\infty \rho_\infty^{-1} F_0 \overline{F}_0 - \rho_0 \rho_\infty^{-1} F_\infty \overline{F}_0 & \text{for } \bullet = 0; \\ d_2 \rho_2^{-1} F_2 \overline{F}_2 & \text{for } \bullet = 2; \\ -d_2 F_2 \overline{F}_\infty - \rho_0^{-1} F_0 \overline{F}_\infty & \text{for } \bullet = \infty. \end{cases} \tag{96}$$

Injecting these computation in (93) and using the relations

$$d_2 F_2 = d_\infty F_0 - d_0 F_\infty \qquad \text{and} \qquad \frac{1}{\rho_2}\overline{F}_2 = \frac{d_\infty}{\rho_\infty d_2}\overline{F}_0 - \frac{d_0}{\rho_0 d_2}\overline{F}_\infty$$

one gets

$$\begin{aligned} 2i\nu\big(V_r^\alpha(\tau,z)\big) &= \frac{d_\infty}{\rho_\infty} F_0 \overline{F}_0 - \frac{\rho_0}{\rho_\infty} F_\infty \overline{F}_0 + d_2 F_2 \overline{F}_\infty + \frac{1}{\rho_0} F_0 \overline{F}_\infty - \frac{d_2}{\rho_2} F_2 \overline{F}_2 \\ &= \frac{d_\infty}{\rho_\infty} F_0 \overline{F}_0 - \frac{\rho_0}{\rho_\infty} F_\infty \overline{F}_0 + \Big(d_\infty F_0 - d_0 F_\infty\Big)\overline{F}_\infty + \frac{1}{\rho_0} F_0 \overline{F}_\infty \\ &\quad - \Big(d_\infty F_0 - d_0 F_\infty\Big)\Big(\frac{d_\infty}{\rho_\infty d_2}\overline{F}_0 - \frac{d_0}{\rho_0 d_2}\overline{F}_\infty\Big) \\ &= 2i\, {}^t\overline{F} \cdot H \cdot F\,, \end{aligned}$$

where F and H stand respectively for the matrices

$$F = \begin{bmatrix} F_\infty \\ F_0 \end{bmatrix} \qquad \text{and} \qquad H = \frac{1}{2i}\begin{bmatrix} -d_0\Big(1 + \frac{d_0}{\rho_0 d_2}\Big) & d_\infty + \frac{1}{\rho_0} + \frac{d_\infty d_0}{\rho_0 d_2} \\ -\frac{\rho_0}{\rho_\infty} + \frac{d_0 d_\infty}{\rho_\infty d_2} & \frac{d_\infty}{\rho_\infty}\Big(1 - \frac{d_\infty}{d_2}\Big) \end{bmatrix}.$$

Because $\rho_2 = \rho_1^{-1}$, one verifies that

$$H = \frac{1}{2i}\begin{bmatrix} \frac{d_0}{d_1}\Big(1-\frac{\rho_1}{\rho_0}\Big) & \frac{1-\rho_0^{-1}-\rho_\infty+\rho_1\rho_\infty\rho_0^{-1}}{d_1} \\ \frac{\rho_1-\rho_0\rho_1-\rho_1\rho_\infty^{-1}+\rho_0\rho_\infty^{-1}}{d_1} & \frac{d_\infty d_{1\infty}}{\rho_\infty d_1}\end{bmatrix} = I\!H_\rho\,.$$

Finally, it follows from (95) that F_∞ and F_0 are nothing else but the components of the map (90) and the proposition follows. □

4.4.5. A normalized version of Veech's map (when $n = 2$ and $\rho_0 = 1$).

According to §3.5.2, when $n = 2$ and $\rho_0 = 1$, setting $X = \begin{bmatrix} -\frac{d_{1\infty}}{d_1} & 1 \\ \rho_\infty & 0\end{bmatrix}$, one has

$$\overline{X}\cdot I\!H_\rho \cdot {}^tX = \begin{bmatrix} 0 & \frac{i}{2} \\ -\frac{i}{2} & 0\end{bmatrix}.$$

Consequently, setting $\boldsymbol{F} = (F_\infty, F_0)$ and

$$\boldsymbol{G} = \big(G_\infty, G_0\big) = \big(F_\infty, F_0\big)\cdot X^{-1} = \Big(F_0, \frac{1}{\rho_\infty}F_\infty + \frac{d_{1\infty}}{\rho_\infty d_1}F_0\Big),$$

one obtains that

$$\Im\mathrm{m}\big(\overline{G_\infty}G_0\big) = \overline{\boldsymbol{G}}\cdot\begin{bmatrix} 0 & \frac{i}{2} \\ -\frac{i}{2} & 0\end{bmatrix}\cdot {}^t\boldsymbol{G} = \overline{\boldsymbol{F}}\cdot I\!H_\rho\cdot {}^t\boldsymbol{F} = v < 0,$$

which implies that the imaginary part of the ratio G_0/G_∞ is negative.

It follows that the map

$$-\frac{G_0}{G_\infty} = -\Big(\frac{1}{\rho_\infty}\frac{F_\infty}{F_0} + \frac{d_{1\infty}}{\rho_\infty d_1}\Big)$$

is a normalized version of Veech's map, with values into the upper half-plane.

5. Flat tori with two conical points

From now on, we focus on the first nontrivial case, namely $g = 1$ and $n = 2$. We fix $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_1 = -\alpha_2 \in]0,1[$. Since it is fixed, we will often omit the subscript α or α_1 in the notations. We want to study the hyperbolic structure on the leaves of Veech's foliation at the level of the moduli space $\mathcal{M}_{1,\alpha} \simeq \mathcal{M}_{1,2}$. We will only consider the most interesting leaves, namely the algebraic ones.

5.1. **Some notations.** In what follows, N stands for an integer bigger than 1.

5.1.1. For any $(a_0, a_\infty) \in \mathbb{R}^2 \setminus \alpha_1\mathbb{Z}^2$, we set

$$r = \big(r_0, r_\infty\big) = \frac{1}{\alpha_1}\big(a_0, a_\infty\big) \in \mathbb{R}^2 \setminus \mathbb{Z}^2$$

and we denote by $\mathscr{F}_r^{\alpha_1}$ (or just by $\mathscr{F}_r$ for short) the leaf $(\xi^\alpha)^{-1}(a_0, a_\infty) = \Xi^{-1}(r)$ of Veech's foliation in the Torelli space. This is the subset of $\mathscr{T}or_{1,2}$ cut out by any one of the following two (equivalent) equations:

$$a_0\tau + \alpha_2 z_2 = a_\infty \qquad \text{or} \qquad z_2 = r_0\tau - r_\infty.$$

We remind the reader that $\mathcal{F}_{[r]}^{\alpha_1}$ (or just $\mathcal{F}_r$ for short) stands for the corresponding leaf in the moduli space of elliptic curves with two marked points:

$$\mathcal{F}_r = \mathcal{F}_{[r]} = p_{1,2}\big(\mathscr{F}_r\big) \subset \mathcal{M}_{1,2}.$$

5.1.2. From a geometric point of view, the most interesting leaves clearly are the leaves $\mathscr{F}_r$ with $r \in \mathbb{Q}^2 \setminus \mathbb{Z}^2$. Indeed, these are exactly the ones which are algebraic subvarieties (we should say 'suborbifolds') of $\mathcal{M}_{1,2}$ and there is such a leaf for each integer $N \geq 2$ (see Corollary 4.17), which is

$$\mathcal{F}_N = \mathcal{F}_N^{\alpha_1} = \mathcal{F}_{(0,-1/N)}\,.$$

An equation of the corresponding leaf $\mathscr{F}_N = \mathscr{F}_{(0,-1/N)}$ in $\mathscr{T}or_{1,2}$ is

$$z_2 = \frac{1}{N}\,. \tag{97}$$

It induces a natural identification $\mathbb{H} \simeq \mathscr{F}_N$ which is compatible with the action of $\Gamma_1(N) \simeq \mathrm{Stab}(\mathscr{F}_{(0,-1/N)})$ (see (82)) hence induces an identification

$$Y_1(N) \simeq \mathcal{F}_N. \tag{98}$$

For computational reasons, it will be useful later to consider other identifications between $\mathcal{F}_N$ and the modular curve $Y_1(N)$ (see §3.1.7 just below). However (98) will be the privileged one. For this reason, we will use the (somewhat abusive) notations

$$Y_1(N) = \mathcal{F}_N \qquad \big(\text{and } \; Y_1(N)^{\alpha_1} = \mathcal{F}_N^{\alpha_1}\big)$$

(the second one to emphasize the fact that $Y_1(N)$ is endowed with the $\mathbb{C}\mathbb{H}^1$-structure corresponding to Veech's one on $\mathcal{F}_N^{\alpha_1}$) to distinguish (98) from the other identifications between $Y_1(N)$ and $\mathcal{F}_N$ that we will consider below.

5.1.3. For any $c \in \mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{i\infty\} \subset \partial\mathbb{H}$, one denotes by $[c]$ the associated cusp of $Y_1(N) = \mathcal{F}_N$. Then the set of cusps

$$C_1(N) = \left\{ [c] \,\middle|\, c \in \mathbb{P}^1(\mathbb{Q}) \right\}$$

is finite and

$$X_1(N) = Y_1(N) \sqcup C_1(N)$$

is a compact smooth algebraic curve (see [13, Chapter I] for instance).

Our goal in this section is to study the hyperbolic structure, denoted by $\mathbf{hyp}^{\boldsymbol{\alpha_1}}_{\mathbf{1,N}}$, of the leaf $Y_1(N)^{\alpha_1} = \mathcal{F}_N^{\alpha_1}$ of Veech's foliation $\mathcal{F}^{\alpha_1}$ on $\mathcal{M}_{1,2}$ in the vicinity of any one of its cusps. More precisely, we want to prove that $\mathrm{hyp}^{\alpha_1}_{1,N}$ extends as a conifold $\mathbb{CH}^1$-structure at such a cusp $\mathfrak{c}$ and give a closed formula for the associated conical angle which will be denoted by

$$\theta_N(\mathfrak{c}) \in \left[0, +\infty\right[^{27}.$$

5.1.4. With this aim in mind, it will be more convenient to deal with the ramified cover $Y(N)$ over $Y_1(N)$ associated to the principal congruence subgroup $\Gamma(N)$. In order to do so, we consider the subgroup

$$G(N) = \Gamma(N) \ltimes \mathbb{Z}^2 \lhd \mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$$

and we denote the associated quotient map by

$$p^N_{1,2} : \mathscr{T}or_{1,2} \to \mathscr{T}or_{1,2}/G(N) =: \mathcal{M}_{1,2}(N)\,. \tag{99}$$

Then from (97), one deduces an identification between the 'level N modular curve $Y(N) = \mathbb{H}/\Gamma(N)$' and the corresponding leaf in $\mathcal{M}_{1,2}(N)$:

$$Y(N) \simeq p^N_{1,2}(\mathscr{F}_N) =: F_N. \tag{100}$$

As above, we will consider this identification as the privileged one and for this reason, it will be indicated by means of the equality symbol. In other terms, we have fixed identifications

$$\begin{aligned} Y_1(N)^{\alpha_1} &= \mathcal{F}_N^{\alpha_1} \quad (\text{or } Y_1(N) = \mathcal{F}_N \text{ for short});\\ \text{and } Y(N)^{\alpha_1} &= F_N^{\alpha_1} \quad (\text{or } Y(N) = F_N \text{ for short}). \end{aligned}$$

One denotes by $C(N)$ the set of cusps of $Y(N)$. Then

$$X(N) = Y(N) \sqcup C(N)$$

is a compact smooth algebraic curve. The complex hyperbolic structure on $Y(N)$ corresponding to Veech's one (up to the identification (100)) will be denoted by $\mathbf{hyp}^{\boldsymbol{\alpha_1}}_{\boldsymbol{N}}$. Under the assumption that it extends as a conifold structure at $\mathfrak{c} \in C(N)$, we will denote by $\vartheta_N(\mathfrak{c})$ the associated conical angle.

[26]By convention, a **complex hyperbolic conifold point of conical angle 0** is nothing else but an usual cusp for a hyperbolic surface (see A.1.1. in Appendix A for more details).

5.1.5. The natural quotient map $Y(N) \to Y_1(N)$ (coming from the fact that $\Gamma(N)$ is a subgroup of $\Gamma_1(N)$) induces an algebraic cover

$$X(N) \longrightarrow X_1(N) \tag{101}$$

which is ramified at the cusps of $X_1(N)$. More precisely, at a cusp $\mathfrak{c} \in C(N)$, a local analytic model for this cover is $z \mapsto z^{N/w_{\mathfrak{c}}}$ where $w_{\mathfrak{c}}$ stands for the **width** of $\mathfrak{c}$, the latter being now seen as a cusp of $X_1(N)$.[28]

It follows that, for any $\mathfrak{c} \in C(N)$, $\mathrm{hyp}_{1,N}^{\alpha_1}$ extends as a $\mathbb{CH}^1$-conifold structure at $\mathfrak{c}$ now considered as a cusp of $Y_1(N)^{\alpha_1}$ if and only if the same holds true, at $\mathfrak{c}$, for the corresponding complex hyperbolic structure $\mathrm{hyp}_N^{\alpha_1}$ on $X(N)^{\alpha_1}$. In this case, one has the following relation between the corresponding cone angles:

$$\theta_N(\mathfrak{c}) = \frac{w_{\mathfrak{c}}}{N} \vartheta_N(\mathfrak{c})\,. \tag{102}$$

In order to get explicit results, it is necessary to have a closed explicit formula for the width of a cusp.

Lemma 5.1. *Assume that* $\mathfrak{c} = [-a'/c'] \in C_1(N)$ *with* $a', c' \in \mathbb{Z}$ *are coprime. Then*

$$w_{\mathfrak{c}} = \frac{N}{\gcd(c', N)}.$$

Proof. The set of cusps of $X(N)$ can be identified with the set of classes $\pm\left[\begin{smallmatrix} a \\ c \end{smallmatrix}\right]$ of the points $\left[\begin{smallmatrix} a \\ c \end{smallmatrix}\right] \in (\mathbb{Z}/N\mathbb{Z})^2$ of order N. To $\mathfrak{c} = [-a'/c']$ is associated $\pm\left[\begin{smallmatrix} a \\ c \end{smallmatrix}\right]$ where a and c stand for the residu modulo N of a' and c' respectively (*cf.* [13, §3.8]). It follows that $\gcd(c', N) = \gcd(c, N)$.

On the other hand, according to [57, §1], the ramification degree of the covering $X(N) \to X_1(N)$ at $\pm\left[\begin{smallmatrix} a \\ c \end{smallmatrix}\right]$ viewed as a cusp of $X_1(N)$ is $\gcd(c, N)$. Since the width of any cusp of $X(N)$ is N, it follows that $w_{\mathfrak{c}} = N/\gcd(c', N)$. □

5.2. **Auxiliary leaves.** For any $(m, n) \in \mathbb{Z}^2 \setminus N\mathbb{Z}^2$ with $\gcd(m, n, N) = 1$, one sets

$$\mathscr{F}_{m,n} = \mathscr{F}_{(m/N, -n/N)}\,.$$

This is leaf of Veech's foliation on $\mathscr{T}or_{1,2}$ cut out by

$$z_2 = \tau m/N + n/N\,.$$

The latter equation induces a natural identification

$$\begin{aligned} \mathbb{H} &\stackrel{\sim}{\longrightarrow} \mathscr{F}_{m,n} \subset \mathscr{T}or_{1,2} \\ \tau &\longmapsto \left(\tau, \frac{m}{N}\tau + \frac{n}{N}\right) \end{aligned} \tag{103}$$

[28] We recall that, $w_{\mathfrak{c}}$ divides N for any $\mathfrak{c} \in C(N)$ hence the map $z \mapsto z^{N/w_{\mathfrak{c}}}$ is holomorphic.

which is compatible with the action of $\Gamma(N) \lhd \Gamma_1(N) \simeq \mathrm{Stab}(\mathscr{F}_{m,n})$ (*cf.* (82)), hence induces a well-defined identification

$$Y(N) \simeq p^N_{1,2}(\mathscr{F}_{m,n}) =: F_{m,n} \subset \mathcal{M}_{1,2}(N). \tag{104}$$

For any $s \in \mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{i\infty\} \subset \partial\mathbb{H}$, one denotes by $[s]$ the associated cusp of $Y(N)$ and by $[s]_{m,n}$ the corresponding cusp for $F_{m,n}$ relatively to the identification (104). Then if one denotes by

$$C_{m,n} = \Big\{ [s]_{m,n} \,|\, s \in \mathbb{P}^1(\mathbb{Q}) \Big\}$$

the set of cusps of $F_{m,n} \simeq Y(N)$, one gets a compactification

$$X(N) \simeq X_{m,n} := F_{m,n} \sqcup C_{m,n}$$

where the identification with $X(N)$ is the natural extension of (104). One will denote by $\mathbf{hyp}^{\boldsymbol{\alpha_1}}_{\boldsymbol{m,n}}$ the complex hyperbolic structure on $X(N)$ corresponding to Veech's one of $F_{m,n}$ up to the preceding identification.

Since $(0, -1/N)$ is a representative for the orbit of $(m/N, -n/N)$ under the action of $\mathrm{SL}_2(\mathbb{Z}) \rtimes \mathbb{Z}^2$ (*cf.* Proposition 4.9), all the leaves $F_{m,n}$ are isomorphic to

$$F_{0,1} = F_N = Y(N)$$

(where the first equality comes from the very definition of $F_{0,1}$ whereas the second one refers to the privileged identification (100)).

What makes considering the whole bunch of leaves $F_{m,n}$ interesting for us is that the natural identifications (104) do depend on (m,n) (even if $F_{m,n}$ coincides with $F_{0,1}$ as a subset of $\mathcal{M}^N_{1,2}$, as it can happen). Thus, we will see that, for any cusp $[s] = [s]_{0,1}$ of $Y(N) = F_{0,1}$, there is a leaf $F_{m,n}$ such that

$$[s]_{0,1} = \big[i\infty\big]_{m,n}.$$

Since the hyperbolic structures of $F_{0,1}$ and of $F_{m,n}$ coincide, this implies that

> *the study of the hyperbolic structure* $\mathrm{hyp}^{\alpha_1}_N = \mathrm{hyp}^{\alpha_1}_{0,1}$ *of* $Y(N) = F_{0,1}$ *in the vicinity of its cusps amounts to the study of the hyperbolic structures* $\mathrm{hyp}^{\alpha_1}_{m,n}$ *of the leaves* $F_{m,n}$*, only in the vicinity of the cusp* $[i\infty]_{m,n}$.

We want to make the above considerations as explicit as possible. Let $\mathfrak{c}$ be a cusp of F_N distinct from $[i\infty]$. There exist $a', c' \in \mathbb{Z}$ with $c' \neq 0$ and $\gcd(a', c') = 1$ such that

$$\mathfrak{c} = \big[-a'/c'\big] = \big[-a'/c'\big]_{0,1}.$$

Since a' and c' are coprime, there exist d' and b' in $\mathbb{Z}$ such that $a'd' - b'c' = 1$. Then one considers the following element of $\mathrm{SL}_2(\mathbb{Z}) \rtimes \mathbb{Z}^2$:

$$M_{a',c'} = \left(\begin{bmatrix} d' & b' \\ c' & a' \end{bmatrix}, \Big(-\lfloor a'/N \rfloor, -\lfloor c'/N \rfloor \Big) \right). \tag{105}$$

It induces an automorphism of the intermediary moduli space $\mathcal{M}_{1,2}(N)$ (defined above in (99)), denoted somewhat abusively by the same notation $M_{a',c'}$. This automorphism leaves the corresponding intermediary Veech's foliation invariant and is compatible with the hyperbolic structure on the leaves.

Setting $a = a' - \lfloor a'/N \rfloor$ and $c = c' - \lfloor c'/N \rfloor$, one verifies that

$$M_{a',c'} \bullet \left(0, -\frac{1}{N}\right) = \left(\frac{c}{N}, -\frac{a}{N}\right),$$

where $\bullet$ stands for the action (65). Thus $M_{a',c'}$ induces an isomorphism $Y(N) = F_{0,1} \xrightarrow{\sim} F_{c,a}$ which extends to an isomorphism between the compactifications $X(N) = X_{0,1} \xrightarrow{\sim} X_{c,a}$, such that

$$M_{a',c'}\Big(\big[-a'/c'\big]\Big) = \big[i\infty\big]_{c,a}.$$

Moreover, $M_{a',c'}$ induces an isomorphism between the $\mathbb{CH}^1$-structures $\mathrm{hyp}_{0,1}^{\alpha_1}$ and $\mathrm{hyp}_{c,a}^{\alpha_1}$. In particular, one deduces the following result:

Proposition 5.2. *Let a' and c' be two coprime integers and denote respectively by a and c their residues modulo N in $\{0, \ldots, N-1\}$.*

(1) *There is an isomorphism of pointed curves carrying a $\mathbb{CH}^1$-structure*

$$\Big(Y(N), \big[-a'/c'\big]\Big) \simeq \Big(F_{c,a}, \big[i\infty\big]_{c,a}\Big).$$

(2) *The two following assertions are equivalent:*
- $\mathrm{hyp}_N^{\alpha_1}$ *extends as a conifold $\mathbb{CH}^1$-structure to $X(N)$;*
- *for every $a, c \in \{0, \ldots, N-1\}$ with $\gcd(a,c,N) = 1$, $\mathrm{hyp}_{c,a}^{\alpha_1}$ extends as a conifold $\mathbb{CH}^1$-structure in the vicinity of the cusp $[i\infty]_{c,a}$ of $F_{c,a}$.*

(3) *When the two equivalent assertions of (2) are satisfied, the conifold angle $\vartheta(-a'/c')$ of hyp_N^{α} at the cusp $[-a'/c']$ of $Y(N)$ is equal to the conifold angle $\vartheta_{c,a}(i\infty)$ of $\mathrm{hyp}_{c,a}^{\alpha_1}$ at the cusp $[i\infty]$ of $F_{c,a}$.*

5.3. **Mano's differential system for algebraic leaves.** We will now focus on the auxiliary algebraic leaves of Veech's foliation considered in Section 5.2. The arguments and results used below are taken from [49, 46] (see also Appendix B).

5.3.1. We fix $m, n \in \{0, \ldots, N-1\}$ such that $(m,n) \neq 0$. For any $\tau \in \mathbb{H}$, one sets:

$$t = t_\tau = \frac{m}{N}\tau + \frac{n}{N}.$$

Hence, correspondingly, one has

$$a_0 = \frac{m}{N}\alpha_1, \qquad a_\infty = -\frac{n}{N}\alpha_1$$

and

$$T(u) = T^{\alpha}(u,\tau) = e^{2i\pi\frac{m}{N}\alpha_1 u}\left(\frac{\theta(u)}{\theta\big(u-t_\tau\big)}\right)^{\alpha_1}.$$

In order to to make the connection with the results in [46], we recall the following notations introduced there:

$$\theta_{m,n}(u) = \theta_{m,n}(u,\tau) = e^{-i\pi\frac{m^2}{N^2}\tau-2i\pi\frac{m}{N}\left(u+\frac{n}{N}\right)}\theta\left(u+\frac{m}{N}\tau+\frac{n}{N},\tau\right).$$

Then setting

$$T_{m,n}(u) = \left(\frac{\theta(u)}{\theta_{m,n}(u)}\right)^{\alpha_1},$$

one verifies that, when the determinations of $T(u)$ and of $T_{m,n}(u)$ are fixed, then up to the change of variable $u \to -u$, these two functions coincide up to multiplication by a non-vanishing complex function of τ. This can be written a little abusively

$$T(u) = \lambda(\tau)\,T_{m,n}(-u) \tag{106}$$

where λ stands for the aforementioned holomorphic function which depends only on τ (and on the integers m, n and N) but not on u.

5.3.2. Since it has values in a projective space, the Veech map stays unchanged if all its components are multiplied by the same non-vanishing function of τ. From (106) and in view of the local expression (90) for the Veech map in terms of elliptic hypergeometric integrals, it follows that the holomorphic map

$$\begin{aligned} V = V_{m,n} \,:\, \mathscr{F}^{\alpha_1}_{m,n} \simeq \mathbb{H} &\longrightarrow \mathbb{P}^1 \\ \tau &\longmapsto \big[V_0(\tau) : V_\infty(\tau)\big] \end{aligned}$$

whose two components are given by (for $\tau \in \mathbb{H}$)

$$V_0(\tau) = \int_{\boldsymbol{\gamma}_0} T_{m,n}(u)du \qquad \text{and} \qquad V_\infty(\tau) = \int_{\boldsymbol{\gamma}_\infty} T_{m,n}(u)du,$$

is nothing else but another expression for the Veech map on $\mathscr{F}_{m,n} \simeq \mathbb{H}$.

We introduce two other holomorphic functions of $\tau \in \mathbb{H}$ defined by

$$W_0(\tau) = \int_{\boldsymbol{\gamma}_0} T_{m,n}(u)\rho'(u)du \qquad \text{and} \qquad W_\infty(\tau) = \int_{\boldsymbol{\gamma}_\infty} T_{m,n}(u)\rho'(u)du$$

(we recall that ρ denotes the logarithmic derivative of θ w.r.t. u, *cf.* §2.2).

The two maps $\tau \mapsto \boldsymbol{\gamma}_\bullet$ for $\bullet = 0, \infty$ form a basis of the space of local sections of the local system over $\mathscr{F}_{m,n}$ whose fibers are the twisted homology groups $H_1(E_{\tau,t}, L_{\tau,t})$'s (see B.3 in Appendix B). Then it follows from [49, 46] (see also

B.3.5 below) that the functions V_0, V_∞, W_0 and W_∞ satisfy the following differential system

$$\frac{d}{d\tau}\begin{bmatrix} V_0 & V_\infty \\ W_0 & W_\infty \end{bmatrix} = M_{m,n}\begin{bmatrix} V_0 & V_\infty \\ W_0 & W_\infty \end{bmatrix}$$

on $\mathbb{H} \simeq \mathscr{F}_{m,n}$, with

(107)

$$M_{m,n} = \begin{bmatrix} A_{m,n} & B_{m,n} \\ C_{m,n} & D_{m,n} \end{bmatrix} = \begin{bmatrix} \alpha_1\left(\frac{\dot{\theta}_{m,n}}{\theta_{m,n}} - \frac{\dot{\theta}'}{\theta'}\right) & \frac{\alpha_1 - 1}{2i\pi} \\ 2i\pi\alpha_1\left(\frac{\ddot{\theta}_{m,n}}{\theta_{m,n}} - \left(\frac{\dot{\theta}_{m,n}}{\theta_{m,n}}\right)^2 - \frac{\ddot{\theta}'}{\theta'} + \left(\frac{\dot{\theta}'}{\theta'}\right)^2\right) & -\alpha_1\left(\frac{\dot{\theta}_{m,n}}{\theta_{m,n}} - \frac{\dot{\theta}'}{\theta'}\right) \end{bmatrix}.$$

The trace of this matrix vanishes and the upper-left coefficient $B_{m,n}$ is constant. Consequently, it follows from a classical result of the theory of linear differential equations (*cf.* Lemma 6.1.1 of [34, §3.6.1] for instance or Lemma A.2.2 in Appendix A) that V_0 and V_∞ form a basis of the space of solutions of the associated second order linear differential equation

$$V^{\bullet\bullet} + \Big(\det\big(M_{m,n}\big) - A^{\bullet}_{m,n}\Big)\, V = 0$$

where the superscript $^\bullet$ indicates the derivative with respect to the variable τ.

