The Poisson boundary of lampshuffler groups

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Lampshuffler groups

Let H be a countable group and denote $\mathrm{FSym}(H)$ the group of finitary permutations of H. We define **the lampshuffler group**

$$\mathrm{Shuffler}(H) = \mathrm{FSym}(H) \rtimes H,$$

where for every $h \in H$ and $f \in \mathrm{FSym}(H)$, we have $(h \cdot f)(x) = hf(h^{-1}x)$, for $x \in H$.

Random walks and the Poisson boundary

Let G be a countable group and μ a probability measure on G. Let $(g_i)_{i \geq 1}$ be a sequence if i.i.d. random variables distributed according to μ. The **μ-random walk on** G is the process $W_0 = e_G$, and $W_n = g_1 g_2 \cdots g_n$, for $n \geq 1$.
Consider the space of infinite trajectories G^{∞} endowed with the probability measure $\mathbb{P}$, which is defined as the push-forward of $\mu^{\mathbb{N}}$ through the map

$$G^{\infty} \to G^{\infty}$$
$$(g_1, g_2, g_3, \ldots) \mapsto (W_1, W_2, W_3, \ldots) := (g_1, g_1g_2, g_1g_2g_3, \ldots).$$

Say that two trajectories $(x_1, x_2, \ldots)$, $(y_1, y_2, \ldots)$ in G^{∞} are **orbit equivalent** if for some $p, N \geq 0$, it holds that $x_n = y_{n+p}$ for every $n \geq N$. Consider the measurable hull of the orbit equivalence relation in G^{∞}. That is, the σ-algebra of measurable subsets of G^{∞} which are unions of the equivalence classes, modulo $\mathbb{P}$-null sets.

Definition

The associated quotient of G^{∞} by this measurable hull is called the **Poisson boundary** of the random walk (G, μ).

Question: Is the Poisson boundary of (G, μ) non-trivial? Can we describe it explicitly (in terms of the geometry of G)?

The Poisson boundary has been completely described (under conditions on μ) for many classes of groups. Notably: free groups [Dynkin–Maljutov, Derrienic], hyperbolic groups [Ancona, Kaimanovich, Chawla–Forghani–Frisch–Tiozzo], discrete subgroups of semi-simple Lie groups [Furstenberg, Ledrappier], wreath products [Erschler, Karlsson–Woess, Sava-Huss, Lyons–Peres], Baumslag-Solitar groups [Kaimanovich, Cuno–Sava-Huss].

Entropy and non-triviality of the boundary

The **entropy** of μ is defined as $H(\mu) = -\sum_{g \in G} \mu(g) \log \mu(g)$. The **Avez asymptotic entropy of** (G, μ) is defined as $h_{\mu} := \lim_{n \to \infty} \frac{H(\mu^{*n})}{n}$.

The Entropy Criterion [Derrienic, Kaimanovich-Vershik]

The Poisson boundary of (G, μ) is trivial if and only if $h_{\mu} = 0$.

Using this together with methods of [Erschler '04], we prove:

Proposition

Consider H a f.g. group and let μ be a non-degenerate probability measure on $\mathrm{Shuffler}(H)$. Suppose that $H(\mu) < \infty$ and that μ induces a transient random walk on H. Then the Poisson boundary of $(\mathrm{Shuffler}(H), \mu)$ is **non-trivial**.

Stabilization of the permutation coordinate

Let G be a f.g. group and μ a probability measure on G. Recall that μ has **finite first moment** if for some (equivalently, every) word length ℓ on G, it holds that $\sum_{g \in G} \ell(g) \mu(g) < \infty$.
Denote by (F_n, S_n) the μ-random walk on $\mathrm{Shuffler}(H)$. We refer to $(F_n)_{n \geq 0}$ as the **permutation coordinate**.

Proposition

Let H be a f.g. group and consider the random walk $(\mathrm{Shuffler}(H), \mu)$. Suppose that μ has finite first moment and that μ induces a transient random walk on H. Then a.s. the permutation coordinate F_n, $n \geq 1$, stabilizes to a limit function $F_{\infty} : H \to H$.

Main Theorem: description of the boundary for $\mathrm{Shuffler}(\mathbb{Z})$

Consider a random walk $(\mathrm{Shuffler}(\mathbb{Z}), \mu)$, such that μ has finite first moment and induces a transient random walk on $\mathbb{Z}$. Then the space of limit functions F_{∞} completely describes the Poisson boundary of $(\mathrm{Shuffler}(\mathbb{Z}), \mu)$.

The conditional entropy criterion

There is a conditional version of the entropy criterion, due to Kaimanovich. This allows not only to determine the (non-)triviality of the Poisson boundary, but also provide complete descriptions of it.

Let G be a group and μ a probability measure on G. Let us denote by $\mathcal{I}$ the sub-σ-algebra of shift-invariant events of G^{∞}.

Definition

A measure space $(B, \mathcal{A}, \lambda)$ endowed with a measurable G-action is called a **μ-boundary** of G if there exists a G-equivariant measurable map $\pi : G^{\infty} \to B$ such that

- $\pi^{-1}(\mathcal{A}) \subseteq \mathcal{I}$ modulo $\mathbb{P}$-null sets, and
- $\lambda = \pi_*(\mathbb{P})$ satisfies $\mu * \lambda = \lambda$ (that is, λ is μ-stationary).

