Mapping class group dynamics on $\mathrm{Aff}(\mathbb{C})$-characters.

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Abstract

We prove that in genus greater than 2, the mapping class group action on $\mathrm{Aff}(\mathbb{C})$-characters is ergodic. This implies that almost every representation $\pi_1 S \longrightarrow \mathrm{Aff}(\mathbb{C})$ is the holonomy of a branched affine structure on S, when S is a closed orientable surface of genus $g \geq 2$.

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Contents

Introduction

Let Γ be the fundamental group of a compact orientable surface S of genus $g \geq 2$. If G is a finite dimensional reductive Lie group (typically $G = \mathrm{PSL}(2, \mathbb{R})$ or $\mathrm{SU}(2)$), one can look at the character variety $\chi(\Gamma, G)$ which is defined to be the quotient $\mathrm{Hom}(\Gamma, G)//G$, in the sense of geometric invariant theory. The mapping class group of S acts on $\chi(\Gamma, G)$ by precomposition, the study of this action was popularized by Goldman in the early 80's. The most classical result in the field, by Goldman, is that the action is ergodic for $G = \mathrm{SU}(2)$ (see [Gol97]). This result was extended to the case where G is compact by Pickrell and Xia, see [PX02]. In this paper we study the case $G = \mathrm{Aff}(\mathbb{C}) = \{z \mapsto az+b \mid (a, b) \in \mathbb{C}^* \times \mathbb{C}\}$. Since $\mathrm{Aff}(\mathbb{C})$ is solvable, the tools from symplectic geometry developed in the reductive case do not apply in our setting. Worst, the character variety is not defined, at least in the sense of geometric invariant theory. This very last point can be solved by defining $\chi(\Gamma, \mathrm{Aff}(\mathbb{C}))$ to be the quotient of $\mathrm{Hom}(\Gamma, \mathrm{Aff}(\mathbb{C})) \setminus \{\text{abelian representations}\}$ by the action of G by conjugation (see Section 1).

$\chi(\Gamma, \mathrm{Aff}(\mathbb{C}))$ has a structure of fiber bundle. It comes from the isomorphism $\mathrm{Aff}(\mathbb{C}) \simeq \mathbb{C}^* \ltimes \mathbb{C}$, a representation $\rho : \Gamma \longrightarrow \mathrm{Aff}(\mathbb{C})$ is the data of a linear part $\alpha : \Gamma \longrightarrow \mathbb{C}^*$ and a translation part $\lambda : \Gamma \longrightarrow \mathbb{C}$ ($\rho = (\alpha, \lambda) \in \mathbb{C}^* \ltimes \mathbb{C}$) , where α is a group homomorphism and λ is a cocyle relation twisted by α. A point in the quotient space will be parametrized by an element in $\mathrm{H}^1(S, \mathbb{C}^*) \simeq (\mathbb{C}^*)^{2g}$ (the linear part) and an element in the projectivized space of $\mathrm{H}^1_\alpha(\Gamma, \mathbb{C}^*) \simeq \mathbb{CP}^{2g-3}$ (the translation part), and this parametrization gives the fiber bundle structure.

In the case where $G = \mathbb{C}$ (the simplest non reductive case), the character variety is $\mathrm{H}^1(S, \mathbb{C}) \simeq \mathbb{C}^{2g}$. The action of the mapping class group on $\mathrm{H}^1(S, \mathbb{C})$ (which happen to be the linear action of $\mathrm{Sp}(2g, \mathbb{Z})$ on $\mathbb{C}^{2g}$) has an invariant non constant continuous function, $\omega \longmapsto \omega \wedge \overline{\omega} \in \mathrm{H}^1(S, \mathbb{R}) \simeq \mathbb{R}$. Hence this action is not ergodic. (A careful study of this action has been carried on by M.Kapovich in [Kap]). The main result of our paper is

Theorem 1. *The mapping class group action on* $\chi(\Gamma, \mathrm{Aff}(\mathbb{C}))$ *is ergodic.*

The mapping class group action preserves this fiber bundle structure, and to prove the theorem we first prove that the induced action on the base is ergodic. Then we observe that the Torelli group stabilizes globally the fibers, and we prove that its action is ergodic in almost every fiber.

- The action on $\mathrm{H}^1(S, \mathbb{C}^*)$ is actually the linear diagonal action of $\mathrm{Sp}(2g, \mathbb{Z})$ on $\mathbb{R}^{2g} \times (\mathbb{R}/\mathbb{Z})^{2g}$. An application of Fourier analysis combined to Moore's theorem gives the ergodicity.

- The Torelli group $\mathcal{I}(S)$ acts preserving the fibers of the fibrations, namely the projectivized spaces of the twisted cohomology group $\mathrm{H}^1_\alpha(\Gamma, \mathbb{C})$. This action happens to be projective an one gets a nice family of representations of the Torelli group :

$$\tau_\alpha : \mathcal{I}(S) \longrightarrow \mathrm{PGL}(2g-2, \mathbb{C})$$

  In Section 3, we provide an explicit computation of the action of a family of Dehn twists along separating curves on $\mathrm{PH}^1_\alpha(\Gamma, \mathbb{C})$. We deduce from this computation that for almost all α, this action is ergodic.

Those two last points together imply the main theorem. A remarkable consequence of the computation is that the mapping class group preserves no symplectic form. Actually it preserves no absolutely continuous measure relatively to the Lebesgue measure, which contrasts with the case where G is reductive, where we have such a symplectic form at hand, by the Goldman's work (see [Gol84]).
Our original motivation was the study of the holonomy of branched affine structures. A direct corollary is that the set of representation arising as the holonomy of such a structure is an open set of full measure of the character variety.

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We introduce notations that will be used all along the paper :

- S is a compact connected oriented surface of genus $g \geq 2$.
- Γ is the fundamental group of the surface S.
- $\mathrm{Aff}(\mathbb{C})$ is the group of complex affine transformations of the complex line.
- $\mathrm{Mod}(S)$ is the mapping class group of S.

1. *Action of the mapping class group on the character variety.*

1.1. Structure of the character variety.

Let us recall the standard presentation for Γ :

$$\Gamma = \langle a_1, b_1, \cdots, a_g, b_g \mid \prod_{i=1}^{g} [a_i, b_i] = 1 \rangle$$

Let $\rho : \Gamma \longrightarrow \mathrm{Aff}(\mathbb{C})$ be a group homomorphism. If we note $\rho(a_i) = A_i z + U_i$ and $\rho(b_i) = B_i z + V_i$, the following holds :

$$\sum_{i=1}^{g} (A_i - 1)V_i + (1 - B_i)U_i = 0.$$

Conversely, every set $(A_i, U_i, B_i, V_i) \in \mathbb{C}^* \times \mathbb{C} \times \mathbb{C}^* \times \mathbb{C}$ verifying the equation above defines a representation of Γ in $\mathrm{Aff}(\mathbb{C})$. Thus $\mathrm{Hom}(\Gamma, \mathrm{Aff}(\mathbb{C}))$ can be seen has an algebraic variety.
The quotient $\mathrm{Hom}(\Gamma, \mathrm{Aff}(\mathbb{C}))$ by the action by conjugation of $\mathrm{Aff}(\mathbb{C})$ is not Haussdorf. Nevertheless, the orbits responsible for this correspond to the degenerate case where the representations are abelian. Removing these ones, one gets a nice quotient.

Definition 2. *The character variety* $\chi(\Gamma, \mathrm{Aff}(\mathbb{C}))$ *is defined to be the quotient of* $\mathrm{Hom}(\Gamma, \mathrm{Aff}(\mathbb{C})) \setminus \{$*abelian representations*$\}$ *by the action by conjugation of* $\mathrm{Aff}(\mathbb{C})$.