Since

$$A^{\bullet}_{m,n} = \alpha_1\left[\frac{\ddot{\theta}_{m,n}}{\theta_{m,n}} - \left(\frac{\dot{\theta}_{m,n}}{\theta_{m,n}}\right)^2 - \frac{\ddot{\theta}'}{\theta'} + \left(\frac{\dot{\theta}'}{\theta'}\right)^2\right]$$

and

$$\det\big(M_{m,n}\big) = -(\alpha_1)^2\left[\frac{\dot{\theta}_{m,n}}{\theta_{m,n}} - \frac{\dot{\theta}'}{\theta'}\right]^2 - \alpha_1(\alpha_1 - 1)\left[\frac{\ddot{\theta}_{m,n}}{\theta_{m,n}} - \left(\frac{\dot{\theta}_{m,n}}{\theta_{m,n}}\right)^2 - \frac{\ddot{\theta}'}{\theta'} + \left(\frac{\dot{\theta}'}{\theta'}\right)^2\right],$$

this differential equation can be written more explicitly

$$V^{\bullet\bullet} - (\alpha_1)^2\left[\left(\frac{\dot{\theta}_{m,n}}{\theta_{m,n}} - \frac{\dot{\theta}'}{\theta'}\right)^2 + \frac{\ddot{\theta}_{m,n}}{\theta_{m,n}} - \left(\frac{\dot{\theta}_{m,n}}{\theta_{m,n}}\right)^2 - \frac{\ddot{\theta}'}{\theta'} + \left(\frac{\dot{\theta}'}{\theta'}\right)^2\right]\cdot V = 0. \tag{108}$$

5.3.3. By a direct computation (left to the reader), one verifies that the matrix (107) and consequently the coefficients of the preceding second order differential equation are invariant by the translation $\tau \mapsto \tau + N$. It follows that the restriction of (108) to the vertical band of width N

$$H_N = \Big\{\tau \in \mathbb{H}\,\Big|\, 0 \leq \mathrm{Re}(\tau) < N\Big\}$$

can be pushed-forward onto a differential equation of the same type on a punctured open neighborhood U^* of the cusp $[i\infty]$ in $Y(N)$.

Let x be the local holomorphic coordinate on $Y(N)$ centered at $[i\infty]$ and related to the variable τ through the formula

$$x = \exp\big(2i\pi\tau/N\big).$$

Then $v(x) = V(\tau(x))$ satisfies a second order differential equation

$$v''(x) + P_{m,n}(x)\cdot v'(x) + Q_{m,n}(x)\cdot v(x) = 0 \tag{109}$$

whose coefficients $P_{m,n}$ and $Q_{m,n}$ are holomorphic on $(\mathbb{C}^*, 0)$.

In [46], Mano establishes the following limits when $\tau \in H_N$ tends to $i\infty$:

$$\frac{\overset{\bullet}{\theta}{}'}{\theta'} \longrightarrow \frac{i\pi}{4} \qquad\qquad \frac{\overset{\bullet\bullet}{\theta}{}'}{\theta'} - \left(\frac{\overset{\bullet}{\theta}{}'}{\theta'}\right) \longrightarrow 0$$

$$\frac{\overset{\bullet}{\theta}_{m,n}}{\theta_{m,n}} \longrightarrow i\pi\left(\frac{m}{N}-\frac{1}{2}\right)^2 \qquad\qquad \frac{\overset{\bullet\bullet}{\theta}_{m,n}}{\theta_{m,n}} - \left(\frac{\overset{\bullet}{\theta}_{m,n}}{\theta_{m,n}}\right)^2 \longrightarrow 0\,.$$

It follows that the coefficient of V in (108) tends to

$$-\left[\alpha_1\cdot i\pi\left(\frac{m^2}{N^2}-\frac{m}{N}\right)\right]^2$$

as τ goes to $i\infty$ in H_N. This implies that the functions $P_{m,n}$ and $Q_{m,n}$ in (109) actually extend meromorphically through the origin.

Since for $x \in \mathbb{C}\setminus[0,+\infty[$ sufficiently close to the origin, one has

$$v'(x) = V^\bullet\big(\tau(x)\big)\cdot\left(\frac{N}{2i\pi x}\right)$$

$$\text{and}\quad v''(x) = V^{\bullet\bullet}\big(\tau(x)\big)\cdot\left(\frac{N}{2i\pi x}\right)^2 - V^\bullet\big(\tau(x)\big)\left(\frac{N}{2i\pi x^2}\right),$$

one gets that the asymptotic expansion of (109) at the origin is written

$$v''(x) + \frac{1}{x}\cdot v'(x) + \left(-\left[\frac{m(m-N)}{2N}\alpha_1\right]^2\cdot\frac{1}{x^2} + O\Big(\frac{1}{x}\Big)\right)\cdot v(x) = 0\,.$$

In particular, this shows that the origin is a regular singular point for (109) and the associated characteristic (or indicial) equation is

$$s(s-1) + s - \left[\frac{m(m-N)}{2N}\alpha_1\right]^2 = s^2 - \left[\frac{m(m-N)}{2N}\alpha_1\right]^2 = 0\,.$$

Thus the two associated characteristic exponents are

$$s_+ = \frac{m(N-m)}{2N}\alpha_1 \qquad\text{and}\qquad s_- = \frac{m(m-N)}{2N}\alpha_1 = -s_+\,,$$

hence the corresponding index is

$$\nu = \nu^N_{m,n} = s_+ - s_- = 2s_+ = \frac{m(N-m)}{N}\alpha_1 .$$

We now have everything in hand to get the result we were looking for.

Proposition 5.3. *Veech's $\mathbb{CH}^1$-structure on $F_{m,n}$ extends to a conifold complex hyperbolic structure at the cusp $[i\infty]_{m,n}$. The associated conifold angle is*

$$\vartheta_N\big([i\infty]_{m,n}\big) = 2\pi m\Big(1-\frac{m}{N}\Big)\alpha_1 .$$

In particular, $[i\infty]_{m,n}$ is a cusp for Veech hyperbolic structure on $F_{m,n}$ (that is, the associated conifold angle is equal to 0) if and only if $m=0$.

Proof. In view of the results and computations above, this is an immediate consequence of Proposition A.2.3. of Appendix A. □

Combining the preceding result with Proposition 5.2, one gets the

Corollary 5.4. *Veech's hyperbolic structure $\mathrm{hyp}_N^{\alpha_1}$ on $Y(N)^{\alpha_1} = F_N = F_{0,1}$ extends to a $\mathbb{CH}^1$-conifold structure on the modular compactification $X(N)^{\alpha_1}$.*

For any coprime $a', c' \in \mathbb{Z}$, the conifold angle at $\mathfrak{c} = [-a'/c'] \in C(N)$ is equal to

$$\vartheta_N\big(\mathfrak{c}\big) = \vartheta_N\big(c\big) = 2\pi c\Big(1-\frac{c}{N}\Big)\alpha_1$$

where c stands for the residue of c' modulo N: $c = c' - \left\lfloor \frac{c'}{N}\right\rfloor \in \big\{0,\ldots,N-1\big\}$.

5.3.4. We remind the reader of the following classical description of the cusps of $F_{0,1} = Y(N)$ (*cf.* [13, §3.8] for instance): the set of cusps $C(N)$ of $Y(N)$ is in bijection with the set of N-order points of the additive group $\big(\mathbb{Z}/N\mathbb{Z}\big)^2$, up to multiplication by -1, the bijection being given by

$$\begin{aligned}\big(\mathbb{Z}/N\mathbb{Z}\big)^2[N]_{/\pm} &\longrightarrow C(N)\\ \pm\big(a,c\big) &\longmapsto \big[-a'/c'\big]\end{aligned}$$

where a' and c' are pairwise prime and congruent to a and c modulo N respectively. From the preceding corollary, it comes that the conifold angle associated to the cusp corresponding to $\pm(a,c)$ with $c \in \{0,\ldots,N-1\}$ is

$$\vartheta_N\big(\pm(a,c)\big) = \vartheta_N\big(c\big) = 2\pi\frac{c(N-c)}{N}\alpha_1 .$$

Note that since $-(a,c) = (N-a, N-c)$ in $\big(\mathbb{Z}/N\mathbb{Z}\big)^2$ and because $\vartheta_N(c) = \vartheta_N(N-c)$, this formula makes sense.

Using the preceding corollary, it is then easy to describe the metric completion $\overline{Y(N)}$ of $Y(N) = F_{0,1}$, the latter being endowed with Veech's $\mathbb{CH}^1$-structure. From the preceding results, it comes that this metric completion is the union of the intermediary leaf $Y(N) = F_{0,1}$ with the subset of its cusps of the form $[-a'/c']$ with c' not a multiple of N. Such cusps correspond to classes $\pm(a,c) \subset (\mathbb{Z}/N\mathbb{Z})^2[N]$ with $c \in \{1,\ldots,N-1\}$. The cusps $[-a'/c']$ with $c' \in N\mathbb{Z}$ are cusps in the classical sense for the $\mathbb{CH}^1$-conifold $\overline{Y(N)}$. The number of these genuine cusps is then equal to $\phi(N)$[29].

At this point, it is easy to give an explicit description of the metric completion of the leaf $Y_1(N)^{\alpha_1} = \mathcal{F}_N^{\alpha_1}$ of Veech's foliation on $\mathcal{M}_{1,2}$:

Corollary 5.5. *Veech's $\mathbb{CH}^1$-structure on $Y_1(N)^{\alpha_1}$ extends to a conifold complex hyperbolic structure on the modular compactification $X_1(N)^{\alpha_1}$.*

For any coprime $a', c' \in \mathbb{Z}$, the conifold angle at $\mathfrak{c} = [-a'/c'] \in C_1(N)$ is equal to

$$\theta_N(\mathfrak{c}) = \theta_N(c) = 2\pi \frac{c(N-c)}{N\gcd(c,N)}\alpha_1 \tag{110}$$

where $c \in \{0,\ldots,N-1\}$ stands for the residue of c' modulo N.

Proof. This follows at once from (102), Lemma 5.1 and Corollary 5.4. □

To conclude this section, we would like to add two remarks.

First, it is to be understood that the preceding corollary completely characterizes Veech's complex hyperbolic structure of the leaf $Y_1(N)^{\alpha_1} = \mathcal{F}_N^{\alpha_1}$ since the latter is completely determined by the conformal structure of $\mathcal{F}_N^{\alpha_1}$ (which is the one of $Y_1(N)$) together with the cone angles at the conifold points, as it follows from a classical result (*cf.* Picard-Heins' theorem in Appendix A).

Since the conifold angles (110) depend linearly on α_1, the family of $Y_1(N)^{\alpha_1}$'s for $\alpha_1 \in [0,1[$ is a real-analytic deformation of the usual modular curve $Y_1(N)$, if one sees it as $Y_1(N)^0$ (as it is natural to do so).

Finally, note that the preceding results refine and generalize (the specialization to the case when $g = 1$ and $n = 2$ of) the main result of [19]. In this article, we prove that when α is rational, Veech's hyperbolic structure of an algebraic leaf of Veech's foliation extends as a conifold structure of the same type to the metric completion of this leaf. Not only our results above give a more precise and explicit version of this result in the case under scrutiny but they also show that the same statements hold true even without assuming that α is rational. This assumption appears to be crucial to make effective the geometric methods à la Thurston used in [19].

[29] Here ϕ stands for Euler's totient function.

5.4. **Some explicit examples.** To illustrate the results obtained above, we first treat explicitly the cases of $Y_1(N)$ for $N = 2,3,4$. These are related to 'classical hypergeometry' and this will be investigated further in Section 6.2. Then we consider the case of $Y_1(5)$ then eventually the one of $Y_1(p)$ for p an arbitrary prime number bigger than or equal to 5.

In what follows, the parameter $\alpha_1 \in]0,1[$ is fixed once and for all. A useful reference for the elementary results on modular curves used below is [13], especially sections §3.7 and §3.8 therein.

5.4.1. **The case of $Y_1(2)$.** The congruence subgroup $\Gamma_1(2)$ has two cusps (namely $[i\infty]$ and $[0]$) and one elliptic point of order 2. Consequently, $Y_1(2)^{\alpha_1}$ is a genuine orbi-leaf of Veech (orbi-)foliation on $\mathcal{M}_{1,2}$. It is the Riemann sphere punctured at two points, say 0 and ∞, with one orbifold point of weight 2, say at 1. The corresponding conifold angles of the associated Veech's $\mathbb{CH}^1$-structure are given in Table 1 below in which the cusps are seen as points of $X_1(2) = \mathbb{P}^1$:

Cusps of $Y_1(2)^{\alpha_1}$	0	1	∞
Conifold angles	$\pi\alpha_1$	π	0

TABLE 1. The cusps and the associated conifold angles of $Y_1(2)^{\alpha_1}$.

5.4.2. **The case of $Y_1(3)$.** This case is very similar to the preceding one: $\Gamma_1(3)$ has two cusps and one elliptic point, but of order 3. Hence $Y_1(3)^{\alpha_1}$ is an orbi-leaf of Veech's (orbi-)foliation. It is $\mathbb{P}^1$ punctured at three points, say 0, 1 and ∞, with one orbifold point of weight 3, say at 1. The corresponding conifold angles are given in the table below.

Cusps of $Y_1(3)^{\alpha_1}$	0	1	∞
Conifold angles	$(4\pi/3)\alpha_1$	$2\pi/3$	0

TABLE 2. The cusps and the associated conifold angles of $Y_1(3)^{\alpha_1}$.

5.4.3. **The case of $Y_1(4)$.** The group $\Gamma_1(4)$ has three cusps and no elliptic point. Hence $Y_1(4)^{\alpha_1}$ is the trice-punctured sphere $\mathbb{P}^1 \setminus \{0,1,\infty\}$. The corresponding conifold angles are given in the table below.

Cusps of $Y_1(4)^{\alpha_1}$	0	1	∞
Conifold angles	$(3\pi/2)\alpha_1$	$\pi\alpha_1$	0

TABLE 3. The cusps and the associated conifold angles of $Y_1(4)^{\alpha_1}$.

The three cases considered above are very particular since they are the only algebraic leaves of Veech's foliation on $\mathcal{M}_{1,2}$ which are isomorphic to the thrice-punctured sphere, hence whose hyperbolic structure can be uniformized by means of Gauß hypergeometric functions. We will return to this later on.

5.4.4. **The case of $Y_1(5)$.** The modular curve $Y(5)$ is of genus 0 and has twelve cusps. Some representatives of these cusps are given in the first row of Table 4 below, the associated conifold angles of $Y(5)^{\alpha_1} = F_{0,1}^{\alpha_1}$ (we use here the notation of §5.2) are given in the second row.

Cusp $\mathfrak{c}$	$i\infty$	0	1	-1	2	-2	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{3}{2}$	$-\frac{3}{2}$	$-\frac{5}{2}$	$-\frac{2}{5}$
$\frac{\vartheta_5(\mathfrak{c})}{2\pi\alpha_1}$	0	$\frac{4}{5}$	$\frac{4}{5}$	$\frac{4}{5}$	$\frac{4}{5}$	$\frac{4}{5}$	$\frac{6}{5}$	$\frac{6}{5}$	$\frac{6}{5}$	$\frac{6}{5}$	$\frac{6}{5}$	0

TABLE 4. The cusps and the associated conifold angles of $Y(5)^{\alpha_1}$.

Since $\Gamma_1(5)$ is generated by adjoining $\tau \mapsto \tau+1$ to $\Gamma(5)$, it comes that among those of Table 4, 0, 1/2, 2/5 and $i\infty$ form a complete set of representatives of the cusps of $\Gamma_1(5)$. Moreover, it can be verified that the quotient map

$$X(5) \longrightarrow X_1(5)\,,$$

which is a ramified covering of degree 5, ramifies at order 5 at the two cusps $[2/5]_{X(5)}$ and $[i\infty]_{X(5)}$ and is étale at the ten others cusps of $X(5)$. It follows that the conifold angles of Veech's complex hyperbolic structure on $Y_1(5)^{\alpha_1}$ are 0, 0, $\frac{12}{5}\pi\alpha_1$ and $\frac{8}{5}\pi\alpha_1$ at $[i\infty]_{X_1(5)}$, $[2/5]_{X_1(5)}$, $[1/2]_{X_1(5)}$ and $[0]_{X_1(5)}$ respectively.

The map $\mathbb{H}^\star \to \mathbb{P}^1$, $\tau \mapsto q^{-1}\prod_{n\geq 1}(1-q^n)^{-5\left(\frac{n}{5}\right)}$ (with $q=e^{2i\pi\tau}$ and where $\left(\cdot\right)$ stands for Legendre's symbol) is known to be a Hauptmodul for $\Gamma_1(5)$ which sends $i\infty$, 2/5, 0 and 1/2 onto 0, ∞, $(11-5\sqrt{5})/2$ and $(11+5\sqrt{5})/2$ respectively.

Cusp in $\mathbb{H}^\star$	$i\infty$	0	$\frac{1}{2}$	$\frac{2}{5}$
Cusp in $X_1(5)\simeq\mathbb{P}^1$	0	$\frac{11-5\sqrt{5}}{2}$	$\frac{11+5\sqrt{5}}{2}$	∞
Conifold angle	0	$\frac{8}{5}\pi\alpha_1$	$\frac{12}{5}\pi\alpha_1$	0

TABLE 5. The cusps and the associated conifold angles of $Y_1(5)^{\alpha_1}$.

5.4.5. **The case of $Y_1(p)$ with p prime.** Now let p be a prime bigger than 3.

It is known that $\Gamma_1(p)$ has genus $\frac{1}{24}(p-5)(p-7)$, no elliptic point and $p-1$ cusps, among which $(p-1)/2$ have width 1, the $(p-1)/2$ other ones having width p. Combining the formalism of Section 5.2 with Proposition 5.3, one easily verifies that the two following assertions hold true:

- the cusps of width 1 of $Y_1(p)^{\alpha_1} = \mathcal{F}_{0,1}$ correspond to the cusps $[i\infty]_{0,k}$ of the leaves $\mathcal{F}_{0,k}$ (associated to the equation $z_2 = k/p$ in $\mathscr{T}or_{1,2}$) for $k = 1,\dots,(p-1)/2$, thus the associated conifold angles are all 0;
- the cusps of width p correspond to the cusps $[i\infty]_{k,0}$ of the leaves $\mathcal{F}_{k,0}$ (associated to the equation $z_2 = k\tau/p$ in the Torelli space) for $k = 1,\dots,(p-1)/2$. For any such k, the associated cusp is $[-p/k]$ and the associated conifold angle is $2\pi k(1-k/p)\alpha_1$.

6. **Miscellanea**

6.1. **Examples of explicit degenerations towards flat spheres with three conical singularities.** The main question investigated above is that of the metric completion of the closed (actually algebraic) leaves of Veech's foliation on $\mathcal{M}_{1,2}$.

In [19], under the supplementary hypothesis that α is rational (but then in arbitrary dimension), the same question has been investigated by geometrical methods. In the particular case when Veech's foliation $\mathcal{F}^{\alpha_1}$ is of (complex) dimension 1, our results in [19] show that, in terms of (equivalence classes of) flat surfaces, the metric completion of an algebraic leaf $\mathcal{F}_N^{\alpha_1}$ in $\mathcal{M}_{1,2}$ is obtained by attaching to it a finite number of points which correspond to flat spheres with three conical singularities, whose associated cone angles can be determined by geometric arguments.

Using some formulae of [47], one can recover the result just mentioned but in an explicit analytic form. We treat succinctly below the case of the cusp $[i\infty]$ of the leaf $\mathcal{F}_{(1/N,0)} \simeq Y_1(N)$ associated to the equation $z_2 = \tau/N$ in $\mathscr{T}or_{1,2}$, for any integer N bigger than or equal to 2. Details are left to the reader in this particular case as well as in the general case (*i.e.* at any other cusp of $Y_1(N)$).

Let α_1 be fixed in $]0,1[$. We consider the flat metric $m_\tau = m_\tau^{\alpha_1} = |\omega_\tau|^2$ on $E_{\tau,\tau/N} = E_\tau \setminus \{[0],[\tau/N]\}$ with conical singularities at $[0]$ and at $[\tau/N]$ where for any $\tau \in \mathbb{H}$, ω_τ stands for the following (multivalued) 1-form on $E_{\tau,\tau/N}$:

$$\omega_\tau(u) = e^{\frac{2i\pi\alpha_1}{N}u}\left[\frac{\theta(u)}{\theta\big(u-\tau/N\big)}\right]^{\alpha_1} du.$$

The map which associates the class of the flat tori $(E_{\tau,\tau/N}, m_\tau)$ in $\mathcal{M}_{1,2}^{\alpha}$ to any τ in Poincaré's upper half-plane uniformizes the leaf $\mathcal{F}_{(1/N,0)}$ of Veech's foliation. Studying the latter in the vicinity of the cusp $[i\infty]$ is equivalent to studying the $(E_{\tau,\tau/N}, m_\tau)$'s when τ goes to $i\infty$ in a vertical band of width 1 of $\mathbb{H}$.

First, we perform the change of variables $u-\tau/N = -v$. Then up to a nonzero constant which does not depend on v, we have

$$\omega_\tau(v) = e^{-\frac{2i\pi\alpha_1}{N}v}\left[\frac{\theta\big(-v+\tau/N\big)}{\theta(v)}\right]^{\alpha_1} dv.$$

We want to look at the degeneration of m_τ when $\tau \to i\infty \in \partial\mathbb{H}$. To this end, one sets $q = \exp(2i\pi\tau)$. Then using the natural isomorphism $E_\tau = \mathbb{C}^*/q^{\mathbb{Z}}$ induced by $v \mapsto x = \exp(2i\pi v)$, one sees that E_τ converges towards the degenerated elliptic curve $\mathbb{C}^*/0^{\mathbb{Z}} = \mathbb{C}^*$ as τ goes to $i\infty$. Moreover, for any fixed $\tau \in \mathbb{H}$, since $dv = (2i\pi x)^{-1}dx$, the 1-form ω_τ writes as follows in the variable x:

$$\omega_\tau(x) = (2i\pi)^{-1} x^{-\frac{\alpha_1}{N}-1} \theta\big(\tau/N - v\big)^{\alpha_1} \theta(v)^{-\alpha_1} dx.$$

Then, from the classical formula

$$\theta(v,\tau) = 2\sin(\pi v)\cdot q^{1/8} \prod_{n=1}^{+\infty} \big(1-q^n\big)\big(1-xq^n\big)\big(1-x^{-1}q^n\big)$$

it follows that

$$\frac{\theta(\tau/N - v)}{\theta(v)} = \frac{\sin(\pi\tau/N - \pi v)}{\sin(\pi v)} \cdot \Theta_N(x,q)$$

with

$$\Theta_N(x,q) = \prod_{n=1}^{+\infty} \frac{\big(1-x^{-1}q^{n+1/N}\big)\big(1-xq^{n-1/N}\big)}{\big(1-xq^n\big)\big(1-x^{-1}q^n\big)}.$$

An important fact concerning the latter function is that as a function of the variable x, $\Theta_N(\cdot,q)$ tends uniformly towards 1 on any compact set as $q \to 0$, that is as τ goes to $i\infty$, for τ varying in any fixed vertical band.

On the other hand, we have

$$\begin{aligned}\frac{\sin(\pi\tau/N - \pi v)}{\sin(\pi v)} &= \frac{e^{i\pi(\tau/N - v)} - e^{-i\pi(\tau/N - v)}}{e^{i\pi v} - e^{-i\pi v}}\\ &= \frac{q^{\frac{1}{2N}} x^{-1/2} - q^{-\frac{1}{2N}} x^{1/2}}{x^{1/2} - x^{-1/2}} = q^{-\frac{1}{2N}} \frac{q^{\frac{1}{N}} - x}{x-1}\end{aligned}$$

hence, up to multiplication by a nonzero constant that does not depend on x and 'up to multi-valuedness', one has

$$\omega_\tau(x) = x^{\frac{-\alpha_1}{N}-1} \left[\frac{x - q^{1/N}}{x-1}\right]^{\alpha_1} \Theta_N(x,q)^{\alpha_1} dx.$$

For $\tau \to i\infty$, one obtains as limit the following multivalued 1-form

$$\omega_{i\infty}(x) = x^{\alpha_1 \frac{N-1}{N} - 1} \big(x-1\big)^{-\alpha_1} dx \tag{111}$$

on $\mathbb{P}^1 \setminus \{0,1,\infty\}$. The associated flat metric $m_{i\infty} = |\omega_{i\infty}|^2$ defines a singular flat structure of bounded area on $\mathbb{P}^1$, with 3 conical singular points at $0, 1$ and ∞, the conical angles of which are respectively

$$\theta_0 = 2\pi\left(1 - \frac{1}{N}\right)\alpha_1, \qquad \theta_1 = 2\pi(1-\alpha_1) \qquad \text{and} \qquad \theta_\infty = \frac{2\pi}{N}\alpha_1. \tag{112}$$

We verify that there exists a positive function $\lambda(\tau)$ such that

$$\lim_{\tau\to i\infty}\int_{E_\tau}\lambda(\tau)|\omega_\tau|^2=\int_{\mathbb{P}^1}|\omega_{i\infty}|^2>0.$$

This shows that the $\mathbb{CH}^1$-structure of $\mathcal{F}_{(1/N,0)}$ is not complete at the cusp $[i\infty]$ and that the (equivalence class of the) flat sphere with three conical singularities of angles as in (112) belongs to the metric completion of the considered leaf in the vicinity of this cusp. When α_1 is assumed to be in $\mathbb{Q}$, the metric completion at this cusp is obtained by adding this flat sphere and nothing else, as it is proved in [19]. The previous analytical considerations show in an explicit manner that this still holds true even without assuming α_1 to be rational.

6.2. **The cases when N is small: relations with classical special functions.** Exactly three of the leaves of Veech's foliation on $\mathcal{M}_{1,2}$ give rise to a hyperbolic conifold of genus 0 with 3 conifold points: the leaves $\mathcal{F}_N = Y_1(N)^{\alpha_1}$ for $N = 2,3,4$, see §5.4 above. Since any such hyperbolic conifold can be uniformized by a classical hypergeometric differential equation (as it is known from the fundamental work of Schwarz recalled in the Introduction), there must be some formulae expressing the Veech map of any one of these three leaves in terms of classical hypergeometric functions. We consider only the case when $N = 2$ in §6.2.1.

Since both $Y_1(5)$ and $Y_1(6)$ are $\mathbb{P}^1$ with four cusps, the leaves $\mathcal{F}_5$ and $\mathcal{F}_6$ correspond to $\mathbb{CH}^1$-conifold structures with four cone points on the sphere. Since any such structure can be uniformized by means of a Heun's differential equation, one deduces that the Veech map of these two leaves can be expressed in terms of Heun functions. We just say a few words about this in §6.2.3.

6.2.1. **The leaf $Y_1(2)^{\alpha_1}$ and classical hypergeometric functions.** It turns out that the case when $N = 2$ can be handled very explicitly by specializing a classical result of Wirtinger. The modular lambda function $\lambda:\tau\mapsto\theta_1(\tau)^4/\theta_3(\tau)^4$ is a Hauptmodul for $\Gamma(2)$: it corresponds to the quotient $\mathbb{H}\to Y(2)=\mathbb{P}^1\setminus\{0,1,\infty\}$. Furthermore, it induces a correspondence between the cusps $[i\infty]$, $[0]$, $[1]$ and the points $0,1$ and ∞ of $X(2)=\mathbb{P}^1$ respectively (*cf.* [8, Chap.VII.§8] if needed).

