TEICHMÜLLER GEOMETRY IN THE HIGHEST TEICHMÜLLER SPACE

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Introduction

The goal of these notes is to explain how the space of reparametrizations of the geodesic flow of a hyperbolic surface can be seen both as the Teichmüller space of the weakly unstable foliation and as a subset of the character variety of the fundamental group of the surface into the group of diffeomorphisms of the circle. This work stemed out of many discussions Bertrand Deroin, who I warmly thank here.

0.1. **Higher Teichmüller theory.** Our main motivation is do develop a framework in which we hope to study *higher Teichmüller theory* with the tools of *classical Teichmüller theory.* Before we get more precise, let us briefly recall what these two terms cover.
Classical Teichmüller theory. Let Σ be a closed oriented surface of genus $g \geq 2$. The *Teichmüller space* of Σ is the space $\mathcal{T}(\Sigma)$ of complex structures on Σ up to isotopy. It carries a properly discontinuous action of the *mapping class group* of Σ, and the quotient under this action is the *moduli space* of Riemann surfaces of genus g.

Teichmüller theory in a broad sense refers to the study of the geometry of the Teichmüller space, which can be traced back to the XIXth century and continued throughout the XXth century, with the works of Ahlfors, Bers, or Wolpert among many others. In a more restrictive sense it could refer to the work of the nazi mathematician Oswald Teichmüller, who proved that $\mathcal{T}(\Sigma)$ is homeomorphic to $\mathbb{R}^{6g-6}$ and constructed its complex structure by studying optimal quasi-conformal maps between Riemann surfaces.

We know since Gauss that a complex structure on Σ is equivalent to a conformal class of Riemannian metrics, and since Poincaré that every such conformal class contains a unique metric of curvature -1. Thus the space $\mathcal{T}(\Sigma)$ is canonically identified with the *Fricke space* $\mathcal{F}(\Sigma)$ of hyperbolic metrics on Σ up to isotopy.

Now, a hyperbolic metric on Σ gives an isometry from the universal cover $\widetilde{\Sigma}$ to the Poincaré half plane $\mathbb{H}^2$ which is equivariant with respect to some representation of the fundamental group Γ of Σ which is *Fuchsian* (i.e. discrete and faithful). This gives an identification between $\mathcal{F}(\Sigma)$ and the space $\mathfrak{X}_{fuchs}(\Gamma, \mathrm{PSL}(2,\mathbb{R})$ of Fuchsian representations up to conjugation.

What makes Teichmüller theory so rich is that the three avatars of the Teichmüller space: $\mathcal{T}(\Sigma)$, $\mathcal{F}(\Sigma)$ and $\mathfrak{X}_{fuchs}(\Gamma, \mathrm{PSL}(2,\mathbb{R}))$ carry different geometric structures that "miraculously" combine together. A stricking example is the fact that Goldman's symplectic form on $\mathfrak{X}_{fuchs}(\Gamma, \mathrm{PSL}(2,\mathbb{R})$ combines with the complex structure of $\mathcal{T}(\Sigma)$ to form the *Weil–Petersson metric*, which lives naturally on $\mathcal{F}(\Sigma)$.

Let us point out that some aspects of Teichmüller theory have been generalized to several infinite dimensional contexts, such as Teichmüller theory of surface of infinite type, Teichmüller theory of foliations and universal Teichmüller theory.
Higher Teichmüller theory. In the 90's, Hitchin discovered that the space $\mathfrak{X}(\Gamma, \mathrm{PSL}(n,\mathbb{R}))$ of representations of Γ into $\mathrm{PSL}(n,\mathbb{R})$ had a distinguished connected component, that we will denote $\mathfrak{X}_{hit}(\Gamma, \mathrm{PSL}(n,\mathbb{R}))$ which coincided with the space $\mathfrak{X}_{fuchs}(\Gamma, \mathrm{PSL}(2,\mathbb{R}))$ for $n = 2$, and which beared some

resemblence with a "higher rank" analog of the Teichmüller space, in particular, it contains a *Fuchsian locus* isomorphic to $\mathcal{T}(\Sigma)$. This analogy was strengthed by Labourie, who proved that representations in Hitchin' component have a very powerful dynamical property that he called *Anosov property*, showing in particular that they are all discrete and faithful. Since then other character varieties $\mathfrak{X}(\Gamma, G)$ have been shown to contain connected components consisting only of Anosov representations. This popularized the term *Higher Teichmüller theory* to refer, depending on the context, to the study of Hitchin or related components of character varieties, of more generally o the study of Anosov representations of surface groups.

Several authors have been working on extending Teichmüller geometry to these higher Teichmüller spaces. An important work in this direction is that of Bridgeman–Canary–Labourie–Sambarino, who used the thermodynamical formalism to construct a Riemannian metric on $\mathfrak{X}_{hit}(\Gamma, \mathrm{PSL}(n, \mathbb{R}))$, which restricts to the Weil–Petersson metric on the Fuchsian locus. Let us briefly sketch how their construction works: they show that one can embed $\mathfrak{X}_{hit}(\Gamma, \mathrm{PSL}(n, \mathbb{R}))$ into the space of Hölder reparametrizations of the geodesic flow of a hyperbolic metric on Σ. There lives the *pressure metric*, which is roughly speaking the second fundamental form of the hypersurface formed by reparametrizations of entropy 1.

What has been missing in higher Teichmüller theory is a higher rank analog of the complex structure of $\mathcal{T}(\Sigma)$. An important open question is for instance whether the pressure metric of Bridgeman–Canary–Labourie–Sambarino is Kähler for some suitable complex structure on $\mathfrak{X}_{hit}(\Gamma, \mathrm{PSL}(n, \mathbb{R}))$.

Here we construct a complex structure on the space of Hölder reparametrizations of the geodesic flow, by identifying it with some foliated Teichüller space. We also develop a *representation theoretic* point of view on that space, identifying it with the space of "Anosov representations" of Γ into the group of diffeomorphisms of the circle. This that many aspects of the classical Teichmüller theory could be generalized to this "highest Teichmüller space" of reparametrizations.

We hope that some of these results will descend to interesting geometric properties of higher Teichmüller spaces after a careful understanding of how these umbed into the space of reparametrizations.

Let us advise the reader that the results presented here are not entirely new. For the most part they seem to be rephrasings of results well-known from hyperbolic dynamists. In particular, the general idea that certain moduli spaces of dynamical systems could be identified with Teichmüller spaces of foliations was introduced by Sullivan in []. In [], Cawley studies introduces the *Teichmüller space of Anosov automorphisms of the torus* and carries a study which is essentially covers what we do here when replacing the geodesic flow of a hyperbolic surface by the suspension flow of a linear automorphism of the torus. It was however very useful to the author writing the many details of these correspondance that he couldn't find elsewhere, and we hope it will be useful to some readers too.

0.2. **Three avatars of a highest Teichmüller space.** Let us now state our results with more precision. We first introduce three avatars of what our "highest Teichmüller space" will be.

0.2.1. *Parametrizations of the geodesic foliation.* Let Σ be a closed hyperbolic surface of genus at least 2. In the sequel, we denote by Γ its fundamental group and by M_Γ its unit tangent bundle. The geodesic flow of Σ is a flow on M_Γ. It turns out that the topological manifold M_Γ and the foliation $\mathcal{G}$ given by the orbits of the geodesic flow are "independent" of the choice of the hyperbolic structure on Σ. We call $\mathcal{G}$ the *geodesic foliation*.

Closed leaves of the geodesic foliation correspond to closed geodesics on Σ and are in one to one correspondance with conjugacy classes of primitive elements in Γ. We denote by $[\Gamma]$ this set.

A reparametrization of the geodesic flow is a continuous flow on M_Γ which is orbit equivalent to the geodesic flow. To avoid as much as possible references to a background hyperbolic metric, we call such flows *parametrizations of the geodesic foliation*.

