Hermitian symmetric spaces of infinite dimension and maximal representations

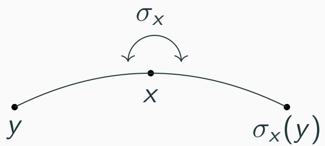
Bruno Duchesne

Institut Élie Cartan

Symmetric spaces of non-compact type

Definition

A symmetric space is a manifold M such that for any $x \in M$, there exists an isometry, σ_x, fixing x with $T_x\sigma_x = -\operatorname{Id}$.



Exemples : Euclidean space $\mathbf{E}^n$, spheres $\mathbf{S}^n$, hyperbolic spaces $\mathbf{H}^n$.

A symmetric space X is of *non-compact type* if it has non-positive sectionnal curvature and no Euclidean factor.

A dictionnary

X symmetric space of non-compact type

$$\updownarrow$$

- $G = \mathrm{Isom}(X)^\circ$ is a connected semi-simple Lie group without compact factor and trivial center.
- $K = \mathrm{Stab}_G(x)$ is a maximal compact subgroup.
- $X \simeq G/K$.

A dictionnary

Classical examples :

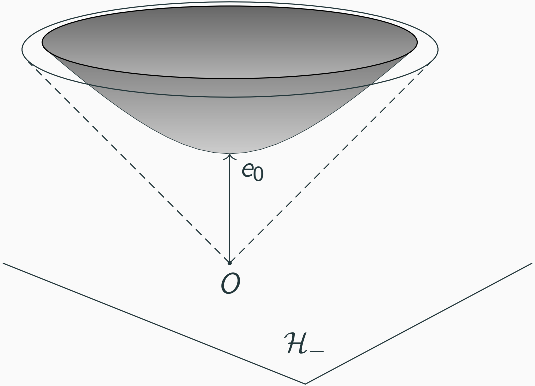
- $\mathrm{SL}_n(\mathbf{R})/\mathrm{SO}_n(\mathbf{R}) \longleftrightarrow$ ellipsoïds centered at 0 and unit volume.

- $\mathbf{H}^n = \mathrm{O}(1,n)/\mathrm{O}(1)\times\mathrm{O}(n) \longleftrightarrow$ upper half of the hyperboloïd.

Infinite dimension ?

Let $\mathcal{H}$ a Hilbert space with basis $(e_i)_{i\in\mathbf{N}}$. $(x, y) = x_0 y_0 - \sum_{i>0} x_i y_i$

$\mathbf{H}^\infty = \{x \in \mathcal{H},\ (x, x) = 1,\ \text{et } x_0 > 0\}$



Infinite dimension

Let $\mathrm{L}^2(\mathcal{H})$ be the set of Hilbert-Schmidt operators of $\mathcal{H}$, those operators M with

$$\sum_{i,j} \langle e_i, Me_j \rangle^2 < \infty.$$

$$\mathrm{GL}^2(\mathcal{H}) = \left\{ A \in \mathrm{GL}(\mathcal{H}),\ A - I \in \mathrm{L}^2(\mathcal{H}) \right\}$$

Then $\mathrm{GL}^2(\mathcal{H}) / \mathrm{O}^2(\mathcal{H})$ is a Riemannian symmetric space of non-positive curvature

Infinite dimension

Let us consider the quadratic form

$$Q(x) = \sum_{i=1}^{p} x_i^2 - \sum_{j>p} x_j^2.$$

Then $\mathcal{X}_{\mathbf{R}}(p,\infty) = \{P \subset \mathcal{H}, \dim(P) = p,\ Q|_P > 0\}$ is a Riemannian symmetric space of non-positive curvature.