6.2.1.1. Specializing a formula obtained by Wirtinger in [82] (see also (1.3) in [77] with $\alpha=(\alpha_1+1)/2, \beta=1/2$ and $\gamma=1$), one obtains the following formula:
(113)

$$F\left(\frac{\alpha_1+1}{2},\frac{1}{2},1;\lambda(\tau)\right)=\frac{2\cos\left(\frac{\pi\alpha_1}{2}\right)\theta_3(\tau)^2\big(1-\lambda(\tau)\big)^{-\frac{\alpha_1}{4}}}{(1-e^{2i\pi\alpha_1})(1-e^{-2i\pi\alpha_1})}\int_{\mathscr{P}(0,\frac{1}{2})}\frac{\theta(u,\tau)^{\alpha_1}}{\theta_1(u,\tau)^{\alpha_1}}du$$

where for any $\tau\in\mathbb{H}$, the integration in the left hand-side is performed along the Pochammer cycle $\mathscr{P}(0,1/2)$ constructed from the segment $[0,1/2]$ in E_τ (see Figure 12 below).

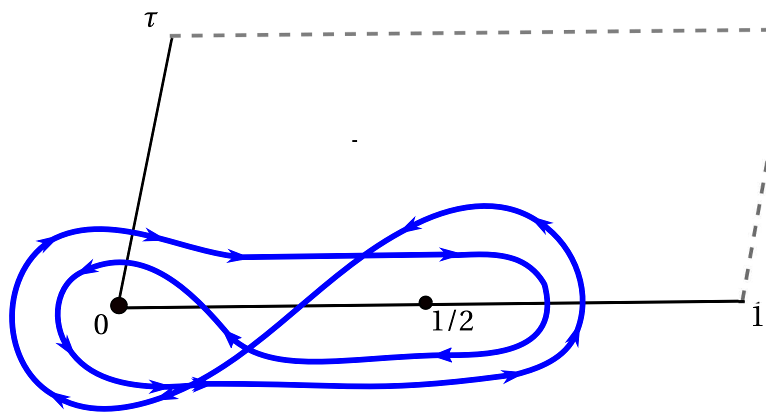


FIGURE 12. The Pochammer contour $\mathscr{P}(0,1/2)$ in E_τ (in blue)

For every τ, the flat metric on E_τ with conical points at [0] and [1/2] associated to the corresponding points of the leaf of equation $z_2 = 1/2$ in $\mathscr{T}or_{1,2}$ is given by $|T^{\alpha_1}(u,\tau)du|^2$ up to normalization, with

$$T^{\alpha_1}(u,\tau) = \big(-\theta(u)/\theta(u-1/2)\big)^{\alpha_1} = \big(\theta(u)/\theta_1(u)\big)^{\alpha_1}.$$

Equation (113) can be written out

$$F\left(\frac{\alpha_1+1}{2},\frac{1}{2},1;\lambda(\tau)\right) = \Lambda_1^{\alpha}(\tau)\int_{\boldsymbol{\gamma}_2} T^{\alpha_1}(u,\tau)du \tag{114}$$

where $\Lambda_1^{\alpha}(\tau)$ is a function of τ and α_1 (easy to make explicit with the help of (113)) and where $\boldsymbol{\gamma}_2$ stands for the element of the twisted homology group $H_1(E_{\tau,1/2}, L_{\tau,1/2})$ obtained after regularizing the twisted 1-simplex $\boldsymbol{\ell}_2$ defined in §3.2.3.

More generally, for any τ and any twisted cycle $\boldsymbol{\gamma}$, there is a formula

$$F_{\boldsymbol{\gamma}}\left(\frac{\alpha_1+1}{2},\frac{1}{2},1;\lambda(\tau)\right) = \Lambda^{\alpha_1}(\tau)\int_{\boldsymbol{\gamma}} T^{\alpha_1}(u,\tau)du \tag{115}$$

where $F_{\boldsymbol{\gamma}}((\alpha_1+1)/2,1/2,1;\cdot)$ is a solution of the hypergeometric differential equation (2) for the corresponding parameters. An important point is that the function $\Lambda_1^{\alpha}(\tau)$ in such a formula is independent of the considered twisted cycle. It follows that the map

$$\tau \longmapsto \left[F\left(\frac{\alpha_1+1}{2},\frac{1}{2},1;\lambda(\tau)\right) : \frac{d}{d\varepsilon}x^{\varepsilon}F\left(\frac{\alpha_1+1}{2}+\varepsilon,\frac{1}{2}+\varepsilon,1+\varepsilon;\lambda(\tau)\right)\Big|_{\varepsilon=0}\right]$$

(whose components form a basis of the associated hypergeometric differential equation, see [85, Chap.III§3]) is nothing else than an expression of the

Veech map $V^{\alpha_1}_{0,1/2} : \mathbb{H} \simeq \mathscr{F}^{\alpha_1}_2 \to \mathbb{P}^1$ in terms of classical hypergeometric functions. As an immediate consequence, one gets that the corresponding conifold structure on $\mathbb{P}^1$ is given by the 'classical hypergeometric Schwarz's map' $S((\alpha_1+1)/2, 1/2, 1; \cdot)$. It follows that the conical angles at the cusps $0, 1$ and ∞ of $X(2) = \mathbb{P}^1$ are respectively 0, $\pi\alpha_1$ and $\pi\alpha_1$.

Note that this is consistent with our results in §5.4.1: the lambda modular function satisfies $\lambda(\tau+1) = \lambda(\tau)/(\lambda(\tau)-1)$ for every $\tau \in \mathbb{H}$ (*cf.* [8]). Thus $\mu = \mu(\lambda) = 4(\lambda-1)/\lambda^2$ is invariant by $\Gamma(2)$ and by $\tau \mapsto \tau+1$, hence is a Hauptmodul for $\Gamma_1(2)$. Veech's hyperbolic conifold structure on $X_1(2)$ is the push-forward by μ of the one just considered on $X(2)$. Moreover, μ is étale at $0, 1$ and ∞, ramifies at the order 2 at $\lambda = 2$ and one has $\mu(0) = \infty$, $\mu(1) = \mu(\infty) = 0$ and $\mu(2) = 1$. It follows that the conifold angles at the cusps 0, 1 and ∞ of $\mathcal{F}^{\alpha_1}_2 = X_1(2)^{\alpha_1} \simeq \mathbb{P}^1$ are $\pi\alpha_1$, π and 0 in perfect accordance with the results given in Table 1.

6.2.1.2. Actually, there is a slightly less explicit but much more geometric approach of the $N = 2$ case. Indeed, for every $\tau \in \mathbb{H}$, the flat metric $m^{\alpha_1}_\tau$ on E_τ with conical points at $[0]$ and $[1/2]$ of respective angles $2\pi(\alpha_1+1)$ and $2\pi(\alpha_1-1)$ is invariant by the elliptic involution (the metric $|T^{\alpha_1}(u,\tau)du|^2$ is easily seen to be invariant by $u \mapsto -u$). Consequently, $m^{\alpha_1}_\tau$ can be pushed-forward by the quotient map $v : E_\tau \to E_\tau/\iota \simeq \mathbb{P}^1$ and gives rise to a flat metric on the Riemann sphere. In the variable u, for the map v, it is convenient to take the map induced by

$$u \longmapsto \frac{\wp(1/2) - \wp(\tau/2)}{\wp(u) - \wp(\tau/2)}\,.$$

Since v ramifies at the second order exactly at the 2-torsion points of E_τ, it follows that the push-forward metric $v_*(m^{\alpha_1}_\tau)$ is 'the' flat metric on $\mathbb{P}^1$ with four conical points at $0, 1, \infty$ and $\lambda(\tau)^{-1} = v((1+\tau)/2)$ whose associated cone angles are respectively $\pi(1+\alpha_1)$, $\pi(1-\alpha_1)$, π and π.

Consequently, in the usual affine coordinate x on $\mathbb{P}^1$, one has

$$v_*\big(m^{\alpha_1}_\tau\big) = \epsilon^{\alpha_1}(\tau)\left| x^{\frac{\alpha_1-1}{2}}(1-x)^{-\frac{\alpha_1+1}{2}}\big(1-\lambda(\tau)x\big)^{-\frac{1}{2}}dx\right|^2 \tag{116}$$

for some positive function ϵ^{α_1} which does not depend on x but only on τ. From (116), one deduces immediately that a formula such as (115) holds true for any twisted cycle $\boldsymbol{\gamma}$ on E_τ. Then one can conclude in the same way as at the end of the preceding paragraph.

6.2.2. **About the $N = 3$ case.** The hyperbolic conifold $Y_1(3)^{\alpha_1}$ is $\mathbb{P}^1$ with three cone points whose conifold angles are $2\pi(2\alpha_1/3), 2\pi/3$ and 0. This $\mathbb{CH}^1$-structure is induced by the hypergeometric equation (2) with $a = (1+\alpha_1)/3$, $b = (1-\alpha_1)/3$

and $c = 1$. On the other hand, the Veech map of $\mathscr{F}_3^{\alpha_1} \simeq \mathbb{H}$ admits as its components the hypergeometric integrals $\int_{\gamma_\bullet} \theta(u)^{\alpha_1}\theta(u-1/3)^{-\alpha_1}du$ with $\bullet = 0, \infty$.

Since $\delta(\tau) = (\eta(\tau)/\eta(3\tau))^{12}$ is a Hauptmodul for $\Gamma_1(3)$ (see case 3B in Table 3 of [9]), it comes that there exists a function $\Delta^{\alpha_1}(\tau)$ depending only on α_1 and on τ, as well as a twisted cycle $\boldsymbol{\beta}$ on E_τ such that a formula of the form

$$F\left(\frac{1+\alpha_1}{3}, \frac{1-\alpha_1}{3}, 1; \delta(\tau)\right) = \Delta^{\alpha_1}(\tau) \int_{\boldsymbol{\beta}} \frac{\theta(u,\tau)^{\alpha_1}}{\theta\big(u-\frac{1}{3},\tau\big)^{\alpha_1}} du \tag{117}$$

holds true for every $\tau \in \mathbb{H}$ and every $\alpha_1 \in]0,1[$ (compare with (113)).

It would be nice to give explicit formulae for Δ^{α_1} and $\boldsymbol{\beta}$. Note that a similar question can be asked in the case when $N = 4$.

6.2.3. **A few words about the case when $N = 5$.** Since $Y_1(5)^{\alpha_1}$ is a four punctured sphere, its $\mathbb{CH}^1$-structure can be recovered by means of the famous Heun equation $\mathrm{Heun}(c, \theta_1, \theta_2, \theta_3, \theta_4, p)$. As is well-known, it is a Fuchsian second-order linear differential equation with four simple poles on $\mathbb{P}^1$. It depends on 6 parameters: the first, c, is the cross-ratio of the four singularities; the next 4 parameters $\theta_1, \ldots, \theta_4$ are the angles corresponding to the exponents of the considered equation at the singular points; finally, p is the so-called *'accessory parameter'* which is the most mysterious one.

In the case of $Y_1(5)^{\alpha_1}$, c is equal to $\omega = (11-5\sqrt{5})/(11+5\sqrt{5})$, hence only depends on the conformal type of $Y_1(5)$. The angles θ_i's are precisely the conifold angles $\theta_i^{\alpha_1}$ of Veech's hyperbolic structure on $Y_1(5)$ (*cf.* Table 5).

It would be interesting to find an expression for the accessory parameter p^{α_1} of the Heun equation associated to $Y_1(5)^{\alpha_1}$ in terms of α_1. Indeed, in this case it might be possible to express the Schwarz map associated to any Heun equation of the form

$$\mathrm{Heun}\big(\omega, \theta_1^{\alpha_1}, \theta_2^{\alpha_1}, \theta_3^{\alpha_1}, \theta_4^{\alpha_1}, p^{\alpha_1}\big)$$

as the ratio of two elliptic hypergeometric integrals. Since the monodromy of such integrals can be explicitly determined (cf. §6.3 below), this could be a way to determine explicitly the monodromy of a new class of Heun equations.

To conclude, note that this approach should also work when $N = 6$ since $Y_1(6)$ is also of genus 0 with four cusps.

6.3. **Holonomy of the algebraic leaves.** We fix an integer $N \geq 2$.

6.3.1. By definition, for $\alpha_1 \in]0,1[$, the **holonomy** of the leaf $Y_1(N)^{\alpha_1}$ is the holonomy of the complex hyperbolic structure it carries. It is a morphism of groups (well defined up to conjugation) which will be denoted by

$$\mathrm{H}_N^{\alpha_1} : \Gamma_1(N) \simeq \pi_1\big(Y_1(N)^{\alpha_1}\big) \longrightarrow \mathrm{PSL}_2(\mathbb{R}) \simeq \mathrm{PU}(1,1)\,. \tag{118}$$ [30]

Its image will be denoted by

$$\mathbf{\Gamma}_1(N)^{\alpha_1} = \mathrm{Im}\big(\mathrm{H}_N^{\alpha_1}\big) \subset \mathrm{PSL}_2(\mathbb{R})\,.$$

and will be called the **holonomy group of $\boldsymbol{Y_1(N)^{\alpha_1}}$**.

It follows from some results in [48] that the $\mathbb{CH}^1$-structure of $Y_1(N)^{\alpha_1}$ is induced by a Fuchsian second-order differential equation[31]. This directly links our work to very classical ones about the monodromy of Fuchsian differential equations. In our situation, the general problem considered by Poincaré at the very beginning of [62] is twofold and can be stated as follows:

(**P1**) for α_1 and N given, determine the holonomy group $\mathbf{\Gamma}_1(N)^{\alpha_1}$;

(**P2**) find all the parameters α_1 and N such that $\mathbf{\Gamma}_1(N)^{\alpha_1}$ is Fuchsian.

To these two problems, we would like to add a third one, namely

(**P3**) among the parameters α_1 and N such that $\mathbf{\Gamma}_1(N)^{\alpha_1}$ is Fuchsian, determine the ones for which this group is arithmetic.

6.3.2. We say a few words about the reason why we believe that the three problems (P1), (P2) and (P3) are important. For this purpose, we recall briefly below the general strategy followed by Deligne and Mostow in [11, 54] and by Thurston in [71] to find new non-arithmetic lattices in $\mathrm{PU}(1,n-1)$, which is nowadays one of the main problems in complex hyperbolic geometry.

For any manifold M carrying a $\mathbb{CH}^n$-structure, one denotes by $\mathbf{\Gamma}(M)$ its holonomy group, namely the image of the associated holonomy representation $\pi_1(M) \to \mathrm{PU}(1,n-1)$.

[30] Since $Y_1(N)$ has orbifold points when $N = 2,3$, it is necessary to consider instead the orbifold fundamental group $\pi_1(Y_1(N)^{\alpha_1})^{\mathrm{orb}}$ in these two cases. We will keep this in mind in what follows and will commit the abuse to speak always of the usual fundamental group.

[31] To be precise, the differential equation considered in Theorem 3.1 of [48] is defined on the cover $Y(N)$ of $Y_1(N)$ but it is easily seen that it can be pushed forward onto $Y_1(N)$.

6.3.2.1. For $n \geq 4$, let $\theta = (\theta_i)_{i=1}^n$ be a n-uplet of angles $\theta_i \in]0, 2\pi[$. The moduli space $\mathcal{M}_{0,\theta}$ of flat spheres with n conical points of angles $\theta_1, \ldots, \theta_n$ identifies to $\mathcal{M}_{0,n}$ and moreover carries a natural complex hyperbolic structure (*cf.* §1.1.3).

For some θ's, which have been completely determined, the associated holonomy group $\mathbf{\Gamma}_\theta = \mathbf{\Gamma}(\mathcal{M}_{0,\theta})$ is a lattice $\mathrm{PU}(1, n-3)$ and some of them are not arithmetic. This has been obtained via the following approach: the metric completion of $\mathcal{M}_{0,\theta}$ is obtained by adding to it several strata that are themselves moduli spaces $\mathcal{M}_{0,\theta'}$ for some angle-data θ' deduced explicitly from θ. Furthermore, the discreteness of the holonomy is also hereditary: if $\mathbf{\Gamma}_\theta$ is discrete, then all the $\mathbf{\Gamma}_{\theta'}$'s corresponding to the θ''s associated to the strata appearing in the metric completion of $\mathcal{M}_{0,\theta}$ must be discrete as well. One ends up with the case when $n = 4$: in this situation, the corresponding holonomy groups $\mathbf{\Gamma}_{\theta''}$'s are triangle subgroups of $\mathrm{PSL}_2(\mathbb{R})$ and the θ'''s corresponding to discrete subgroups are known. This allows to find an explicit finite list of original θ's for which $\mathbf{\Gamma}_\theta$ may be discrete. At this point, there is still some work to do to verify that these angle-data give indeed complex hyperbolic lattices and to determine the arithmetic ones but this was achieved in [54].

6.3.2.2. Our results proven in [19] show that a very similar picture occurs for the metric completion of an algebraic leaf of Veech's foliation in $\mathcal{M}_{1,n}$ for any $n \geq 2$. Hence a strategy similar to the one outlined above is possible when looking at algebraic leaves with discrete holonomy group in $\mathrm{PU}(1, n-1)$.

Let $\mathcal{F}_N^\alpha$ be an algebraic leaf of Veech's foliation on $\mathcal{M}_{1,n}$. As already mentioned in §4.2.6, it is proven in [19] that the metric completion of $\mathcal{F}_N^\alpha$ can be inductively constructed by adjoining strata $\mathcal{S}'$ which are covers of moduli spaces of flat surfaces $\mathcal{S}$. These moduli spaces can be of two different kinds: either $\mathcal{S}$ is an algebraic leaf of Veech's foliation $\mathcal{F}^{\alpha'}$ of $\mathcal{M}_{1,n'}$ with $n' < n$, for some particular n'-uplet α' constructed from N and α; or $\mathcal{S}$ is a moduli space of flat spheres $\mathcal{M}_{0,\tilde{\theta}}$ for a $\tilde{n}$-uplet $\tilde{\theta}$ (with $\tilde{n} \leq n+1$) which also depends only on N and α.

In this situation, the property of having a discrete holonomy group happens to be hereditary as well. Consequently, a necessary condition for $\mathbf{\Gamma}(\mathcal{F}_N^\alpha)$ to be discrete is that $\mathbf{\Gamma}(\mathcal{S}') = \mathbf{\Gamma}(\mathcal{S})$ be discrete as well, this for any stratum $\mathcal{S}'$ appearing in the metric completion of $\mathcal{F}_N^\alpha$. Since all the genus 0 holonomy groups $\mathbf{\Gamma}_\theta$ which are discrete are known, we only have to consider the groups $\mathbf{\Gamma}(\mathcal{S})$ for the genus 1 strata $\mathcal{S}$. Arguing inductively, one ends up, as in the genus 0 case, to considering the holonomy groups of the 1-dimensional strata of $\overline{\mathcal{F}_N^\alpha}$ which are of genus 1.

The preceding discussion shows that in order that a strategy similar to the one used in the genus 0 case succeeds in the genus 1 case that we are considering in this paper, it is crucial to have a perfect understanding of the holonomy groups of the algebraic leaves of Veech's foliation on $\mathcal{M}_{1,2}$. From this perspective, the three questions (P1),(P2) and (P3) appear to be particularly relevant.

6.3.3. It is easy to deduce from the results obtained above a vast class of parameters α_1 for which $\mathbf{\Gamma}_1(N)^{\alpha_1}$ is a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$.

6.3.3.1. It follows from Poincaré's uniformization theorem that the holonomy group $\mathbf{\Gamma}_1(N)^{\alpha_1}$ is Fuchsian as soon as $Y_1(N)^{\alpha_1}$ is an orbifold, that is as soon as for any cusp $\mathfrak{c}$ of $Y_1(N)^{\alpha_1}$, the associated conifold angle $\theta_N^{\alpha_1}(\mathfrak{c})$ is an integral part of 2π.

Now it has been shown above (see (110)) that for such a cusp $\mathfrak{c}$, one has

$$\theta_N^{\alpha_1}(\mathfrak{c}) = 2\pi \frac{c(c-N)}{N\,\mathrm{gcd}(c,N)}\alpha_1$$

for a certain integer $c \in \{0,\dots,N-1\}$ depending on $\mathfrak{c}$. Then, setting N' as the least common multiple of the integers $\frac{c(N-c)}{\mathrm{gcd}(c,N)}$ for $c = 1,\dots,N-1$, we get the

Corollary 6.1. *If $\alpha_1 = \frac{N}{N'\ell}$ with $\ell \in \mathbb{N}_{>0}$, then $\mathbf{\Gamma}_1(N)^{\alpha_1}$ is Fuchsian.*

6.3.3.2. It is more than likely that the preceding result only gives a partial answer to (P2). Indeed, it is well-known that there exist triangle subgroups of $\mathrm{PSL}_2(\mathbb{R})$ which are not of orbifold type (*i.e.* not all the angles of 'the' corresponding hyperbolic triangle are integral parts of π) but such that the $\mathbb{CH}^1$-holonomy of the associated $\mathbb{P}^1$ with three conifold points is Fuchsian, see [29, p. 572], [39, Theorem 2.3] or [54, Theorem 3.7][32].

The situation is certainly similar for the holonomy groups $\mathbf{\Gamma}_1(N)^{\alpha_1}$: for N fixed, there are certainly more parameters α_1 whose associated holonomy group is discrete than the ones given by Corollary 6.1 which correspond to the cases when Veech's $\mathbb{CH}^1$-structure on $X_1(N)^{\alpha_1}$ actually is of orbifold type.

A complete answer to (P2) would be very interesting but, for the moment, we do not see how this problem can be attacked in full generality. A difficulty inherent to this problem is that there is no known explicit finite type representation of $\Gamma_1(N)$ as a group for N arbitrary, except when $N = p$ for a prime number p and, even in this case, the known set of generators of $\Gamma_1(p)$ is quite complicated, see [18][33]. Note that this is in sharp contrast with the corresponding situation in the genus 0 case, where the ambient space is always $\mathcal{M}_{0,4} \simeq \mathbb{P}^1 \setminus \{0,1,\infty\}$ whose topology, if not trivial, is particularly simple.

[32] Beware that two cases have been forgotten in reference [54].

[33] Actually the results contained in [18] concern the $\Gamma(p)$'s for p prime but they straightforwardly apply to the $\Gamma_1(p)$'s since $\Gamma_1(N) = \langle \Gamma(N), \tau \mapsto \tau+1 \rangle$ for any N).

6.3.4. According to a well-known result of Takeuchi [69, Theorem 3], there only exist a finite number of triangle subgroups of PU(1,1) which are arithmetic. It is natural to expect that a similar situation does occur among the groups $\mathbf{\Gamma}_1(N)^{\alpha_1}$ which are discrete. However, we are not aware of any conceptual approaches to tackles a question such as (P3) for the moment. For instance, determining the holonomy groups of Corollary 6.1 which are arithmetic when $N \geq 5$ seems out of reach for now.[34] Here again, the main reason being the inherent complexity of the congruences subgroups $\Gamma_1(N)$ in their whole.

The situation is not as bad for (P1), at least if one considers the problem for a fixed N and from a computational perspective. Indeed, we have now at disposal integral represention for the components of the developing map of $Y_1(N)^{\alpha_1}$ and this can be used, as in the classical hypergeometric case (see for instance [85, Chap.IV. §5]), to determine explicitly the corresponding holonomy group $\mathbf{\Gamma}_1(N)^{\alpha_1}$. More precisely, it follows from our results in §4.4 that, setting $T_N(u,\tau) = \theta(u,\tau)^{\alpha_1}/\theta(u-1/N,\tau)^{\alpha_1}$, the map

$$F_N : \tau \longmapsto \begin{bmatrix} F_N^\infty(\tau) \\ F_N^0(\tau) \end{bmatrix} = \begin{bmatrix} \int_{\boldsymbol{\gamma}_\infty} T_N(u,\tau)\,du \\ \int_{\boldsymbol{\gamma}_0} T_N(u,\tau)\,du \end{bmatrix}$$

is the developing map of the lift of Veech's hyperbolic structure on $Y_1(N)^{\alpha_1}$ to its universal covering $\mathscr{F}_N \simeq \mathbb{H}$. Consequently, to any projective transform $\widehat{\tau} = (a\tau+b)/(c\tau+d)$ corresponding to an element $g = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma_1(N)$ will correspond a matrix $M(g)$ such that $F_N(\widehat{\tau}) = M(g)\cdot F_N(\tau)$ for every $\tau \in \mathbb{H}$. Since $\Gamma_1(N) \simeq \pi_1(Y_1(N)^{\alpha_1})$, it comes that the map $g \mapsto M(g)$ induces the holonomy representation (118) (up to a suitable conjugation which can be determined explicitly from §3.5.2, see also §4.4.5).

Using Mano's connexions formulae presented in §3.5.1, it is essentially a computational task to determine explicitly $M(g)$ from g if the latter is given. This is what we explain in §6.3.5 just below. Then, once a finite set of explicit generators $g_1,\ldots,g_\ell$ of $\Gamma_1(N)$ is known, one can compute the matrices $M(g_1),\ldots,M(g_\ell)$ which generate $\mathbf{\Gamma}_1(N)^{\alpha_1}$ (modulo conjugation) and then study this group, for instance by means of algebraic methods.

6.3.5. **Some explicit connection formulae.** Let $\alpha_1 \in]0,1[$ be fixed. Below, we use the formulae of §3.5.1 to obtain some lemmata which can be used to compute explicitly the image of a given element of $\Gamma_1(N)$ in $\mathbf{\Gamma}_1(N)^{\alpha_1}$. We end by illustrating our method with two explicit computations in §6.3.6.4

[34]Note that since $\mathbf{\Gamma}_1(N)^{\alpha_1}$ are triangle subgroups of $\mathrm{PSL}_2(\mathbb{R})$ for $N = 2,3,4$ (see §6.2), the three problems (P1),(P2) and (P3) can be completely solved in these cases.

6.3.5.1. For $a=(a_0,a_\infty)\in\mathbb{R}^2\setminus\alpha_1\mathbb{Z}^2$, one sets $r=\alpha_1^{-1}(a_0,a_\infty)\in\mathbb{R}^2\setminus\mathbb{Z}^2$ and

$$\omega_a(u,\tau)=\exp\big(2i\pi a_0 u\big)\theta(u,\tau)^{\alpha_1}\theta\big(u-(r_0\tau-r_\infty),\tau\big)^{-\alpha_1}du. \tag{119}$$

As seen before, the map

$$\begin{aligned} F_a\,:\,\mathbb{H} &\longrightarrow \mathbb{C}^2\\ \tau \longmapsto F_a(\tau) &= \begin{bmatrix} F_a^\infty(\tau)\\ F_a^0(\tau)\end{bmatrix} = \begin{bmatrix} \int_{\gamma_\infty}\omega_a(u,\tau)\\ \int_{\gamma_0}\omega_a(u,\tau)\end{bmatrix}\end{aligned}$$

can be seen as an affine lift of the Veech map $V_a^{\alpha_1}:\mathscr{F}_a\to\mathbb{CH}^1$ of the leaf

$$\mathscr{F}_a=\big\{(\tau,z_2)\in\mathscr{T}or_{1,2}\,\big|\,a_0\tau-\alpha_1 z_2=a_\infty\big\}\simeq\mathbb{H}$$

of Veech's foliation on the Torelli space $\mathscr{T}or_{1,2}$.

In order to determine the hyperbolic holonomy of an algebraic leaf $Y_1(N)^{\alpha_1}$ it is necessary to establish some connection formulae for the function F_a. By this, we mean two very slightly distinct things:

- first, given a modular transformation $\widehat{\tau}=(m\tau+n)/(p\tau+q)$, we want to express $F_a(\widehat{\tau})$ in terms of $F_{\widehat{a}}(\tau)$ for a certain $\widehat{a}$ (which is easy to determine explicitly);
- second, we want to relate $F_a(\tau)$ and $F_{a''}(\tau)$ for any τ, when a and a'' are congruent modulo $\alpha_1\mathbb{Z}^2$.