Let $\rho \in \mathrm{Hom}(\Gamma, \mathrm{Aff}(\mathbb{C}))$ be a representation, one can look at its linear part (obtained from ρ just by post composing by the natural group homomorphism $\mathbb{C}^* \ltimes \mathbb{C} \longrightarrow \mathbb{C}^*$). This allows us to define :

$$l : \mathrm{Hom}(\Gamma, \mathrm{Aff}(\mathbb{C})) \longrightarrow \mathrm{Hom}(\Gamma, \mathbb{C}) = \mathrm{Hom}(H_1(S, \mathbb{Z}), \mathbb{C})$$

which factors through $\chi(\Gamma, \mathrm{Aff}(\mathbb{C}))$, because two conjugate representations have the same linear part.

Proposition 3. *The map* $L : \chi(\Gamma, \mathrm{Aff}(\mathbb{C})) \longrightarrow H^1(S, \mathbb{C}^*)$ *is a projective fibration with fiber* $\mathbb{CP}^{2g-3}$.

Proof. The map l restricted to $\mathrm{Hom}(\Gamma, \mathrm{Aff}(\mathbb{C})) \setminus l^{-1}(\{\mathrm{Id}\})$ is a vector bundle with fiber $\mathbb{C}^3$. Furthermore, $l^{-1}(\{\alpha\}) = Z^1_\alpha(\Gamma, \mathbb{C})$ where

$$Z^1_\alpha(\Gamma, \mathbb{C}) = \{\lambda : \Gamma \longrightarrow \mathbb{C} \mid \forall \gamma, \gamma' \in \Gamma \ \lambda(\gamma \cdot \gamma') = \lambda(\gamma) + \alpha(\gamma)\lambda(\gamma')\}$$

The vector space $Z^1_\alpha(\Gamma,\mathbb{C})$ is the set of cochains of the cohomology of Γ twisted by α. The action of $\mathrm{Aff}(\mathbb{C})$ by conjuguation stabilizes the fibers $l^{-1}(\{\alpha\}) = Z^1_\alpha(\Gamma,\mathbb{C})$. Let $\rho := z \mapsto az+b$ and $\lambda \in Z^1_\alpha(\Gamma,\mathbb{C})$. We have $\rho\cdot\lambda = b(1-\alpha)+a\lambda$, so the quotient of $Z^1_\alpha(\Gamma,\mathbb{C})$ by the action of $\mathrm{Aff}(\mathbb{C})$ is the projective space of $Z^1_\alpha(\Gamma,\mathbb{C})/\mathbb{C}\cdot(1-\alpha) = \mathrm{H}^1_\alpha(\Gamma,\mathbb{C})$. Twisted homology theory (see [Hat02], p.327) ensures that as soon as $\alpha \neq \mathrm{Id}$, $\dim_{\mathbb{C}} \mathrm{H}^1(\Gamma,\mathbb{C}) = 2g-2$, so the fiber is isomorphic to $\mathbb{CP}^{2g-3}$.

□

From now on, χ will be the variety of $\mathrm{Aff}(\mathbb{C})$-characters.

Let H is a subgroup of $\mathbb{C}^*$. We define

$$\chi_H = \{\rho \in \chi \mid \mathrm{Im}(L(\rho)) \subset H\}$$

One will say that a representation ρ is

1. **unitary** (or euclidean) if it belongs to $\chi_{\mathbb{U}}$.
2. **real** if it belongs to $\chi_{\mathbb{R}^*}$.
3. **almost real** if there exists a subgroup of finite index Γ' in Γ such that $L(\rho)(\Gamma') \subset \mathbb{R}^*$.
4. **abelian** is the image of ρ is an abelian subgroup of $\mathrm{Aff}(\mathbb{C})$.
5. **strictly affine** in any other case.

1.2. The $\mathrm{Mod}(S)$ action.

The mapping class group of a closed surface S is classically defined as

$$\mathrm{Mod}(S) = \mathrm{Homeo}^+(S)/\mathrm{Homeo}_0(\mathrm{S})$$

Any element of $\mathrm{Mod}(S)$ defines an element of $\mathrm{Out}(\Gamma) = \mathrm{Aut}(\Gamma)/\mathrm{Inn}(\Gamma)$. By a theorem of Dehn-Nielsen-Baer

$$\mathrm{Mod}(S) \simeq \mathrm{Out}^+(\Gamma)$$

where $\mathrm{Out}^+(\Gamma)$ is the subgroup of elements in $\mathrm{Out}(\Gamma)$ preserving the fundamental class in $\mathrm{H}^2(\Gamma \simeq \pi_1 S, \mathbb{Z})$.

Notice now that any element of $\mathrm{Aut}(\Gamma)$ acts on $\mathrm{Hom}(\Gamma, \mathrm{Aff}(\mathbb{C}))$ by precomposition. This action induces an action of $\mathrm{Out}(\Gamma)$ on the character variety. An important remark which will be detailed later is that this action preserves the fiber bundle structure described in the previous section.

Proposition 4. *1. Let H be a subgroup of $\mathbb{C}^*$. Then the $\mathrm{Mod}(S)$-action preserves χ_H.*

2. The $\mathrm{Mod}(S)$-action preserves the set of almost-real representations.

3. The $\mathrm{Mod}(S)$-action preserves the set of strictly affine representations.

Remark This action preserves no measure *a priori*. Still χ is a differentiable manifold and even tough the Lebesgue measure is not canonically defined, it makes sense to say that a subset A has measure zero (just say that is Lebesgue measure in any chart is zero). In a more general setting, an action by diffeomorphisms on a manifold will be said to be ergodic if any invariant subset has zero measure or full measure in the sense defined previously.

1.3. The symplectic representation.

The mapping class group acts naturally on $\mathrm{H}_1(S, \mathbb{Z})$, preserving the symplectic intersection form. Up to the choice of a symplectic basis of $\mathrm{H}_1(S, \mathbb{Z})$, one gets a linear representation of $\mathrm{Mod}(S)$ in $\mathrm{Sp}(2g, \mathbb{Z})$:

$$\Psi : \mathrm{Mod}(S) \longrightarrow \mathrm{Sp}(2g, \mathbb{Z}).$$

Let us denote by $\mathcal{I}(S)$ the kernel of this representation. This group is usually called the Torelli group. It is the subgroup of $\mathrm{Mod}(S)$ acting trivially on the homology of S.

Theorem 5. *The image of the symplectic representation is* $\mathrm{Sp}(2g, \mathbb{Z})$.

This theorem was originally proved by Poincaré. A modern proof of this theorem can be found in [FM12].
This way $\mathrm{Mod}(S)$ acts on $\mathrm{Hom}(\mathrm{H}_1(S, \mathbb{Z}), \mathbb{C}^*)$ by precomposition by the image of the symplectic representation. This means that for $f \in \mathrm{Mod}(S)$, the following diagram commutes :

$$\begin{array}{ccc} \chi & \xrightarrow{\quad f \quad} & \chi \\ \downarrow L & & \downarrow L \\ \mathrm{H}^1(S, \mathbb{C}^*) & \xrightarrow[\Psi(f)]{} & \mathrm{H}^1(S, \mathbb{C}^*) \end{array}$$

1.4. The Torelli group action on the fibers.

Proposition 6. *The Torelli group $\mathcal{I}(S)$ preserves the fibers of L, and acts on them by projective transformations.*

Proof. Let f be an automorphism whose class in $\mathrm{Mod}(S)$ belongs to $\mathcal{I}(S)$. f acts linearly on $Z^1_\alpha(\Gamma, \mathbb{C})$, preserving the line generated by $1 - \alpha$. Thus f defines a linear automorphism $\mathrm{H}^1_\alpha(S, \mathbb{C})$, and so a projective transformation of $\mathrm{PH}^1_\alpha(S, \mathbb{C})$. □

2. *Ergodicity of the* $\mathrm{Sp}(2g, \mathbb{Z})$*-action on* $(\mathbb{C}^*)^{2g}$.