Two parametrizations of the geodesic foliation are *conjugate* if they are conjugate by a homeomorphism preserving each geodesic leaf. This equivalence relation is not closed on the space of reparametrizations. We call two parametrizations *weakly conjugate* if one is the uniform limit of conjugates of the other. The space of parametrizations of the geodesic foliation modulo weak conjugation is denoted

$$\mathrm{Par}(\mathcal{G})\ .$$

The subset of equivalence classes of parametrizations which are Hölder regular is denoted by

$$\mathrm{Par}^h(\mathcal{G})\ .$$

If φ is a parametrization of the geodesic foliation, the *period map* of φ is the function

$$L_\varphi : [\Gamma] \to \mathbb{R}_{>0}$$

associating to a closed geodesic γ the time φ takes to run through γ.

There is a natural "scaling" action of $\mathbb{R}_{>0}$ on $\mathrm{Par}(\mathcal{G})$. The *topological entropy* $h_{top}(\varphi)$ of a flow φ provides a way to normalize parametrizations. Indeed, the function h_{top} is well-defined on $\mathrm{Par}(\mathcal{G})$, continuous, positive and homogeneous of degree -1 with respect to the scaling action. Thus every parametrization φ admits a unique scaling of entropy 1. We denote by

$$\mathrm{Par}_1(\mathcal{G})$$

the space of parametrizations of entropy 1 up to weak equivalence, and by $\mathrm{Par}_1^h(\mathcal{G})$ its intersection with $\mathrm{Par}^h(\mathcal{G})$.

0.2.2. *Anosov actions on the circle.* Recall that the hyperbolic surface Σ is isometric to the quotient $j(\Gamma)\backslash\mathbb{D}$ where $\mathbb{D}$ denotes Poincaré's hyperbolic disc and $j : \Gamma \to \mathrm{Isom}_+(\mathbb{D})$ is a *Fuchsian representation* (i.e. discrete and faithful). The representation j provides an analytic action of Γ on the unit circle $\mathbb{S}^1 = \partial\mathbb{D}$. This action has *maximal Euler class*, i.e. the twisted product bundle $\Sigma \times_j \mathbb{S}^1$ is isomorphic to the unit tangent bundle of Σ.

Let $\mathrm{Diff}(\mathbb{S}^1)$ denote the group of diffeomorphisms of $\mathbb{S}^1$ of class $\mathcal{C}^1$ and $\mathrm{Diff}^h(\mathbb{S}^1)$ the subgroup of diffeomorphisms with Hölder derivatives. We endow $\mathrm{Diff}(\mathbb{S}^1)$ with the $\mathcal{C}^1$ topology. By a theorem of Matsumoto, a morphism $\rho : \Gamma \to \mathrm{Diff}(\mathbb{S}^1)$ has maximal Euler class if and only if it is semi-conjugate to a Fuchsian representation j. We call ρ an *Anosov representation* into $\mathrm{Diff}(\mathbb{S}^1)$ or an *Anosov action* on $\mathbb{S}^1$ if it is conjugate to a Fuchsian representation j via a bi-Hölder homeomorphism. We will see in Section **??** that this is equivalent for the action of ρ to be topologically conjugated to j and *expanding* (see for instance [1] for the relevance of expanding properties in one dimensional dynamics).

Let ρ be an Anosov action on the circle. for every $\gamma \in \Gamma$, the diffeomorphism $\rho(\gamma)$ has a unique attracting fixed point on $\mathbb{S}^1$ that we denote abusively γ_+. The derivative of $\rho(\gamma)$ at its attracting fixed point is less than 1 and is invariant by conjugation of γ by a diffeomorphism. We define the *period map* of ρ as the map

$$\begin{array}{rccl} L_\rho : & [\Gamma] & \to & \mathbb{R}_{>0} \\ & [\gamma] & \mapsto & -\log\left(\rho(\gamma)'(\gamma_+)\right) \end{array} .$$

The space $\mathrm{Hom}(\Gamma, \mathrm{Diff}(\mathbb{S}^1))$ of morphisms from Γ to $\mathrm{Diff}(\mathbb{S}^1)$ inherits a topology from that of $\mathrm{Diff}(\mathbb{S}^1)$. The group $\mathrm{Diff}(\mathbb{S}^1)$ acts continuously on $\mathrm{Hom}(\Gamma, \mathrm{Diff}(\mathbb{S}^1))$ by conjugation. This action may not be proper. We denote by $\mathfrak{X}(\Gamma, \mathrm{Diff}(\mathbb{S}^1))$ the largest Hausdorff quotient of $\mathrm{Hom}(\Gamma, \mathrm{Diff}(\mathbb{S}^1))/\mathrm{Diff}(\mathbb{S}^1)$. We also denote by $\mathfrak{X}_{an}(\Gamma, \mathrm{Diff}(\mathbb{S}^1))$ the open subset of equivalence classes of Anosov representations, and by $\mathfrak{X}_{an}(\Gamma, \mathrm{Diff}^h(\mathbb{S}^1))$ the subset of equivalence classes of Anosov representations with values in $\mathrm{Diff}^h(\mathbb{S}^1)$.

0.2.3. *Teichmüller space of the weakly stable foliation.* The geodesic flow of a hyperbolic surface is a well-known example of an *Anosov flow*. In particular, it has a *weakly unstable foliation* $\mathcal{W}^u$ of dimension 2, which contains the geodesic foliation. This foliation is "independent" of the choice of a hyperbolic structure.

A *foliated Riemannian metric* g on $\mathcal{W}^u$ is the data of a scalar product on each tangent space to $\mathcal{W}^u$ which is of class $\mathcal{C}^\infty$ along the leaves and varies transversally continuously for the $\mathcal{C}^\infty$ topology. We call it *transversally Hödler* if there exists $\alpha > 0$ such that the metrics on two ε-close leaves are ε^α-close for the $\mathcal{C}^\infty$ topology.

Two metrics g and h are *conformally equivalent* if there is a continuous function σ on M_Γ such that $h = e^\sigma g$. A *foliated conformal structure* on $\mathcal{W}^u$ is a conformal equivalence class of Riemannian metrics on $\mathcal{W}^u$. It is *transversally Hölder* if it admits a transversally Hölder representative.

A *foliated homotopy* $(h_t)_{0\leq t\leq 1}$ is a continuous family of continuous self maps of M_Γ preserving the leaves of $\mathcal{W}^u$ and such that $h_0 = \mathrm{Id}$. Two foliated conformal structures $[g_1]$ and $[g_2]$ are *homotopic* if there exists a foliated homotopy (h_t) such that $h_1^*[g_2] = g_1$.

Let $[g]$ be a conformal structure on $\mathcal{W}^u$, and let $[\gamma]$ be a closed geodesic. The weakly unstable leaf containing $[\gamma]$ is conformally equivalent (for the conformal structure $[g]$) to $l\backslash\mathbb{D}$ for some hyperbolic isometry l. The *period map* of $[g]$ is the map

$$L_{[g]} : [\Gamma] \to \mathbb{R}_{>0}$$

associating to γ the translation length of l. The period map is homotopy invariant.

The *Teichmüller space* of the foliation $\mathcal{W}^u$ is the space of foliated conformal structures on $\mathcal{W}^u$ modulo homotopy. We denote it by $\mathcal{T}(\mathcal{W}^u)$. We denote by $\mathcal{T}^h(\mathcal{W}^u)$ the subset of homotopy classes of transversally Hölder conformal structures.

Teichmüller spaces of 2-dimensional foliations were introduced by Sullivan in []. There, he proves that those Teichmüller spaces (in particular $\mathcal{T}(\mathcal{W}^u)$ have the geometry of a (possibly infinite dimensional) complex manifold biholomorphic to a bounded domain in a complex Banach space.

Note that we could imagine several variations in the definition of a Teichmüller space. For instance, replacing homotopy equivalence by isotopy equivalence, or weakening the leafwise regularity of our conformal structures. In the classical theory, all these definitions are equivalent thanks to the following classical theorems:

Theorem 0.1. *Any two smooth structures on a topological surface are isotopic.*

Theorem 0.2. *Two homeomorphisms of a surface which are homotopic are isotopic.*

We do not know whether the equivalent results exist for 2-dimensional foliations and try to avoid entering into those details here.