Connection formulae of the first type will be said of **modular type** whereas those of the second type will be said of **translation type**.

6.3.5.2. **Connection formulae of modular type.** Consider the following two elements of $\mathrm{SL}_2(\mathbb{Z})$ whose classes generate the modular group:

$$T=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}\qquad\text{and}\qquad S=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}.$$

For $a=(a_0,a_\infty)\in\mathbb{R}^2\setminus\alpha_1\mathbb{Z}^2$, one sets

$$a'=\big(a_0,a_\infty-a_0\big)\qquad\text{and}\qquad\widetilde{a}=\big(a_\infty,-a_0\big).$$

Then the restriction of T (resp. of S) to $\mathscr{F}_{a'}$ (resp. to $\mathscr{F}_{\widetilde{a}}$) induces an isomorphism from this leaf onto $\mathscr{F}_a$. Moreover, it follows from [76, Theorem 0.7] that both isomorphisms are compatible with the $\mathbb{CH}^1$-structures of these leaves. The point is that in order to determine the holonomy of any leaf $Y_1(N)^{\alpha_1}$, we need to make this completely explicit.

Each of the two matrices T and S induces an automorphism of the Torelli space $\mathscr{T}or_{1,n}$ that will be designated slightly abusively by the same letter. In the natural affine coordinates (τ, z_2) on the Torelli space (see §4.2.2 above), these two automorphisms are written

$$T(\tau, z_2) = \big(\tau+1, z_2\big) \qquad \text{and} \qquad S(\tau, z_2) = \big(-1/\tau, -z_2/\tau\big).$$

We recall that $\rho = \rho(a)$ stands for

$$(\rho_0, \rho_\infty) = \big(\rho_0(a), \rho_\infty(a)\big) = \big(\exp(2i\pi a_0), \exp(2i\pi a_\infty)\big)$$

with corresponding notations for ρ' and $\widetilde{\rho}$, that is

$$\begin{aligned} \rho' =& \rho(a') = \big(\rho'_0, \rho'_\infty\big) = \big(\rho_0, \rho_\infty \rho_0^{-1}\big) \\ \text{and} \quad \widetilde{\rho} =& \rho(\widetilde{a}) = \big(\widetilde{\rho}_0, \widetilde{\rho}_\infty\big) = \big(\rho_\infty, \rho_0^{-1}\big). \end{aligned}$$

To save space, we will denote by $I\!H_a$ the matrix $I\!H_{\rho(a)}$ (*cf.* (40)) below. We recall that it is the matrix of Veech's form on the target space of the map F_a.

To state our result concerning connection formulae of modular type, we will assume that

$$\big(a_0, -a_\infty\big) \in \alpha_1 \, [0,1[^2 . \tag{120}$$

This condition can be interpreted geometrically as follows: (120) is equivalent to the fact that, for any $\tau \in \mathbb{H}$, the point $(a_0\tau - a_\infty)/\alpha_1$, which is a singular point of the multivalued holomorphic 1-form $m_r(u,\tau)$, see (119), belongs to the standard fundamental domain $[0,1[_\tau = [0,1[+[0,1[\tau \subset \mathbb{C}$ of E_τ.

Remark that this condition is not really restrictive. Indeed, considering the action (65), it comes easily that for any $a \in \mathbb{R}^2 \setminus \alpha_1\mathbb{Z}^2$, there exists a^* in the same $(\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2)$-orbit (hence such that $\mathscr{F}_a \simeq \mathscr{F}_{a^*}$) which satisfies (120).

Lemma 6.2. *Assume that condition* (120) *holds true.*

(1) *For every* $\tau \in \mathbb{H}$*, one has*

$$F_a(\tau+1) = T_a \cdot F_{a'}(\tau)$$

with

$$T_a = \begin{bmatrix} 1 & \rho_\infty/\rho_0 \\ 0 & 1 \end{bmatrix}. \tag{121}$$

(2) *There exists a function* $\tau \mapsto \sigma_a(\tau)$ *such that for every* $\tau \in \mathbb{H}$*, one has*

$$F_a(-1/\tau) = \sigma_a(\tau)\,\big(S_a \cdot F_{\widetilde{a}}(\tau)\big)$$

with

$$S_a = \begin{bmatrix} 1-\rho_\infty & \rho_0^{-1} \\ -\rho_0 & 0 \end{bmatrix}. \tag{122}$$

The following notations will be convenient in the proof below: for $\bullet = 0, \infty$ and $\tau \in \mathbb{H}$, we denote by $\boldsymbol{\gamma}_\bullet(\tau)$ the twisted 1-cycle constructed in §3 with τ seen as a point of $\mathscr{F}_a$: the ambient torus is E_τ which is punctured at $[0]$ and $[z_2]$ with $z_2 = \alpha_1^{-1}(a_0\tau - a_\infty)$. And for any superscript *, we will write $\boldsymbol{\gamma}_\bullet^\star(\tau)$ for the corresponding cycles but when a has been replaced by $a^\star$. For instance, $\boldsymbol{\gamma}'_0(\tau)$ is the twisted cycle on the torus E_τ, punctured at $[0]$ and $[z'_2]$ with $z'_2 = (a'_0\tau - a'_\infty)/\alpha_1 = r_0\tau + (r_0 - r_\infty)$, obtained by the regularization of $]0,1[$.

Proof. We first treat the case of the transformation $\tau \mapsto \tau + 1$ that will be used to deal with the second one after.

On the one hand, one has

$$\omega_a(u, \tau+1) = \omega_{a'}(u, \tau). \tag{123}$$

On the other hand, the map associated to $E_\tau \to E_{\tau+1}$ lifts to the identity in the variable u. Consequently, one has (see Figure 13 below)

$$\boldsymbol{\gamma}_\infty(\tau+1) = \boldsymbol{\gamma}'_\infty(\tau) + \rho'_\infty \boldsymbol{\gamma}'_0(\tau) \qquad \text{and} \qquad \boldsymbol{\gamma}_0(\tau+1) = \boldsymbol{\gamma}'_0(\tau). \tag{124}$$

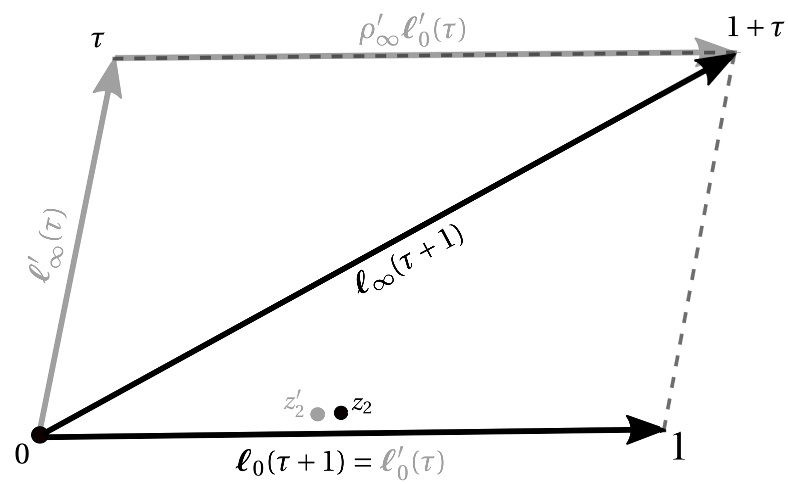


FIGURE 13. Relations between the locally-finite twisted 1-simplices $\boldsymbol{\ell}_0(\tau+1), \boldsymbol{\ell}_\infty(\tau+1), \boldsymbol{\ell}'_0(\tau)$ and $\boldsymbol{\ell}'_\infty(\tau)$ which give (124) after regularization (*cf.* §3.3.1.2). The point z_2 has been assumed to be of the form $\epsilon\tau + 1/N$ with $N = 2$ and $\epsilon > 0$ small (*i.e.* the pictured case corresponds to $a = (\epsilon, -1/2)$).

The two looked forward relations $F_a^\infty(\tau+1) = F_{a'}^\infty(\tau) + (\rho_\infty/\rho_0) \cdot F_{a'}^0(\tau)$ and $F_a^0(\tau+1) = F_{a'}^0(\tau)$ then follow immediately from (123) and (124).

Note that the following relation holds true

$$\mathbb{I}H_{a'} = {}^t\overline{T_a} \cdot \mathbb{I}H_a \cdot T_a\,,$$

which is coherent with the fact that T induces an isomorphism between the $\mathbb{CH}^1$-structure of the leaves $\mathscr{F}_{a'}$ and $\mathscr{F}_a$ of Veech's foliation on the Torelli space.

We now turn to the case of S. Considering the principal determination of the square root, one has

$$\theta\Big(-\frac{u}{\tau}, -\frac{1}{\tau}\Big) = i \cdot \sqrt{\frac{\tau}{i}} \cdot e^{\frac{i\pi u^2}{\tau}} \cdot \theta(u,\tau)$$

for every $\tau \in \mathbb{H}$ and every $u \in \mathbb{C}$ (see [8, (8.5)]).

By a direct computation (see also [46, Prop. 6.1]), one gets

$$\omega_a\big(-u/\tau, -1/\tau\big) = \sigma_a(\tau) \cdot \omega_{\widetilde{a}}(u,\tau) \tag{125}$$

with $\sigma_a(\tau) = -\tau^{-1}\exp\big(-i\pi(a_0 + a_\infty\tau)^2/(\alpha_1\tau)\big)$.

Now one needs to determine the action of S_* on the twisted 1-cycles $\widetilde{\boldsymbol{\gamma}}_\infty$ and $\widetilde{\boldsymbol{\gamma}}_0$. To this end, one remarks that the following decomposition holds true

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

This relation will allow us to express the action of S_* as a composition of three maps of the type of the one previously determined. More explicitly, from the preceding matricial decomposition we get that the isomorphism between $\mathscr{F}_{\widetilde{a}}$ and $\mathscr{F}_a$ induced by the restriction of S factorizes as follows

$$\mathscr{F}_{\widetilde{a}} \xrightarrow{\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]} \mathscr{F}_{a^\star} \xrightarrow{\left[\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right]} \mathscr{F}_{a'} \xrightarrow{\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]} \mathscr{F}_a\,, \qquad \mathscr{F}_{\widetilde{a}} \xrightarrow{\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]} \mathscr{F}_a \tag{126}$$

where $a^\star = (a_\infty, a_\infty - a_0)$.

From what has been done before, we get that, in the suitable corresponding basis, the matrices of the linear transformations on the twisted 1-cycles associated to the first and the last isomorphisms in (126) are T_a and $T_{a^\star}$ respectively.

By an analysis completely similar and symmetric to the one we did to treat the first case, one gets that the isomorphism between $\mathscr{F}_{a^\star}$ and $\mathscr{F}_{a'}$ in the middle of (126) induces the following transformation for the corresponding twisted 1-cycles:

$$\begin{bmatrix} \boldsymbol{\gamma}'_\infty\big(\frac{\tau}{1-\tau}\big) \\ \boldsymbol{\gamma}'_0\big(\frac{\tau}{1-\tau}\big) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\rho'_0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{\gamma}^*_\infty(\tau) \\ \boldsymbol{\gamma}^*_0(\tau) \end{bmatrix}.$$

From (126) and what follows, it comes that for every $\tau \in \mathbb{H} \simeq \mathscr{F}_{\widetilde{a}}$, the action of S on twisted 1-cycles is given by

$$\begin{bmatrix}\boldsymbol{\gamma}_\infty(-1/\tau)\\ \boldsymbol{\gamma}_0(-1/\tau)\end{bmatrix} = \begin{bmatrix}1 & \frac{\rho_\infty}{\rho_0}\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ -\rho_0 & 1\end{bmatrix}\begin{bmatrix}1 & \frac{\rho_\infty^*}{\rho_0^*}\\ 0 & 1\end{bmatrix}\begin{bmatrix}\widetilde{\boldsymbol{\gamma}}_\infty(\tau)\\ \widetilde{\boldsymbol{\gamma}}_0(\tau)\end{bmatrix}$$

that is by

$$\begin{bmatrix}\boldsymbol{\gamma}_\infty(-1/\tau)\\ \boldsymbol{\gamma}_0(-1/\tau)\end{bmatrix} = \begin{bmatrix}1-\rho_\infty & \rho_0^{-1}\\ -\rho_0 & 0\end{bmatrix}\begin{bmatrix}\widetilde{\boldsymbol{\gamma}}_\infty(\tau)\\ \widetilde{\boldsymbol{\gamma}}_0(\tau)\end{bmatrix}. \tag{127}$$

The second part of the lemma thus follows by combining (125) with (127). □

6.3.5.3. **Connection formulae of translation type.** Since our main interest is in the algebraic leaves of $\mathscr{F}^{\alpha_1}$, we will establish connection formulae of translation type only for the maps F_a's associated to such leaves. The general case does not present more difficulty but is not of interest to us.

Let N be a fixed integer strictly bigger than 1. One sets

$$\mu = e^{\frac{2i\pi}{N}\alpha_1}.$$

For $m, n \in \mathbb{Z}^2$ with $(m, n, N) = 1$, remember the notation $\mathscr{F}_{m,n}$ of Section 5.2:

$$\mathscr{F}_{m,n} = \mathscr{F}_{(m/N,-n/N)} = \Big\{(\tau, z_2) \in \mathscr{T}or_{1,2} \;\Big|\; z_2 = (m/N)\tau + n/N\Big\}.$$

The lifted holonomy associated to this leaf of Veech's foliation is

$$a^{m,n} = \Big(\frac{m}{N}\alpha_1, -\frac{n}{N}\alpha_1\Big)$$

whose associated linear holonomy is given by

$$\rho^{m,n} = \big(\rho_0^{m,n}, \rho_1^{m,n}, \rho_\infty^{m,n}\big) = \big(\mu^m, \mu^N, \mu^{-n}\big) = \Big(e^{\frac{2i\pi m}{N}\alpha_1}, e^{2i\pi\alpha_1}, e^{-\frac{2i\pi n}{N}\alpha_1}\Big).$$

From now on, we use the notations $\omega_{m,n}$ and $F_{m,n}$ for $\omega_{a^{m,n}}$ and $F_{a^{m,n}}$ respectively: for $\tau \in \mathbb{H}$, one has

$$F_{m,n}(\tau) = \begin{bmatrix}\int_{\boldsymbol{\gamma}_\infty}\omega_{m,n}(u,\tau)\\ \int_{\boldsymbol{\gamma}_0}\omega_{m,n}(u,\tau)\end{bmatrix} \quad \text{with} \quad \omega_{m,n}(u,\tau) = \frac{e^{\frac{2i\pi m\alpha_1}{N}}\theta(u,\tau)^{\alpha_1}}{\theta\big(u-\big(\frac{m\tau+n}{N}\big),\tau\big)^{\alpha_1}}du. \tag{128}$$

Then, using the notations of Section 3.4, one sets (see (40)):

$$I\!H_{m,n} = I\!H_{\rho^{m,n}} = \frac{1}{2i}\begin{bmatrix}\frac{(\mu^m-1)(1-\mu^{N-m})}{\mu^N-1} & \frac{1-\mu^{-m}-\mu^{-n}+\mu^{N-m-n}}{\mu^N-1}\\ \frac{\mu^N-\mu^{N+m}-\mu^{N+n}+\mu^{m+n}}{\mu^N-1} & \frac{(\mu^n-1)(1-\mu^{N-n})}{\mu^N-1}\end{bmatrix}.$$

It is the matrix of Veech's hermitian form in the basis $(F_{m,n}^\infty, F_{m,n}^0)$.

In the lemma below, we use the notations of §3.5.1.

Lemma 6.3. (1) *For any* $\tau \in \mathbb{H}$*, one has*

$$F_{m,n-N}(\tau) = B_{m,n} \cdot F_{m,n}(\tau)$$

with $B_{m,n} = \mu^{-\frac{N}{2}} \cdot \mathrm{HT2}_{\rho^{m,n}}$*;*

(2) *There exists a function* $\tau \mapsto \eta_{m,n}(\tau)$ *such that for every* $\tau \in \mathbb{H}$*, one has:*

$$F_{m-N,n}(\tau) = \eta_{m,n}(\tau)\, A_{m,n} \cdot F_{m,n}(\tau)$$

with $A_{m,n} = \mu^{-\frac{N}{2}-n} \cdot \mathrm{VT2}_{\rho^{m,n}}$.

Proof. The relation $\omega_{m,n-N} = \mu^{-N/2}\omega_{m,n}$ follows easily from the quasi-periodicity property (17) of θ. On the other hand, one can write $F_{m,n-N} = \langle \omega_{m,n-N}, \boldsymbol{\gamma}_{m,n-N}\rangle$ with ${}^t\boldsymbol{\gamma}_{m,n-N} = (\boldsymbol{\gamma}^{\infty}_{m,n-N}, \boldsymbol{\gamma}^{0}_{m,n-N})$. From §3.5.1, it comes that $\boldsymbol{\gamma}_{m,n-N} = \mathrm{HT2}_{\rho^{m,n}} \cdot \boldsymbol{\gamma}_{m,n}$ where $\mathrm{HT2}_{\rho^{m,n}}$ stands for the 2×2 matrix $\mathrm{HT2}_{\rho}$ defined in (45) with $\rho = \rho^{m,n}$. The first connection formula follows immediately.

From (17) again, one deduces that the following relation holds true: $\omega_{m-N,n} = \mu^{-\frac{N}{2}-n}\exp(i\pi\tau\alpha_1(1-2m/N))\omega_{m,n}$. On the other hand, one has $\boldsymbol{\gamma}_{m-N,n} = \mathrm{VT2}_{\rho^{m,n}} \cdot \boldsymbol{\gamma}_{m,n}$. Setting $\eta_{m,n}(\tau) = e^{i\pi\tau\alpha_1(1-2m/N)})$, the second formula follows. □

Note that what is actually interesting in the preceding lemma is that the matrices $B_{m,n}$ and $A_{m,n}$ can be explicited.

Indeed, one has $B_{m,n} = (\mu^{-\frac{N}{2}}\beta^{i,j}_{m,n})^2_{i,j=1}$ with

$$\begin{aligned}
\beta^{1,1}_{m,n} &= \mu^{2N-n} - \mu^{2N+m-n} + \mu^{N+m} + \mu^{N-m-n} - \mu^{N-n},\\
\beta^{1,2}_{m,n} &= \mu^{2N-m-n} + \mu^{2N-2n} - \mu^{2N-n} - 2\,\mu^{N-m-n} + \mu^{N} + \mu^{N-m-2n} - \mu^{N-n},\\
\beta^{2,1}_{m,n} &= -\,\mu^{N+2m} - \mu^{m} + 2\,\mu^{N+m} + 2 - \mu^{N} - \mu^{-m}
\end{aligned}$$

and $\beta^{2,2}_{m,n} = -\,\mu^{N+m} - \mu^{N-m} + \mu^{N+m-n} - \mu^{N-n} + 2\,\mu^{N} + 2\,\mu^{-m} + \mu^{-n} - \mu^{-m-n} - 1$.

(Verification: the following relation holds true: ${}^t\overline{B_{m,n}} \cdot I\!H_{m,n-N} \cdot B_{m,n} = I\!H_{m,n}$).

The matrix $A_{m,n}$ is considerably simpler. One has:

$$A_{m,n} = \mu^{-\frac{N}{2}-n}\begin{bmatrix} 1 & 0 \\ \mu^{n}(\mu^{m-N}-1) & \mu^{n-N}\end{bmatrix}.$$

(Verification: the following relation holds true: ${}^t\overline{A_{m,n}} \cdot I\!H_{m-N,n} \cdot A_{m,n} = I\!H_{m,n}$).

6.3.6. **Effective computation of the holonomy of** $Y_1(N)^{\alpha_1}$**.** We now explain how the connection formulae that we have just established can be used to compute the holonomy group $\boldsymbol{\Gamma}_1(N)^{\alpha_1}$ of $Y_1(N)^{\alpha_1}$ in an effective way.

6.3.6.1. As a concrete model for this ‘hyperbolic conicurve’, we choose the quotient of the leaf $\mathscr{F}_{0,1}$ (cut out by $z_2 = 1/N$ in the Torelli space) by its stabilizer. Actually, we will use the natural isomorphism $\mathbb{H} \simeq \mathscr{F}_{0,1}$ to see $Y_1(N)^{\alpha_1}$ as the standard modular curve $\mathbb{H}/\Gamma_1(N)$. From this point of view, the developing map of the associated $\mathbb{CH}^1$-conifold structure is nothing else but the map $F_{0,1}$ considered above (*cf.* (128) with $m = 0$ and $n = 1$). It follows that the $\mathbb{CH}^1$-holonomy of $Y_1(N)^{\alpha_1}$ can be determined through the connection formulae satisfied by $F_{0,1}$. Note that the corresponding hermitian matrix $I\!H_{0,1}$ simplifies and has a relatively simple expression (*cf.* also §3.5.2):

$$I\!H_{0,1} = \frac{1}{2i}\begin{bmatrix} 0 & \mu^{-1} \\ -\mu & \frac{(\mu-1)(1-\mu^{N-1})}{\mu^N-1} \end{bmatrix}.$$

6.3.6.2. Let $g\cdot\tau = (p\tau+q)/(r\tau+s)$ be the image of $\tau \in \mathbb{H}$ by an element $g = \left[\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right]$ of $\Gamma_1(N)$. From the two lemmata proved above, it comes that there exists a matrix $\Lambda'(g) \in \mathrm{Aut}(I\!H_{0,1})$ as well as a non-vanishing function $\lambda_g(\tau)$ such that

$$F_{0,1}(g\cdot\tau) = \lambda_g(\tau)\,\Lambda'(g)\cdot F_{0,1}(\tau)\,. \tag{129}$$

Moreover, one can ask $\Lambda'(g)$ to have coefficients in $\mathbb{R}(\mu)$. The map $\Lambda' : g \mapsto \Lambda'(g)$ is a representation of $\Gamma_1(N)$ in $\mathrm{Aut}(I\!H_{0,1}) \cap \mathrm{PSL}_2(\mathbb{R}(\mu))$.

Then, conjugating this representation by the matrix

$$Z = \sqrt{2}\begin{bmatrix} \mu^{-1} & -\frac{\mu^{N-1}-1}{\mu^N-1} \\ 0 & 1 \end{bmatrix}$$

(*cf.* §3.5.2), one gets a normalized representation of $\Gamma_1(N)$ in $\mathrm{PSL}_2(\mathbb{R})$

$$\begin{aligned} \Lambda = \Lambda_N^{\alpha_1} : \Gamma_1(N) &\longrightarrow \mathrm{PSL}_2(\mathbb{R}) \\ g &\longmapsto \Lambda(g) = Z^{-1}\cdot\Lambda'(g)\cdot Z \end{aligned} \tag{130}$$

for the considered $\mathbb{CH}^1$-holonomy. It is a deformation of the standard inclusion of the projectivization $\mathbf{\Gamma}_1(N)$ of $\Gamma_1(N)$ as a subgroup of $\mathrm{PSL}_2(\mathbb{Z}) \subset \mathrm{PSL}_2(\mathbb{R})$ which is analytic with respect to the parameter $\alpha_1 \in]0,1[$.

6.3.6.3. We now explain how to compute $\Lambda(g)$ explicitly for $g = \left[\begin{smallmatrix} p & q \\ m & n \end{smallmatrix}\right] \in \Gamma_1(N)$.

Writing g as a word in $T = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$ and $S = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$, one can use Lemma 6.2 to get that

$$F_{0,1}\left(\frac{a\tau+b}{c\tau+d}\right) = M(g)\cdot F_{m,n}(\tau) \tag{131}$$

where $M(g)$ is a product (which can be made explicit) of the matrices $T_{a'}$ and $S_{a''}$ (see formulae (121) and (122) respectively) for some a' and a'' easy to determine.

Next, the fact that g belongs to $\Gamma_1(N)$ implies in particular that $m = m'N$ and $n = 1 + n'N$ for some integers m', n'. One can then use Lemma 6.3 and construct a function $\lambda_g(\tau)$ and a matrix $N(g)$ which is a product of m' (resp. n') matrices of the type $\mathrm{VT2}_{\hat{m},\hat{n}}$ (resp. $\mathrm{HT2}_{\tilde{m},\tilde{n}}$), for some $(\hat{m},\hat{n})$'s and $(\tilde{m},\tilde{n})$'s, which are easy to make explicit, such that

$$F_{m,n}(\tau) = \lambda_g(\tau) N(g) \cdot F_{0,1}(\tau)\,. \tag{132}$$

Then setting $\Lambda'(g) = M(g) \cdot N(g)$, one gets (129) from (131) and (132).

Below, we illustrate the method just described by computing explicitly the image by Λ of two simple elements of $\Gamma_1(N)$.

Remark 6.4. We have described above an algorithmic method to compute $\Lambda(g)$ when g is given. It would be interesting to have a closed formula for $\Lambda(g)$ in terms of the coefficients of g. Such formulae have been obtained by Graf [25, 26] and more recently (and independently) by Watanabe [77, 80] in the very similar case of the *'complete elliptic hypergeometric integrals'* which are the hypergeometric integrals associated to $\Gamma(2)$ of the following form

$$\int_{\boldsymbol{\gamma}} \theta(u,\tau)^{\beta_0}\theta_1(u,\tau)^{\beta_1}\theta_2(u,\tau)^{\beta_2}\theta_3(u,\tau)^{\beta_3}du\,,$$

where $\boldsymbol{\gamma}$ stands for a twisted cycles supported in $E_\tau \setminus E_\tau[2]$ (the β_i's being fixed real parameters summing up to 0).

6.3.6.4. **Two explicit computations (N arbitrary).** We consider the two following elements of $\Gamma_1(N)$:

$$T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad \text{and} \qquad U_N = \begin{bmatrix} 1 & 0 \\ -N & 1 \end{bmatrix}.$$

We want to compute their respective image by Λ in $\mathrm{SL}_2(\mathbb{R})$.

The case of T is very easy to deal with. From the first point of Lemma 6.2, one has

$$\Lambda'(T) = \begin{bmatrix} 1 & \rho_\infty^{0,1}/\rho_0^{0,1} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \mu^{-1} \\ 0 & 1 \end{bmatrix}.$$

After conjugation by Z, one gets

$$\Lambda(T) = Z^{-1} \cdot \Lambda'(T) \cdot Z = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in \mathrm{PSL}_2(\mathbb{R})\,.$$

To deal with U_N, one begins by writing

$$U_N = S \cdot T^N \cdot S^{-1}.$$

In what follows, we write $=_\tau$ to designate an equality which holds true up to multiplication by a function depending on τ.

