The Poisson boundary of wreath products

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Wreath products

Let A, B be countable groups, and consider their **wreath product**

$$A \wr B := \bigoplus_B A \rtimes B.$$

Here $\bigoplus_B A = \{f : B \to A \mid f \text{ of finite support}\}$ and B acts on $f \in \bigoplus_B A$ as follows: $(b \cdot f)(x) = f(b^{-1}x)$, $x, b \in B$.

Random walks and the Poisson boundary

Let G be a countable group and μ a probability measure on G. Let $(g_i)_{i\geq 1}$ be a sequence if i.i.d. random variables distributed according to μ. The **μ-random walk on** G is the process $W_0 = e_G$, and $W_n = g_1 g_2 \cdots g_n$, for $n \geq 1$.
Consider the space of infinite trajectories G^∞ endowed with the probability measure $\mathbb{P}$, which is defined as the push-forward of $\mu^{\mathbb{N}}$ through the map

$$G^\infty \to G^\infty$$
$$(g_1, g_2, g_3, \ldots) \mapsto (W_1, W_2, W_3, \ldots) := (g_1, g_1g_2, g_1g_2g_3, \ldots).$$

Say that two trajectories $(x_1, x_2, \ldots)$, $(y_1, y_2, \ldots)$ in G^∞ are **orbit equivalent** if for some $p, N \geq 0$, it holds that $x_n = y_{n+p}$ for every $n \geq N$. Consider the measurable hull of the orbit equivalence relation in G^∞. That is, the σ-algebra of measurable subsets of G^∞ which are unions of the equivalence classes, modulo $\mathbb{P}$-null sets.

Definition
The associated quotient of G^∞ by this measurable hull is called the **Poisson boundary** of the random walk (G, μ).

Question: Is the Poisson boundary of (G, μ) non-trivial? Can we describe it explicitly (in terms of the geometry of G)?

The Poisson boundary has been completely described (under conditions on μ) for many classes of groups: free groups [Dynkin–Maljutov, Derriennic], hyperbolic groups [Ancona, Kaimanovich, Chawla–Forghani–Frisch–Tiozzo], discrete subgroups of semi-simple Lie groups [Furstenberg, Ledrappier], wreath products [Erschler, Karlsson–Woess, Sava-Huss, Lyons–Peres], Baumslag-Solitar groups [Kaimanovich, Cuno–Sava-Huss].

Random walks on wreath products

Consider μ a probability measure on $A \wr B$, and the associated random walk $\{(f_n, X_n)\}_{n\geq 1}$. We call f_n the lamp configuration at instant n and X_n the position in the base group at instant n.

Definition [Stabilization]
We say that the lamp configuration stabilizes a.s. if for every $b \in B$, there is $N \geq 1$ such that $f_n(b) = f_N(b)$ for all $n \geq N$.

Theorem [Kaimanovich-Vershik '83]
Suppose that μ is a non-degenerate finitely supported probability measure on $A \wr B$, that induces a transient random walk on B. Then the lamp configuration stabilizes a.s., and the Poisson boundary is non-trivial.

The same conclusion holds for infinitely supported measures with a finite first moment [Kaimanovich; Karlsson-Woess; Erschler].

Main Theorem [Frisch - S.]

Let A be a countable group and let μ be a probability measure on $G = A \wr \mathbb{Z}^d$, $d \geq 1$. Suppose that

1. $H(\mu) = \sum_{g\in G} -\mu(g)\log(\mu(g)) < \infty$, and
2. the lamp configuration stabilizes a.s. along sample paths.

Then the Poisson boundary of $(A \wr \mathbb{Z}^d, \mu)$ is the space of infinite lamp configurations $(A^{\mathbb{Z}^d}, \nu)$, where ν is the hitting measure.