0.3. **Main results.** The purpose of these notes is to clarify the relation between the three spaces described above. This is summarized in the following theorems:

Theorem 0.3. *There exists a continuous map*

$$\mathrm{DF} : \mathfrak{X}_{an}(\Gamma, \mathrm{Diff}(\mathbb{S}^1)) \to \mathrm{Par}_1(\mathcal{G})$$

such that

$$L_{\mathrm{DF}(\rho)} = L_\rho$$

for all ρ. This map is a surjective and restricts to a bijection between $\mathfrak{X}_{an}(\Gamma, \mathrm{Diff}^h(\mathbb{S}^1))$ and $\mathrm{Par}_1^h(\mathcal{G})$.

Theorem 0.4. *There exists a continuous map*

$$\mathrm{CF} : \mathcal{T}(\mathcal{W}^u) \to \mathrm{Par}_1(\mathcal{G})$$

such that

$$L_{\mathrm{CF}([g])} = L_{[g]}$$

for all $[g]$. This map restricts to a bijection between $\mathcal{T}^h(\mathcal{W}^u)$ and $\mathrm{Par}_1^h(\mathcal{G})$.

Note that the sets $\mathrm{Par}_1^h(\mathcal{G})$ and $\mathcal{T}^h(\mathcal{W}^u)$ are respectively dense in $\mathrm{Par}_1(\mathcal{G})$ and $\mathcal{T}(\mathcal{W}^u)$. By Theorem 0.4, one can thus see $\mathrm{Par}_1(\mathcal{G})$ and $\mathcal{T}(\mathcal{W}^u)$ as two natural completions of the space of Hölder parametrizations. We do not know whether the two completions coincide (i.e. whether CF is a bijection).

1. Cocycles, cohomology and parametrizations

In this section, we gather some classical results about cocycles along a continuous flow and their relation with reparametrizations of that flow. This leads to the description of the space $\mathrm{Par}(\mathcal{G})$ as a convex cone in an infinite dimensional Banach space. This description seems to date back to Bowen.

In order to stick to our objective, we restrict ourselves to reparametrizations of the geodesic flow of a closed negatively curved surface. However, all the results here could work in the very general setting of parametrizations of a 1-dimensional lamination, except for a few results which use the density of closed orbits and hold for any topologically transitive Anosov flow.

1.1. **Geodesic, stable and unstable foliation.** In all the paper, Σ denotes a closed oriented surface of genus at least 2 and Γ its fundamental group. Recall that Γ is *hyperbolic* in the sense of Gromov. its boundary at infinity $\partial_\infty\Gamma$ is a topological circle with a canonical Hölder structure. Let $\widetilde{M_\Gamma}$ denote the set of cyclically oriented triples of distinct points of $\partial_\infty\Gamma$. The group Γ acts properly discontinuously and cocompactly on $\widetilde{M_\Gamma}$. We denote the quotient by M_Γ.

For every $y \neq z \in \partial_\infty\Gamma$, let us denote by $\widetilde{W^s}(z)$ the set $\{(x_-, x_t, x_+) \in \widetilde{M_\Gamma} \mid x_+ = z\}$, by $\widetilde{W^u}(y)$ the set $\{(x_-, x_t, x_+) \in \widetilde{M_\Gamma} \mid x_- = y\}$ and by $\widetilde{G}(y,z)$ the set $\widetilde{W^u}(y) \cap \widetilde{W^s}(z)$. Note that the cyclic order on $\partial_\infty\Gamma$ induces an order on $\widetilde{G}(y,z)$ given by

$$(y,t,z) \leq (y,s,z) \iff (y,t,s,z) \text{ are cyclically ordered.}$$

The sets $\widetilde{\mathcal{G}}(y,z)$ are the leaves of a one dimensional Hölder foliation $\widetilde{\mathcal{G}}$ of $\widetilde{M_\Gamma}$ which is preserved by the action of Γ and thus induces a Hölder foliation $\mathcal{G}$ of M_Γ that we call the *geodesic foliation.* Similarly, the sets $\widetilde{\mathcal{W}^s}(z)$ and $\widetilde{\mathcal{W}^u}(y)$ respectively induce Hölder foliations of M_Γ of dimension 2 called the *weakly stable* and *weakly unstable foliations* and denoted $\mathcal{W}^s$ and $\mathcal{W}^u$.

Remark 1.1. If x is a point in M_Γ, we will denote by $\mathcal{G}(x)$, $\mathcal{W}^s(x)$ and $\mathcal{W}^u(x)$ the geodesic, stable and unstable leaves passing through x.

We can now define a (continuous) *parametrization* of the geodesic foliation to be a continuous flow on M_Γ whose orbits are the leaves of the geodesic foliation. More precisely:

Definition 1.2. A continuous parametrization of the geodesic foliation $\mathcal{G}$ on M_Γ is a continuous continuous flow $(\varphi_t)_{t\in\mathbb{R}}$ on M_Γ whose orbits are the leaves of the foliation $\mathcal{G}$ and which respects the orientation of $\mathcal{G}$.

The parametrization (φ_t) is *Hölder* if each φ_t is bi-Hölder the map $(t,x) \to (t, \varphi_t(x))$ is a Hölder homeomorphism of $M_\Gamma \times \mathbb{R}$.

Remark 1.3. A parametrization φ lifts to a Γ-equivariant flow on $\widetilde{M_\Gamma}$ that we still denote φ. More generally, any object on M_Γ which lifts naturally to $\widetilde{M_\Gamma}$ will called the same when lifted. This will avoid some unnecessarily heavy notations.

Every non-trivial element $\gamma \in \Gamma$ has a unique attracting fixed point $\gamma_+ \in \partial_\infty\Gamma$ and a unique repelling repelling fixed point $\gamma_- \in \partial_\infty\Gamma\backslash\{\gamma_+\}$.

Let $x \in \partial_\infty\Gamma$ be a point on the geodesic $\mathcal{G}(\gamma_-, \gamma_+)$. We define $L_\varphi(\gamma)$ as the positive number t such that

$$(\gamma_-, \gamma \cdot x, \gamma_+) = \varphi_t(\gamma_-, x, \gamma_+)) \ .$$

One easily verifies that $L_{\widetilde{\varphi}}(\gamma)$ does not depend on the choice of x, is invariant by conjugation of γ and verifies $L_\varphi(\gamma^n) = nL_\varphi(\gamma)$.

Definition 1.4. Let φ be a continuous parametrization of the geodesic foliation. The map

$$L_\varphi : [\Gamma] \to \mathbb{R}_{>0}$$

is called the *period map* of φ.

Definition 1.5. Two continuous parametrizations (φ_t) and (ψ_t) of the geodesic foliation are called *conjugate* if there exists a continuous function $f : M_\Gamma \to \mathbb{R}$ such that the flows (φ_t) and (ψ_t) are conjugated by the homeomorphism

$$h : x \mapsto \varphi_{f(x)}(x).$$

One can provide the space of all parametrizations of the geodesic foliation with the topology of uniform convergence on compact sets of $M_\Gamma \times \mathbb{R}$. Conjugation of flows defines an equivalence relation on this space. We define $\mathrm{Par}(\mathcal{G})$ as the quotient of the space of all parametrizations by the closure of this equivalence relation. The main purpose of this section is to describe the geometry of $\mathrm{Par}(\mathcal{G})$.

1.1.1. *Main example: geodesic flow of a negatively curved surface.* Assume Σ is endowed with a Riemannian metric of negative curvature. Then there is a well-known identification of M_Γ with $T_1\Sigma$, the unit tangent bundle to Σ, through which the geodesic flow is a parametrization of the geodesic foliation. The geodesic flow is Anosov and the foliations $\mathcal{W}^s$ and $\mathcal{W}^u$ are precisely the weakly stable and unstable foliations of this Anosov. Though different negatively curved metrics give rise to different flows [2], these flows are orbit equivalent and have the same weakly stable and unstable foliations.

Instead of considering parametrizations of the geodesic foliation, we could have fixed a negatively curved metric on Σ and considered *reparametrizations* of its geodesic flow (this approach is more common in the litterature). However, for our purpose, we prefer to emphasize that we have no privileged choice of a negatively curved metric or parametrization of the geodesic foliation.