One has the identification

$$\mathcal{X}_{\mathbf{R}}(p,\infty) = \mathrm{O}(p,\infty)/\,\mathrm{O}(p) \times \mathrm{O}(\infty).$$

Infinite dimension

Let us consider the following Hermitian form

$$Q(x) = \sum_{i=1}^{p} |x_i|^2 - \sum_{j>p} |x_i|^2.$$

Then $\mathcal{X}_{\mathbf{C}}(p,\infty) = \{P \subset \mathcal{H}, \dim(P) = p,\ Q|_P > 0\}$ is a Hermitian symmetric space of non-positive curvature. One has the identification

$$\mathcal{X}_{\mathbf{C}}(p,\infty) = \mathrm{U}(p,\infty)/\,\mathrm{U}(p) \times \mathrm{U}(\infty).$$

Representations of $\mathsf{SL}_2(\mathbf{R})$

Let $\mathcal{H} = \mathsf{L}^2(S^1)$. For $g \in \mathsf{SL}_2(\mathbf{R})$ and $f \in \mathcal{H}$, one define

$$\pi_s(g)(f) = \mathsf{Jac}(g)^{1/2+s} f \circ g^{-1}.$$

For $p \in \mathbf{N}$ and $s \in (p - 1/2, p + 1/2)$, π_s preserves a quadratic form of signature (p, ∞). One obtains an action of $\mathsf{SL}_2(\mathbf{R})$ on $\mathcal{X}_{\mathbf{R}}(p, \infty)$.

Action the Cremona group

Let C be the group of birational transformations $P^2(\mathbf{C})$.

There exists a rich action of C on $\mathbf{H}^{\infty}$ by isometries. Thanks to this action, one can show that C satisfies the Tits alternative (Cantat 2012) and has many normal subgroups (Cantat-Lamy 2013).

Classification of symmetric spaces of infinite dimension

What are all symmetric spaces of infinite dimension ? Can one classify them ?

Does the strategy of Élie Cartan still work in infinite dimension ?

There is no classification of Banach algebras.

Classification of infinite dimensional symmetric spaces

Theorem (D. 2015)

Let (M, g) be a symmetric space with non-positive curvature operator M, then it is isometric to the Hilbertian product

$$M \simeq \prod_i^2 M_i$$

Where each M_i is irreducible of finite dimension or isometric to one of the following :

$$GL^2_\infty(\mathbf{R})/O^2_\infty(\infty), \quad U^{*\,2}(\infty)/Sp^2(\infty), \quad U^2(p,\infty)/U^2(p)\times U^2(\infty),$$

$$O^2(p,\infty)/O^2(p)\times O^2(\infty),$$

$$O^{*\,2}(\infty)/U^2(\infty), \quad Sp^2_\infty(\mathbf{R})/U^2(\infty), \quad Sp^2(p,\infty)/Sp^2(p)\times Sp^2(\infty),$$

$$GL^2_\infty(\mathbf{C})/U^2(\infty), \quad O^2_\infty(\mathbf{C})/O^2(\infty), \quad Sp^2_\infty(\mathbf{C})/Sp^2(\infty).$$

Classification of infinite dimensional symmetric spaces

The *rank* is the maximal dimension of a flat subspace.

Corollary

Let (M, g) be a symmetric space with non-positive curvature operator, irreducible with rank $p < \infty$ and infinite dimension then M is isometric to

$$O(p,\infty)/O(p) \times O(\infty), \quad U(p,\infty)/U(p) \times U(\infty),$$
$$Sp(p,\infty)/Sp(p) \times Sp(\infty).$$

A citation of Gromov

"This spaces look to me as cute and sexy as their finite dimensional siblings but they have been for years shamefully neglected by geometers and algebraists a like."

– Gromov, *Asymptotic invariants of infinite groups.*

Margulis superrigidity

Lett G be a Lie group, a lattice of G is a discrete subgroup Γ of finite covolume.

Theorem (Margulis 1974)
Let G, H be two semi-simple Lie groups with finite center and no compact factors. Let $\Gamma < G$ be an irreducible lattice and $\rho\colon \Gamma \to H$, a representation with Zariski dense image.

If $\mathrm{Rank}_{\mathbf{R}}(G) \geq 2$ then there exists a representation $\overline{\rho}\colon G \to H$ such that ρ is the restriction of $\overline{\rho}$ to H.

Superrigidity in infinite dimension

Theorem (D. 2015)
Let G be a semi-simple Lie group with finite center and no compact factor with $\operatorname{Rank}_{\mathbf{R}}(G) \geq 2$. Let Γ be an irreducible lattice of G without torsion. Let $\mathcal{Y}$ be a simply connected Riemannian manifold with non-positive curvature and finite rank.