Remarks and additional results

Previous results
The main theorem generalizes previous results under the following hypotheses:

- The projection of the random walk to $\mathbb{Z}^d$ has non-zero mean and $\sum_{g\in G} |g|\mu(g) < \infty$ [Kaimanovich '01],
- $d \geq 5$ and $\sum_{g\in G} |g|^3\mu(g) < \infty$ [Erschler '11],
- $d \geq 3$ and $\sum_{g\in G} |g|^2\mu(g) < \infty$ [Lyons-Peres '21].

The two conditions of the main theorem hold in particular when $\sum_{g\in G} |g|\mu(g)$ and the projection to $\mathbb{Z}^d$ is transient. Moreover, we prove:

Theorem [Frisch - S.]
Let A, B be countable groups and let μ be a probability measure on $G = A \wr B$. Suppose that

1. $\sum_{g\in G} |g|\mu(g) < \infty$, and
2. the projection of the random walk to B is transient.

Then the Poisson boundary of $(A \wr B, \mu)$ is the space of infinite lamp configurations (A^B, ν), where ν is the hitting measure.

The conditional entropy criterion

Definition
A probability space (X, λ) endowed with a measurable G-action is called a **μ-boundary** of G if there exists a measurable map $\pi : G^\infty \to X$ if

- $\pi \circ T = \pi$, where $T : G^\infty \to G^\infty$ is the shift map $T\{w_n\}_n = \{w_{n+1}\}_n$, and
- $\lambda = \pi_*(\mathbb{P})$ (and then λ is μ-stationary, i.e. $\mu * \lambda = \lambda$).

Remark: (X, λ) is a μ-boundary if and only if it is a G-equivariant quotient of the Poisson boundary.

Definition
If $\mathbf{X} = (X, \lambda)$ is a μ-boundary, then for λ-a.e. $\xi \in X$ there exists the conditional probability $\mathbb{P}^\xi$, which satisfies $\mathbb{P} = \int_X \mathbb{P}^\xi \, d\lambda(\xi)$.
Let us define

$$H_{\mathbf{X}}(w_n) = \int_X \sum_{g\in G} -\mathbb{P}^\xi(w_n = g)\log(\mathbb{P}^\xi(w_n = g))\, d\lambda(\xi).$$

The conditional entropy criterion [Kaimanovich]
Let μ be a probability measure on G with $H(\mu) < \infty$, and consider $\mathbf{X} = (X, \lambda)$ a μ-boundary of G. Then

$$\lim_{n\to\infty} \frac{H_{\mathbf{X}}(w_n)}{n} = 0 \iff \mathbf{X} \text{ is the Poisson boundary of } (G, \mu).$$

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Proof of the main Theorem for a particular (degenerate) random walk on $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}^d$

Consider $A = \mathbb{Z}/2\mathbb{Z}$, and μ supported on elements that move the person and/or modify lamps in the semigroup $\{\mathbf{v} \in \mathbb{Z}^d \mid v_i \geq 0, \text{ for } i = 1, \ldots, d\}$, with $H(\mu) < \infty$. Denote $\{(f_n, X_n)\}_n$ the μ-random walk on $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}^d$. Let $\varepsilon > 0$.

- Abelian groups have a trivial Poisson boundary, and hence $H(X_n) \leq \varepsilon n$ for n large enough.
- We can find $R \geq 1$ and $K > 0$ such that the R-large increments $\beta_n(R) = \left\{(X_j, g_j) \mid \text{ if } |g_j| > R\right\}$ satisfy $H(\beta_n(R)) < \varepsilon n + K$, for n large enough.
- The base group $\mathbb{Z}^d$ is partitioned into 3 sets: the first two sets are positions b where $f_n(b) = f_\infty(b)$ is already stabilized, and positions b where $\beta_n(R)$ reveals the value $f_n(b)$.
- The third set is the positions where the lamp configuration is uncertain, which has size at most CR^d.
- We conclude using the conditional entropy criterion, since

$$H_{A^B}(X_n, f_n) \leq H(X_n) + H(\beta_n(R)) + H_{A^B}(f_n \mid \beta_n(R)) \leq 2\varepsilon n + K + CR^d \log(2).$$