Then, for any $\tau \in \mathbb{H} \simeq \mathscr{F}_{0,1}$, setting $\tau' = -1/\tau$, one has

$$
\begin{aligned}
F_{0,1}\left(\frac{\tau}{1-N\tau}\right) = F_{0,1}\left(\frac{-1}{\tau'+N}\right) &=_\tau S_{0,1}\cdot F_{-1,0}(\tau'+N)\\
&=_\tau S_{0,1}\cdot T_{-1,0}\cdot F_{-1,-1}(\tau'+N-1)\\
&=_\tau S_{0,1}\cdot T_{-1,0}\cdots T_{-1,-N+1}\cdot F_{-1,-N}(\tau')\\
&=_\tau S_{0,1}\cdot T_{-1,0}\cdots T_{-1,N-1}\cdot B_{-1,0}\cdot F_{-1,0}(\tau')\\
&=_\tau S_{0,1}\cdot T_{-1,0}\cdots T_{-1,N-1}\cdot B_{-1,0}\cdot \big(S_{1,0}\big)^{-1}\cdot F_{0,1}(\tau)\,.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\Lambda'(U_N) &= S_{0,1}\cdot T_{-1,0}\cdots T_{1,-N+1}\cdot B_{-1,0}\cdot\big(S_{0,1}\big)^{-1}\\
&= S_{0,1}\cdot \begin{bmatrix}1 & \mu\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & \mu^2\\ 0 & 1\end{bmatrix}\cdots\begin{bmatrix}1 & \mu^N\\ 0 & 1\end{bmatrix}\cdot B_{-1,0}\cdot\big(S_{0,1}\big)^{-1}\\
&= S_{0,1}\cdot \begin{bmatrix}1 & \frac{\mu(1-\mu^N)}{(1-\mu)}\\ 0 & 1\end{bmatrix}\cdot B_{-1,0}\cdot\big(S_{0,1}\big)^{-1}.
\end{aligned}
$$

Since

$$
S_{0,1} = \begin{bmatrix}1-\mu^{-1} & 1\\ -1 & 0\end{bmatrix}
$$

and $\quad B_{-1,0} = \mu^{1-N/2}\begin{bmatrix}\mu^{2N-1}-\mu^{2N-2}+\mu^{N-2}+\mu^N-\mu^{N-1} & \mu^{2N}-\mu^N\\ -\frac{(\mu-1)^2(\mu^{N-1}+1)}{\mu^2} & \mu^{N-1}-\mu^N+1\end{bmatrix},$

an explicit computation gives

$$
\Lambda'(U_N) = \mu^{\frac{N}{2}}\cdot\begin{bmatrix}1 & \frac{(1-\mu)^2}{\mu^2}\\ \frac{\mu^2(1-\mu^{-N})}{1-\mu} & 1-\mu-\mu^{-N}+2\mu^{1-N}\end{bmatrix}
$$

(Remark: for $\alpha_1\to 0$, one has $\mu\to 1$ hence $\Lambda'(U_N)\to\left[\begin{smallmatrix}1 & 0\\ -N & 1\end{smallmatrix}\right]$, as expected).

One deduces the following explicit expression for $\Lambda(U_N) = Z^{-1}\Lambda'(U_N)Z$:

$$
\Lambda(U_N) = \mu^{\frac{N}{2}}\begin{bmatrix}\frac{1+\mu^2-\mu^N-\mu^{2-N}}{(\mu-1)(\mu^N-1)} & \frac{\lambda(U_N)}{(\mu-1)(\mu^N-1)^2\mu}\\ \frac{\mu(\mu^{-N}-1)}{\mu-1} & -\frac{-3\mu^{N+1}+6\mu+\mu^{2-N}+\mu^{2+N}+\mu^N-2-3\mu^{1-N}+\mu^{-N}-2\mu^2}{(\mu-1)(\mu^N-1)}\end{bmatrix}
$$

with

$$
\begin{aligned}
\lambda(U_N) = -1-5\mu^{N+1}-2\mu^4-\mu^{2+2N}-\mu^{2N}+2\mu^N-2\mu^{3+N}+\mu^{2+N}\\
+\mu^{4-N}-\mu^2+\mu^{N+4}+\mu^{2-N}-3\mu^{3-N}+2\mu+3\mu^{1+2N}+5\mu^3\,.
\end{aligned}
$$

(Remark : one has $\Lambda^*(U_N) = \mu^{-\frac{1}{2}}\Lambda(U_N)\in \mathrm{SL}_2(\mathbb{R})$).

A necessary condition so that the $\mathbb{CH}^1$-holonomy of $Y_1(N)^{\alpha_1}$ is discrete is that $\Lambda^*(U_N)$ together with the fixed parabolic element $\Lambda(T) = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$ generates a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$. There are many papers dealing with this problem. For instance, in [20], Gilman and Maskit give an explicit algorithm to answer this question. However, if this algorithm can be used quite effectively to solve any given explicit case, the complexity of $\Lambda^*(U_N)$ seems to make its use too involved to describe precisely the set of parameters $\alpha_1 \in]0,1[$ and $N \in \mathbb{N}_{\geq 2}$ so that $\langle \Lambda^*(U_N), \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right] \rangle$ be a lattice in $\mathrm{SL}_2(\mathbb{R})$.

6.4. **Volumes.** We recall that $Y_1(N)^{\alpha_1}$ stands for the modular curve $Y_1(N)$ endowed with the pull-back by Veech's map of the standard hyperbolic structure of $\mathbb{CH}^1$. In particular, the hyperbolic metric which is considered on $Y_1(N)^{\alpha_1}$ has constant curvature equals to -1. We denote by $\mathrm{Vol}_1^{\alpha_1}(N)$ the volume (writing the 'area' would be more accurate) of $Y_1(N)^{\alpha_1}$.

6.4.1. According to the version for compact hyperbolic surfaces with conical singularities of Gauß-Bonnet's Theorem (*cf.* Theorem A.1 in Appendix A) and in view of our results in §5.3.4, one has

$$\mathrm{Vol}_1^{\alpha_1}(N) = 2\pi \left[2g_1(N) - 2 - \sum_{\mathfrak{c} \in C_1(N)} \left(1 - \frac{\theta_N(\mathfrak{c})}{2\pi}\right)\right]^{35} \tag{133}$$

where

- $g_1(N)$ stands for the genus of the compactified modular curve $X_1(N)$;
- for any $\mathfrak{c} \in C_1(N)$, $\theta_N(\mathfrak{c})$ denotes the conifold angle of $X_1(N)^{\alpha_1}$ at $\mathfrak{c}$.

Since $\theta_N(\mathfrak{c})$ depends linearly on α_1 for every $\mathfrak{c}$ (*cf.* (110)), it follows that

$$\mathrm{Vol}_1^{\alpha_1}(N) = A(N) + B(N)\,\alpha_1$$

for two arithmetic constants $A(N), B(N)$ depending only on N.

We recall that the following closed formula holds true

$$g_1(N) = g\big(X_1(N)\big) = 1 + \frac{N^2}{24} \prod_{p|N} \left(1 - p^{-2}\right) - \frac{1}{4} \sum_{0<d|N} \phi(d)\phi(N/d)$$

for any $N \geq 5$, with $g_1(M) = 0$ for $M = 1, \ldots, 4$ (see [37]).

On the other hand, we are not aware of any general closed formula, in terms of N, for a set of representatives $[-a_i/c_i]$ with $i = 1, \ldots, |C_1(N)|$ of the set of cusps $C_1(N)$ of $Y_1(N)$. Consequently, obtaining closed formulae for $A(N)$ and $B(N)$ in terms of N does not seem easy in general. However, there are algorithmic methods which determine explicitly such a set of representatives. Then determining $\mathrm{Vol}_1^{\alpha_1}(N)$ reduces to a computational task once N has been given.

[35] Actually, this formula is only valid when $N \geq 4$. Indeed, $Y_1(N)$ has an orbifold point when $N = 2, 3$ and it has to be taken into account when computing $\mathrm{Vol}_1^{\alpha_1}(N)$ in these two cases.

6.4.2. Since the two values $A(N)$ and $B(N)$ depend heavily on the arithmetic properties of N, one can expect to be able to say more about them when N is simple from this point of view, for instance when N is prime.

Let p be a prime number bigger than or equal to 5. Then

$$g_1(p) = \frac{1}{24}(p-5)(p-7)$$

and there is an explicit description of the conifold points and of the associated conifold angles of $X_1(p)^{\alpha_1}$ (see §5.4.5). In the case under scrutiny, formula (133) specializes into

$$\mathrm{Vol}_1^{\alpha_1}(p) = 2\pi\Big(2g_1(p) - 2 + (p-1)\Big) - 2\pi\alpha_1 \sum_{k=1}^{(p-1)/2} k\Big(1 - \frac{k}{p}\Big)$$

and after a simple computation, one obtains the nice formula

$$\mathrm{Vol}_1^{\alpha_1}(p) = \frac{\pi}{6}\big(p^2-1\big)\,(1-\alpha_1)\ .$$

6.4.3. Besides being nice and even if it only concerns the leaves associated to prime numbers, the preceding formula could be helpful regarding the determination of the Ω^{α_1}-volume $\mathrm{Vol}^{\alpha_1}\big(\mathcal{M}_{1,2}\big)$ of the moduli space $\mathcal{M}_{1,2}$ (see §1.1.6). We say a few words about this in the lines below.

First of all, using (65), one proves easily that the $(\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2)$-orbit of the holonomy data $\alpha_1(0,-1/p)$ associated to $Y_1(p)^{\alpha_1}$ is $\mathcal{O}_p^{\alpha_1} = \frac{\alpha_1}{p}\big(\mathbb{Z}^2 \setminus p\mathbb{Z}^2\big)$. One denotes by $\mathbb{1}_p^{\alpha_1}$ the characteristic function of this subset of $\mathrm{Im}(\xi^{\alpha_1}) = \mathbb{R}^2 \setminus \alpha_1\mathbb{Z}^2$.

Then remark that $p^{-2}\mathbb{1}_p^{\alpha_1} da$ converges towards the standard Lebesgue measure $da = da_0 \wedge da_\infty$ on $\mathbb{R}^2 \setminus \alpha_1\mathbb{Z}^2$ as p tends to $+\infty$ among primes. Denoting by $d\mu_p^{\alpha_1}$ the $(1,1)$-form on $Y_1(p)^{\alpha_1}$ associated to its hyperbolic conifold structure, the preceding remark implies that $p^{-2}\mathbb{1}_p^{\alpha_1} da \wedge d\mu_p^{\alpha_1}$ converges (in some sense) to Veech's volume form Ω^{α_1} on $\mathcal{M}_{1,2}$ when p goes to infinity.

In [76], Veech conjectured that $\mathrm{Vol}^{\alpha_1}(\mathcal{M}_{1,2}) = \int_{\mathcal{M}_{1,2}} \Omega^{\alpha_1}$ is finite. Assuming that this holds true, one can apply dominated convergence theorem and eventually get that the following equalities hold true:

$$\mathrm{Vol}^{\alpha_1}\big(\mathcal{M}_{1,2}\big) = \int_{\mathcal{M}_{1,2}} \Omega^{\alpha_1} = \lim_{p\to+\infty} p^{-1}\mathrm{Vol}_1^{\alpha_1}(p) = \frac{\pi}{6}\,(1-\alpha_1)\ .$$

We plan to come back on this in the future and give rigorous proofs of all the preceding assertions.

Appendix A : 1-dimensional complex hyperbolic conifolds

We define and state a few basic results concerning $\mathbb{CH}^1$-conifolds below. The general notion of conifolds is rather abstract (see [71, 43] or [19, Appendix B]) but greatly simplifies in the case under scrutiny.

We denote by $\mathbb{D}$ the unit disk in the complex plane. As the upper half-plane $\mathbb{H}$, it is a model of the complex hyperbolic space $\mathbb{CH}^1$.

A.1. Basics. The map $f : \mathbb{H} \to \mathbb{D}^*$, $w \mapsto e^{iw}$ is (a model of) the universal cover of the punctured disk $\mathbb{D}^* = \mathbb{D} \setminus \{0\}$. We denote by $\widetilde{\mathbb{D}}^*$ the upper-half plane endowed with the pull-back by f of the standard hyperbolic structure on $\mathbb{D}$.

A.1.1. $\mathbb{CH}^1$-cones. For any $\theta \in]0, +\infty[$, the translation $t_\theta : w \mapsto w + \theta$ leaves invariant the complex hyperbolic structure of $\widetilde{\mathbb{D}}^*$ (since it is a lift of the rotation $z \mapsto e^{i\theta} z$ which is an automorhism of $\mathbb{D}$ fixing the origin). It follows that the complex hyperbolic structure of $\widetilde{\mathbb{D}}^*$ factors through the action of t_θ. The quotient $\mathfrak{C}_\theta^* = \widetilde{\mathbb{D}}^* / \langle t_\theta \rangle$ carries an hyperbolic structure which is not metrically complete. Its metric completion, denoted by $\mathfrak{C}_\theta$, is obtained by adjoining only one point to $\mathfrak{C}_\theta^*$, called the **apex** and denoted by 0. By definition, $\mathfrak{C}_\theta$ (resp. $\mathfrak{C}_\theta^*$) is the **(punctured) $\mathbb{CH}^1$-cone of angle** θ.

It will be convenient to also consider the case when $\theta = 0$. By convention, we define $\mathfrak{C}_0^*$ as $\mathbb{H} / \langle \tau \mapsto \tau + 1 \rangle$ when $\mathbb{H}$ is endowed with its standard hyperbolic structure. It is nothing else but $\mathbb{D}^*$ but now endowed with the hyperbolic structure given by the uniformization (and by restriction from the standard one of $\mathbb{D}$). Note that $\mathfrak{C}_0^*$ is nothing else than a neighborhood of what is classically called a **cusp** in the theory of Riemann surfaces.

As is well-known, a $\mathbb{CH}^1$-structure on an orientable smooth surface Σ can be seen geometrically as a (class for a certain equivalence relation of a) pair (D, μ) where $\mu : \pi_1(\Sigma) \to \mathrm{Aut}(\mathbb{CH}^1)$ is a representation (the **holonomy representation**) and $D : \widetilde{\Sigma} \to \mathbb{CH}^1$ a μ-equivariant étale map (the **developing map**). With this formalism, it is easy to give concrete models of the $\mathbb{CH}^1$-cones defined above.

For any $\theta > 0$, one defines $D_\theta(z) = z^\theta$, and μ_θ stands for the character associating $e^{i\theta}$ to the class of a small positively oriented circle around the origin in $\mathbb{D}$. We see D_θ as a multivalued map from $\mathbb{D}$ to itself. Its monodromy μ_θ leaves the standard hyperbolic structure of $\mathbb{D}$ invariant. Consequently, the pair (D_θ, μ_θ) defines a $\mathbb{CH}^1$-structure on $\mathbb{D}^*$ and one verifies promptly that it identifies with the one of the punctured $\mathbb{CH}^1$-cone $\mathfrak{C}_\theta^*$. To define $\mathfrak{C}_0^*$ this way, one can take $D_0(z) = \log(z)/(2i\pi)$ as a developing map and as holonomy representation, we take the parabolic element $\mu_0 : x \mapsto x + 1$ of the automorphism group of $\mathrm{Im}(D_0) = \mathbb{H}$ (μ_0 is nothing else but the monodromy of D_0).

By computing the pull-backs of the standard hyperbolic metric on their target space by the elementary developing maps considered just above, one gets the following characterization of the $\mathbb{CH}^1$-cones in terms of the corresponding hyperbolic metrics: $\mathfrak{C}_0^*$ and $\mathfrak{C}_\theta^*$ for any $\theta > 0$ can respectively be defined as the hyperbolic structure on $\mathbb{D}^*$ associated to the metrics

$$ds_0 = \frac{|dz|}{|z|\log|z|} \qquad \text{and} \qquad ds_\theta = 2\theta\frac{|z|^{\theta-1}|dz|}{1-|z|^{2\theta}}.$$

Note that for any positive integer k, $\mathfrak{C}_{2\pi/k}$ is the orbifold quotient of $\mathbb{D}$ by $z \mapsto e^{2i\pi/k}z$. In particular, $\mathfrak{C}_{2\pi}$ and $\mathfrak{C}_{2\pi}^*$ are nothing else than $\mathbb{D}$ and $\mathbb{D}^*$ respectively, hence most of the time it will be assumed that $\theta \neq 2\pi$.

One verifies that among all the $\mathbb{CH}^1$-cones, the one of angle 0 is characterized geometrically by the fact that the associated holonomy is parabolic, or metrically, by the fact that $\mathfrak{C}_0^*$ is complete. Finally, we mention that the area of the $\mathfrak{C}_\theta^*$ is locally finite at the apex 0 for any $\theta \geq 0$.

A.1.2. $\mathbb{CH}^1$- conifold structures. Let S be a smooth oriented surface and let $(s_i)_{i=1}^n$ be a n-uplet of pairwise distinct points on it. One sets $S^* = S \setminus \{s_i\}$. A $\mathbb{CH}^1$-structure on S^* naturally induces a conformal structure or, equivalently, a structure of Riemann surface on S^*. When endowed with this structure, we will denote S^* by X^* and s_i by x_i for every $i = 1, \ldots, n$.

We will say that the hyperbolic structure on X^* **extends as** (or just **is** for short) **a** $\mathbb{CH}^1$**- conifold (structure)** on X if, for every puncture x_i, there exists $\theta_i \geq 0$ and a germ of pointed biholomorphism $(X^*, x_i) \simeq (\mathfrak{C}_\theta^*, 0)$ which is compatible with the $\mathbb{CH}^1$-structures on the source and on the target. In this case, each puncture x_i will be called a **conifold point** and θ_i will be the associated **conifold** (or **cone**) **angle**. Remark that our definition differs from the classical one since we allow some cone angles θ_i to vanish. The punctures with conifold angle 0 are just cusps of X.

Note that when the considered hyperbolic structure on X^* is conifold then its metric completion (for the distance induced by the $\mathbb{CH}^1$-structure) is obtained by adding to X^* the set of conifold points of positive cone angles.

An important question is the existence (and possibly the unicity) of such conifold structures when X is assumed to be compact. In this case, as soon as the genus g of X and the number n of punctures verify $2g-2+n > 0$, it follows from **Poincaré-Koebe uniformization theorem** that there exists a Fuchsian group Γ such that $\mathbb{H}/\Gamma \simeq X^*$ as Riemann surfaces with cusps (and Γ is essentially unique). Actually, this theorem generalizes to any $\mathbb{CH}^1$-orbifold structures on X (see *e.g.* Theorem IV.9.12 in [17] for a precise statement). It implies in particular that such a structure, when it exists, is uniquely characterized by the conformal type of X^* and by the cone angles at the orbifold points.

It turns out that the preceding corollary of Poincaré's uniformization theorem generalizes to the class of $\mathbb{CH}^1$-conifolds. Indeed, long before Troyanov proved his theorem (recalled in §1.1.5) concerning the existence and the unicity of a flat structure with conical singularities on a surface (we could call such a structure a '$\mathbb{E}^2$*-conifold structure*'), Picard had established the corresponding result for compact complex hyperbolic conifolds of dimension 1:

Theorem A.1.2. *Assume that* $2g-2+n>0$ *and let* $(X,(x_i)_{i=1}^n)$ *be a compact* n*-marked Riemann surface of genus* g*. Let* $(\theta_i)_{i=1}^n \in [0,\infty[^n$ *be an angle datum.*

(1) *The following two assertions are equivalent:*
   - *there exists a hyperbolic conifold structure on* X *with a conical singularity of angle* θ_i *at* x_i*, for* $i=1,\ldots,n$*;*
   - *the* θ_i*'s are such that the following inequality is satisfied:*

$$2\pi\big(2g-2+n\big)-\sum_{i=1}^{n}\theta_i>0\,. \tag{134}$$

(2) *When the two preceding conditions are satisfied, then the corresponding conifold hyperbolic metric on* X *is unique (if normalized in such a way that its curvature be -1) and the area of* X *is equal to the LHS of* (134).

Actually, the preceding theorem has been obtained by Picard at the end of the 19th century under the assumption that $\theta_i>0$ for every i (see [60] and the references therein). For the extension to the case when some hyperbolic cusps are allowed (*i.e.* when some of the angles θ_i vanish), we refer to [30, Chap.II] although it is quite likely that this generalization was already known to Poincaré.

A.2. Second order differential equations and $\mathbb{CH}^1$-conifold structures. Given a $\mathbb{CH}^1$-structure on a punctured Riemann surface X^*, the question is to verify whether it actually extends as a conifold structure at the punctures. This can be achieved by looking at the associated Schwarzian differential equation.

We detail below some aspects of the theory of second order differential equations which are needed for this. Most of the material presented below is very classical and well-known (the reader can consult [84, 64] among the huge amount of references which address the issue).

A.2.1. Since we are concerned by a local phenomenon, we will work locally and assume that $X^*=\mathbb{D}^*$. In this case, the considered $\mathbb{CH}^1$-structure on X^*, which we will denote by $\mathfrak{X}^*$ for convenience, is characterized by the data of its developing map $D: X^*\to\mathbb{CH}^1$ alone. Let x be the usual coordinates on $\mathbb{D}$. Although D is a multivalued function of x, its monodromy lies in $\mathrm{Aut}(\mathbb{CH}^1)$

hence is projective. It follows that the **Schwarzian derivative of D with respect to x**, defined as

$$\{D,x\} = \left(\frac{D''(x)}{D'(x)}\right)' - \frac{1}{2}\left(\frac{D''(x)}{D'(x)}\right)^2 = \frac{D'''(x)}{D'(x)} - \frac{3}{2}\left(\frac{D''(x)}{D'(x)}\right)^2,$$

is non-longer multivalued. In other words, there exists a holomorphic function Q on X^* such that the following **Schwarzian differential equation** holds true:

$$(\mathcal{S}\mathfrak{X}^*) \qquad \{D,x\} = Q(x).$$

It turns out that the property for $\mathfrak{X}^*$ to extend as a $\mathbb{CH}^1$-conifold at the origin can be deduced from this differential equation as we will explain below.

Note that, since any function of the form $(aD+b)/(cD+d)$ with $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{SL}_2(\mathbb{C})$ satisfies $(\mathcal{S}\mathfrak{X}^*)$, this differential equation (or, in other terms, the function Q) alone does not characterizes $\mathfrak{X}^*$. This $\mathbb{CH}^1$-structure is characterized by the data of an explicit model U of $\mathbb{CH}^1$ as a domain in $\mathbb{P}^1$ (for instance $U = \mathbb{D}$ or $U = \mathbb{H}$) and by a $\mathrm{Aut}(U)$-orbit of U-(multi)valued solutions of $(\mathcal{S}\mathfrak{X}^*)$.

A.2.2. We now recall some very classical material about Fuchsian differential equations (see [84, 34, 64] among many references).

As is well-known, given $R \in \mathcal{O}(X^*)$, the Schwarzian differential equation

$$(\mathcal{S}_R) \qquad \{S,x\} = R(x)$$

is associated to the class of second-order differential equations of the form

$$(\mathcal{E}_{p,q}) \qquad F'' + pF' + qF = 0$$

for any function $p,q \in \mathcal{O}(X^*)$ such that $R = 2(q - p'/2 - p/4)$. Given two such functions p and q, the solutions of $(\mathcal{S}_R)$ are the functions of the form F_1/F_2 for any basis (F_1, F_2) of the space of solutions of $(\mathcal{E}_{p,q})$.

In what follows, we fix such an equation $(\mathcal{E}_{p,q})$ and will work with it. The reason for doing so is twofold: first, it is easier to deal with such a linear equation than with $(\mathcal{S}_R)$ which involves a non-linear Schwarzian derivative. Secondly, it is through some second-order linear differential equations that we are studying Vecch's $\mathbb{CH}^1$-structures on the algebraic leaves of Veech's foliation on $\mathcal{M}_{1,2}$ in this text (see §5.3 for more details).

We recall that $(\mathcal{E}_{p,q})$ (resp. $(\mathcal{S}_R)$) is **Fuchsian** (at the origin) if p,q are (resp. R is) meromorphic at this point with $p(x) = O(x^{-1})$ and $q(x) = O(x^{-2})$ (resp. $R(x) = O(x^{-2})$ for x close to 0 in $\mathbb{D}^*$. In this case, defining p_0 and q_0 as the complex numbers such that $p(x) = p_0 x^{-1} + O_0(1)$ and $q(x) = q_0 x^{-2} + O_0(x^{-1})$, one can construct the quadratic equation

$$s(s-1) + sp_0 + q_0 = 0$$

which is called the **characteristic equation** of $(\mathcal{E}_{p,q})$. Its two roots ν_+ and ν_- (we assume that $\Re e(\nu_+) \geq \Re e(\nu_-)$) are the two **characteristic exponents** of this equation and their difference $\nu = \nu_+ - \nu_-$ is the associated **projective index**[36]. The latter can also be defined as the complex number such that $R(x) = \frac{1-\nu^2}{2}x^{-2} + O(x^{-1})$ in the vicinity of the origin, which shows that it is actually associated to the Schwarzian equation $(\mathcal{S}_R)$ rather than to $(\mathcal{E}_{p,q})$.

It is known that one can give a normal form of a solution of $(\mathcal{S}_R)$ in terms of ν : generically (and this will be referred to as the **standard case**), there is a local invertible change of coordinate $x \mapsto y = y(x)$ at the origin so that y^ν provides a solution of $(\mathcal{S}_R)$ on a punctured neighborhood of 0. Another case is possible only when $\nu = n \in \mathbb{N}$. In this case, known as the **logarithmic case**, a solution of $(\mathcal{S}_R)$ could be of the form $y^{-n} + \log(y)$. These results (which are simple consequences of Frobenius theorem for Fuchsian second-order differential equations, see for instance [84, §2.5])[37] are summarized in Table 6.

Case \ Index $\boldsymbol{\nu}$	$\boldsymbol{\nu} \notin \mathbb{N}$	$\boldsymbol{\nu} = \boldsymbol{n} \in \mathbb{N}^*$	$\boldsymbol{\nu} = \mathbf{0}$
Standard	y^ν	y^n	—
Logarithmic	—	$y^{-n} + \log y$	$\log y$

TABLE 6.

We will use this result to determine when the $\mathbb{CH}^1$-structure $\mathfrak{X}^*$ extends as a conifold structure at the origin by means of some analytical considerations about the associated Schwarzian differential equation $(\mathcal{S}\mathfrak{X}^*)$.

Before turning to this, we would like to state another very classical (and elementary) result about Fuchsian differential systems and equations that we use several times in this text (for instance in §5.3 above or in B.3.5 below).

Let

$$(\mathscr{S}) \qquad\qquad Z' = M \cdot Z$$

be a meromorphic linear 2×2 differential system on $(\mathbb{C}, 0)$: $M = (M_{i,j})_{i,j=1}^2$ is a matrix of (germs of) meromorphic functions at the origin and the unknown $Z = {}^t(F, G)$ is a 2×1 matrix whose coefficients F and G are (germs of) holomorphic functions at a point $x_0 \in (\mathbb{C}, 0)$ distinct from 0.

[36] Note that ν is actually only defined up to sign in full generality.

[37] See also [64, Théorème IX.1.1] for the sketch of a more direct proof.

Lemma A.2.2. *Assume that $M_{1,2}$ does not vanish identically. Then:*

(1) *the space of first components F of solutions $Z = {}^t(F,G)$ of $(\mathscr{S})$ coincides with the space of solutions of the second-order differential equation*

$$(\mathscr{E}_{\mathscr{S}}) \qquad F'' + p\,F' + q\,F = 0$$

with

$$p = -\mathrm{Tr}(M) - \frac{M'_{12}}{M_{12}} \qquad \text{and} \qquad q = \det(M) - M'_{11} + M_{11}\frac{M'_{12}}{M_{12}};$$

(2) *the differential equation $(\mathscr{E}_{\mathscr{S}})$ is Fuchsian if and only if M has a pole of order at most 1 at the origin. In this case, the characteristic exponents of $(\mathscr{E}_{\mathscr{S}})$ coincide with the eigenvalues of the residue matrix of M at 0.*

Proof. This is a classical result which can be proved by straightforward computations (see *e.g.* [34, Lemma 6.1.1, §3.6.1] for the first part). □

A.2.3. We now return to the problematic mentioned in **A.2.1.** above.

We first consider the models of $\mathbb{CH}^1$-cones considered in **A.1**. By some easy computations, one gets that

$$\{D_s(x), x\} = \frac{1-s^2}{2x^2}$$

for any $s \geq 0$ (we recall that $D_0(x) = \log(x)$ and $D_s(x) = x^s$ for $s > 0$).