1.2. **Cocycles.** Let us fix a continuous parametrization φ of the geodesic foliation. We recall here a few classical facts about continuous cocycles along φ.

Definition 1.6. A continuous cocycle along φ is a continuous function

$$c : M_\Gamma \times \mathbb{R} \to \mathbb{R}$$

such that

$$c(x, t + s) = c(x, t) + c(\varphi_t x), s)$$

for all $x \in M_\Gamma$ and all $t, s \in \mathbb{R}$.

The following examples relate cocycles to continuous functions on one side and to reparametrizations of φ on the other side.

Example 1.7. Let $f : M_\Gamma \to \mathbb{R}$ be a continous function. Then the function

$$\begin{array}{rccc} c_f : & M_\Gamma \times \mathbb{R} & \to & \mathbb{R} \\ & (x,t) & \mapsto & \int_0^t f(\varphi(u,x))\mathrm{d}u \end{array}$$

is a cocycle. We call such a cocycle an *integral cocycle*.

Example 1.8. Let ψ be another parametrization of the geodesic foliation. One can associate to the pair (φ, ψ) the cocycle $c_{\varphi\to\psi}$ along φ defined by

$$\psi_{c_{\varphi\to\psi}(x,t)}(x) = \varphi_t(x) \ .$$

In other words, $c_{\varphi\to\psi}(x,t)$ is the time taken by the flow ψ to move from x to $\varphi_t(x)$. We call such a cocycle a *reparametrization cocycle*. When the base flow φ is fixed once and for all, we simply denote this cocycle by c_ψ.

The space $Z(\varphi)$ of cocycles along φ is a Banach space for the norm

$$\|c\|_\infty = \sup_{0\leq t\leq 1} \sup_{x\in M_\Gamma} |c(x,t)| \ .$$

Definition 1.9. A *Livšic coboundary* is a cocycle c for which there exists a continuous function F such that

$$c(x,t) = F(\varphi_t(x)) - F(x)$$

for all $x \in M_\Gamma$ and $t \in \mathbb{R}$. Two cocycles c_1 and c_2 are called *Livšic cohomologous* if $c_1 - c_2$ is a Livsic coboundary.

The following propositions reduce the study of cohomology classes of cocycles to that of integral cocycles.

Proposition 1.10. *Let c_f be the integral cocycle associated to a continuous function f. Then c_f is a Livsic coboundary if and only if f is a derivative along φ, i.e. there exists a continuous function $F : M_\Gamma \to \mathbb{R}$ such that $\frac{1}{\varepsilon}(F(\varphi_\varepsilon(x)) - F(x))$ converges uniformly to f.*

Proposition 1.11. *Let c be a cocycle along φ and let T be a positive number. Then c is Livsic cohomologous to the integral cocycle c_T associated to the function $x \mapsto \frac{1}{T}c(x,T)$. In particular, every cocycle is Livsic cohomologous to an integral cocycle.*

Proof. Let c be a cocycle along φ. Define

$$F : x \mapsto \int_0^T c(x,u)\mathrm{d}u \ .$$

For all $x \in M_\Gamma$ and all $t \in \mathbb{R}$, we have

$$\begin{aligned}
\int_0^t c(\varphi_u(x), T)\mathrm{d}t &= \int_0^t c(x, u+T) - c(x,u)\mathrm{d}u \\
&= \int_t^{t+T} c(x,u)\mathrm{d}u - \int_0^T c(x,u)\mathrm{d}u \\
&= \int_0^T c(x, t+u)\mathrm{d}u - \int_0^T c(x,u)\mathrm{d}u \\
&= \int_0^T c(x,t) + c(\varphi_t(x), u\mathrm{d}u - \int_0^T c(x,u), \mathrm{d}u \\
&= Tc(x,t) + F(\varphi_t(x)) - F(x) \ .
\end{aligned}$$

The integral cocycle associated to the function $x \mapsto c(x,T)$ is thus cohomologous to Tc. □

The space of Livsic coboundaries is not closed in $Z(\varphi)$. We call *weak coboundary* a uniform limit of Livsic coboundaries and we say that two cocycles are *weakly cohomologous* if they differ by a weak coboundary. We denote by $B(\varphi)$ the space of weak coboundaries and by $H^1(\varphi)$ the quotient $Z(\varphi)/B(\varphi)$ with the Banach norm

$$\|[c]\| \stackrel{\text{def}}{=} \inf_{c' \sim c} \|c'\|_\infty \ .$$

Let $Z'(\varphi)$ be the space of continuous functions on M_Γ with the supremum norm and $B'(\varphi) \subset Z'(\varphi)$ the closure of the subspace of functions which are derivatives along φ. Propositions 1.10 and 1.11 imply the following

Corollary 1.12. *The space $H^1(\varphi)$ is isometric to the quotient $Z'(\varphi)/B'(\varphi)$ with the quotient norm*

$$\|[f]\| = \inf_{g \in B'(\varphi)} \|f - g\|_\infty \ .$$

Proof. Consider the linear maps

$$\begin{array}{rccc} A: & Z'(\varphi) & \to & Z(\varphi) \\ & f & \mapsto & c_f \end{array}$$

and

$$\begin{array}{rccc} B: & Z(\varphi) & \to & Z'(\varphi) \\ & c & \mapsto & c(\cdot, 1) \ . \end{array}$$

A and B are both continuous of operator norm at most 1, and are inverses up to coboundaries by Propositions 1.10 and 1.11. They thus factor to isometries between $Z'(\varphi)/B'(\varphi)$ and $Z(\varphi)/B(\varphi)$. □

1.3. **The dual space of invariant measures.** Recall that the dual of the space $Z'(\varphi)$ is the space of *signed Borel measures* on M_Γ, i.e. linear forms of the form

$$\mu : f \mapsto \int_{M_\Gamma} f \mathrm{d}\mu^+ - \int_{M_\Gamma} f \mathrm{d}\mu^-$$

where μ^+ and μ^- are finite Borel measures on M_Γ. It contains the closed subspace of signed Borel measures *invariant* by the flow φ, which are characterized by the following proposition:

Proposition 1.13. *A signed Borel measure μ on M_Γ is φ-invariant if and only if*

$$\mu(f) = 0$$

for all $f \in B'(\varphi)$.

The dual statement readily follows from the Hahn–Banach theorem:

Corollary 1.14. *Let f be a continuous function on M_Γ. Then f belongs to $B'(\varphi)$ if and only if $\mu(f) = 0$ for all invariant finite Borel measure μ.*

It follows that the space of invariant signed Borel measures on M_Γ is isomorphic to the dual of $H^1(\varphi)$. This also gives a characterization of the quotient norm on $H^1(\varphi)$. For every continuous function f and every $T > 0$, define

$$I_T f : x \mapsto \frac{1}{T}\int_0^T f(\varphi_t(x))\mathrm{d}t = \frac{1}{T} c_f(x,T) \ .$$

Note that $I_T f$ is Livsic cohomologous to f by Proposition 1.11. Let $\mathcal{M}^1(\varphi)$ denote the space of φ-invariant probability measures on M_Γ.

Lemma 1.15. *Let f be a continuous function on M_Γ. Then the following equalities hold:*

$$\inf_{g \in B'} \sup_{M_\Gamma}(f-g) = \sup_{\mu \in \mathcal{M}^1(\varphi)} \int_{M_\Gamma} f\mathrm{d}\mu = \lim_{T\to+\infty} \sup_{M_\Gamma} I_T f \ .$$

Corollary 1.16. *Let c be a continuous cocycle. Then we have the equalities*

$$\|[c]\| = \sup_{\mu\in\mathcal{M}^1(\varphi)} |\mu(c)| = \lim_{T\to+\infty} \frac{1}{T}\|c(\cdot,T)\|_\infty \ ,$$

where $\mu(c) = \int f\mathrm{d}\mu$ for any function f such that c is cohomologous to c_f.

Proof of Lemma 1.15. Note first that the function

$$T \mapsto \sup_{x\in M_\Gamma} \int_0^T f(\varphi_t(x))\mathrm{d}x$$

is subadditive. Thus $\sup_{M_\Gamma} I_T f$ converges as T goes to $+\infty$.