The choice of a symplectic basis $a_1, b_1, ..., a_g, b_g$ of $\mathrm{H}_1(S, \mathbb{Z})$ identifies $\mathrm{H}^1(S, \mathbb{C}^*)$ and $(\mathbb{C}^*)^{2g}$ via the map

$$\alpha \longrightarrow (\alpha(a_1), \alpha(b_1), ..., \alpha(a_g), \alpha(b_g))$$

The exponential map identifies $\mathrm{H}^1_\alpha(S, \mathbb{C}) \simeq (\mathbb{C}^*)^{2g}$ with $\mathbb{T}^{2g} \times \mathbb{R}^{2g}$ in such a way that the $\mathrm{Sp}(2g, \mathbb{Z})$-action on $\mathrm{H}^1_\alpha(S, \mathbb{C})$ the diagonal action by linear transformations on $\mathbb{T}^{2g} \times \mathbb{R}^{2g}$. Let us recall the following theorem :

Proposition 7. *The* $\mathrm{Sp}(2g,\mathbb{Z})$*-action on* $\mathbb{R}^{2g}$ *is ergodic.*

It is a corollary of Moore's theorem, which states that if Γ is a lattice in a semi-simple Lie group G and H is a closed non-compact subgroup of G, then the Γ-action on G/H is ergodic. The original proof of this theorem can be found in [Moo66].

Proposition 8. *The* $\mathrm{Sp}(2g,\mathbb{Z})$*-action on* $(\mathbb{C}^*)^{2g}$ *is ergodic with respect to the Lebesgue measure.*

Proof. A point in $\mathbb{T}^{2g} \times \mathbb{R}^{2g}$ will be identified by the coordinates (θ, x). Let A be a mesurable $\mathrm{Sp}(2g,\mathbb{Z})$-invariant set. Let us write the Fourier expansion of φ_A, the characteristic function of A :

$$\varphi_A(\theta, x) = \sum_{p\in\mathbb{Z}^{2g}} a_p(x) e^{2i\pi\langle p,\theta\rangle}$$

Since φ_A is $\mathrm{Sp}(2g,\mathbb{Z})$-invariant, one has

$$\forall\gamma \in \mathrm{Sp}(2g,\mathbb{Z}),\ \forall x \in \mathbb{R}^{2g}\ \forall p \in \mathbb{Z}^{2g}\ a_{{}^t\gamma p}(x) = a_p(\gamma x)$$

Let $B(z,n) = \{x \in \mathbb{R}^{2g} \mid z \text{ appears in the Fourier expansion of } \varphi_A(x,\cdot) \text{ with multiplicity } n\}$ The $B(z,n)$ are measurable $\mathrm{Sp}(2g,\mathbb{Z})$-invariant sets, so they are either of full measure or of measure zero, because the $\mathrm{Sp}(2g,\mathbb{Z})$-action on $\mathbb{R}^{2g}$ is ergodic. Moreover, the number of couples (z,n) such that $|z| > \delta$ for some fixed δ and $B(z,n)$ is of full measure is finite . So the number of couples (z,n) such that $B(z,n)$ is of full measure is at most countable. The intersection of all these $B(z,n)$ is still of full measure and on this set E the set of Fourier coefficients(counted with multiplicity) of φ_A is constant. Let $a \in \mathbb{C}$ non zero such that a is one of the coefficient of the Fourier expansion of φ_A at $x \in E$ appearing n times. One defines the map :

$$T_a : E \longrightarrow \mathcal{P}(\mathbb{Z}^{2g})$$

which associates to x the set of points p such that $a_p(x) = a$.
As the image of T_a is included in the set of finite subsets of $\mathbb{Z}^{2g}$ (which is countable) and since $a \neq 0$, there exists a subset $K \subset \mathcal{P}(\mathbb{Z}^{2g})$ such that $D = T^{-1}(K)$ has positive measure. Let us assume that K is different from $\{0\}$. Then $\mathrm{Stab}(K) \subset \mathrm{Sp}(2g,\mathbb{Z})$ stabilizes the subvector space generated by K in $\mathbb{R}^{2g}$. If V is complementary to W, $\mathrm{Stab}(K)$ stabilizes the fibers of the fibration $V \oplus W \longrightarrow W$.
If $V \neq \mathbb{R}^{2g}$, Fubini's theorem ensures that there exists a partition of $W = W_1 \coprod W_2$ such that both $D_1 = (W_1 \times V) \cap D$ and $D_2 = (W_2 \times V) \cap D$ have positive measure. Moreover, both are $\mathrm{Stab}(K)$-invariant. Let $\gamma \in \mathrm{Sp}(2g,\mathbb{Z})$, $\gamma(D_1)$ and $\gamma(D_2)$ are $\gamma\mathrm{Stab}(K)\gamma^{-1}$-invariant. If γ_1 and γ_2 are such that $\gamma_1\mathrm{Stab}(K)\gamma_1^{-1} = \gamma_2\mathrm{Stab}(K)\gamma_2^{-1}$ then $\gamma_1(D_1) = \gamma_2(D_1)$ and $\gamma_1(D_2) = \gamma_2(D_2)$. The sets $\bigcup_{\gamma\in\mathrm{Sp}(2g,\mathbb{Z})} \gamma(D_1)$ and $\bigcup_{\gamma\in\mathrm{Sp}(2g,\mathbb{Z})} \gamma(D_2)$ form a non trivial $\mathrm{Sp}(2g,\mathbb{Z})$-invariant partition of $\mathbb{R}^{2g}$ which is impossible because this action is ergodic. If K generates $\mathbb{R}^{2g}$, $\mathrm{Stab}(K)$ is a finite group. Then K is conjugated in $\mathrm{Gl}_{2g}(\mathbb{R})$ to a subgroup of isometries for any scalar product, and so preserves a fibration in spheres, which is impossible by a similar argument. We have then proved that for all $p \in \mathbb{Z}^{2g}$ non zero and for almost all x, $a_p(x) = 0$. So φ_A only depends on x, and as the $\mathrm{Sp}(2g,\mathbb{Z})$-action is ergodic on $\mathbb{R}^{2g}$, φ_A is constant almost everywhere. Hence $\mathrm{Sp}(2g,\mathbb{Z})$ acts ergodically on $(\mathbb{C}^*)^{2g}$.

□

3. *The Torelli group action on* $\mathrm{PH}^1_\alpha(\Gamma, \mathbb{C})$.

Let us fix once and for all a point $p \in S$ in such a way that we identify $\pi_1(S, p)$ and Γ. Any diffeomorphism f of S fixing the point p defines canonically an automorphism of Γ whose class in $\mathrm{Out}(\Gamma)$ is the class of f^* in $\mathrm{Mod}(S)$.

3.1. Action of a Dehn twist on $\mathrm{H}^1_\alpha(\Gamma, \mathbb{C})$

Proposition 9. *Any Dehn twist along a separating curve belongs to $\mathcal{I}(S)$.*

Proof. If T_δ is the Dehn twist along a simple curve δ then its action on homology is

$$T_\delta \cdot a = a + i(a, \delta) \cdot [\delta]$$

where i is the algebraic intersection form. If δ is separating, then $[\delta] = 0$ and then T_δ's action on homology is trivial. Hence T_δ belong to $\mathcal{I}(S)$.

□

We now explain how one can make an effective calculation of the action of a Dehn twist along a separating curve.