It follows that a necessary condition for the origin to be a conifold point for the $\mathbb{CH}^1$-structure $\mathfrak{X}^*$ is that the Schwarzian differential equation $(\mathcal{S}\mathfrak{X}^*)$ must be Fuchsian at this point, *i.e.* that $Q(x) = O(x^{-2})$ in the vicinity of 0.

A natural guess at this point would be that the preceding condition is also sufficient. It turns out that it is the case indeed:

Proposition A.2.3. *The two following assertions are equivalent:*

(1) *the $\mathbb{CH}^1$-structure $\mathfrak{X}^*$ extends as a conifold structure at the origin;*

(2) *the Schwarzian differential equation $(\mathcal{S}\mathfrak{X}^*)$ is Fuchsian.*

Proving this result is not difficult. We provide a proof below for the sake of completeness. We will denote the monodromy operator acting on (germs at the origin of) multivalued holomorphic functions on $(\mathbb{D}^*, 0)$ by M_0.

We will need the following

Lemma A.2.3. *For any positive integer n and any Moebius transformation $g \in \mathrm{PGL}_2(\mathbb{C})$, the multivalued map $D(x) = g(x^{-n} + \log(x))$ is not the developing map of a $\mathbb{CH}^1$-structure on a punctured open neighborhood of the origin in $\mathbb{C}$.*

Proof. The monodromy around the origin of such a (multivalued) function D is projective. Let T_0 stand for the matrix associated to it. On the one hand, T_0 is parabolic with $g(0) \in \mathbb{P}^1$ as its unique fixed point. On the other hand, the image of any punctured small open neighborhood of the origin by D is a punctured open neighbourhood of $g(0)$. These two facts imply that there does not exist a model U of $\mathbb{CH}^1$ in $\mathbb{P}^1$ (as an open domain) such that D has values in U and $T_0 \in \mathrm{Aut}(U)$. This shows in particular that D can not be the developing map of a $\mathbb{CH}^1$-structure on any punctured open neighborhood of $0 \in \mathbb{C}$. □

Proof of Proposition A.2.3. According to the discussion which precedes the Proposition, (1) implies (2), hence the only thing remaining to be proven is the converse implication. We assume that $(\mathcal{S}\mathfrak{X}^*)$ is Fuchsian and let ν be its index. We will consider the different cases of Table 6 separately.

We assume first that ν is not an integer. Then there exists a local change of coordinates $x \mapsto y = y(x)$ fixing the origin such that y^ν is a solution of $(\mathcal{S}\mathfrak{X}^*)$. Consequently, the developing map $D : X^* \to \mathbb{D}$ of $\mathfrak{X}^*$ can be written $D = (ay^\nu + b)/(cy^\nu + d)$ for some complex numbers a, b, c, d such that $ad - bc = 1$.

Clearly, $b/d \in \mathbb{D}$, hence up to post-composition by an element of $\mathrm{Aut}(\mathbb{D}) = \mathrm{PU}(1,1)$ sending b/d onto 0, one can assume that $b = 0$. By assumption, the monodromy of D belongs to $\mathrm{PU}(1,1)$. Since it has necessarily the origin as a fixed point, it follows that this monodromy is given by

$$M_0(D) = e^{2i\pi\mu} D$$

for a certain real number μ. On the other hand, one has

$$M_0(D) = \frac{aM_0(y^\nu)}{cM_0(y^\nu) + d} = \frac{ae^{2i\pi\nu} y^\nu}{ce^{2i\pi\nu} y^\nu + d}.$$

From the two preceding expressions for $M_0(D)$ and since $a \neq 0$, one deduces that the relations

$$e^{2i\pi\mu} \tfrac{aY}{cY+d} = \tfrac{ae^{2i\pi\nu} Y}{ce^{2i\pi\nu} Y+d} \iff e^{2i\pi\mu}\big(ce^{2i\pi\nu} Y + d\big) = e^{2i\pi\nu}\big(cY + d\big)$$

hold true as rational/polynomial identities in Y. Because $e^{2i\pi\nu} \neq 1$ by assumption, it follows that $c = 0$ and $\nu \in \mathbb{R}^+ \setminus \mathbb{N}$. Consequently, one has $D(x) = \widetilde{y}(x)^\nu$ for a certain multiple $\widetilde{y}$ of y. It is a local biholomorphism which induces an isomorphism of $\mathbb{CH}^1$-structures $\mathfrak{X}^* \simeq \mathfrak{C}^*_{2\pi\nu}$. This proves (1) in this case.

Assume now that $\nu = 0$. Then $\log(y)/(2i\pi)$ is a solution of $(\mathcal{S}\mathfrak{X}^*)$ for a certain local coordinate y fixing the origin. In this case, it is more convenient to take $\mathbb{H}$ as the target space of the developing map D of $\mathfrak{X}^*$. Since the monodromy of D is parabolic, one can assume that its fixed point is $i\infty$, which implies that D can be written $D = a\log(y)/(2i\pi) + b$ with $a, b \in \mathbb{C}$ and $a \neq 0$. Setting $\beta = \exp(2i\pi b/a) \neq 0$ and replacing y by βy, one can assume that $b = 0$.

Moreover, since D has monodromy in $\mathrm{PSL}_2(\mathbb{R})$, a must be real and positive. Computing the pull-back by D of the hyperbolic metric of $\mathbb{H}$, one gets

$$D^*\left(\frac{|dz|}{\big|\Im\mathrm{m}(z)\big|}\right)=\frac{|dD|}{\big|\Im\mathrm{m}(D)\big|}=\frac{\frac{a}{2\pi}\frac{|dy|}{|y|}}{\frac{a}{2\pi}\big|\Re\mathrm{e}\big(\log(y)\big)\big|}=\frac{|dy|}{|y|\log|y|}$$

which shows that y induces an isomorphism of $\mathbb{CH}^1$-structures $\mathfrak{X}^*\simeq\mathfrak{C}_0^*$.

We now consider the case when $\nu=n\in\mathbb{N}^*$ and y^n is a solution of $(\mathcal{S}\mathfrak{X}^*)$ for a certain local coordinate $y=y(x)$ fixing the origin. As above, one can assume that the developing map D of $\mathfrak{X}^*$ is written $D=ay^n/(cy^n+d)$. In this case, the monodromy argument used previously does not apply but one can conclude directly by remarking that since n is an integer, there exists another local coordinate $\widetilde{y}$ at the origin such that the relation $ay^n/(cy^n+d)=\widetilde{y}^n$ holds true identically. This shows that $\mathfrak{X}^*$ is isomorphic to $\mathfrak{C}^*_{2\pi n}$.

Finally, the last case of Table 6, namely the logarithmic case with $\nu\in\mathbb{N}^*$, does not occur according to Lemma A.2.3., hence we are done. □

Appendix B : the Gauß-Manin connection associated to Veech's map

Many properties of the hypergeometric function $F(a,b,c;\cdot)$ hence of the associated $\mathbb{CH}^1$-valued multivalued Schwarz map $S(a,b,c;\cdot)$ can be deduced from the hypergeometric differential equation (2).

Let $\mathscr{F}_a^\alpha$ be a leaf of Veech's foliation in the Torelli space $\mathscr{T}or_{1,n}$. As shown in §4.4.3, Veech's map $V_a^\alpha:\mathscr{F}_a^\alpha\to\mathbb{CH}^{n-1}$ has an expression $V_a^\alpha=[v_\bullet]$ whose components $v_\bullet=\int_{\gamma_\bullet}T_a^\alpha(u)du$, with $\bullet=\infty,0,3,\ldots,n$ are elliptic hypergeometric integrals. A very natural approach to the study of V_a^α is by first constructing the differential system satisfied by these.

Something very similar has been done in the papers [46] and [49] but in the more general context of isomonodromic deformations of linear differential systems on punctured elliptic curves. The results of these two papers can be specialized to the case we are interested in, but this requires a little work in order to be made completely explicit. This is what we do in this appendix.

We first introduce the Gauß-Manin connection in a general context and then specialize and make everything explicit in the case of punctured elliptic curves.

B.1. Basics on Gauß-Manin. In this subsection, we present general facts relative to the construction of the Gauß-Manin connection ∇^{GM}. We first define it analytically in B.1.2. Then we explain how it can be computed by means of relative differential forms, see B.1.3. We conclude in B.1.4 by stating the comparison theorem which asserts that, under reasonable hypotheses, one can construct∇^{GM} by considering only algebraic relative differential forms.

The material presented below is well-known hence no proof is given. Classical references are the paper [36] by Katz and Oda and the book [10] by Deligne.

Another more recent and useful reference is the book [3] by André and Baldassarri, in particular the third chapter. Note that the general strategy followed in this book goes by 'dévissage' and reduces the proofs of most of the main results to a particular ideal case, called an *'elementary coordinatized fibration'* by the authors (*cf.* [3, Chap. 3, Definition 1.3]). We think it is worth mentioning that the specific case of punctured elliptic curves we are interested in is precisely of this kind, see Remark B.2.4 below.

B.1.1. Let $\pi : \mathscr{X} \longrightarrow S$ be a family of Riemann surfaces over a complex manifold S. This means that π is a holomorphic morphism whose fibers $X_s = \pi^{-1}(s)$, $s \in S$, all are (possibly non-compact) Riemann surfaces. We assume that π is smooth and as nice as needed to make everything we say below work well.

Let Ω be a holomorphic 1-form on $\mathscr{X}$ and for any $s \in S$, denote by ω_s its restriction to the fiber X_s: $\omega_s = \Omega|_{X_s}$. Then one defines differential covariant operators by setting

$$\nabla(\eta) = d\eta + \Omega \wedge \eta \qquad \Big(\text{resp. } \nabla_s(\eta) = d\eta + \omega_s \wedge \eta\Big)$$

for any (germ of) differential form η on $\mathscr{X}$ (resp. on X_s, for any $s \in S$).

The associated kernels

$$L = \mathrm{Ker}\big(\nabla : \mathscr{O}_{\mathscr{X}} \to \Omega^1_{\mathscr{X}}\big) \qquad \text{and} \qquad L_s = \mathrm{Ker}\big(\nabla_s : \mathscr{O}_{X_s} \to \Omega^1_{X_s}\big)$$

are local systems on $\mathscr{X}$ and X_s respectively, such that $L|_{X_s} = L_s$ for any $s \in S$.

B.1.2. Let B be the first derived direct image of L by π:

$$B = R^1\pi_*(L).$$

It is the sheaf on S the stalk of which at $s \in S$ is the first group of twisted cohomology $H^1(X_s, L_s)$. We assume that $\pi : \mathscr{X} \to S$ and Ω are such that B is a local system on S, of finite rank denoted by r. Then, tensoring by the structural sheaf of S, one obtains

$$\mathscr{B} = B \otimes_{\mathbb{C}} \mathscr{O}_S\,.$$

It is a locally free sheaf of rank r on S. Moreover, there exists a unique connection on $\mathscr{B}$ whose kernel is B. The latter is known as the **Gauß-Manin connection** and will be denoted by

$$\nabla^{GM} : \mathscr{B} \to \mathscr{B} \otimes \Omega^1_S.$$

We have thus given an analytic definition of the Gauß-Manin connection in the relative twisted context. Note that this definition, although rather direct, is not constructive at all. We will remedy to this below.

B.1.3. We recall that the sheaves $\Omega^\bullet_{\mathscr{X}/S}$ of **relative differential forms** on $\mathscr{X}$ are the ones characterized by requiring that the following short sequences of $\mathscr{O}_{\mathscr{X}}$-sheaves are exact:

$$0 \longrightarrow \pi^*\Omega^\bullet_S \longrightarrow \Omega^\bullet_{\mathscr{X}} \longrightarrow \Omega^\bullet_{\mathscr{X}/S} \longrightarrow 0\,.$$

More concretely, let $s_1, \ldots, s_n$ stand for local holomorphic coordinates on a small open subset $U \subset S$ and let z be a vertical local coordinate on an open subset $\widetilde{U} \subset \pi^{-1}(U)$ étale over U. Then there are natural isomorphisms

$$\Omega^0_{\mathscr{X}/S}\big|_{\widetilde{U}} \simeq \mathscr{O}_{\widetilde{U}} \qquad \text{and} \qquad \Omega^1_{\mathscr{X}/S}\big|_{\widetilde{U}} \simeq \mathscr{O}_{\widetilde{U}} \cdot dz. \tag{135}$$

For any local section η of $\Omega^\bullet_{\mathscr{X}}$, we denote by $\eta_{\mathscr{X}/S}$ the section of $\Omega^\bullet_{\mathscr{X}/S}$ it induces. With the above notation, assuming that η is a holomorphic 1-form on $\widetilde{U}$, then the local isomorphism (135) identifies $\eta_{\mathscr{X}/S}$ with $\eta - \eta(\partial_z)dz$.

Since the exterior derivative d commutes with the pull-back by π, one obtains the **relative de Rham complex** $(\Omega^\bullet_{\mathscr{X}/S}, d)$. One verifies easily that the connexion ∇_Ω on $\Omega^\bullet_{\mathscr{X}}$ induces a connexion $\nabla_{\mathscr{X}/S}$ on the relative de Rham complex, so that any square of $\mathscr{O}_{\mathscr{X}}$-sheaves as below is commutative:

$$\begin{array}{ccccc}
\Omega^\bullet_{\mathscr{X}} & \longrightarrow & \Omega^\bullet_{\mathscr{X}/S} & \longrightarrow & 0 \\
\downarrow{\scriptstyle\nabla} & & \vdots{\scriptstyle\nabla_{\mathscr{X}/S}} & & \\
\Omega^{\bullet+1}_{\mathscr{X}} & \longrightarrow & \Omega^{\bullet+1}_{\mathscr{X}/S} & \longrightarrow & 0\,.
\end{array}$$

In the local coordinates $(s_1, \ldots, s_n, z)$ considered above, writing $\Omega = \omega + \varphi dz$ for a holomorphic function φ and a 1-form ω such that $\omega(\partial_z) = 0$ (*i.e.* $\omega = \Omega_{\mathscr{X}/S}$ with the notation introduced above), it comes that $\nabla_{\mathscr{X}/S}$ satisfies

$$\nabla_{\mathscr{X}/S}(f) = \sum_{i=1}^{n} \big(\partial f/\partial s_i\big) ds_i + \omega f \tag{136}$$

for any holomorphic function f on $\widetilde{U}$ and is characterized by this property.

By definition, $(\Omega^\bullet_{\mathscr{X}/S}, \nabla_{\mathscr{X}/S})$ is the **relative twisted de Rham complex** associated to π and Ω. Under some natural assomptions, the direct images $\pi_*\Omega^\bullet_{\mathscr{X}/S}$ are coherent sheaves of $\mathscr{O}_S$-modules and $\nabla_{\mathscr{X}/S}$ gives rise to a connection on S

$$\pi_*\big(\nabla_{\mathscr{X}/S}\big) : \pi_*(\mathscr{O}_{\mathscr{X}}) \longrightarrow \pi_*\big(\Omega^1_{\mathscr{X}/S}\big)\,.$$

Note that $\pi_*(\mathscr{O}_{\mathscr{X}})$ is nothing else but $\mathscr{O}_{\mathscr{X}}$ seen as a $\mathscr{O}_S$-module by means of π. For this reason, we will abusively write down the preceding connection as

$$\nabla_{\mathscr{X}/S} : \mathscr{O}_{\mathscr{X}} \longrightarrow \pi_*\big(\Omega^1_{\mathscr{X}/S}\big)\,. \tag{137}$$

B.1.4. On the other hand, the map

$$U \longmapsto \mathbf{H}^1\Big(\pi^{-1}(U), \big(\Omega^\bullet_{\mathscr{X}/S}, \nabla_{\mathscr{X}/S}\big)\big|_U\Big)$$

defines a presheaf (of hypercohomology groups) on S. The associated sheaf is denoted by $R^1\pi_*(\Omega^\bullet_{\mathscr{X}/S}, \nabla)$. Its stalk at any $s \in S$ coincides with $\mathbf{H}^1(X_s, (\Omega^\bullet_{X_s}, \omega_s))$ hence is naturally isomorphic to $H^1(X_s, L_s)$ (see §3.1.8).

It follows that one has a natural isomorphism

$$\mathscr{B} \simeq R^1\pi_*\big(\Omega^\bullet_{\mathscr{X}/S}, \nabla\big).$$

We make the supplementary assumption that π is affine (this implies in particular that the fibers X_s can no more be assumed to be compact). Then it follows (see [3, Chapt.III,§2.7]) that $R^1\pi_*\big(\Omega^\bullet_{\mathscr{X}/S}, \nabla\big)$ hence $\mathscr{B}$ identifies with the cokernel of the connection $\pi_*\big(\nabla_{\mathscr{X}/S}\big)$, denoted by $\nabla_{\mathscr{X}/S}$ for short, see (137).

In other terms, one has a natural isomorphism of $\mathscr{O}_S$-sheaves

$$\mathscr{B} \simeq \frac{\pi_*\Omega^1_{\mathscr{X}/S}}{\nabla_{\mathscr{X}/S}\big(\mathscr{O}_{\mathscr{X}}\big)}. \tag{138}$$

For a local section $\eta_{\mathscr{X}/S}$ of $\Omega^1_{\mathscr{X}/S}$, we denote by $[\eta_{\mathscr{X}/S}]$ its class in $\mathscr{B}$, or equivalently, its class modulo $\nabla_{\mathscr{X}/S}(\mathscr{O}_{\mathscr{X}})$.

By means of the latter isomorphism, one can give an effective description of the action of the Gauß-Manin connection. Let v be a vector field over the open subset $U \subset S$ of T_S (*i.e.* an element of $\Gamma(U, T_S)$). Then

$$\nabla^{GM}_v = \big\langle \nabla^{GM}(\cdot), v\big\rangle$$

is a derivation of the $\mathscr{O}_U$-module $\Gamma(U, \mathscr{B})$. An element of this space of sections is represented by the class $[\eta_{\mathscr{X}/S}]$ (that is $\eta_{\mathscr{X}/S}$ modulo $\nabla_{\mathscr{X}/S}\mathscr{O}(\widetilde{U})$) of a relative 1-form $\eta_{\mathscr{X}/S} \in \Gamma(\tilde{U}, \Omega^1_{\mathscr{X}/S})$. Let $\tilde{\eta}$ be a section of $\Omega^1_{\mathscr{X}}$ over $\tilde{U}$ such that $\tilde{\eta}_{\mathscr{X}/S} = \eta_{\mathscr{X}/S}$. Then, for any lift $\tilde{v}$ of v over $\tilde{U}$, one has

$$\nabla^{GM}_v\big([\eta_{\mathscr{X}/S}]\big) = \big[\nabla_{\tilde{v}}(\tilde{\eta})_{\mathscr{X}/S}\big].$$

Finally, we mention that when not only π but also S is supposed to be affine (as will hold true in the case we will interested in, *cf.* B.3 below), then there is a more elementary description of the RHS of the isomorphism (138). Indeed, in this case, according to [3, p.117], $\mathscr{B}$ identifies with the $\mathscr{O}_S$-module attached to the first cohomology group of the complex of global sections

$$\mathscr{O}(\mathscr{X}) \to \Omega^1_{\mathscr{X}/S}(\mathscr{X}) \to \Omega^2_{\mathscr{X}/S}(\mathscr{X}) \to \cdots.$$

If additionally S is assumed to be of dimension 1, then $\Omega^2_{\mathscr{X}/S}$ is trivial, hence one obtains the following generalization of (25) in the relative case:

$$\mathscr{B} \simeq \mathscr{O}_S \otimes_{\mathbb{C}} \frac{H^0\big(\mathscr{X}, \Omega^1_{\mathscr{X}/S}\big)}{\nabla_{\mathscr{X}/S}\big(H^0\big(\mathscr{X}, \mathscr{O}_{\mathscr{X}}\big)\big)}. \tag{139}$$

B.1.5. Assume that the fibers X_s's are punctured Riemann surfaces and that $\mathscr{X}$ can be compactified in the vertical direction into a family $\overline{\pi} : \overline{\mathscr{X}} \to S$ of compact Riemann surfaces. The original map π is the restriction of $\overline{\pi}$ to $\mathscr{X}$ which is nothing else but the complement of a divisor $\mathscr{Z}$ in $\overline{\mathscr{X}}$.

Instead of considering holomorphic (relative) differential forms on $\mathscr{X}$ as above, one can make the same constructions using rational (relative) differential forms on $\overline{\mathscr{X}}$ with poles on $\mathscr{Z}$. More concretely, one makes all the constructions sketched above starting from the sheaves of $\mathscr{O}_{\overline{\mathscr{X}}}(*\mathscr{Z})$-modules $\Omega^{\bullet}_{\overline{\mathscr{X}}}(*\mathscr{Z})$ on $\mathscr{X}$.

A fundamental result of the field, due to Deligne in its full generality, is that the twisted comparison theorem mentioned in §3.1.9 can be generalized to the relative case, at least when $\mathscr{Z}$ is a relative divisor with normal crossing over S (see [10, Théorème 6.13] or [3, Chap.4, Theorem 3.1] for precise statements).

In the particular case of relative dimension 1, this gives the following version of the isomorphism (138):

$$\mathscr{B} \simeq \frac{\pi_* \Omega^1_{\mathscr{X}/S}(*\mathscr{Z})}{\nabla_{\mathscr{X}/S}\big(\mathscr{O}_{\mathscr{X}}(*\mathscr{Z})\big)}.$$

When S is also assumed affine, one gets the following generalization of (26):

$$\mathscr{B} \simeq \mathscr{O}_S \otimes_{\mathbb{C}} \frac{H^0\big(\mathscr{X}, \Omega^1_{\mathscr{X}/S}(*\mathscr{Z})\big)}{\nabla_{\mathscr{X}/S}\big(H^0\big(\mathscr{X}, \mathscr{O}_{\mathscr{X}}(*\mathscr{Z})\big)\big)}. \tag{140}$$

B.1.6. We now explain how the material introduced above can be used to construct differential systems satisfied by generalized hypergeometric integrals.

Let $\check{B}$ be the dual of B. It is the local system on S whose fiber $\check{B}_s$ at s is the twisted homology group $H_1(X_s, L_s^\vee)$. Let $\check{\nabla}^{GM}$ be the **dual Gauß-Manin connection** on the associated sheaf $\check{\mathscr{B}} = \mathscr{O}_S \otimes \check{B}$. We recall that, by definition, it is the connection the solutions of which form the local system $\check{B}$. It can also be characterized by the following property: for any local sections with the same definition domain b and $\check{\beta}$ of $\mathscr{B}$ and $\check{\mathscr{B}}$ respectively, one has

$$d\langle b, \check{\beta}\rangle = \Big\langle \nabla^{GM}(b), \check{\beta}\Big\rangle + \Big\langle b, \check{\nabla}^{GM}\big(\check{\beta}\big)\Big\rangle. \tag{141}$$

B.1.7. We use again the notations of B.1.1. Let T be a global (but multivalued) function on $\mathscr{X}$ satisfying $\check{\nabla}(T) = dT - \Omega T = 0$. For any $s \in S$, one denotes its restriction to X_s by T_s. Let I be the local holomorphic function defined on a small open subset $U \subset S$ as the following generalized hypergeometric integral depending holomorphically on s:

$$I(s) = \int_{\boldsymbol{\gamma}_s} T_s \cdot \eta^s . \tag{142}$$

Here the $\boldsymbol{\gamma}_s$'s stand for $L_s^\vee$-twisted 1-cycles which depend analytically on $s \in U$ and $s \mapsto \eta^s$ is a holomorphic 'section of $\Omega^1_{\mathscr{X}}$ over U', *i.e.* $\eta^s \in \Omega^1(X_s)$ for every $s \in U$ and the dependency with respect to s is holomorphic. From what has been said above, the value $I(s)$ actually depends only on the twisted homology classes $[\boldsymbol{\gamma}_s]$ and on the class $[\eta^s_{\mathscr{X}/S}]$ of η^s in $H^0(X_s, \Omega^1_{X_s})/\nabla_s(\mathscr{O}(X_s))$.

In other terms, for every $s \in U$, one has

$$I(s) = \Big\langle [\boldsymbol{\gamma}_s] \, , \, [\eta^s_{\mathscr{X}/S}] \Big\rangle . \tag{143}$$

To simplify the discussion, assume now that S is affine and of dimension 1 (as it will be the case in B.3 below). Now s has to be understood as a global holomorphic coordinate on $U = S$. Setting $\sigma = \partial/\partial s$, one denotes the associated derivation by $\nabla^{GM}_\sigma(\cdot) = \langle \nabla^{GM}(\cdot)\, , \sigma \rangle$. We define $\check{\nabla}^{GM}_\sigma$ similarly.

In most of the cases (if not all), the twisted 1-cycles $\boldsymbol{\gamma}_s$'s appearing in such an integral are locally obtained by topological deformations. In this case, it is well known (*cf.* [11, Remark (**3.6**)]) that $s \mapsto [\boldsymbol{\gamma}_s]$ is a section of $\check{B}$ hence belongs to the kernel of $\check{\nabla}^{GM}$, *i.e.* $\check{\nabla}^{GM}(\boldsymbol{\gamma}_s) \equiv 0$.

Let $\widetilde{\sigma}$ be a fixed lift of σ over U. Then from (141) and (143), it follows that

$$\begin{aligned} I'(s) = \frac{d}{ds} \int_{\boldsymbol{\gamma}_s} T_s \cdot \eta^s &= \frac{d}{ds} \Big\langle [\boldsymbol{\gamma}_s] \, , \, [\eta^s_{\mathscr{X}/S}] \Big\rangle \\ &= \Big\langle [\boldsymbol{\gamma}_s] \, , \, \nabla^{GM}_\sigma [\eta^s_{\mathscr{X}/S}] \Big\rangle = \int_{\boldsymbol{\gamma}_s} T_s \cdot \nabla_{\widetilde{\sigma}}\big(\eta^s\big) \end{aligned}$$

for every $s \in U$. More generally, for any integer n, one has

$$I^{(n)}(s) = \Big\langle [\boldsymbol{\gamma}_\sigma] \, , \, \big(\nabla^{GM}_\sigma\big)^n \, [\eta^s_{\mathscr{X}/S}] \Big\rangle = \int_{\boldsymbol{\gamma}_s} T_s \cdot \nabla^n_{\widetilde{\sigma}}\big(\eta^s\big) \tag{144}$$

where $\nabla^n_{\widetilde{\sigma}}$ stands for the n-th iterate of $\nabla_{\widetilde{\sigma}}$ acting on the sheaf of 1-forms on $\mathscr{X}$.

To make the writing simpler, if μ is a section of $\Omega^1_{\mathscr{X}}$, we will denote the section $[\mu_{\mathscr{X}/S}]$ of $H^0(\mathscr{X}, \Omega^1_{\mathscr{X}/S})/\nabla_{\mathscr{X}/S}(H^0(\mathscr{X}, \mathscr{O}_{\mathscr{X}}))$ that it induces just by $[\mu]$ below.