The inequality

$$\sup_{\mu\in\mathcal{M}^1(\varphi)} \int_{M_\Gamma} f\mathrm{d}\mu \le \inf_{g\in B'(\varphi)} \sup_{M_\Gamma}(f-g)$$

follows for the inequality $\int_{M_\Gamma} f\mathrm{d}\mu \le \sup_{M_\Gamma} f$ and the fact that $\int_{M_\Gamma} g\mathrm{d}\mu = 0$ for $g \in B'(\varphi)$.

The inequality

$$\inf_{g\in B'(\varphi)} \sup_{M_\Gamma}(f-g) \le \lim_{T\to+\infty} \sup_{M_\Gamma} I_T f$$

follows from the fact that $I_T f$ is cohomologous to f for all T.

Finally, let x_T be a point where $I_T f$ achieves its supremum and consider the probability measure μ_T defined by

$$\int_{M_\Gamma} g\mathrm{d}\mu_T = \frac{1}{T}\int_0^T g(\varphi_t(x_T))\mathrm{d}t \ .$$

Let ν be an accumulation point of μ_T for the vague topology. Then ν is φ-invariant and we have

$$\int_{M_\Gamma} f \mathrm{d}\nu = \lim_{T\to+\infty} I_T f(x_T) = \lim_{T\to+\infty} \sup_{M_\Gamma} I_T f\ .$$

We conclude that

$$\lim_{T\to+\infty} \sup_{M_\Gamma} I_T f \leq \sup_{\mu\in\mathcal{M}^1(\varphi)} \int f \mathrm{d}\mu\ .$$

□

Proof of Corollary 1.16. By Proposition 1.11, we can assume without loss of generality that c is the integral cocycle associated to a continuous function c_f.

Set $N(f) = \inf_{g\in B'} \sup_{M_\Gamma}(f-g)$. By Lemma 1.15, we have

$$\max(N(f), N(-f)) = \sup_{\mu\in\mathcal{M}^1(\varphi)} |\mu(c)| = \lim_{T\to+\infty} \frac{1}{T} \|c(\cdot, T)\|_\infty\ .$$

On one side, we have

$$\max(N(f), N(-f)) \leq \inf_{g\in B'} \|f-g\|_\infty = \|[c]\|\ .$$

On the other side, we have

$$\lim_{T\to+\infty} \frac{1}{T} \|c(\cdot, T)\|_\infty \geq \|[c]\|$$

since $\frac{1}{T}c(\cdot, T)$ is cohomologous to f. Hence $\|[c]\| = \max(N(f), N(-f))$. □

1.4. **Positive cocycles and reparametrizations.** We still fix a parametrization φ of the geodesic foliation. Let ψ be another parametrization. Recall that the *reparametrization cocycle* c_ψ along φ is defined by the relation

$$\psi_{c_{\varphi\to\psi(x,t)}}(x) = \varphi_t(x)\ . \tag{1}$$

Such a cocycle is *positive*, i.e. $c_{\varphi\to\psi}(x,t) > 0$ for $t > 0$. Conversely, if c is a positive cocycle, then the relation (1) defines a unique parametrization ψ such that $c_{\varphi\to\psi} = c$.

Proposition 1.17. *Two parametrizations ψ_1 and ψ_2 are conjugate if and only if their reparametrization cocycles are Livsic cohomologous.*

Proof. Let h be a geodesic preserving homeomorphism such that

$$h(\psi_1(t,x)) = \psi_2(t, h(x))\ .$$

Define $F(x)$ as the time s such that

$$\psi_2(s,x) = h(x)\ .$$

We then have

$$\begin{aligned}
\psi_2(c_{\psi_2}(x,t) + F(\varphi(t,x)), x) &= \psi_2(F(\varphi(t,x)), \varphi(t,x)) \quad \text{by definition of } c_{\Psi_2}\\
&= h(\varphi(t,x)) \quad \text{by definition of } F\\
&= h(\psi_1(c_{\psi_1}(x,t), x) \quad \text{by definition of } c_{\psi_1}\\
&= \psi_2(c_{\psi_1}(x,t), h(x))\\
&= \psi_2(c_{\psi_1}(x,t) + F(x), x)\ .
\end{aligned}$$

Therefore,

$$c_{\psi_1}(x,t) - c_{\psi_2}(x,t) = F(\varphi(x,t)) - F(x) \ .$$

Conversely, assume there exists a continuous function F such that

$$c_{\psi_1}(x,t) - c_{\psi_2}(x,t) = F(\varphi(x,t)) - F(x) \ .$$

Set $h(x) = \psi_2(F(x), x)$. Given $x \in M_\Gamma$ and $s \in \mathbb{R}$, let t be such that $c_{\psi_1}(x,t) = s$. We then have

$$\begin{aligned}
\psi_2(s, h(x)) &= \psi_2(s + F(x), x) \\
&= \psi_2(c_{\psi_1}(x,t) + F(x), x) \\
&= \psi_2(c_{\psi_2}(x,t) + F(\varphi_t(x)), x) \quad \text{by definition of } F \\
&= \psi_2(F(\varphi_t(x)), \varphi_t(x)) \quad \text{by definition of } c_{\psi_2} \\
&= h(\varphi_t(x)) \quad \text{by definition of } h \\
&= h(\psi_1(s,x)) \ .
\end{aligned}$$

Symmetrically, the map $g : x \mapsto \psi_1(-F(x), x)$ satisfies

$$\psi_{1s} \circ g = g \circ \psi_{2s} \ .$$

One can verify that h and g are inverses of each other. They thus provide the required conjugation. □

Definition 1.18. We call two parametrizations ψ_1 and ψ_2 *weakly conjugate* if ψ_2 is a uniform limit of conjugates of ψ_1. (Here and elsewhere, uniform limit means limit for the compact-open topology.)

One easily verifies that the map $\psi \mapsto c_\psi$ is a homeomorphism from the space of parametrizations of the geodesic foliation (with the totoplogy of uniform convergence on compact sets) to the space of positive cocycles. Thus Proposition 1.17 implies the following corollary:

Corollary 1.19. *Two parametrizations ψ_1 and ψ_2 are weakly conjugate if and only if their reparametrization cocycles are weakly cohomologous. Moreover, the map*

$$[\psi] \mapsto [c_\psi]$$

Is a homeomorphism from the space $\mathrm{Par}(\mathcal{G})$ *of weak conjugacy classes of parametrizations to the domain $H^1_+(\varphi)$ of weak cohomology classes of positive cocycles.*

Remark 1.20. This corollary asserts in particular that weak conjugacy of flows is indeed an equivalence relation, and that $\mathrm{Par}(\mathcal{G})$ is the largest Hausdorff quotient of the space of parametrizations up to (strong) conjugacy.

The domain $H^1_+(\psi)$ is a convex open cone in $H^1_+(\psi)$. The following proposition gives further characterizations of it:

Proposition 1.21. *Let c be a cocycle along φ. The following are equivalent:*

(1) *c is cohomologous to a positive cocycle,*
(2) *c is cohomologous to the integral cocycle c_f associated to a positive function f,*
(3) *There exists $T > 0$ such that $c(x,T)$ is positive for all x,*

(4) *There exist constants* $A, B > 0$ *such that*
$$c(x,t) \geq At - B$$
for all $x \in M_\Gamma$ *and all* $t \geq 0$,

(5) $\mu(c) > 0$ *for all* $\mu \in \mathcal{M}^1(\varphi)$,

(6) *There exists a constant* $A > 0$ *such that* $\mu(c) > A$ *for all* $\mu \in \mathcal{M}^1(\varphi)$.

We call a cocycle satisfying these properties and expanding cocycle.

Proof. The equivalence between (1) and (2) follows from Proposition 1.11. The equivalence between (2), (3) and (6) follows from applying Lemma 1.15 to the cocycle $-c$. The equivalence between (5) and (6) follows from the vague compactness of $\mathcal{M}^1(\varphi)$. Finally, the equivalence between (3) and (4) follows from
$$c(x, nT) = \sum_{k=0}^{n-1} c(\varphi_{kT}(x), T)~.$$
□

Corollary 1.19 gives a homeomorphism
$$\mathrm{Id}_\varphi : \mathrm{Par}(\mathcal{G}) \to H^1_+(\varphi)~,$$
depending on the choice of a background parametrization φ. Let us now describe the coordinate changes with respect to different background parametrizations.