Lemma 10. *Let δ be a separating curve in S such that $p \notin \delta$, and $[\delta] \in \pi_1(S, p)$ be a representative of δ's free homotopy class. Then there exists $\mu \in Z^1_\alpha(\pi_1(S, p), \mathbb{C})$ such that for all $\gamma \in \pi_1(S, p)$ and $\lambda \in Z^1_\alpha(\pi_1(S, p), \mathbb{C})$*

$$\lambda(T_\delta(\gamma)) = \mu(\gamma)\lambda([\delta]) + \lambda(\gamma)$$

Proof. Let $p \in S$ be the base point of $\pi_1 S$ in such a way that all the closed curves we will look at will be based at p, except if it is explicitly mentioned. Let δ be a simple separating closed curve in S such that an embedded annulus around δ does not contains p. Such a curve exists in any free homotopy class of a simple closed curve. Let T_δ be the Dehn twist along δ. Let $[\gamma]$ be a class in $\pi_1 S$ and $\gamma \in [\gamma]$ such that the number of intersection of γ with δ is minimal. Let $q_1, ..., q_k$ be the intersection points of γ and δ in the order in which they appear. Let $q_0 \in \delta$, c an arbitrary path from p to q_0 and t the closed curve going from p to q_0 through c, then going through δ once and coming back to p through c.
Let β_i be the closed curve going from p to q_i through γ, going through δ (in the positive sens if $(-1)^{i+1} = 1$ and in the negative sens if $(-1)^{i+1} = -1$) until q_0 and going back to p through the path c. Hence(It is a simple verification) :

$$T_\delta([\gamma]) = [\gamma] \prod_{i=1}^{k} [\beta_i]^{-1} [t]^{\epsilon(i)} [\beta_i]$$

Remark that this formula holds whenever δ is non separating. Using this latest hypothesis, one finds

$$T^n_\delta([\gamma]) = T^{n-1}([\gamma]) \prod_{i=1}^{k} T^{n-1}([\beta_i]^{-1}) [t]^{\epsilon(i)} T^{n-1}([\beta_i])$$

Let us compute $\lambda(T^n_\delta([\gamma])$

$$\lambda(T_\delta^n([\gamma]) = \lambda(T^{n-1}([\gamma])) + \alpha([\gamma])\lambda([t]) \cdot \sum_{i=1}^{n} \epsilon(i)\alpha([\beta_i])$$

and $\mu(\gamma) = \sum_{i=1}^{n} \epsilon(i)\alpha([\beta_i])$.

□

3.2. Action of a subgroup generated by two Dehn twists.

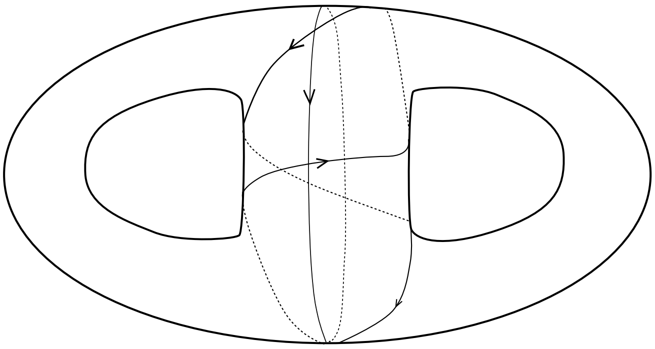


Figure 1: The curves δ_1 and δ_2

Let us consider the curves δ_1 and δ_2 from figure 3.2. The Dehn twists along those curves generate a group $G \subset \mathcal{I}(S)$.
Let T_i be the automorphism of Γ induced by the Dehn twist along δ_i. T_i acts on $Z^1_\alpha(\Gamma, \mathbb{C})$ preserving the line generated by $(1-\alpha)$. Lemma 10 ensures that the action of T_i is $T_i \cdot \lambda = \lambda + \varphi_i \cdot \mu_i$ where $\mu_i \in Z^1_\alpha(\Gamma, \mathbb{C})$ and φ_i are such that $\varphi_i(\mu_i) = 0$.

Proposition 11.

1. $\mu_1(\delta_2) = (1-\alpha(a_1)^{-1}) \cdot (1-\alpha(a_2)^{-1})$
2. $\mu_2(\delta_1) = (1-\alpha(a_1)) \cdot (1-\alpha(a_2))$
3. $\mu_1(\delta_1) = 0$
4. $\mu_2(\delta_2) = 0$

Proof. The two last inequalities directly follow from the fact that a simple closed curve does not auto-intersect.
Let us write $\mu_1(\delta_2) = \sum_{i=1}^{n} \epsilon(i)\alpha([\beta_i])$ according to Proposition 11. Let us compute the β_i using the algorithm described in the proof of lemma 10. One can choose q_0 and c the path from t_0 to p to be the part of δ_2 going from p to q_0 (cf. figure 3.2). This way β_1 is null-homotopic.

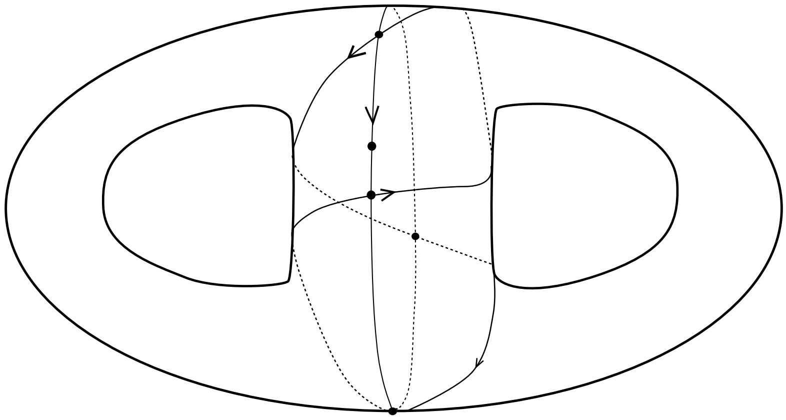


Figure 2: Combinatorics of the intersections between δ_1 and δ_2

β_2 is the curve built following δ_2 from p to q_2 then going to q_0 following δ_1 in reverse and going back to p along δ_2 in reverse. This gives the following curve :

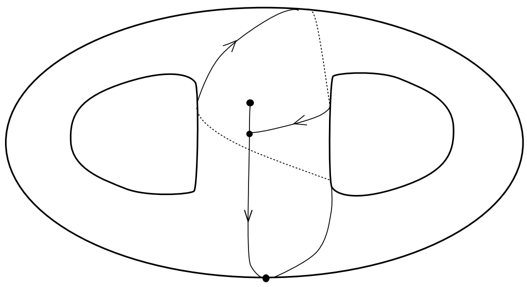


Figure 3: The curve β_2

The curve β_2 is homologuous to a_1^{-1}. Proceeding with the algorithm, one finds :

- β_1 is homologuous to 0.
- β_2 is homologuous to a_1^{-1}.
- β_3 is homologuous to $a_1^{-1}a_2^{-1}$.
- β_4 is homologuous to a_2^{-1}

This gives $\mu_1(\delta_2) = 1-\alpha(a_1)^{-1}+\alpha(a_1)^{-1}\alpha(a_2)^{-1}-\alpha(a_2)^{-1}$. A likewise calculation gives the value of $\mu_2(\delta_1)$.

□

Proposition 12. $[\mu_1]$ *and* $[\mu_2] \in \mathrm{H}^1_\alpha(\Gamma,\mathbb{C})$ *form a basis of* $\mathrm{H}^1_\alpha(\Gamma,\mathbb{C})$ *for all* α *in a dense set open set of full measure.*

Proof. Assume there exists constants a, b, c such that

$$a\mu_1 + b\mu_2 + c(1-\alpha) = 0$$

Evaluating on δ_1 et δ_2 , one finds $0 = a\mu_1(\delta_2) = b\mu_2(\delta_1)$. For α in a set of full measure(the set of α such that $(1-\alpha(a_1)^{-1})(1-\alpha(a_2)^{-1})$ and $(1-\alpha(a_1))(1-\alpha(a_2))$ do not vanish), $a = b = 0$, and so $c = 0$.