If Γ acts by isometries on $\mathcal{Y}$ without fixed points in $\partial\mathcal{Y}$ then Γ stabilizes a totally geodesic subspace of $\mathcal{Y}$ isometric to a product of factors of $\mathcal{X}_G$.

Ideas : Existence of a Γ-equivariant harmonic map $\mathcal{X}_G \to \mathcal{Y}$ then a Bochner type inequality due to Mok-Siu-Yeung.

In rank 1

Let $g \in \mathrm{Isom}(\mathcal{X})$, the *translation length*

$$\ell_{\mathcal{X}}(g) = \inf_{x \in \mathcal{X}} d(gx, x).$$

Theorem (Monod-Py)

For each $t \in (0,1]$ there is, up to conjugacy, exactly one irreducible continuous representation $\rho_t \colon \mathrm{Isom}(\mathbf{H}^n) \to \mathrm{Isom}(\mathbf{H}^\infty)$ such that $\ell_{\mathbf{H}^\infty}(\rho_t(g)) = t\ell_{\mathbf{H}^n}(g)$.

Moreover, there is an equivariant harmonic map $\mathbf{H}^n \to \mathbf{H}^\infty$ that is totally geodesic if and only if $t = 1$. The group $\mathrm{Isom}(\mathbf{H}^n)$ acts cocompactly on the convex hull of the image if this map.

Hermitian symmetric space

A symmetric space (M, g) is *Hermitian* is there is a complex structure J that is invariant under the connected component of the isometry group.

The Kähler form is $\omega(X, Y) = g(X, JY)$.

Examples : $\mathcal{X}_{\mathbf{C}}(p, \infty)$ and $\mathcal{X}_{\mathbf{R}}(2, \infty)$ (associated to $\mathrm{PU}(p, \infty)$ and $\mathrm{PO}^{+}(2, \infty)$) are Hermitian.

Bounded cohomology

The *bounded cohomology* $\mathrm{H}_b^n(G, \mathbf{R})$ of a group G is the cohomology of the complex $\mathrm{C}_b^n(G, \mathbf{R})^G =$

$$\left\{ f : G^{n+1} \to \mathbf{R} | f \text{ is } G\text{-invariant}, \sup_{(g_0,\ldots,g_n)\in G^{n+1}} |f(g_0, \ldots, g_n)| < \infty \right\}$$

whose coboundary operator is defined by the formula

$$df(g_0, \ldots, g_{n+1}) = \sum_{i=0}^{n+1} (-1)^i f(g_0, \ldots, \hat{g}_i, \ldots, g_{n+1}).$$

If G is a locally compact group, one defines the *continuous bounded cohomology* $\mathrm{H}_{cb}^2(G, \mathbf{R})$.

Continuous bounded of simple Lie groups

Let G be a simple Lie group of non-compact type and finite center.Then $\mathrm{H}^2_{cb}(G, \mathbf{R}) \neq 0$ if and only G is Hermitian and in that case

$$\mathrm{H}^2_{cb}(G, \mathbf{R}) = \mathbf{R}\kappa_G^{cb}$$

Where κ_G^{cb} is the *bounded Kähler class*.

The bounded Kähler class

The *bounded Kähler class* of an Hermitian simple Lie group G is the class $\kappa_G^b \in \mathrm{H}_b^2(G, \mathbf{R})$ defined by the cocycle

$$C_\omega^x(g_0, g_1, g_2) = \frac{1}{\pi} \int_{\Delta(g_0 x, g_1 x, g_2 x)} \omega$$

where x is any base point in the corresponding symmetric space $\mathcal{X}_G$.

Continuous bounded cohomology and lattices

Let Γ be a lattice in $\mathsf{SU}(1,n)$.