Let φ_1, φ_2 be two parametrizations of the geodesic foliation. Given a cocycle c along φ_1, define
$$\begin{array}{rccc} c \circ c_{\varphi_2 \to \varphi_1} : & M_\Gamma \times \mathbb{R} & \to & \mathbb{R} \\ & (x,t) & \mapsto & c(x, c_{\varphi_2\to\varphi_1}(x,t))~. \end{array}$$

Proposition 1.22. *The following holds:*

(1) *If* c *is a cocycle along* φ_1, *then* $c \circ c_{\varphi_2\to\varphi_1}$ *is a cocycle along* φ_2.

(3) *If* c *and* c' *are cocycles along* φ, *then* $c \circ c_{\varphi_2\to\varphi_1}$ *and* $c' \circ c_{\varphi_2\to\varphi_1}$ *are Livsic (resp. weakly) cohomologuous if an only if* c *and* c' *are Livsic (resp. weakly) cohomologuous.*

(4) *If* ψ *is another parametrization, then*
$$c_{\varphi_1\to\psi} \circ c_{\varphi_2\to\varphi_1} = c_{\varphi_2\to\psi}~.$$

It follows that the map $c \mapsto c \circ c_{\varphi_2\to\varphi_1}$ induces a map $I_{\varphi_1\to\varphi_2} : H^1(\varphi_1) \to H^1(\varphi_2)$. This map is an isomorphism of Banach spaces and maps $H^1_+(\varphi_1)$ to $H^1_+(\varphi_2)$. Finally, we have
$$\mathrm{Id}_{\varphi_2} \circ \mathrm{Id}_{\varphi_1}^{-1} = I_{\varphi_1\to\varphi_2}~.$$

In conclusion, the space $\mathrm{Par}(\mathcal{G})$ has the structure of a convex Banach cone. More precisely, it is endowed with a family of homeomorphisms to open convex cones in Banach spaces whose transition maps are linear isomorphisms. We all the (I_φ) the *affine charts* of $\mathrm{Par}(\mathcal{G})$.

1.5. **Period map for cocycles.** We fix again a background parametrization φ. Recall that closed leaves in $\mathcal{G}$ are in bijection with primitive conjugacy classes in Γ, and that for $[\gamma] \in [\Gamma]$, the period $L_\varphi(\gamma)$ is the time taken by the flow φ to go through γ.

Definition 1.23. Let c be a cocycle along φ. The *period map* of c is the map

$$\begin{array}{rccl} L_c: & [\Gamma] & \to & \mathbb{R} \\ & \gamma & \mapsto & c(x_\gamma, L_\varphi(\gamma)) \end{array}$$

where x_γ is any point on the closed geodesic γ.

Example 1.24. If c_ψ is the reparametrization cocycle associated to ψ, then

$$L_{c_\psi} = L_\psi \ .$$

One easily verifies that L_c only depends on the weak cohomology class of c. In fact, we have

Proposition 1.25. *For every $\gamma \in [\Gamma]$ and every $c \in Z(\varphi)$,*

$$L_c(\gamma) = L_\varphi(\gamma)\delta_\gamma(c)$$

where δ_γ is the φ-invariant probability measure supported on the closed geodesic γ.

The closing lemma for the geodesic flow of a hyperbolic surface has the following consequence:

Lemma 1.26. *The convex hull of the $(\delta_\gamma)_{\gamma\in[\Gamma]}$ is dense in $\mathcal{M}^1(\varphi)$ for the vague topology.*

Thus, Proposition 1.21 and 1.16 imply the following

Corollary 1.27 (Corollary of Proposition 1.21)**.** *A cocycle c is expanding if and only if there exists a constant $A > 0$ such that*

$$L_c \geq A L_\varphi \ .$$

Corollary 1.28 (Corollary of Corollary 1.16)**.** *Two cocycles c and c' along φ are weakly cohomologous if and only if $L_c \equiv 0$.*

Note that this last corollary is strengthened by Livšic's theorem under a Hölder regularity hypothesis, see Theorem 6.4.

1.6. **Scaling and the space $\mathbf{P}\mathrm{Par}(\mathcal{G})$.** Let ψ be a parametrization of the geodesic foliation and $\lambda \in \mathbb{R}_{>0}$. The *scaled flow* ψ^λ is defined by

$$\psi^\lambda_t(x) = \psi_{t/\lambda}(x) \ .$$

Scaling defines an action of $\mathbb{R}_{>0}$ on the space $\mathrm{Par}(\mathcal{G})$. We denote by $\mathbf{P}\mathrm{Par}(\mathcal{G})$ the quotient of $\mathrm{Par}(\mathcal{G})$ by this action and call it the *projective space of parametrizations.*

Let φ be a background parametrization of $\mathcal{G}$. We then have

$$c_{\varphi\to\psi^\lambda} = \lambda c_{\varphi\to\psi} \ .$$

In other words, the affine chart $I_\varphi : \mathrm{Par}(\mathcal{G}) \to H^1(\varphi)$ conjugates the scaling action with the scalar multiplication. The space $\mathbf{P}\mathrm{Par}(\mathcal{G})$ is thus an open domain in the projective space over a Banach space.

Proposition 1.29. *The domain* $\mathbf{P}\mathrm{Par}(\mathcal{G}) \subset \mathbf{P}H^1(\varphi)$ *is a* weakly proper convex domain*, i.e. it intersects every projective line in a proper interval (and interval which is not the complement of a point).*

Remark 1.30. In this infinite dimensional setting, "weakly proper" is weaker that the stronger property of being bounded in some affine chart.

Recall that a (weakly) proper convex domain Ω carries a natural projectively invariant metric called the *Hilbert metric*, defined by

$$d(x,y) = \frac{1}{2}\log[a,x,b,y] \ ,$$

where a and b are the endpoints of the intersection of Ω with the projective line spanned by x and y, and $[a,x,b,y]$ the cross-ratio of those four points. We call the Hilbert metric on $\mathbf{P}\mathrm{Par}(\mathcal{G})$ the *Hilbert–Thurston distance* because, as stated in the next theorem, it is a symmetrization of the distance introduced by Thurston on the Teichmüller space. We denote it by d_{HT}.

Theorem 1.31. *Let φ and ψ be two parametrizations of the geodesic foliation. The Hilbert–Thurston distance between (the projective classes of) φ and ψ is given by*

$$d_{HT}(\varphi,\psi) = \frac{1}{2}\left(\log \sup_{\gamma\in[\Gamma]} \frac{L_\psi(\gamma)}{L_\varphi(\gamma)} + \log \sup_{\gamma\in[\Gamma]} \frac{L_\varphi(\gamma)}{L_\psi(\gamma)}\right) \ .$$

The proofs of both Proposition 1.29 and Theorem 1.31 follow from the following computation:

Lemma 1.32. *Let f and g be two points in $H^1_+(\varphi)$. Assume that we don't have $L_f < L_g$. Then*

$$\sup\{\lambda \geq 0 \mid (1-\lambda)f + \lambda g \in H^1_+(\varphi)\} = \frac{1}{1 - \inf_{\gamma\in\Gamma} \frac{L_g(\gamma)}{L_f(\gamma)}} \ .$$

Proof. For some $\lambda > 0$, we have

$$\begin{aligned} L_{(1-\lambda)f+\lambda g} &= L_f + \lambda(L_g - \lambda L_f) \\ &= L_f(1 - \lambda(1 - \frac{L_g}{L_f})) \end{aligned}$$

Assume first that

$$\lambda \leq \frac{1-\varepsilon}{1 - \frac{L_g(\gamma)}{L_f(\gamma)}}$$

for all $\gamma \in [\Gamma]$ such that $L_g(\gamma) < L_f(\gamma)$. For some $\gamma \in \Gamma$, if $L_g(\gamma) \geq L_f(\gamma)$, then

$$L_{(1-\lambda)f+\lambda g}(\gamma) \geq L_f(\gamma) \ ,$$

and otherwise,

$$L_{(1-\lambda)f+\lambda g}(\gamma) \geq \varepsilon L_f(\gamma) \ .$$

In any case, we get

$$L_{(1-\lambda)f+\lambda g} \geq \varepsilon L_f \geq \varepsilon' L_\varphi \ ,$$

hence $(1-\lambda) + \lambda g$ belongs to $H^1_+(\varphi)$ by Corollary 1.27.