□

Matrices of T_1 and T_2 in this basis are :

$$\begin{pmatrix} 1 & (1-\alpha(a_1))(1-\alpha(a_2)) \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ (1-\alpha(a_1)^{-1})(1-\alpha(a_2)^{-1}) & 1 \end{pmatrix}$$

3.3. A criterion for ergodicty.

Lemma 13 (Jorgensen)**.** *If two matrices A and B generate a non-elementary discrete subgroup of* $\mathrm{PSL}(2,\mathbb{C})$ *then*

$$|\mathrm{Tr}(A)^2-4|+|\mathrm{Tr}(ABA^{-1}B^{-1})-2|\geq 1$$

This lemma is proven in [Jør76].

Let us compute the quantity of the lemma for $A=\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ and $B=\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}$.

$$\mathrm{Tr}(ABA^{-1}B^{-1})=2+(ab)^2$$

$$\mathrm{Tr}(A)=2$$

So if A and B generate a non-elementary subgroup and if $|ab|<1$, $\langle A,B\rangle$ is not discrete. One the other hand, it is clear that when a and b are nonzero, the group generated by A and B is non-elementary. In that case, A acts by translations on $\mathbb{CP}^1$, the only point of finite orbit for A is the point at infinity. But since $b\neq 0$, B send the point at infinity on 0 which has infinite orbit for the action of A.

Proposition 14. *If H is a non-discrete and non-elementary subgroup of* $\mathrm{SL}(2,\mathbb{C})$*, then $\overline{H}$ is either all* $\mathrm{SL}(2,\mathbb{C})$ *or conjugate to* $\mathrm{SL}(2,\mathbb{R})$*, a $\mathbb{Z}/2\mathbb{Z}$-extension of* $\mathrm{SL}(2,\mathbb{R})$.

This proposition can be found in [Kap09](p.69).

Lemma 15. *Let H be a subgroup of* $\mathrm{SL}(n+1,\mathbb{C})$ *such that the action of $\overline{H}$ on $\mathbb{CP}^n$ is transitive. Then the action of H on $\mathbb{CP}^n$ is ergodic.*

Proof. This lemma is a consequence of Lebesgue regularity lemma.

□

4. *Proof of the main theorem in genus 2.*

The set U of elements $\alpha\in \mathrm{H}^1(S,\mathbb{C}^*)$ such that $|(1-\alpha(a_1))(1-\alpha(a_2))(1-\alpha(a_1)^{-1})(1-\alpha(a_2)^{-1})|<1$ and $(1-\alpha(a_1))(1-\alpha(a_2))(1-\alpha(a_1)^{-1})(1-\alpha(a_2)^{-1})\notin\mathbb{R}$ has positive measure (it contains an open set of $(\mathbb{C}^*)^4$ with 2 analytic submanifolds of codimension 1 removed). Since the mapping class group action on $\mathrm{H}^1(S,\mathbb{C}^*)\simeq(\mathbb{C}^*)^4$ is ergodic , $V=\mathrm{Mod}(S)\cdot U$ has full measure.

Proposition 16. *For all $\alpha\in V$, the Torelli group action on $\mathrm{PH}^1_\alpha(\Gamma,\mathbb{C})$ is ergodic.*

Proof. Consider $\alpha\in V$. Then there exists $\beta\in U$ and $\phi\in\mathrm{Mod}(S)$ such that $\phi\cdot\beta=\alpha$. Recall that $G\subset\mathcal{I}(S)$ is the group generated by the Dehn twists along δ_1 and δ_2. Precomposing by ϕ gives a projective isomorphism :

$$\phi^*:\mathrm{PH}^1_\beta(\Gamma,\mathbb{C})\longrightarrow\mathrm{PH}^1_\alpha(\Gamma,\mathbb{C})$$

such that the action of the groups G and $\phi G\phi^{-1}$ (on $\mathrm{PH}^1_\beta(\Gamma,\mathbb{C})$ and $\mathrm{PH}^1_\alpha(\Gamma,\mathbb{C})$ respectively) are conjugated by ϕ^*. If $\beta \in U$, the G-action on $\mathrm{PH}^1_\beta(\Gamma,\mathbb{C}) \simeq \mathbb{CP}^1$ is the action of a group with closure isomorphic to PSU(2) or PSL$(2,\mathbb{C})$. Lemma 15 ensures that this action is ergodic, so the $\phi G\phi^{-1}$ action on $\mathrm{PH}^1_\alpha(\Gamma,\mathbb{C})$ is ergodic since it is conjugated to G through a projective isomporhism.

□

One can take as the Lebesgue measure on $\chi \setminus L^{-1}(\mathrm{Id})$ the measure $m = \mu \otimes \nu_\alpha$ where μ is the Lebesgue measure on $\mathrm{H}^1(S,\mathbb{C}^*)$ and $(\nu_\alpha)_{\alpha\in \mathrm{H}^1(S,\mathbb{C}^*)}$ is a familly of measures on $\mathrm{PH}^1_\alpha(\Gamma,\mathbb{C})$ depending analytically on α.

We are now ready to end the proof of the main theorem in genus 2. Let $A \subset \chi \setminus L{-}1(\mathrm{Id})$ be an invariant measurable subset for the Mod(S) action. If $\mu(L(A)) = 0$, then $m(A) = 0$. Thus we can assume $\mu(L(A)) > 0$. Since the Mod(S) action on $\mathrm{H}^1(S,\mathbb{C}^*)$ is ergodic, $L(A)$ has full measure. Put $A_\alpha = A \cap \mathrm{PH}^1_\alpha(\Gamma,\mathbb{C})$. Fubini theorem implies that

$$m(A\cap B) = \int_{L(A\cap B)} \nu_\alpha(A_\alpha \cap B)d\mu$$

where B is any measurable subset.
If $m(A) > 0$, there exists $\epsilon > 0$ and a set with positive measure $W \subset L(A)$ for which $\nu_\alpha(A_\alpha) > \epsilon$. Remind that the set V has full measure so $\mu(W\cap V) > 0$. Since $\mu(W\cap V) > 0$, $\mathrm{Mod}(S)\cdot(W\cap V)$ has full measure. But if $\alpha \in \mathrm{Mod}(S)\cdot(W\cap V) \subset V$, $\nu_\alpha(A_\alpha) > 0$ because it contains the image of a A_β of a map $\phi \in \mathrm{Mod}(S)$ sending β on α. But since α belongs to V, $\nu_\alpha(A_\alpha) > 0$ and the Torelli group action on $\mathrm{PH}^1_\alpha(\Gamma,\mathbb{C})$ is ergodic, A_α has full measure. So for almost all α, $\nu_\alpha(A_\alpha \cap B) = \nu_\alpha(\mathrm{PH}^1_\alpha(\Gamma,\mathbb{C}) \cap B)$ and

$$m(A\cap B) = m(B)$$

So A has full measure, which proves that the action is ergodic.

5. *Higher genus.*

We proved in section 2 that the mapping class group action on $\mathrm{H}^1(S,\mathbb{C}^*)$ is ergodic. In genus 2, the strategy is still to study the Torelli group action in the fibers $\mathrm{PH}^1_\alpha(\Gamma,\mathbb{C})$. To be more precise, we prove that for almost all α, this action is ergodic giving explicit formulas for the action of some specific Dehn twists. Let $p \in S$ be the base point of $\pi_1 S = \Gamma$. Any diffeomorphism f fixing p whose action on $\mathrm{H}_1(S,\mathbb{Z})$ is trivial acts linearly on $\mathrm{H}^1_\alpha(\Gamma,\mathbb{C})$ in such a way that the action of the class of f in Mod(S) is the projectivized action of f on $\mathrm{PH}^1_\alpha(\Gamma,\mathbb{C})$. In this section we prove that we can find a subgroup of diffeomorphisms fixing p whose action on $\mathrm{H}^1_\alpha(\Gamma,\mathbb{C})$ is ergodic.