By hypothesis, the twisted cohomology groups $H^1(X_s, L_s)$ are all of the same finite dimension $N > 0$. It follows that there is a non-trivial $\mathscr{O}(U)$-linear relation between the classes of the $\nabla_\sigma^k(\eta^s)$'s for $k = 0,\dots,N$, *i.e.* there exists $(A_0,\dots,A_N) \in \mathscr{O}(U)^{N+1}$ non-trivial and such that the following relation

$$A_0(s)\cdot\big[\eta^s\big] + A_1(s)\big[\nabla_\sigma\big(\eta^s\big)\big] + \cdots + A_N(s)\big[\nabla_\sigma^N\big(\eta^s\big)\big] = 0$$

holds true for every $s \in U$. Since the value of the k-th derivative $I^{(k)}$ at s actually depends only on the class of $\nabla_\sigma^k(\eta^s)$ (see (144)), one obtains that the function I satisfies the following linear differential equation on U:

$$A_0 \cdot I + A_1 \cdot I' + \cdots + A_N \cdot I^{(N)} = 0\,.$$

Note that the function I defined in (142) is not the only solution of this differential equation. Indeed, it is quite clear that this equation is also satisfied by any function of the form $s \mapsto \int_{\boldsymbol{\beta}_s} T_s\cdot\eta^s$ as soon as $s\mapsto \boldsymbol{\beta}_s$ is a section of $\check{B}$.

B.2. The Gauß-Manin connection on a leaf of Veech's foliation. We are now going to specialize the material presented in the preceding subsections to the case of punctured elliptic curves we are interested in.

In what follows, as before, $\alpha_1,\dots,\alpha_n$ stand for fixed real numbers bigger than -1 that sum up to 0: one has $\alpha_i \in]-1,\infty[$ for $i = 1,\dots,n$ and $\sum_{i=1}^n \alpha_i = 0$.

B.2.1. Forgetting the last variable z_{n+1} induces a projection from $\mathscr{T}or_{1,n+1}$ onto $\mathscr{T}or_{1,n}$. For our purpose, it will be convenient to see this space rather as a kind of covering space of the 'universal curve' over the target Torelli space. For this reason, we will write u instead of z_{n+1} and take this variable as the first one.

In other terms, we consider

$$\mathscr{C}\mathscr{T}or_{1,n} = \Big\{\big(u,\tau,z\big)\in\mathbb{C}\times\mathscr{T}or_{1,n}\ \big|\ u\in\mathbb{C}\setminus\cup_{i=1}^n\big(z_i+\mathbb{Z}_\tau\big)\Big\}\simeq\mathscr{T}or_{1,n+1}$$

and the corresponding projection $\mathscr{C}\mathscr{T}or_{1,n}\to\mathscr{T}or_{1,n} : (u,\tau,z)\to(\tau,z)$.

We define two automorphisms of $\mathscr{C}\mathscr{T}or_{1,n}$ by setting

$$T_1(u,\tau,z) = \big(u+1,\tau,z\big) \qquad\text{and}\qquad T_\tau(u,\tau,z) = \big(u+\tau,\tau,z\big)$$

for any element (u,τ,z) of this space.

The group spanned by T_1 and T_τ is isomorphic to $\mathbb{Z}^2$ and acts discontinuously without fixed points on $\mathscr{C}\mathscr{T}or_{1,n}$. The associated quotient, denoted by $\mathscr{E}_{1,n}$, is nothing but the **universal n-punctured elliptic curve** over $\mathscr{T}or_{1,n}$. This terminology is justified by the fact that the projection onto $\mathscr{T}or_{1,n}$ factorizes and gives rise to a fibration

$$\pi : \mathscr{E}_{1,n} \longrightarrow \mathscr{T}or_{1,n}$$

the fiber of which at $(\tau,z)\in\mathscr{T}or_{1,n}$ is the n-punctured elliptic curve $E_{\tau,z}$.

There is a partial vertical compactification

$$\overline{\pi} : \overline{\mathscr{E}}_{1,n} \longrightarrow \mathscr{T}or_{1,n}$$

whose fiber at (τ, z) is the unpunctured elliptic curve E_τ. The latter extends π, is smooth and proper and comes with n canonical sections (for $k = 1, \ldots, n$):

$$\begin{aligned}[k]_{1,n} : \mathscr{T}or_{1,n} &\longrightarrow \overline{\mathscr{E}}_{1,n} \\ \big(\tau, z\big) &\longmapsto [z_k] \in E_\tau\,.\end{aligned}$$

In particular, because of the normalization $z_1 = 0$, $[1]_{1,n}$ is nothing else but the zero section $[0]_{1,n}$ which associates $[0] \in E_\tau$ to (τ, z): one has $[1]_{1,n} = [0]_{1,n}$.

B.2.2. Recall the expression (28) for the function T considered in Section 3:

$$T(u, \tau, z) = \exp\big(2i\pi\alpha_0 u\big)\prod_{k=1}^{n}\theta\big(u - z_k, \tau\big)^{\alpha_k}\,.$$

Contrary to §3 where τ and z were assumed to be fixed and only u was allowed to vary, we want now all the variables u, τ and z to be free. In other terms, we now see T as a multivalued holomorphic function on $\mathscr{C}\mathscr{T}or_{1,n}$.

Let Ω stand for the total logarithmic derivative of T on $\mathscr{C}\mathscr{T}or_{1,n}$:

$$\Omega = d\log T = (\partial \log T/\partial u)du + (\partial \log T/\partial \tau)d\tau + \sum_{j=2}^{n}(\partial \log T/\partial z_j)dz_j\,.$$

A straightforward computation shows that

$$\Omega = \omega + \sum_{k=1}^{n}\alpha_k\Big[\eta(u - z_k)d\tau - \rho(u - z_k)dz_k\Big] \tag{145}$$

where

- $\omega = (2i\pi\alpha_0 + \delta)du$ stands for the logarithmic total derivative of T with respect to the single variable u (thus $\delta = \sum_{k=1}^{n}\alpha_k\rho(u - z_k)$ see (29) in Section 3) but now considered as a holomorphic 1-form on $\mathscr{C}\mathscr{T}or_{1,n}$;
- we have set for any $(u, \tau, z) \in \mathscr{C}\mathscr{T}or_{1,n}$:

$$\eta(u) = \eta(u, \tau) = \partial \log\theta(u, \tau)/\partial\tau = \frac{1}{4i\pi}\frac{\theta''(u, \tau)}{\theta(u, \tau)}\,.$$

After easy computations, one deduces from the functional equations (17) that for every $\tau \in \mathbb{H}$ and every $u \in \mathbb{C} \setminus \mathbb{Z}_\tau$, one has

$$\rho(u + 1) = \rho(u) \qquad\qquad \eta(u + 1) = \eta(u) \tag{146}$$

$$\rho(u + \tau) = \rho(u) - 2i\pi \qquad\qquad \eta(u + \tau) = \eta(u) - \rho(u) + i\pi\,. \tag{147}$$

In Section 3, we have shown that when τ is assumed to be fixed, ω is $\mathbb{Z}_\tau$-invariant. It follows that, on $\mathscr{C}\mathscr{T}or_{1,n}$, one has:

$$T_1^*(\omega) = \omega \qquad \text{and} \qquad T_\tau^*(\omega) = \omega + \big(2i\pi\alpha_0 + \delta\big)d\tau\,. \tag{148}$$

We set

$$\widetilde{\omega} = \Omega - \omega = \sum_{k=1}^{n} \alpha_k \eta\big(u - z_k\big) d\tau - \sum_{k=1}^{n} \alpha_k \rho\big(u - z_k\big) dz_k .$$

It follows immediately from (146) that $T_1^*(\widetilde{\omega}) = \widetilde{\omega}$. With (148), this gives us

$$T_1^*\big(\Omega\big) = \Omega . \tag{149}$$

On the other hand, using (147) and the fact that $\sum_{k=1}^n \alpha_k = 0$, one has

$$T_\tau^*\big(\widetilde{\omega}\big) = \widetilde{\omega} - \delta d\tau + 2i\pi \sum_{k=1}^{n} \alpha_k dz_k .$$

Combining the latter equation with (148), one eventually obtains

$$T_\tau^*(\Omega) = \Omega + 2i\pi\Big(\alpha_0 d\tau + \sum_{k=2}^{n} \alpha_k dz_k\Big). \tag{150}$$

B.2.3. The fact that Ω is not T_τ-invariant prevents this 1-form from descending onto $\mathscr{E}_{1,n}$. However, viewed the obstruction $T_\tau^*(\Omega) - \Omega$ explicited just above, this will not be the case over a leaf of Veech's foliation on the Torelli space.

More precisely, let $a = (a_0, a_\infty) \in \mathbb{R}^2$ be such that the leaf $\mathscr{F}_a = \mathscr{F}_{(a_0,a_\infty)}$ of Veech's foliation on $\mathscr{T}or_{1,n}$ is not empty. Remember that this leaf is cut out by the equation

$$a_0\tau + \sum_{j=2}^{n} \alpha_j z_j = a_\infty . \tag{151}$$

Let $\mathscr{E}_a$ and $\mathscr{C}\mathscr{T}or_a$ stand for the restrictions of $\mathscr{E}_{1,n}$ and $\mathscr{C}\mathscr{T}or_{1,n}$ over $\mathscr{F}_a$ respectively. Clearly, $\mathscr{C}\mathscr{T}or_a$ is invariant by T_1 and T_τ. Moreover, from (151), it comes that $a_0 d\tau + \sum_{j=2}^n \alpha_j dz_j = 0$ when restricting to $\mathscr{F}_a$.

Thus, denoting by Ω_a the restriction of Ω to $\mathscr{C}\mathscr{T}or_a$, it follows from (149) and (150) that

$$T_1^*\big(\Omega_a\big) = \Omega_a \qquad \text{and} \qquad T_\tau^*\big(\Omega_a\big) = \Omega_a .$$

This means that Ω_a descends to $\mathscr{E}_a$ as a holomorphic 1-form. We denote again its push-forward onto $\mathscr{E}_a$ by Ω_a.

Looking at (145), it is quite clear that for any $(\tau, z) \in \mathscr{F}_a$, one has

$$\Omega_a|_{E_{\tau,z}} = \omega_a(\cdot, \tau, z) \tag{152}$$

where the right-hand side is the rational 1-form (29) on $E_{\tau,z} = \pi^{-1}(\tau, z)$.

With the help of Ω_a we are going to make the same constructions as in Section 3 but relatively over the leaf $\mathscr{F}_a$.

B.2.4. We now specialize the constructions and results of B.1 by taking

$$\mathscr{X} = \mathscr{E}_a, \qquad S = \mathscr{F}_a \qquad \text{and} \qquad \Omega = \Omega_a.$$

The covariant operator $\nabla_{\Omega_a} : \eta \mapsto d\eta + \Omega_a \wedge \eta$ induces an integrable connexion on $\Omega^\bullet_{\mathscr{E}_a}$. Its kernel L_a is a local system of rank 1 on $\mathscr{E}_a$. Moreover, it follows immediately from (152) that given (τ, z) in $\mathscr{F}_a$, its restriction to $E_{\tau,z} = \pi^{-1}(\tau, z)$ coincides with the local system $L_{\omega(\cdot,\tau,z)}$ associated to the 1-form $\omega(\cdot, \tau, z)$ on $E_{\tau,z}$ constructed in §3.2, denoted here by $L_{\tau,z}$ for simplicity.

On the leaf $\mathscr{F}_a \subset \mathscr{T}or_{1,n}$, one considers the local system $B_a = R^1\pi_*(L_a)$ whose stalk at (τ, z) is nothing else but $H^1(E_{\tau,z}, L_{\tau,z})$. The associated sheaf $\mathscr{B}_a = \mathscr{O}_{\mathscr{F}_a} \otimes_{\mathbb{C}} B_a$ is locally free and of rank n according to Theorem 3.3.

We are interested in the Gauß-Manin connection

$$\nabla_a^{GM} : \mathscr{B}_a \to \mathscr{B}_a \otimes \Omega^1_{\mathscr{F}_a}$$

which we would like to make explicit.

Let $\overline{\mathscr{E}}_a$ and $[k]_a$ (for $k = 1, \ldots, n$) stand for the restrictions of $\overline{\mathscr{E}}_{1,n}$ and of $[k]_{1,n}$ over $\mathscr{F}_a$. For any $k = 1, \ldots, n$, the image of $\mathscr{F}_a$ by $[k]_a$ is a divisor in $\overline{\mathscr{E}}_a$, denoted by $Z[k]_a$. Consider their union

$$\mathscr{Z}_a = \bigcup_{k=1}^{n} Z[k]_a.$$

It is a relative divisor in $\overline{\mathscr{E}}_a$ with simple normal crossing (the $Z[k]_a$'s are smooth and pairwise disjoint!), hence Deligne's comparison theorem of B.1.5 applies: there is an isomorphism of $\mathscr{O}_{\mathscr{F}_a}$-sheaves

$$\mathscr{B}_a \simeq \mathscr{O}_{\mathscr{F}_a} \otimes_{\mathbb{C}} \frac{H^0\big(\mathscr{E}_a, \Omega^1_{\mathscr{E}_a/\mathscr{F}_a}(*\mathscr{Z}_a)\big)}{\nabla_{\mathscr{E}_a/\mathscr{F}_a}\big(H^0\big(\mathscr{E}_a, \mathscr{O}_{\mathscr{E}_a}(*\mathscr{Z}_a)\big)\big)}. \tag{153}$$

Remark B.2.4. Actually, the geometrical picture we have can be summarized by the following commutative diagram

$$\begin{array}{ccccc} \mathscr{E}_a & \hookrightarrow & \overline{\mathscr{E}}_a & \hookleftarrow & \mathscr{Z}_a \\ & {\scriptstyle \pi_a} \searrow & \downarrow {\scriptstyle \overline{\pi}_a} & \swarrow & \\ & & \mathscr{F}_a & & \end{array}$$

where the two horizontal arrows are complementary inclusions. Since the restriction of $\overline{\pi}_a$ to $\mathscr{Z}_a$ is obviously an étale covering, this means that $\pi_a : \mathscr{E}_a \to \mathscr{F}_a$ is precisely what is called an *'elementary fibration'* in [3]. Even better, quotienting by the elliptic involution over $\mathscr{F}_a$ (which exists since the latter is affine), one sees that $\overline{\pi}_a$ factorizes through the relative projective line $\mathbb{P}^1_{\mathscr{F}_a} \to \mathscr{F}_a$. In the terminology of [3], this means that the elementary fibration π_a can be *'coordinatized'*.

B.2.5. At this point, we use Theorem 3.3 to obtain a relative version of it.

We consider the horizontal non-reduced divisor supported on $\mathscr{Z}_a$:

$$\mathscr{Z}'_a = \mathscr{Z}_a + Z[0]_a = 2Z[0]_a + \sum_{k=2}^{n} Z[k]_a \,.$$

For dimensional reasons, it follows immediately from Theorem 3.3 that

$$\mathscr{B}_a \simeq \mathscr{O}_{\mathscr{F}_a} \otimes_{\mathbb{C}} \frac{H^0\big(\mathscr{E}_a, \Omega^1_{\mathscr{E}_a/\mathscr{F}_a}(\mathscr{Z}'_a)\big)}{\nabla_{\mathscr{E}_a/\mathscr{F}_a}\big(H^0\big(\mathscr{E}_a, \mathscr{O}_{\mathscr{E}_a}(\mathscr{Z}'_a)\big)\big)}\,.$$

Recall the 1-forms

$$\varphi_0 = du\,, \qquad \varphi_1 = \rho'(u,\tau,z)du \quad \text{and} \quad \varphi_k = \big(\rho(u-z_k,\tau) - \rho(u,\tau)\big)du$$

(with $k = 2, \ldots, n$) considered in §3.3.2. We now consider them with (τ, z) varying in $\mathscr{F}_a$. Then these appear as elements of $H^0(\mathscr{E}_a, \Omega^1_{\mathscr{E}_a/\mathscr{F}_a}(\mathscr{Z}'_a))$. Moreover, they span this space and if $[\varphi_0], \ldots, [\varphi_n]$ stand for their associated classes up to the image of $H^0(\mathscr{E}_a, \mathscr{O}_{\mathscr{E}_a}(\mathscr{Z}'_a))$ by $\nabla_{\mathscr{E}_a/\mathscr{F}_a}$, it follows from Theorem 3.3 that $[\varphi_0], \ldots, [\varphi_{n-1}]$ form a basis of $\mathscr{B}_a$ over $\mathscr{O}_{\mathscr{F}_a}$. In other terms, one has

$$\mathscr{B}_a \simeq \mathscr{O}_{\mathscr{F}_a} \otimes \big(\oplus_{i=0}^{n-1} \mathbb{C}[\varphi_i]\big)\,.$$

From the preceding trivialization, one deduces that

$$\nabla^{GM} \begin{pmatrix} [\varphi_0] \\ \vdots \\ [\varphi_{n-1}] \end{pmatrix} = M \begin{pmatrix} [\varphi_0] \\ \vdots \\ [\varphi_{n-1}] \end{pmatrix}$$

for a certain matrix $M \in GL_n(\Omega^1_{\mathscr{F}_a})$ which completely characterizes the Gauß-Manin connection. We explain below how M can be explicitly computed.

B.2.6. Knowing ∇^{GM} is equivalent to knowing the action of any $\mathscr{O}_{\mathscr{F}_a}$- derivation

$$\nabla^{GM}_{\sigma} = \big\langle \nabla^{GM}\,, \sigma \big\rangle : \mathscr{B}_a \longrightarrow \mathscr{B}_a$$

for any vector field σ on $\mathscr{F}_a$. Since τ and $z_2, \ldots, z_{n-1}$ are global affine coordinates on $\mathscr{F}_a$, $T\mathscr{F}_a$ is a locally free $\mathscr{O}_{\mathscr{F}_a}$-module with $(\partial/\partial\tau, \partial/\partial z_2, \ldots, \partial/\partial z_{n-2})$ as a basis. It follows that the Gauß-Manin connection we are interested in is completely determined by the n 'Gauß-Manin derivations'

$$\nabla^{GM}_{\tau} := \nabla^{GM}_{\partial/\partial\tau} \qquad \text{and} \qquad \nabla^{GM}_{z_i} := \nabla^{GM}_{\partial/\partial z_i} \quad \text{for } i = 2, \ldots, n-1. \tag{154}$$

Let U be a non-empty open sub-domain of $\mathscr{F}_a$ and set $\widetilde{U} = \pi^{-1}(U)$.

For $\widetilde{\eta} \in \Gamma(\widetilde{U}, \Omega^1_{\mathscr{E}_a})$, we recall the following notations:

- $\widetilde{\eta}_{\mathscr{E}_a/\mathscr{F}_a}$ stands for the class of η in $\Gamma(\widetilde{U}, \Omega^1_{\mathscr{E}_a/\mathscr{F}_a})$.
- $[\widetilde{\eta}_{\mathscr{E}_a/\mathscr{F}_a}]$ stands for the class of $\eta_{\mathscr{E}_a/\mathscr{F}_a}$ modulo the image of $\nabla_{\mathscr{E}_a/\mathscr{F}_a}$.

Let μ be a section of $\pi_*\Omega^1_{\mathscr{E}_a/\mathscr{F}_a}$ over U. To compute $\nabla^{GM}_\xi(\mu)$ with $\xi = \tau$ or $\xi = z_i$ with $i \in \{2, \ldots, n-1\}$, we first consider a relative 1-form $\eta_{\mathscr{E}_a/\mathscr{F}_a}$ over $\widetilde{U}$ such that $[\eta_{\mathscr{E}_a/\mathscr{F}_a}] = \mu$ (here we use the isomorphism (153)).

In the coordinates $u, \tau, z = (z_2, \ldots, z_{n-1})$ on $\mathscr{E}_a$, one can write explicitly

$$\eta_{\mathscr{E}_a/\mathscr{F}_a} = N(u, \tau, z)du$$

for a holomorphic function N such that for any $(\tau, z) \in U$, the map $u \mapsto N(u, \tau, z)$ is a rational function on E_τ, with poles at $[0]$ and $[z_2], \ldots, [z_n]$ exactly, where

$$z_n = \frac{1}{\alpha_n}\Big(a_\infty - a_0\tau - \sum_{k=2}^{n-1} \alpha_k z_k\Big).$$

Consider the following 1-form

$$\Xi = du + \frac{\rho(u,\tau)}{2i\pi} d\tau$$

which is easily seen to be invariant by T_1 and T_τ.

Then one defines

$$\widetilde{\eta} = N \cdot \Xi = N(u, \tau, z)\Big(du + \frac{\rho(u,\tau)}{2i\pi} d\tau\Big). \tag{155}$$

Using the fact that $N(u, \tau, z)$ is $\mathbb{Z}_\tau$-invariant with respect to u when $(\tau, z) \in U$ is fixed, one verifies easily that the 1-form $\widetilde{\eta}$ defined just above is invariant by T_1 and T_τ hence descends to a section of $\pi_*\Omega^1_{\mathscr{E}_a}$ over U, again denoted by $\widetilde{\eta}$[38].

The vector fields

$$\zeta_\tau = \frac{\partial}{\partial \tau} - \frac{\rho}{2i\pi}\frac{\partial}{\partial u} \qquad \text{and} \qquad \zeta_i = \frac{\partial}{\partial z_i} \quad \text{for } i = 2, \ldots, n-1 \tag{156}$$

all are invariant by T_1 and by T_τ hence descend to rational vector fields on $\overline{\mathscr{E}}_a$ with poles along $\mathscr{Z}_a$, all denoted by the same notation. Clearly, one has $\pi_*(\zeta_\tau) = \partial/\partial\tau$ and $\pi_*(\zeta_i) = \partial/\partial z_i$ for $i = 2, \ldots, n-1$.

We now have at our disposal everything we need to compute the actions of the derivations (154) on $\mu \in \Gamma(U, \pi_*\Omega^1_{\mathscr{E}_a/\mathscr{F}_a})$: for $\star \in \{\tau, z_2, \ldots, z_{n-1}\}$, one has

$$\nabla^{GM}_\star \mu = \Big[\big\langle \nabla\widetilde{\eta}, \zeta_\star \big\rangle_{\mathscr{E}_a/\mathscr{F}_a}\Big] = \Big[\big\langle d\widetilde{\eta} + \Omega_a \wedge \widetilde{\eta}, \zeta_\star \big\rangle_{\mathscr{E}_a/\mathscr{F}_a}\Big]$$

and the right hand side can be explicitly computed with the help of the explicit formulae (145), (155) and (156).

[38] More conceptually, the map $N(u, \tau, z)du \mapsto N(u, \tau, z)(du + (2i\pi)^{-1}\rho(u, \tau)d\tau)$ can be seen as a splitting of the epimorphism of sheaves $\Omega^1_{\mathscr{E}_a} \to \Omega^1_{\mathscr{E}_a/\mathscr{F}_a}$.

We will not make the computations of the $\nabla^{GM}_{\star}[\varphi_k]$ explicit in the general case but only in the case when $n=2$ just below.

B.3. The Gauß-Manin connection for elliptic curves with two conical points. One specializes now in the case when $n=2$. Then the leaf $\mathscr{F}_a$ is isomorphic to $\mathbb{H}$, hence the $\mathscr{O}_{\mathscr{F}_a}$-module of derivations on $\mathscr{F}_a$ is $\mathscr{O}_{\mathscr{F}_a}\cdot(\partial/\partial\tau)$. Thus in this case, the Gauß-Manin connection is completely determined by ∇^{GM}_{τ}.

We will use below the following convention about the partial derivatives of a function N holomorphic in the variables u and τ: we will denote by N_u or N' (resp. N_τ or $\overset{\bullet}{N}$) the partial derivative of N with respect to u (resp. to τ). The notation N' will be used to mean that we consider N as a function of u with τ fixed (and vice versa for $\overset{\bullet}{N}$).

B.3.1. As in B.2.6, let η be a section of $\pi_*\Omega^1_{\mathscr{E}_a/\mathscr{F}_a}$ over a small open subset $U\subset\mathscr{F}_a\simeq\mathbb{H}$. It is written

$$\eta = N(u,\tau)du$$

for a holomorphic function N which, for any $\tau\in U$, is rational on E_τ, with poles at $[0]$ and $[t]$ exactly, with

$$t = t_\tau = \frac{a_0}{\alpha_1}\tau - \frac{a_\infty}{\alpha_1}.$$

Then one has (with $\widetilde{\eta} = N\cdot\Xi = N(u,\tau)(du+(2i\pi)^{-1}\rho(u,\tau)d\tau)$):

$$\nabla_a\widetilde{\eta} = \nabla_a\big(N\cdot\Xi\big) = dN\wedge\Xi + N\cdot\nabla_a\Xi$$

and since $\langle\Xi,\zeta_\tau\rangle=0$, it follows that

$$\big\langle\nabla_a\widetilde{\eta},\zeta_\tau\big\rangle = \big\langle dN,\zeta_\tau\big\rangle\cdot\Xi + N\cdot\big\langle d\Xi,\zeta_\tau\big\rangle + N\cdot\big\langle\Omega_a\wedge\Xi,\zeta_\tau\big\rangle. \tag{157}$$

Easy computations give

$$\begin{aligned}\big\langle dN,\zeta_\tau\big\rangle &= N_\tau-(2i\pi)^{-1}\rho\cdot N_u,\\ \big\langle d\Xi,\zeta_\tau\big\rangle &= -(2i\pi)^{-1}\rho_u\cdot\Xi\\ \text{and}\quad \big\langle\Omega_a\wedge\Xi,\zeta_\tau\big\rangle &= \big(\Omega_\tau-(2i\pi)^{-1}\rho\cdot\Omega_u\big)\cdot\Xi.\end{aligned}$$

Injecting these into (157) and since $\Xi_{\mathscr{E}_a/\mathscr{F}_a}=du$, one finally gets

$$\big\langle\nabla_a\widetilde{\eta},\zeta_\tau\big\rangle_{\mathscr{E}_a/\mathscr{F}_a} = N_\tau du + \Omega_\tau N du - (2i\pi)^{-1}\nabla_{\mathscr{E}_a/\mathscr{F}_a}\big(\rho N\big) \tag{158}$$

where $\nabla_{\mathscr{E}_a/\mathscr{F}_a} = d_u(\cdot)+\Omega_u du\wedge\cdot$ stands for the vertical covariant derivation

$$\begin{aligned}\nabla_{\mathscr{E}_a/\mathscr{F}_a}:\mathscr{O}_{\mathscr{E}_a/\mathscr{F}_a} &\longrightarrow \Omega^1_{\mathscr{E}_a/\mathscr{F}_a}\\ F=F(u,\tau) &\longmapsto F_u du + F\Omega_u du.\end{aligned}$$

It follows essentially from (158) that the differential operator

$$\begin{aligned} \widetilde{\nabla}_\tau : \Omega^1_{\mathscr{E}_a/\mathscr{F}_a} &\longrightarrow \Omega^1_{\mathscr{E}_a/\mathscr{F}_a} \\ N du &\longmapsto N_\tau du + \Omega_\tau N du - \frac{1}{2i\pi} \nabla_{\mathscr{E}_a/\mathscr{F}_a}\big(\rho N\big) \end{aligned} \tag{159}$$

is a $\pi^{-1}\mathscr{O}_{\mathscr{F}_a}$-derivation which is nothing else but a lift of the Gauß-Manin derivation ∇_τ^{GM} we are interested in. The fact that $\widetilde{\nabla}_\tau$ is explicit will allow us to determine explicitly the action of ∇_τ^{GM} below.

Remark B.3.1. It is interesting to compare our formula (159) for $\widetilde{\nabla}_\tau$ to the corresponding one in [49], namely the specialization when $\lambda = 0$ of the one for the differential operator ∇_τ given just before Proposition 4.1 page 3878 in [49]. The latter is not completely explicit since in order to compute $\nabla_\tau N du$ with N as above it is necessary to introduce a deformation $N(u,\tau,\lambda)$ of $N = N(u,\tau)$ which is meromorphic with respect to λ. However such deformations $\varphi_i(u,\tau,\lambda)du$ are explicitly given for the $N_i = \varphi_i(u,\tau,0)du$'s (*cf.* [49, p. 3875]), hence Mano and Watanabe's formula can be used to effectively determine the Gauß-Manin connection. Note that our arguments above show that $\widetilde{\nabla}_\tau$ is a lift of the Gauß-Manin derivation ∇_τ^{GM} indeed. The corresponding statement is not justified in [49] and is implicitly left to the reader.