The restriction map $i^* : \mathrm{H}^2_{cb}(\mathsf{SU}(1,n),\mathbf{R}) \to \mathrm{H}^2_b(\Gamma,\mathbf{R})$

The transfer map $T_b^* : \mathrm{H}^2_b(\Gamma,\mathbf{R}) \to \mathrm{H}^2_{cb}(\mathsf{SU}(1,n),\mathbf{R})$

The bounded Kähler class

The *bounded Kähler class* of the groups $G = \mathrm{PU}(p, \infty)$ and $G = \mathrm{PO}^+(2, \infty)$ is the class $\kappa_G^b \in \mathrm{H}_b^2(G, \mathbf{R})$ defined by the cocycle

$$C_\omega^x(g_0, g_1, g_2) = \frac{1}{\pi} \int_{\Delta(g_0 x, g_1 x, g_2 x)} \omega$$

where x is any base point in the corresponding symmetric space $\mathcal{X}$.

$$\|\kappa_G^b\|_\infty = \mathrm{rank}(\mathcal{X})$$

Toledo invariant and maximal representations

Let G be a lattice in $\mathrm{SU}(1,n)$ and let κ_n^{cb} be the bounded Kähler class of $\mathrm{SU}(1,n)$.

Definition

Let $G \in \{\mathrm{PO}(2,\infty), \mathrm{PU}(p,\infty)\}$ and let $\rho : \Gamma \to G$ be an homomorphism. The *Toledo invariant* of the representation ρ is the number i_ρ such that

$$T_b^* \rho^* \kappa_G^b = i_\rho \kappa_n^{cb}$$

Milnor-Wood inequality : $|i_\rho| \leq \mathrm{rank}(\mathcal{X})$. The representation is *maximal* when there is equality.

A bit of history

Let $\Gamma < \mathrm{SU}(1,n)$ and $\rho\colon \Gamma \to G$ be a maximal representation where G is Hermitian.

- [Goldman 1988]Γ is cocompact, $n = 1$, $G = \mathrm{SU}(1,1)$. Then maximal representations are Fuchsian.
- [Toledo 1989]Γ is cocompact, $n = 1$, $G = \mathrm{SU}(1,n)$. Then there is an invariant complex geodesic line.
- [Burger-Iozzi-Wienhard 2003] $n = 1$. Then the Zariski closure is of tube type (e.g. $\mathrm{SU}(k,k)$).
- [Pozzetti 2015] $G = \mathrm{SU}(k,l)$. Then the image is not Zariski-dense for $k \neq l$.
- [Koziarz-Maubon 2017] Γ is cocompact. Necessarily $G = \mathrm{SU}(k,l)$ with $l \geq kn$ and the representation is rigid.

One example

A maximal representation is obtained this way :

$$\Gamma \to \mathrm{SU}(1, n) \to \mathrm{SU}(p, pn)$$

where $\mathrm{SU}(1, n) \to \mathrm{SU}(p, pn)$ is the diagonal inclusion.

Rigidity for maximal representations

Theorem (D.-Lécureux-Pozzetti)
Let Γ be a lattice of $\mathrm{SU}(1,n)$ with $n \geq 1$ and $\rho\colon \Gamma \to \mathrm{PU}(p,\infty)$ be a maximal representation. If $p \leq 2$ then there exists a finite dimensional totally geodesic subspace $\mathcal{Y} \subset \mathcal{X}_{\mathbf{C}}(p,\infty)$ that is Γ-invariant.

More generally, there is no Zariski-dense maximal representation $\Gamma \to \mathrm{PU}(p,\infty)$.

Steps of the proof

Steps :

1. Existence of a boundary map $\partial\mathcal{X}_{\mathbf{C}}(1, n) \to \partial\mathcal{X}_{\mathbf{C}}(p, \infty)$.
2. This boundary map sends chains to chains.
3. Geometry of chains.

Difficulties :

1. The space is no more locally compact.
2. There is no Zariski topology.

Geometrically dense maximal representations

Let Σ be a torus with one puncture and Γ_Σ.

Theorem (BLP)

There are geometrically dense maximal representations $\rho : \Gamma_\Sigma \to \mathrm{PO}_{\mathbf{R}}(2, \infty)$.

Boundaries at infinity

Let $\mathcal{X}$ a symmetric space of non-positive curvature. The *boundary at infinity* $\partial\mathcal{X}$ is the set of classes of geodesic rays that are at bounded distance.