In a way similar to genus 2, one builds $2g-2$ curves $(\delta_i,\eta_i)_{1\leq i\leq g-1}$ with the following properties :

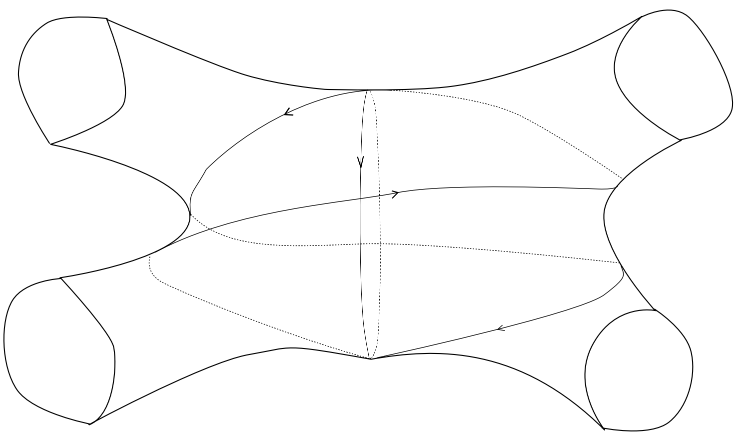
1. For all $i \neq j$,the curve δ_i (respectively η_i) is disjoint from the curves δ_j and η_j.
2. For a generic $\alpha \in \mathrm{H}^1(S,\mathbb{C}^*)$ (in an open dense subset of full measure), the classes $[\mu_1],[\nu_1],\cdots,[\mu_{g-1}],[\nu_{g-1}]$ form a basis of $\mathrm{H}^1_\alpha(\Gamma,\mathbb{C})$.

3. Both the action of T_{δ_i} and T_{η_i} stabilize the projective line associated to the plane $[\mu_i], [\nu_i]$.

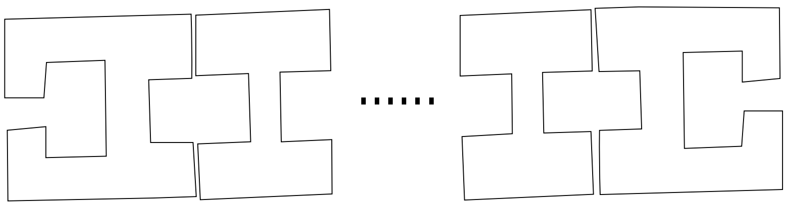
4. The group generated by T_{δ_i} and T_{η_i} acts projectively, the action is ergodic on the stabilized projective line for all i and for α in an open set.

5. The $g-1$ groups $G_i = \langle T_{\delta_i}, T_{\eta_i} \rangle$ commute, this way the $G = G_1 \cdots G_{g-1}$ action is a diagonal action on $\mathbb{C}^{2g-2} \simeq \mathrm{H}^1_\alpha(\Gamma, \mathbb{C})$.

Take the genus 2 surface from figure 3.2 and cut it twice along simple closed curves, in a way to get a four holed sphere with boundary :



Take $g-1$ copies of this sphere, S_1, S_2, ..., S_{g-1}, each one carrying 2 marked simple closed curves δ_i and η_i. Let us glue them back along the following patern :



This way one gets a genus g surface with the announced family of curves. Take a point p disjoint from all the curves, and for each curve a path going from p to a point of this curve. For δ_1, let $\tilde{\delta_1}$ be the curve built going from p to δ_1 through the chosen path, doing one turn of δ_1 and coming back to p. One builds for each δ_i and η_i a curve $\tilde{\delta_i}$ and $\tilde{\eta_i}$ in a similar way. Let $i \neq 1$, $T_{\delta_i}(\tilde{\delta_1}) = \gamma\tilde{\delta_1}\gamma^{-1}$ for some $\gamma \in \Gamma$ homologuous to δ_i. $\gamma \in \mathrm{D}\Gamma$ since δ_i is separating, so for all $\lambda \in \mathrm{H}^1_\alpha(\Gamma, \mathbb{C})$, $\lambda(T_{\delta_i}(\tilde{\delta_1})) = \lambda(\tilde{\delta_1})$.
The same way one can define, associated to $\tilde{\delta_i}, \tilde{\eta_i}$ the cocycles μ_i, ν_i such that :

$$T_{\delta_i} \cdot \lambda = \lambda + \lambda(\tilde{\delta_i})\mu_i$$

$$T_{\eta_i} \cdot \lambda = \lambda + \lambda(\tilde{\eta_i})\nu_i$$

for all $\lambda \in \mathrm{H}^1_\alpha(\Gamma, \mathbb{C})$.

Let us assume from now on that α is generic in the following sense : the field generated by the images of α has transcendental dimension $2g$. The set of such α has full Lebesgue measure.

Proposition 17.

1. *For all i, there exists two homology classes a_i and b_i such that*
   - $\mu_i(\eta_i) = (1-\alpha(a_i)^{-1}) \cdot (1-\alpha(b_i)^{-1})$
   - $\nu_i(\delta_i) = (1-\alpha(a_i)) \cdot (1-\alpha(b_i))$
   - $\mu_i(\delta_i) = 0$
   - $\nu_i(\eta_i) = 0$
2. *The classes $[\mu_1], [\nu_1], \cdots, [\mu_{g-1}], [\nu_{g-1}]$ generate $\mathrm{H}^1_\alpha(\Gamma, \mathbb{C})$.*
3. *For all $1 \leq i \leq g-1$, the action of the group generated by T_{δ_i} and T_{η_i} stabilizes the vector space generated by $[\mu_i]$ and $[\nu_i]$.*

Proof.

1. The first point is exactly proposition 11 extended to higher genus. The proof works the same way, applying lemma 10.
2. One writes a relation of linear dependence :

$$\sum_i u_i \mu_i + v_i \nu_i = k(1-\alpha)$$

   Evaluating in $\tilde{\delta_i}$ and $\tilde{\eta}_i$, one finds that all the coefficients u_i et v_i are zero, which implies $k = 0$.

3. Last point is a direct consequence of the remarks above the proposition. If $i \neq j$, then $\mu_i(T_{\delta_i}\tilde{\delta_j}) = \mu_i(\tilde{\delta_j})$, but since $\tilde{\delta_j}$ is homotopic to a curve disjoint from δ_i, $\mu_i(\tilde{\delta_j}) = 0$. It works the same with the curves η_i, in such a way that the vector space generated by the $[\mu_i]$ and $[\nu_i]$ is stabilized by the action of $G_i = \langle T_{\delta_i}, T_{\eta_i} \rangle$.

□

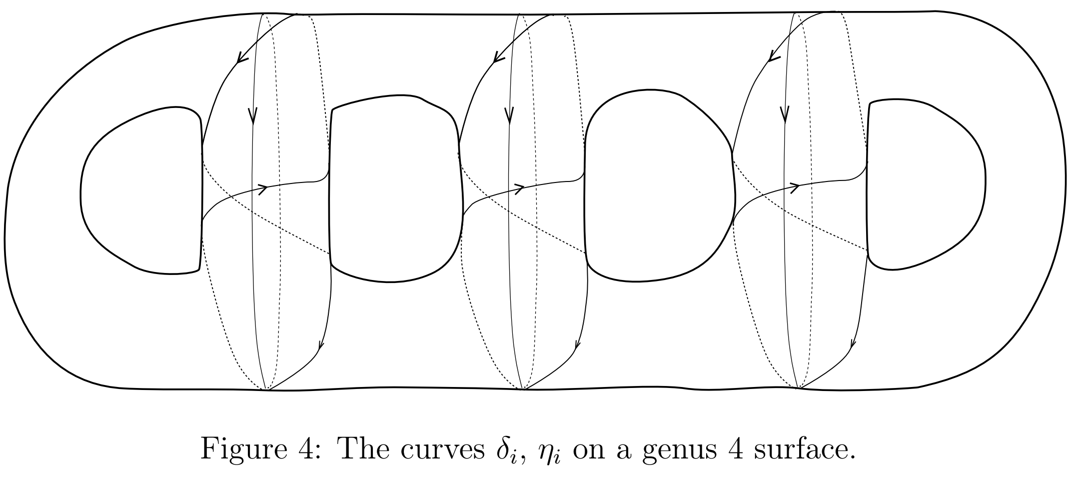


Figure 4: The curves δ_i, η_i on a genus 4 surface.