The conditional entropy criterion [Kaimanovich]

Let μ be a probability measure on G with finite entropy, and consider $\mathbf{B} = (B, \mathcal{A}, \nu)$ a μ-boundary of G. Suppose that for every $\varepsilon > 0$ there exists a random sequence of finite subsets $\{Q_{n,\varepsilon}\}_{n \geq 1}$ of G such that

1. the random set $Q_{n,\varepsilon}$ is a measurable function with respect to $\mathcal{A}$, for every $n \geq 1$;
2. $\limsup_{n \to \infty} \frac{1}{n} \log |Q_{n,\varepsilon}| < \varepsilon$ almost surely; and
3. $\limsup_{n \to \infty} \mathbb{P}\left(W_n \in Q_{n,\varepsilon}\right) > 0$, where $\{W_n\}_{n \geq 1}$ is the trajectory of the μ-random walk.

Then $\mathbf{B}$ coincides with the Poisson boundary of (G, μ).

The displacement of a permutation

Definition

Let $\sigma \in \mathrm{FSym}(\mathbb{Z})$. The **displacement** of σ is defined as $\mathrm{Disp}(\sigma) = \sum_{k \in \mathbb{Z}} |\sigma(k) - k|$.

We will need the following lemma.

Lemma

Let $\varepsilon > 0$ and $C > 0$. Consider for $n \geq 1$,

$$D_n := \left\{ \sigma \in \mathrm{FSym}(\mathbb{Z}) \mid \mathrm{supp}(\sigma) \subseteq [-\varepsilon n, \varepsilon n] \text{ and } \mathrm{Disp}(\sigma) < C\varepsilon n \right\}.$$

Then there is $K \geq 0$ such that

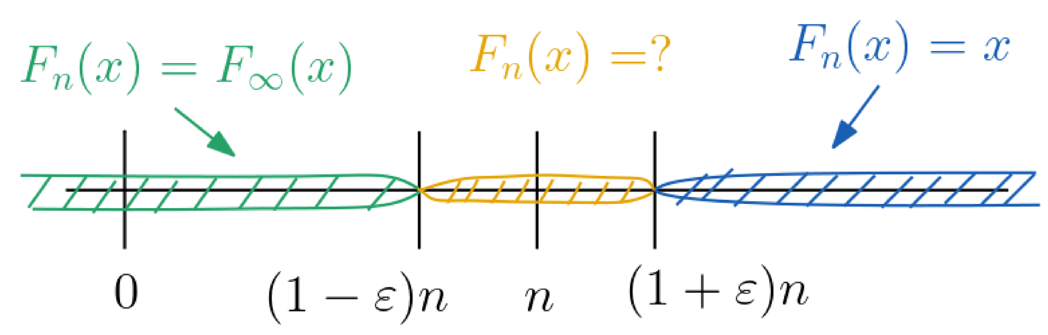
$$\limsup_{n \to \infty} \frac{1}{n} \log |D_n| < K\varepsilon.$$

Proof: Let us denote $J_{n\varepsilon} := [-\varepsilon n, \varepsilon n]$.

- For $\sigma \in D_n$ and $k \in J_{n\varepsilon}$, define
$$d_k := |\sigma(k) - k| \text{ and } s_k = \begin{cases} 1, & \text{if } \sigma(k) \geq k, \text{ and} \\ -1, & \text{if } \sigma(k) < k. \end{cases}$$
- Note that σ is completely determined by the values $\{d_k, s_k \mid k \in J_{n\varepsilon}\}$.
- There are $2^{2\varepsilon n+1}$ possible values for $\{s_k \mid k \in J_{n\varepsilon}\}$.
- On the other hand, the values $\{d_k \mid k \in J_{n\varepsilon}\}$ satisfy $\sum_{k \in J_{n\varepsilon}} d_k < C\varepsilon n$. Counting the possible values of $\{d_k\}$ is equivalent to distributing identical balls into distinguishable boxes.
- We conclude that $|D_n| \leq 2^{2\varepsilon n+1} \cdot \binom{(C+2)\varepsilon n + 1}{C\varepsilon n}$, and the lemma follows from Stirling's approximation.

Proof of the main theorem

- Let us assume that $\mu(f, x) > 0$ implies $x = +1 \in \mathbb{Z}$. That is, every increment adds 1 to the $\mathbb{Z}$ coordinate.
- Let $\varepsilon > 0$ and condition on the limit function F_{∞}. Denote by (F_n, S_n) the μ-random walk on $\mathrm{Shuffler}(\mathbb{Z})$.
- Our assumption on μ implies that $S_n = n$ a.s. Thus, we just need to estimate the value of F_n.
- The first moment hypothesis guarantees that the permutation increments σ_n belong to $\mathrm{Sym}([-\varepsilon n, \varepsilon n])$, for large enough n. This implies that the permutation coordinate F_n satisfies
$$F_n(x) = \begin{cases} x, & \text{if } x > (1+\varepsilon)n, \text{ and} \\ F_{\infty}(x), & \text{if } x < (1-\varepsilon)n. \end{cases}$$
- Denote by φ_n the restriction of F_n to the interval $[(1-\varepsilon)n, (1+\varepsilon)n]$. A rough estimate for the possible values of φ_n is $(2\varepsilon n + 1)!$. However, this is not good enough to apply the conditional entropy criterion.
- Solution: the displacement $\mathrm{Disp}(\sigma_n)$ of the permutation increments σ_n has finite first moment (as a random variable on $\mathbb{Z}$).
- Using this together with the law of large numbers, we get that $\mathrm{Disp}(\varphi_n) < C\varepsilon n$, for some fixed $C > 0$. This allows us to use the lemma above and hence apply the conditional entropy criterion.



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