For $\mathbf{H}^n_{\mathbf{C}} = \mathcal{X}_{\mathbf{C}}(1, n)$, $\partial\mathbf{H}^n_{\mathbf{C}} \simeq \{\text{isotropic lines}\}$.

For $\mathcal{X}_{\mathbf{C}}(p, \infty)$, $\partial\mathcal{X}_{\mathbf{C}}(p, \infty)$ has a structure of spherical building. Each cell corresponds to a flag of isotropic subspaces.

Let $\mathcal{I}_p = \{\text{maximal isotropic subspaces}\}$.

Existence of boundary map

Theorem (BLP)

Let $\Gamma < \mathrm{SU}(1,n)$ *be a countable subgroup,* $B = \partial H^n_{\mathbf{C}}$ *and* $p \in \mathbf{N}$.

If Γ *acts geometrically densely on* $\mathcal{X}_{\mathbf{K}}(p,\infty)$ *with* $p \leq 2$*, then there is a measurable* Γ*-equivariant map* $\phi\colon B \to \mathcal{I}_{\mathbf{p}}$*. Moreover, for almost all pair* $(b,b') \in B^2$*,* $\phi(b)$ *and* $\phi(b')$ *are transverse.*

If $\Gamma \to \mathrm{PO}_{\mathbf{K}}(p,\infty)$ *is a representation with a Zariski-dense image, then there is a measurable* Γ*-equivariant map* $\phi\colon B \to \mathcal{I}_{\mathbf{p}}$*. Moreover, for almost all pair* $(b,b') \in B^2$*,* $\phi(b)$ *and* $\phi(b')$ *are transverse.*

Chains at infinity

A *chain* in $\partial H^n_{\mathbf{C}}$ is the boundary of a complex geodesic.

A *chain* in $\mathcal{I}_p$ corresponds to the boundary of a totally geodesic copy of $\mathcal{X}_{\mathbf{C}}(p,p)$.

Cartan and Bergmann invariants

The *Cartan invariant* is a map $c\colon (\partial H^n_{\mathbf{C}})^3 \to [-1,1]$ such that $|c(\xi_1,\xi_2,\xi_3)|$ is maximal iff ξ_1,ξ_2,ξ_3 lie in a common complex geodesic.

The *Bergmann invariant* is a map $\beta\colon \mathcal{I}_p^3 \to [-p,p]$ such that $|\beta(\xi_1,\xi_2,\xi_3)|$ is maximal iff ξ_1,ξ_2,ξ_3 lie in a common copy of $\partial\mathcal{X}_{\mathbf{C}}(p,p)$.

Lemma

For every $V \in \mathcal{I}_p$, the cocycle C^V_β defined by

$$C^V_\beta(g_0,g_1,g_2) = \beta_{\mathbf{C}}(g_0V, g_1V, g_2V)$$

represents the bounded Kähler class.

Chains to chains

Theorem

Let $\Gamma < \mathrm{SU}(1,n)$ *be a lattice. Assume that a representation* $\rho : \Gamma \to \mathrm{PU}(p,\infty)$ *is maximal and admits an equivariant boundary map* $\phi : \partial\mathcal{X}_{\mathbf{C}}(1,n) \to \mathcal{I}_p(p,\infty)$*. Then the boundary map* ϕ *almost surely maps chains to chains.*

Geometry of chains

Theorem

Let $n \geq 2$ and let $\Gamma < \mathrm{SU}(1,n)$ be a complex hyperbolic lattice, and let $\rho : \Gamma \to \mathrm{PO}_{\mathbf{C}}(p,\infty)$ be a maximal representation. If there is a ρ-equivariant measurable map $\phi \colon \partial H^n_C \to \mathcal{I}_p$ then there is a finite dimensional totally geodesic Hermitian symmetric subspace $\mathcal{Y} \subset \mathcal{X}(p,\infty)$ that is invariant by Γ. Furthermore, the representation $\Gamma \to \mathrm{Isom}(\mathcal{Y})$ is maximal.

Idea : One can reconstruct ∂H^n_C with finitely many chains. So, the same is true for the essential image of ϕ.