We now have everything we need to prove :

Theorem 18. *The action of the mapping class group on χ is ergodic in genus $g \geq 2$.*

Proof. Let G be the group generated by the T_{δ_i}, T_{η_i}. $G = G_1 \times \cdots \times G_{g-1}$ since the G_i commute. The G_i action on the subvector space generated by $[\mu_i]$ and $[\nu_i]$ is the action of the group generated by the matrices :

$$\begin{pmatrix} 1 & (1-\alpha(a_i))(1-\alpha(b_i)) \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ (1-\alpha(a_i)^{-1})(1-\alpha(b_i)^{-1}) & 1 \end{pmatrix}$$

Applying Jorgensen's lemma, there exists an open set U of $\mathrm{H}^1_\alpha(\Gamma, \mathbb{C})$ for which for all i, the action of G_i on the vector space generated $[\mu_i]$ and $[\nu_i]$ est ergodic (since the action of its closure is transitive). This implies (according to Fubini's theorem) that the action of G on $\mathrm{H}^1_\alpha(\Gamma, \mathbb{C})$ is ergodic, hence the action of the Torelli group is ergodic on $\mathrm{PH}^1_\alpha(\Gamma, \mathbb{C})$ for $\alpha \in U$. Proposition 16 implies it is ergodic on $\mathrm{PH}^1_\alpha(\Gamma, \mathbb{C})$ for α in a dense set of full measure. Applying Fubini's theorem and using the fact that the action of $\mathrm{Mod}(S)$ is ergodic on $\mathrm{H}^1(S, \mathbb{C}^*)$, one finds that the action of $\mathrm{Mod}(S)$ on χ is ergodic.

□

Remark From the description of the image of the group G in $\mathrm{PH}^1_\alpha(\Gamma, \mathbb{C})$, one can see that the mapping class group preserves no measure in the class of Lebesgue measure. In particular this implies that there is no **invariant** symplectic form.

6. *Euclidean characters.*

Let us look at the action of the mapping class group on $\chi_{\mathbb{U}}$.
Let $\rho : \Gamma \longrightarrow \mathrm{Aff}(\mathbb{C})$ be a euclidean representation. One can naturally associate to ρ a flat bundle in $\mathbb{CP}^1$ over S the following way : let $\tilde{S}$ be a universal cover of S, Γ acts on $\tilde{S} \times E$:

$$\gamma \cdot (x, z) = (\gamma \cdot x, \rho(\gamma)(z))$$

The bundle associated to ρ is the quotient $F_\rho = \tilde{S} \times E/\Gamma$. The foliation $\tilde{S} \times E$ factors through the quotient and defines a flat connection. Remark that this construction can be made for any representation.

Whenever ρ is euclidean, one can define a volume form μ_x , $x \in S$ on the fibers since the standard volume form on E is preserved by the action of Γ (since ρ is euclidean). One can build on F_ρ a 2-form ω_ρ vanishing on the leaves of the foliation and equal to μ_x in each fiber. Moreover the form ω is closed, since it is the form dz in the coordinates (x, z).

Proposition 19. *Let s be a section of the bundle F_ρ.*

$$\mathrm{v}(\rho) = \int_S s^*\omega$$

does not depend on the choice of the section s. It is the volume of the representation ρ .

Proof. E being convex, two sections s_1 and s_2 are homotopic through s_t. Notice that $\int_S s^*\omega$ is the volume of the graph of ρ. The proposition is a corollary of Stokes theorem applied to the image of the homotopy s_t in $[0,1]\times F_\rho$.

□

The volume defines a function $\mathrm{v} : \mathrm{Hom}(\Gamma, \mathrm{Iso}_+(\mathbb{C})) \longrightarrow \mathbb{R}$. Let us study the restriction of this function to $Z^1_\alpha(\Gamma,\mathbb{C})$ for a given $\alpha \neq 1$. The volume of a cocycle in $\lambda \in Z^1_\alpha(\Gamma,\mathbb{C})$ is the volume of the associated representation.

This form can also be defined in a totally homological way. If α and β are two elements of $\mathrm{H}^1(S,\mathbb{C}^*)$, one can define an algebraic product :

$$\wedge : \mathrm{H}^1_\alpha(\Gamma,\mathbb{C}) \times \mathrm{H}^1_\beta(\Gamma,\mathbb{C}) \longrightarrow \mathrm{H}^2_{\alpha\beta}(\Gamma,\mathbb{C})$$

$\mathrm{H}^2_\alpha(\Gamma,\mathbb{C}) = 0$ as soon as $\alpha \neq 1$. The bilinear form

$$\begin{array}{rccc} \wedge_\alpha & : & \mathrm{H}^1_\alpha(\Gamma,\mathbb{C}) \times \mathrm{H}^1_\alpha(\Gamma,\mathbb{C}) & \longrightarrow & \mathbb{C} \\ & & (\lambda,\mu) & \longmapsto & \lambda \wedge \overline{\mu} \end{array}$$

identifying canonically $\mathrm{H}^2(\Gamma,\mathbb{C})$ and $\mathbb{C}$. See [DM86] for more details (where everything is done is the case of holed spheres, nevertheless it still holds in our setting).

Proposition 20. *Take $\alpha \in \mathrm{H}^1(S,\mathbb{C}^*)$*

1. *For $\lambda \in Z^1_\alpha(\Gamma,\mathbb{C})$, $\mathrm{v}(\lambda)$ only depends on the class of λ in $H^1_\alpha(\Gamma,\mathbb{C})$.*
2. *The induced function $\mathrm{v} : H^1_\alpha(\Gamma,\mathbb{C}) \longrightarrow \mathbb{R}$ is a non-degenerate hermitian form.*
3. *For all α the signature of the form is $(g-1, g-1)$.*

Proof.

1. Remark that if $f := az+b \in \mathrm{Aff}(\mathbb{C})$, the map

$$\begin{array}{rccc} \Psi & : & \tilde{S}\times E & \longrightarrow & \tilde{S}\times E \\ & & (x,z) & \longmapsto & (x,f(z)) \end{array}$$

é induces an affine isomorphism between the bundles F_ρ and $F_{f\rho f^{-1}}$for any representation ρ. From the definition of the forms ω one gets

$$\Psi_*\omega_\rho = |a|^2\omega_{f\rho f^{-1}}$$

Any two representation define the same element in $H^1_\alpha(\Gamma,\mathbb{C})$ if and only if they are conjugated by a translation. In this case, they have the same volume. The formula above ensures us that v is an hermitian form.

2. The fact that the form is non degenerate is just Poincaré duality in twisted cohomology.

3. Assume α is real. Then conjugation is an order 2 endomorphism of $\mathrm{H}^1_\alpha(\Gamma,\mathbb{C})$ such that $\mathrm{v}(\overline{\lambda}) = -\mathrm{v}(\lambda)$ for every $\lambda \in \mathrm{H}^1_\alpha(\Gamma,\mathbb{C})$. Since v is non-degenerate, its signature is $(g-1, g-1)$. An argument of connectivity extends the property to arbitrary α.

□

Proposition 21. 1. *The action of the mapping class group preserves* $\chi_{\mathbb{U}}^{+}$, $\chi_{\mathbb{U}}^{-}$ *and* $\chi_{\mathbb{U}}^{0}$.