Finally, it is fair to mention a notable feature of Mano-Watanabe's operator ∇_τ that our $\widetilde{\nabla}_\tau$ does not share: for $i \in \{0,1,2\}$, $\nabla_\tau N_i$ is a rational 1-form on E_τ, with polar divisor $\geq 2[0] + [t_\tau]$, hence can be written as a linear combination in N_0, N_1 and N_2. This is not the case for the $\widetilde{\nabla}_\tau N_i$'s. For instance, $\widetilde{\nabla}_\tau N_1$ has a pole of order four at [0] (see also B.3.3 below).

B.3.2. Some explicit formulae. In the case under study, we have

$$T(u,\tau) = e^{2i\pi a_0 u}\theta(u)^{\alpha_1} \cdot \theta(u-t)^{-\alpha_1}$$

(with $t = (a_0/\alpha_1)\tau - (a_\infty/\alpha_1)$) hence

$$\Omega = d\log T = \Omega_u du + \Omega_\tau d\tau$$

with

$$\begin{aligned} \Omega_u &= \partial \log T/\partial u = 2i\pi a_0 + \alpha_1\big(\rho(u) - \rho(u-t)\big) \\ \text{and} \quad \Omega_\tau &= \partial \log T/\partial \tau = \frac{\alpha_1}{4i\pi}\left(\frac{\theta''(u)}{\theta(u)} - \frac{\theta''(u-t)}{\theta(u-t)}\right) + a_0\rho(u-t)\,. \end{aligned} \tag{160}$$

For $i = 0,1,2$, one writes $\varphi_i = N_i(u)du$ with

$$N_0(u) = 1\,, \qquad N_1(u) = \rho'(u) \qquad \text{and} \qquad N_2(u) = \rho(u-t) - \rho(u)\,.$$

The following functions will appear in our computations below:

$$P(u) = P(u,\tau) = \frac{\theta''(u)}{\theta(u)} - \frac{\theta''(u-t)}{\theta(u-t)} - 2\big(\rho(u)-\rho(u-t)\big)\cdot\rho(u)$$

$$\text{and} \qquad \mu(u) = \mu(u,\tau) = -\frac{1}{2}\left(\frac{\theta'''(u)}{\theta(u)} - \frac{\theta''(u)\theta'(u)}{\theta(u)^2}\right).$$

Lemma B.3.2.1. *For any fixed $\tau\in\mathbb{H}$, $P(u)$ is $\mathbb{Z}_\tau$-invariant and one has*

$$P = \left[\rho'(t)+\rho(t)^2 - \frac{\theta'''}{\theta'}\right]\cdot N_0 + 2\cdot N_1 + 2\rho(t)\cdot N_2 \tag{161}$$

as an elliptic function of u.

Proof. Using (17) and (146), one verifies easily that for τ fixed, $P(\cdot,\tau)$ is $\mathbb{Z}_\tau$-invariant and, viewed as a rational function on E_τ, its polar divisor is $2[0]+[t]$. By straightforward computations, one verifies that $P(\cdot)$ has the same polar part as the right-hand-side of (161). By evaluating at one point (for instance at $u=0$), the lemma follows. □

By straightforward computations, one verifies that the following holds true:

Lemma B.3.2.2. *For $\tau\in\mathbb{H}$ fixed, the meromorphic function*

$$u \longmapsto \mu(u)+\rho(u)\rho'(u)$$

is an elliptic function, i.e. is $\mathbb{Z}_\tau$-invariant in the variable u.

B.3.3. Computation of $\nabla^{GM}_{\boldsymbol{\tau}}[\boldsymbol{\varphi}_0]$. Since N_0 is constant, the partial derivatives $\partial N_0/\partial u$ and $\partial N_0/\partial\tau$ both vanish. Then from (158), it comes

$$\begin{aligned}
\widetilde{\nabla}_\tau\varphi_0 &= \left[\Omega_\tau - \frac{1}{2i\pi}\Big(\rho_u+\Omega_u\cdot\rho\Big)\right]du \\
&= \left[\frac{\alpha_1}{4i\pi}\left(\frac{\theta''(u)}{\theta(u)} - \frac{\theta''(u-t)}{\theta(u-t)}\right) + a_0\rho(u-t)\right. \\
&\qquad\qquad \left. - \frac{1}{2i\pi}\left(\rho'(u) + \left(2i\pi\frac{a_0}{\alpha_1} + \alpha_1\big(\rho(u)-\rho(u-t)\big)\right)\cdot\rho(u)\right)\right]du \\
&= \frac{a_0}{\alpha_1}du - \frac{1}{2i\pi}\rho'(u)du + \frac{\alpha_1}{4i\pi}P(u)du.
\end{aligned}$$

It follows from Lemma B.3.2.1. that

$$\widetilde{\nabla}_\tau\varphi_0 = \frac{\alpha_1}{4i\pi}\left(\rho'(t)+\rho(t)^2-\frac{\theta'''}{\theta'}\right)\cdot\varphi_0 + \frac{\alpha_1-1}{2i\pi}\cdot\varphi_1 + \left(a_0+\frac{\alpha_1}{2i\pi}\rho(t)\right)\cdot\varphi_2$$

thus in (twisted) cohomology, because $2i\pi a_0[\varphi_0] = \alpha_1[\varphi_2]$ (*cf.* (37)), one deduces that the following relation holds true:

(162)
$$\nabla_\tau^{GM}[\varphi_0] = \left(2i\pi\frac{a_0^2}{\alpha_1} + a_0\rho(t) + \frac{\alpha_1}{4i\pi}\left(\rho'(t) + \rho(t)^2 - \frac{\theta'''}{\theta'}\right)\right)[\varphi_0] + \frac{\alpha_1 - 1}{2i\pi}[\varphi_1].$$

B.3.4. Computation of $\nabla_\tau^{GM}[\varphi_1]$. From (158), it comes

$$\widetilde{\nabla}_\tau\varphi_2 = \widetilde{\nabla}_\tau\big(\rho' du\big) = \Big[\,\dot{\rho}' + \Omega_\tau\,\rho' - \frac{1}{2i\pi}\Big(\rho\cdot\rho'' + (\rho')^2 + \Omega_u\cdot\rho\rho'\Big)\Big]du.$$

By construction, for any $\tau \in \mathbb{H}$ fixed, the right-hand-side is a rational 1-form on E_τ. It follows from [49] that there exist three 'constants depending on τ', $A_i(\tau)$ with $i = 0,1,2$ and a rational function $\Phi(\cdot) = \Phi(\cdot,\tau)$ depending on τ, all to be determined, such that

$$\widetilde{\nabla}_\tau\varphi_2 = A_0(\tau)\cdot\varphi_0 + A_1(\tau)\cdot\varphi_1 + A_2(\tau)\cdot\varphi_2 - \frac{1}{2i\pi}\nabla_{\mathscr{E}_a/\mathscr{F}_a}\Phi.$$

Using (160) and the following formulae

$$\begin{aligned}
\rho(u) &= \theta'(u)/\theta(u)\\
\rho'(u) &= \theta''(u)/\theta(u) - \big(\theta'(u)/\theta(u)\big)^2\\
\rho''(u) &= \theta'''(u)/\theta(u) - 3\theta''(u)\theta'(u)/\theta(u)^2 + 2\big(\theta'(u)/\theta(u)\big)^3\\
\text{and }\ \dot{\rho}'(u) &= \frac{1}{4i\pi}\left[\frac{\theta^{(4)}(u)}{\theta(u)} - \left(\frac{\theta''(u)}{\theta(u)}\right)^2 - 2\frac{\theta'''(u)\theta'(u)}{\theta(u)^2} + 2\frac{\theta''(u)\theta'(u)^2}{\theta(u)^3}\right]
\end{aligned}$$

one verifies by lengthy but straightforward computations that one has

$$\begin{aligned}
A_0(\tau) &= -\,a_0\mu(t) - \frac{\alpha_1}{4i\pi}\Big(\mu'(t) + 2\rho(t)\mu(t) - 3\,\mu'(0)\Big);\\
A_1(\tau) &= -\,a_0\rho(t) - \frac{\alpha_1}{4i\pi}\left(\rho'(t) + \rho(t)^2 - \frac{\theta'''}{\theta'}\right);\\
A_2(\tau) &= a_0\rho'(t) - \frac{\alpha_1}{2i\pi}\mu(t)\\
\text{and}\quad \Phi(u) &= \mu(u) + \rho(u)\rho'(u).
\end{aligned}$$

Since $\Phi(u)$ is rational according to Lemma B.3.2.2., one has $[\nabla_{\mathscr{E}_a/\mathscr{F}_a}\Phi] = 0$ in (twisted) cohomology and because $2i\pi a_0[\varphi_0] = \alpha_1[\varphi_2]$, one obtains that

$$\nabla_\tau^{GM}[\varphi_1] = \left(A_0(\tau) + 2i\pi\frac{a_0}{\alpha_1}A_2(\tau)\right)\cdot[\varphi_0] + A_1(\tau)\cdot[\varphi_1]$$

uniformly with respect to $\tau \in \mathbb{H}$, that is, more explicitly

$$\nabla_\tau^{GM}[\varphi_1] = \left(2i\pi\frac{a_0^2}{\alpha_1}\rho'(t) - 2a_0\mu(t) - \frac{\alpha_1}{4i\pi}\Big(\mu'(t) + 2\rho(t)\mu(t) - 3\,\mu'(0)\Big)\right)\cdot[\varphi_0]$$

$$- \left(a_0\rho(t) + \frac{\alpha_1}{4i\pi}\left(\rho'(t) + \rho(t)^2 - \frac{\theta'''}{\theta'}\right)\right)\cdot[\varphi_1]\,. \tag{163}$$

B.3.5. The Gauß-Manin connection ∇^{GM} and the differential equation satisfied by the components of Veech's map. From (162) and (163), one deduces the

Thorem B.3.5. *The action of the Gauß-Manin derivation ∇_τ^{GM} in the basis formed by $[\varphi_0]$ and $[\varphi_1]$ is given by*

$$\nabla_\tau^{GM}\begin{pmatrix}[\varphi_0]\\ [\varphi_1]\end{pmatrix} = \begin{pmatrix}M_{00} & M_{01}\\ M_{10} & M_{11}\end{pmatrix}\cdot\begin{pmatrix}[\varphi_0]\\ [\varphi_1]\end{pmatrix} \tag{164}$$

with

$$\begin{aligned}
M_{00} =& 2i\pi\frac{a_0^2}{\alpha_1} + a_0\rho(t) + \frac{\alpha_1}{4i\pi}\left(\rho'(t) + \rho(t)^2 - \frac{\theta'''}{\theta'}\right);\\
M_{01} =& \frac{\alpha_1 - 1}{2i\pi};\\
M_{10} =& 2i\pi\frac{a_0^2}{\alpha_1}\rho'(t) - 2a_0\mu(t) - \frac{\alpha_1}{4i\pi}\Big(\mu'(t) + 2\rho(t)\mu(t) - 3\,\mu'(0)\Big)\\
\textit{and}\quad M_{11} =& -a_0\rho(t) - \frac{\alpha_1}{4i\pi}\left(\rho'(t) + \rho(t)^2 - \frac{\theta'''}{\theta'}\right).
\end{aligned}$$

Consequently, according to B.1.5, for any horizontal family of twisted 1-cycles $\tau \mapsto \boldsymbol{\gamma}(\tau)$, if one sets

$$F_0(\tau) = \int_{\boldsymbol{\gamma}(\tau)} T(u,\tau)du \qquad \text{and} \qquad F_1(\tau) = \int_{\boldsymbol{\gamma}(\tau)} T(u,\tau)\rho'(u,\tau)du$$

then $F = {}^t(F_0, F_1)$ satisfies the differential system

$$\dot{F} = dF/d\tau = MF \tag{165}$$

where $M = M(\tau)$ is the 2×2 matrix appearing in (164).

At this point, we recall the definition of Veech's map: it is the map

$$V : \mathscr{F}_a \simeq \mathbb{H} \longrightarrow \mathbb{P}^1\,, \quad \tau \longmapsto V(\tau) = \begin{bmatrix} v_0(\tau)\\ v_\infty(\tau)\end{bmatrix} \tag{166}$$

with for every $\tau \in \mathbb{H}$:

$$v_0(\tau) = \int_{\Upsilon_0} T(u,\tau)du \qquad \text{and} \qquad v_\infty(\tau) = \int_{\Upsilon_\infty} T(u,\tau)du.$$

Then applying Lemma 6.1.1 of [34, §3.6.1] (see also Lemma A.2.2. above) to the differential system (165), one obtains the

Corollary B.3.5. *The components v_0 and v_∞ of Veech's map of the leaf $\mathscr{F}_a$ form a basis of the space of solutions of the following linear differential equation*

$$\overset{\bullet\bullet}{v} - \left(2i\pi a_0^2/\alpha_1\right) \overset{\bullet}{v} + \left(\det M(\tau) + \overset{\bullet}{M}_{11}\right) v = 0. \tag{167}$$

The coefficient of $\overset{\bullet}{v}$ in (167) being constant, the functions

$$\widetilde{v}_\star(\tau) = \exp\left(- i\pi(a_0^2/\alpha_1)\cdot\tau\right) v_\star(\tau) \qquad \text{with } \star = 0, \infty$$

satisfy a linear second order differential equation in reduced form and can be taken as the components of Veech's map (166).

From our point of view, the second-order Fuchsian differential equation (167) is for elliptic curves with two punctures what Gauß hypergeometric differential equation (2) is for $\mathbb{P}^1$ with four punctures.

Finally, in the case when $a = (a_0, a_\infty) = \alpha_1(m/N, -n/N)$ with $N \geq 2$ and $(m,n) \in \{0,\ldots,N-1\}^2 \setminus \{(0,0)\}$, we have $t = (m/N)\tau + (n/N)$, thus

$$T(u) = e^{\frac{2i\pi m}{N}\alpha_1}\theta(u)^{\alpha_1}\theta\big(u - (m/N)\tau - n/N\big)^{-\alpha_1}.$$

Specializing Theorem B.3.5. and Corollary B.3.5. to this case, we let the readers verify that one recovers (the special case of) Mano's differential system considered in §5.3.

References

[1] AHLFORS L. V., *The complex analytic structure of the space of closed Riemann surfaces.* In 'Analytic functions', pp. 45–66, Princeton Univ. Press, 1960.

[2] AOMOTO K., KITA M., Theory of hypergeometric functions. With an appendix by Toshitake Kohno. Springer Monographs in Mathematics. Springer-Verlag, Tokyo, 2011.

[3] ANDRÉ Y., BALDASSARRI F., De Rham Cohomology of Differential Modules on Algebraic Varieties. Progress in Mathematics Vol. 189. Birkhäuser Basel, 2001.

[4] BERGER A., *Über die zur dritten Stufe gehörigen hypergeometrischen Integrale am ellipti--schen Gebilde*. Monatsh. f. Math. **17** (1906), 137–160 & 179–206.

[5] BOWDITCH B., *Singular Euclidean structures on surfaces.* J. London Math. Soc. **44** (1991), 553–565.

[6] CAMACHO C., LINS NETO A., Geometric theory of foliations. Birkhäuser Boston, 1985.

[7] ČERMÀK A., *Některé poznámky k theorii hypergeometrické funkce na základě thetafun--kcí*. Časopis **36** (1907), 441-460.

[8] CHANDRASEKHARAN K., Elliptic functions. Grundlehren der Mathematischen Wissenschaften 281. Springer-Verlag, Berlin, 1985.
[9] CONWAY J., NORTON S., *Monstrous moonshine*. Bull. London Math. Soc. **11** (1979), 308–339.
[10] DELIGNE P., Équations différentielles à points singuliers réguliers. Lecture Notes in Mathematics, Vol. 163, Springer-Verlag, Berlin-New York, 1970.
[11] DELIGNE P., MOSTOW G., *Monodromy of hypergeometric functions and nonlattice inte--gral monodromy*. Inst. Hautes Études Sci. Publ. Math. **63** (1986), 5–89.
[12] DELIGNE P., MOSTOW G., Commensurabilities among lattices in PU$(1,n)$. Annals Math. Studies **132**. Princeton University Press, Princeton, 1993.
[13] DIAMOND F., SHURMAN J., A first course in modular forms. Graduate Texts in Mathematics, 228. Springer-Verlag, New York, 2005.
[14] DOMBROWSKI H., *Fastautomorphe Funktionen zweiten Grades*. Math. Annalen **149** (1963), 71–06.
[15] DUTKA J., *The early history of the hypergeometric function*. Arch. Hist. Exact Sci. **31** (1984), 15–34.
[16] EPSTEIN D., *Curves on 2-manifolds and isotopies*. Acta Math. **115** (1966), 83–107.
[17] FARKAS H., KRA I., Riemann surfaces. Second edition. Graduate Texts in Mathematics 71. Springer-Verlag, 1992.
[18] FRASCH H., *Die Erzeugenden der Hauptkongruenzgruppen für Primzahlstufen*. Math. Ann. **108** (1933), 229–252.
[19] GHAZOUANI S., PIRIO L., *Moduli spaces of flat tori with prescribed holonomy*. Preprint arXiv:1604.01812 (2016).
[20] GILMAN J., MASKIT B., *An algorithm for 2-generator Fuchsian groups*. Michigan Math. J. **38** (1991), 13–32.
[21] GOLDMAN W. *Invariant functions on Lie groups and Hamiltonian flows of surface group representations*. Inventiones math. **85** (1986), 263–302.
[22] GOLDMAN W. Complex hyperbolic geometry. Oxford Mathematical Monographs. Oxford Univ. Press, New York, 1999.
[23] GOLDMAN W. *Locally homogeneous geometric manifolds*. In 'Proceedings of the I.C.M. Volume II', Hind. Book Agency, 2010, 717–744.
[24] GRAF F., *O určení grupy hypergeometrické differenciální rovnice*. Časopis **36** (1907), 354–360.
[25] GRAF F., *O určení základních substitucí hypergeometrické grupy pomocí Wirtingerovy formule*. Časopis **37** (1908), 8–13.
[26] GRAF F., *Všeobecném určeni číselných koefficientů grupy hypergeometrické rovnice dif--ferenciálni*. Rozpravy **17**, Nr. 32 (1908).
[27] GRAF F., *O degeneraci Wirtingerovy formule*. Rozpravy **19**, Nr. 29, (1910).
[28] GRAY J., Linear differential equations and group theory from Riemann to Poincaré. Modern Birkhäuser Classics. Birkhäuser Boston, 2008.
[29] GREENBERG L., *Maximal Fuchsian groups*. Bull. Amer. Math. Soc. **69** (1963), 569–573.
[30] HEINS M., *On a class of conformal metrics*. Nagoya Math. J. **21** (1962), 1–60.
[31] ITO K.-K., *The elliptic hypergeometric functions associated to the configuration space of points on an elliptic curve. I. Twisted cycles*. J. Math. Kyoto Univ. **49** (2009), 719–733.
[32] ITO K.-K., *Twisted Poincaré lemma and twisted* Č*ech-de Rham isomorphism in case dimension =1*. Kyoto J. Math. **50** (2010), 193–204.

[33] KOKOTOV A., *Polyhedral surfaces and determinant of Laplacian.* Proc. Amer. Math. Soc. **141** (2013), 725–735.

[34] IWASAKI K., KIMURA H., SHIMOMURA S. AND YOSHIDA M., From Gauss to Painlevé. A modern theory of special functions. Aspects of Mathematics, E16. Friedr. Vieweg & Sohn, Braunschweig, 1991.

[35] KAMPÉ DE FÉRIET M., La fonction hypergéométrique. Mém. Sci. math. **85**, Gauthier-Villars, 1937.

[36] KATZ N., ODA T., *On the differentiation of de Rham cohomology classes with respect to parameters.* J. Math. Kyoto Univ. **8** (1968), 199–213.

[37] KIM C. H., KOO J. K., *On the genus of some modular curves of level N*. Bull. Aust. Math. Soc. **54** (1996), 291–297.

[38] KITA M., YOSHIDA M., *Intersection theory for twisted cycles.* Math. Nachr. **166** (1994), 287–304.

[39] KNAPP A., *Doubly generated Fuchsian groups.* Michigan Math. J. **15** (1969), 289–304.

[40] KODAIRA K., *On Compact Analytic Surfaces II*. Ann. of Math **77** (1963), 563–626.

[41] LIVNÉ R., On Certain Covers of the Universal Elliptic Curve. Ph.D. Thesis, Harvard University, 1981.

[42] LOOIJENGA E., *Uniformization by Lauricella functions—an overview of the theory of Deligne-Mostow*. In: 'Arithmetic and geometry around hypergeometric functions', 207–244, Progr. Math., 260, Birkhäuser, Basel, 2007.

[43] MCMULLEN C., *Braid groups and Hodge theory*. Math. Ann. **355** (2013), 893–946.

[44] MANIN Y., *Sixth Painlevé equation, universal elliptic curve, and mirror of* $\mathbb{P}^2$. In: Geometry of differential equations, 131–151, Amer. Math. Soc. Transl. Ser. 2, 186, 1998.

[45] MANGLER W., *Die Klassen von topologischen Abbildungen einer geschlossenen Fläche auf sich.* Math. Z. **44** (1939), 541–554.

[46] MANO T., *The Riemann-Wirtinger integral and monodromy-preserving deformation on elliptic curves*. Int. Math. Res. Not. (2008), Art. ID rnn110, 19 pp.

[47] MANO T., *Monodromy preserving deformation of linear differential equations on a rational nodal curve.* Kumamoto J. Math. **24** (2011), 1–32.

[48] MANO T., *On a generalization of Wirtinger's integral.* Kyushu J. Math. **66** (2012), 435–447.

[49] MANO T., WATANABE H., *Twisted cohomology and homology groups associated to the Riemann-Wirtinger integral*. Proc. Amer. Math. Soc. **140** (2012), 3867–3881.

[50] MANO T., *Some remarks on "Twisted cohomology and homology groups associated to the Riemann-Wirtinger integral"*. Personnal e-mail (February 24, 2015)

[51] MASUR H., SMILLIE J., *Hausdorff dimension of sets of nonergodic measured foliations*. Ann. of Math. (2) **134** (1991), 455–543.

[52] MAZZOCCO M., *Picard and Chazy solutions to the Painlevé VI equation.* Math. Ann. **321** (2001), 157–195.

[53] MOSTOW G., *Generalized Picard lattices arising from half-integral conditions.* Inst. Hautes Études Sci. Publ. Math. **63** (1986), 91–106.

[54] MOSTOW G., *On discontinuous action of monodromy groups on the complex n-ball.* J. Amer. Math. Soc. **1** (1988), 555–586.

[55] NAG S., *The Torelli spaces of punctured tori and spheres.* Duke Math. J. **48** (1981), 359–388.

[56] NAG S., The complex analytic theory of Teichmüller spaces. Canadian Math. Soc. Series Monog. John Wiley & Sons, New York, 1988.

[57] OGG A., *Rational points on certain elliptic modular curves.* In 'Analytic number theory', Proc. Sympos. Pure Math. XXIV, pp. 221–231. Amer. Math. Soc., Providence, 1973.
[58] PETR K., *Poznámka o integrálech hypergeometrické differenciální rovnice.* Časopis **38** (1909), 294–306
[59] PICARD É., *Mémoire sur la théorie des fonctions algébriques de deux variables.* Journal de Liouville **5** (1889), 135–319.
[60] PICARD É., *De l'integration de l'équation* $\nabla u = e^u$ *sur une surface de Riemann fermée.* J. Reine Angew. Math. **130** (1905), 243-258.
[61] PICK G., *Zur Theorie der hypergeometrischen Integrale am elliptischen Gebilde.* Monatsh. für Math. u. Phys. **18** (1907), 317–320.
[62] POINCARÉ H., *Sur les groupes des équations linéaires.* Acta Math. **4** (1884), 201–312.
[63] PUTMAN A., *Cutting and pasting in the Torelli group.* Geom. Topol. **11** (2007), 829–865.
[64] DE SAINT-GERVAIS H. P., Uniformisation des surfaces de Riemann. ENS Éditions, 2010.
[65] SCHNEPS L., *Special loci in moduli spaces of curves.* In 'Galois groups and fundamental groups', 217–275, Math. Sci. Res. Inst. Publ., 41, Cambridge Univ. Press, 2003.
[66] SCHWARZ H.A., *Ueber diejenigen Fälle, in welchen die Gaussische hypergeometrische Reihe eine algebraische Function ihres vierten Elementes darstellt*. J. Reine Angew. Math. **75** (1873), 292–335.
[67] SHIODA T., *On elliptic modular surfaces.* J. Math. Soc. Japan **24** (1972), 20–59.
[68] SPIRIDONOV V. P., *Essays on the theory of elliptic hypergeometric functions.* Math. Surveys **63** (2008), 405–472.
[69] TAKEUCHI K, *Arithmetic triangle groups.* J. Math. Soc. Japan **9** (1977), 91–106.
[70] TEICHMÜLLER O., *Bestimmung der extremalen quasikonformen Abbildungen bei geschlossenen orientierten Riemannschen Flächen.* Abh. Preuss. Akad. Wiss. Math.-Nat. Kl. (1943). No. 4, 42 pages.
[71] THURSTON W., *Shapes of polyhedra and triangulations of the sphere.* In 'The Epstein birthday schrift', 511–549, Geom. Topol. Monogr., 1, Coventry, 1998.
[72] THURSTON W., Three-dimensional geometry and topology. Vol. 1, Princeton Mathematical Series 35, Princeton University Press, 1997.
[73] TROYANOV M., *Les surfaces euclidiennes à singularités coniques.* Enseign. Math. **32** (1986), 79–94.
[74] VAN DEN BERG B., On the Abelianization of the Torelli group. PhD thesis, University of Utrecht (2003)
[75] VASSILIEV V.A., Applied Picard-Lefschetz Theory. Mathematical Surveys and Monographs, 97. American Mathematical Society, Providence, 2002.
[76] VEECH W., *Flat surfaces.* Amer. J. Math. **115** (1993), 589–689.
[77] WATANABE H., *Transformation relations of matrix functions associated to the hypergeometric function of Gauss under modular transformations.* J. Math. Soc. Japan **59** (2007),113–126.
[78] WATANABE H., *Twisted homology and cohomology groups associated to the Wirtinger integral.* J. Math. Soc. Japan **59** (2007), 1067–1080.
[79] WATANABE H., *Linear differential relations satisfied by Wirtinger integrals.* Hokkaido Math. J. **38** (2009), 83–95.
[80] WATANABE H., *On the general transformation of the Wirtinger integral.* Osaka J. Math. **51** (2014), 425–439.
[81] WEIL A., *Modules des surfaces de Riemann.* Séminaire Bourbaki **4** (1956-1958), 413–419

[82] WIRTINGER W., *Zur Darstellung der hypergeometrischen Function durch bestimmte Integrale*. Akad. Wiss. Wien. **111** (1902), 894–900.

[83] WIRTINGER W., *Eine neue Verallgemeinerung der hypergeometrischen Integrale.* Wien. Ber. **112** (1903), 1721–1733.

[84] YOSHIDA M., Fuchsian differential equations. With special emphasis on the Gauß--Schwarz theory. Aspects of Mathematics, E11. Friedr. Vieweg & Sohn, 1987.

[85] YOSHIDA M., Hypergeometric functions, my love. Modular interpretations of configu--ration spaces. Aspects of Mathematics, E32. Friedr. Vieweg & Sohn, 1997.

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