Assume now that there exists γ such that $L_g(\gamma) < L_f(\gamma)$ and

$$\lambda > \frac{1}{1 - \frac{L_g(\gamma)}{L_f(\gamma)}} \ .$$

For this γ we then have

$$L_{(1-\lambda)f+\lambda g}(\gamma) < 0 \ .$$

Thus $(1-\lambda)f + \lambda g$ does not belong to $H^1_+(\varphi)$. We conclude that

$$\begin{aligned}\sup\{\lambda \mid (1-\lambda)f + \lambda g \in H^1_+(\varphi)\} &= \inf_{\gamma\in[\Gamma], L_g(\gamma)<L_f(\gamma)} \frac{1}{1 - \frac{L_g(\gamma}{L_f(\gamma)}} \\ &= \frac{1}{1 - \inf_{\gamma\in[\Gamma]} \frac{L_g(\gamma)}{L_f(\gamma)}}.\end{aligned}$$

□

Proof of Proposition 1.29. Let $[f]$ and $[g]$ be two distinct points in $\mathrm{Par}(\mathcal{G})$. Then L_f is not a multiple of L_g. We thus have

$$\inf \frac{L_f}{L_g} < \sup \frac{L_f}{L_g} \ .$$

Up to multiplying f by a scalar, we can assume that

$$\inf \frac{L_f}{L_g} < 1 < \sup \frac{L_f}{L_g} \ .$$

The set $I = \{\lambda \in \mathbb{R} \mid (1-\lambda)f + \lambda g \in H^1_+(\varphi)\}$ is an open interval since $H^1_+(\varphi)$ is open and convex. By Lemma 1.32, it is bounded from above and from below by inverting the roles of f and g. This shows that $\mathbf{P}\mathrm{Par}(\mathcal{G})$ is weakly proper. □

Proof of Theorem 1.31. Let $[f]$ and $[g]$ be two distinct points in $\mathrm{Par}(\mathcal{G})$. Assume again that

$$\inf \frac{L_f}{L_g} < 1 < \sup \frac{L_f}{L_g} \ .$$

By Lemma 1.32, we have

$$\{\lambda \in \mathbb{R} \mid (1-\lambda)f + \lambda g \in H^1_+(\varphi)\} = (\lambda_-, \lambda_+) \ ,$$

where

$$\lambda_+ = \frac{1}{1 - \inf_{\gamma\in[\Gamma]} \frac{L_g(\gamma)}{L_f(\gamma)}}$$

and

$$\lambda_+ = \frac{1}{1 - \inf_{\gamma\in[\Gamma]} \frac{L_f(\gamma)}{L_g(\gamma)}} - 1 \ .$$

By definition of the Hilbert distance, we have

$$\begin{aligned} d_{HT}(f,g) &= \frac{1}{2}\log\left(\frac{\lambda_+}{\lambda_+ - 1}\frac{\lambda_- + 1}{\lambda_-}\right) \\ &= \frac{1}{2}\log\left(\frac{1}{\inf_{\gamma\in[\Gamma]}\frac{L_g(\gamma)}{L_f(\gamma)}}\frac{1}{\inf_{\gamma\in[\Gamma]}\frac{L_f(\gamma)}{L_g(\gamma)}}\right) \\ &= \frac{1}{2}\left(\log\left(\sup_{\gamma\in[\Gamma]}\frac{L_f(\gamma)}{L_g(\gamma)}\right) + \log\left(\sup_{\gamma\in[\Gamma]}\frac{L_g(\gamma)}{L_f(\gamma)}\right)\right) \ . \end{aligned}$$

□

In Section **??** we will introduce a positive continuous function h_{top} on $\mathrm{Par}(\mathcal{G})$ which is homogeneous of degree -1 (i.e. $h_{top}(\varphi^\lambda) = \frac{1}{\lambda}h_{top}(\varphi)$). Experts will recognize the *topological entropy* of a flow. We will denote by $\mathrm{Par}_1(\mathcal{G})$ set of parametrization $\varphi \in \mathrm{Par}(\mathcal{G})$ such that $h_{top}(\varphi) = 1$. By homogeneity of h_{top} every parametrization φ admits a unique scaling of entropy 1. Thus $\mathrm{Par}_1(\mathcal{G})$ can be identified with $\mathbf{P}\mathrm{Par}(\mathcal{G})$.

2. Buseman cocycles, foliated 1-forms, and horocycles

2.0.1. *Buseman cocycles.* Let $\widetilde{N}_\Gamma^s$ (resp. $\widetilde{N}_\Gamma^u$) denote the space of pairs of points $(x,y) \in \widetilde{M}_\Gamma^2$ such that x and y are contained in the same stable (resp. unstable) leaf.

Definition 2.1. A *stable Buseman cocycle* (resp. *unstable Buseman cocycle*) on M_Γ is a Γ-invariant continuous function B on $\widetilde{N}_\Gamma^s$ (resp. $\widetilde{N}_\Gamma^u$) such that for all $x,y,z \in \widetilde{M}_\Gamma$ belonging to the same weakly stable (resp. weakly unstable) leaf, we have

$$B(x,z) = B(x,y) + B(y,z) \ .$$

Let φ be a parametrization of the geodesic flow. Then every Buseman cocycle B on M_Γ induces a cocycle along φ defined by

$$c(x,t) = B(x, \varphi_t(x)) \ .$$

We will say that a cocycle c is *stably (resp. unstably) Buseman* if it is associated to a stable (resp. unstable) Buseman cocycle via this construction. We have the following characterization :

Proposition 2.2. *Let c be a continuous cocycle along φ. The following are equivalent:*

- *c is stably (resp. unstably) Buseman,*
- *the function*
$$(x,y) \mapsto c(x,t) - c(y,t)$$
converges uniformly on every compact subset of $\widetilde{N}_\Gamma^s$ (resp. $\widetilde{N}_\Gamma^u$) when t goes to $+\infty$ (resp. $-\infty$).

More over, the associated Buseman cocycle is determined by

$$B(x,y) = \lim_{t\to\pm\infty} c(x,t) - c(y,t) \ .$$

In particular it is unique.

Corollary 2.3. *If φ is a Hölder parametrization of the geodesic flow, then every Hölder cocycle along φ is stably and unstably Buseman.*

We leave the proofs of the previous proposition and corollary as an exercise.

Definition 2.4. A stable (resp. unstable) Buseman cocycle B is a *coboundary* if there exists a continuous function F on M_Γ such that

$$B(x, y) = F(y) - F(x)$$

for all $(x, y) \in \widetilde{N}^s_\Gamma$ (resp. $\widetilde{N}^u_\Gamma$). Two Buseman cocycles are Livšic cohomologous if their difference is a coboundary.

One easily verifies that two Buseman cocycles are Livsic cohomologous if and only if their associated cocycles along φ are Livsic cohomologous.

2.1. Horocyclic foliations.

Definition 2.5. A parametrization φ of the geodesic foliation is said to admit a *stable (resp. unstable) horocyclic foliation* or, for short *stable (resp. unstable) horocycles* if there exists a 1-dimensional continuous foliation $\mathcal{H}^s$ (resp. $\mathcal{H}^u$) of M_Γ such that

- Every leaf of $\mathcal{H}^s$ (resp. $\mathcal{H}^u$) is contained in a leaf of $\mathcal{W}^s$ (resp. $\mathcal{W}^u$),
- The foliation $\mathcal{H}^s$ (resp. $\mathcal{H}^u$) is transverse to $\mathcal{G}$ in $\mathcal{W}^s$ (resp. in $\mathcal{W}^u$), i.e. locally, each leaf of $\mathcal{H}^s$ and each leaf of $\mathcal{G}$ contained in the same weakly stable leaf intersect at exactly one point,
- $\mathcal{H}^s$ (resp. $\mathcal{H}^u$) is preserved by the flow φ.