2. *The Torelli group acts on* $\mathrm{PH}^1_{\alpha}(\Gamma, \mathbb{C})$ *by transformations belonging to* $\mathrm{PU}(\wedge_{\alpha})$.

Proof. Just let a lift of a diffeomorphism to $\tilde{S}$ fixing a base point act on $\tilde{S} \times E$ to see that two representations differing from f^* define the same volume form. □

The representation of the Torelli group in the case of punctured spheres. We have defined a family of representation indexed by $\mathrm{H}^1(S, \mathbb{U})$ of the Torelli group in $\mathrm{PU}(\wedge_{\alpha}) \simeq \mathrm{PU}(g-1, g-1)$. Very little is known about this representation except for the fact that for almost all parameters, its image is not discrete. This family was originally discovered by Chueshev in the early 90's, see [Chu90]. Let us now assume that S has a finite number of punctures. One can still build an Hermitian form on $\mathrm{H}^1_{\alpha}(\Gamma, \mathbb{C})$: Veech shows in [Vee93] that the signature of the $\wedge_{\alpha}$ depends on α. Moreover, one can pick α in order that $\wedge_{\alpha}$ has signature $(1, n)$. The Torelli group still defines a representation in $\mathrm{PU}(1, n)$.
It is an important question in complex hyperbolic geometry to build lattices in the isometry group. It is natural here to ask if these representation might lead to new constructions of lattices in $\mathrm{PU}(1, n)$.

7. Link with branched affine structures and open problems.

The original framework of this work was the study of affine branched structures, especially their holonomy representations. A complex projective structure on a surface S is an atlas of charts in $\mathbb{CP}^1$ where the transition maps are the restriction of elements in $\mathrm{PSL}(2, \mathbb{C}) = \mathrm{Aut}(\mathbb{CP}^1)$. One can also think of a projective structure as a $(\mathbb{CP}^1, \mathrm{PSL}(2, \mathbb{C}))$-structure in the sense of (X, G)-structures defined by Thurston. If S is a surface endowed with a projective structure, one can pull this structure back to its universal cover $\tilde{S}$, in such a way this structure factors through the quotient $S = \tilde{S}/\Gamma$. Since $\tilde{S}$ is simply connected, any projective chart can be fully extended to $\tilde{S}$. This defines a local diffeomorphsim

$$\mathrm{dev} : \tilde{S} \longrightarrow \mathbb{CP}^1$$

which is unique up to postcomposition by an element of $\mathrm{PSL}(2, \mathbb{C})$. Since the structure factors trough, there exists a morphism $\mathrm{hol} : \Gamma \longrightarrow \mathrm{PSL}(2, \mathbb{C})$ called the *holonomy* such that for every $\gamma \in \Gamma$ and $x \in \tilde{S}$ we have

$$\mathrm{dev}(\gamma \cdot x) = \mathrm{hol}(\gamma)(\mathrm{dev}(x))$$

Given a type of (X, G)-structure, one might ask what are the group homomorphism which can arise as the holonomy map of a (X, G)-structure.

Translations surfaces and periods of abelian differentials. A translation surface is an atlas of charts in $\mathbb{C}$ with transition maps being translations. Since such structures can only arise when S is a torus, one has to allow singularities : a finite set of points can carry a conical structure with angle being a integer multiple of 2π. See [Zor06] for a survey on the subject. The holonomy map of such a structure is a morphism $\omega : \Gamma \longrightarrow \mathbb{C}$ which factors through $\omega : \mathrm{H}_1(S, \mathbb{Z}) \longrightarrow \mathbb{C}$ since $\mathbb{C}$ is abelian. In this case, the holonomy problem is totally solved since the 20's (see [Hau20]) by the following theorem :

Theorem 22 (Haupt, 1920). *An element $\omega \in \mathrm{H}^1(S, \mathbb{C}) = \mathrm{Hom}(\mathrm{H}_1(S, \mathbb{Z}), \mathbb{C})$ is the holonomy map of a translation surface (or equivalently is the periods of an abelian differential over a Riemann surface) if and only if the two following conditions hold :*

1. $\mathcal{I}(\omega) \cdot \mathcal{R}(\omega) > 0$
2. *If the image of ω in $\mathbb{C}$ is a lattice Λ, then*

$$\mathcal{I}(\omega) \cdot \mathcal{R}(\omega) > \mathrm{vol}(\mathbb{C}/\Lambda)$$

A proof of this theorem exploiting mapping class group dynamics has been given in [Kap].

Holonomy of complex projective structures. The holonomy problem is also solved in the case of complex projective structures. Let us recall the theorem due to Gallo, Kapovich and Marden (see [GKM00]) :

Theorem 23. *A group homomorphism $\rho : \Gamma \longrightarrow \mathrm{PSL}(2, \mathbb{C})$ is the holonomy of a complex projective structure if and only if the two following conditions hold :*

1. *ρ lifts to $\mathrm{SL}(2, \mathbb{C})$.*
2. *The image of ρ is a non-elementary subgroup of $\mathrm{PSL}(2, \mathbb{C})$.*

We also can permit that our projective structures carry singular points which are locally branched projective covering. Translation surfaces are particular cases of branched projective structures, whose holonomy lives in the subgroup of translations. In this case the holonomy problem is answered by Haupt's theorem. Now one can look at complex affine structures, which are $(\mathbb{C}, \mathrm{Aff}(\mathbb{C}))$-structures with branched points.

Complex (branched) affine structures, holonomy and open problems. A complex affine structure is defined to be a Riemann surface S with an non constant holomorphic function

$$\mathrm{dev} : \tilde{S} \simeq \mathbb{H} \longrightarrow \mathbb{C}$$

equivariant with respect to a representation $\rho : \Gamma \longrightarrow \mathrm{Aff}(\mathbb{C})$. One can check that this definition is equivalent to the usual definition with charts and transition maps living in $\mathrm{Aff}(\mathbb{C})$. We ask the following question : which representation $\rho : \Gamma \longrightarrow \mathrm{Aff}(\mathbb{C})$ can be realized has the holonomy map of a branched complex affine structure ? A nice argument of Ehresmann popularized by Thurston ensures that the set of geometric holonomies (which are realized by a branched affine structure) is an open set of the character variety. Another remark is that whenever one can realize one representation, one can realize all its image by the action of the mapping class group. So we have a nice corollary of theorem 18 :

Corollary 24. *The subset consisting of representations which can be realized by a branched complex affine structure is an open set of full measure.*

We give here a list of questions arising from the study of these affine structures which seems interesting to the author :

1. Characterize the representations which are the holonomy of a branched affine structure.
2. Build explicit models realizing a given holonomy.
3. Describe more precisely the action of the mapping class group on χ and $\chi_{\mathbb{U}}^{+}$. Does there exists an analogous theorem to Ratner's, or is it possible to find orbits whose closure is not homogeneous ?
4. Study the dynamics of the directional foliation in the case where the holonomy lies in $\mathbb{R}^* \ltimes \mathbb{C}$. Can phenomenons different from those known in the case of translation surfaces happen ?
5. Study the family of representations of the Torelli group $\tau_\alpha : \mathcal{I}(S) \longrightarrow \mathrm{PGL}(2g-2, \mathbb{C})$. For which parameter α the image of this representation is discrete ? When α is unitary, can one build this way lattices in $\mathrm{PU}(g-1, g-1)$?
6. Explore the case where the singularities are arbitrary.
7. Study the dynamics of the isoholonomic foliation of the moduli space of branched affine complex structures. Is it ergodic ?

Recall that a strictly affine representation is a nonabelian representation which is not unitary and whose angles of linear parts generate an infinite group of $\mathbb{R}/\mathbb{Z}$. About the holonomy problem, the following conjecture seems reasonable :

Conjecture 25. *Every strictly affine representation is the holonomy of a branched affine structure.*

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