Example 2.6. The geodesic flow of a negatively curved metric on Σ is *Anosov.* It thus admits both stable and unstable horocycles given by the strongly stable and unstable foliations of the flow.

Remark 2.7. The subtlety of this section is that a continuous parametrization of an Anosov flow may not admit strongly stable or unstable foliations.

The relation with Buseman cocycles is given by the following proposition:

Proposition 2.8. *Let φ and ψ be parametrizations of the geodesic foliation. Assume that φ admits stable (resp. unstable) horocycles. Then ψ admits stable (resp. unstable) horocycles if and only if the reparametrization cocycle $c_{\varphi\to\psi}$ is stably (resp. unstably) Buseman.*

Proof. Exercise. □

Definition 2.9. Let φ be a parametrization of the geodesic foliation admiting stable (resp. unstable) cocycles. We will say that φ admits a *horocyclic flow* if there exists a flow h on M_Γ whose orbit foliation is the stable (resp. unstable) horocyclic foliation, and such that φ and h have the following comuting property:

$$\varphi_t \circ h_s \circ \varphi_{-t} = h_{e^{-t}s}$$

(resp.

$$\varphi_t \circ h_s \circ \varphi_{-t} = h_{e^{t}s} \quad)$$

for all $(s, t) \in \mathbb{R}^2$.

The pair (φ, h) gives a locally free action of the affine group on M_Γ whose orbit foliation is $\mathcal{W}^s$ (resp. $\mathcal{W}^u$). We call it a *stable (resp. unstable) affine action.*

Example 2.10. The geodesic and horocyclic flows of a hyperbolic metric on Σ give a stable affine action.

A stable (resp. unstable) affine action (φ, h) induces in particular a *foliated smooth structure* (i.e. a transversally continuous family of smooth structures on the leaves of) $\mathcal{W}^s$ (resp. $\mathcal{W}^u$) as well as two vector fields X and Y tangent to $\mathcal{W}^s$ (resp. $\mathcal{W}^u$) that generate respectively the flow φ and h and satisfy the comutation relation

$$[X, Y] = -Y$$

(resp.

$$[X, Y] = Y \quad) .$$

We will come back extensively on stable affine actions and their relation to foliated hyperbolic structures in Section 4. For now, we only need the existence of a stable affine action and its associated foliated smooth structure.

2.2. **stable 1-forms.** In this section, we assume that the weakly stable (resp. unstable) foliation is provided with an affine action (φ, h). One can take φ and h to be respectively the geodesic and horocyclic flow of a hyperbolic metric on Σ. In particular, the weakly stable (resp. unstable) leaves carry a smooth structure. Let X and Y be the vector fields on $\mathcal{W}^s$ (resp. $\mathcal{W}^u$) generating the flows φ and h respectively.

Definition 2.11. A *stable* 1*-form* (resp. *unstable* 1*-form*) of class $\mathcal{C}^k$ on M_Γ is a family of 1-forms of class $\mathcal{C}^k$ on the leaves of $\widetilde{\mathcal{W}}^s$ (resp. $\widetilde{\mathcal{W}}^u$), preserved by Γ, and depending continuously on the leaf for the $\mathcal{C}^k$ topology. It is called *closed* if it is closed on each leaf.

(Recall that a continuous 1-form is closed if it is locally the differential of a $\mathcal{C}^1$ function.)

A closed stable (resp. unstable) 1-form α gives rise to a stable (resp. unstable) Buseman cocycle B_α defined by

$$B_\alpha(x, y) = \int_x^y \alpha \ .$$

The associated cocycle c_α along φ is the integral cocycle associated to the function $\alpha(X)$ (i.e. $c_\alpha(x, t)$ is the integral of α from x to $\varphi(t, x)$).

Proposition 2.12. *Every (stable or unstable) Buseman cocycle is Livsic cohomologous to the Buseman cocycle associated to a closed (stable or unstable)* 1*-form of class* $\mathcal{C}^\infty$*.*

Proof. Let B be a stable Buseman cocycle. We construct α by "smoothening" B. To do so, we can for instance choose a probability law ν on $\mathbb{R}^2$ with smooth density with respect to Lebesgue and compact support. For $(x, y) \in \widetilde{N}^s_\Gamma$, define

$$\nu_* B(x, y) = \int_{\mathbb{R}^2} B(x, \exp(tX) \circ \exp(sY) \cdot y) \mathrm{d}\nu(t, s) \ .$$

(Watch out that $\nu_* B$ is *not* a Buseman cocycle.) The function $\nu_* B(x, \cdot)$ is smooth on each stable leaf. Moreover, for x_1 and x_2 in the same stable leaf, we have

$$\nu_* B(x_1, \cdot) = \nu_* B(x_2, \cdot) + B(x_1, x_2) ,$$

so the 1-form $\alpha = \mathrm{d}\nu_* B(x, \cdot)$ does not depend on x. This defines a closed foliated 1-form of class $\mathcal{C}^\infty$ on M_Γ.

Let B_α be the Buseman cocycle associated to α. For x_0, x and y in the same stable leaf, we have

$$\begin{aligned} B_\alpha(x,y) &= \nu_* B(x_0, y) - \nu_* B(x_0, x) \\ &= B(x_0, y) - B(x_0, x) + \nu_* B(y,y) - \nu_* B(x,x) \\ &= B(x,y) + \nu_* B(y,y) - \nu_* B(x,x) . \end{aligned}$$

The function

$$x \mapsto \nu_* B(x,x) = \int_{\mathbb{R}^2} B(x, \exp(tX) \circ \exp(sY) \cdot x) \mathrm{d}\nu(t,s)$$

is continuous and Γ-invariant. Therefore, B and B_α are cohomologous. □

Let $Z(\mathcal{W}^s)$ denote the space of foliated closed 1-forms of class $\mathcal{C}^0$. We provide $Z(\mathcal{W}^s)$ with the supremum norm:

$$\|\alpha\|_\infty = \sup_M \max\{|\alpha(X)|, |\alpha(Y)|\} .$$

Let $B(\mathcal{W}^s)$ denote the closure of the subspace of exact 1-forms. Finally, let $H^1(\mathcal{W}^s)$ denote the quotient $Z(\mathcal{W}^s)/B(\mathcal{W}^s)$, provided with the quotient norm.

The map

$$\begin{array}{ccc} Z(\mathcal{W}^s) & \to & \mathcal{C}^0(M, \mathbb{R}) \\ \alpha & \mapsto & \alpha(X) \end{array}$$

induces a linear map $\Pi : H^1(\mathcal{W}^s) \to H^1(\varphi)$.

Proposition 2.13. *The map* Π *is an isometric bijection.*

Proof. Let us prove that Π preserves the norm. Since by Proposition 2.12, every Hölder function f is in the image of Π, we obtain that Π has dense image. Since the domain of Π is a Banach space, we will conclude that Π is an isomorphism.

Let α be a closed foliated 1-form on $\mathcal{W}^s$. Then, by definition of the norms, one clearly has

$$\|\alpha\|_\infty \geq \sup_M |\alpha(X)| .$$

It follows that

$$\|\Pi([\alpha])\| \leq \|[\alpha]\| .$$

To prove the converse inequality, let us define

$$\alpha_T = \frac{1}{T} \int_0^T \varphi_t^* \alpha \mathrm{d}t .$$

The form α_T is cohomologous to α.

Now, one has

$$\alpha_T(X)(x) = \frac{1}{T} \int_0^T \alpha(X)(\varphi_t(x)) .$$

By Corollary 1.16, we thus have

$$\|\alpha_T(X)\|_\infty \underset{T\to+\infty}{\longrightarrow} \|[\alpha(X)]\| = \|\Pi(\alpha)\| \ .$$

Meanwhile, since $\varphi_{t*}Y = e^{-t}Y$, one has

$$|\alpha_T(Y)(x)| \leq \frac{1}{T}\int_0^T e^{-t}\,\|\alpha(Y)\|_\infty \,\mathrm{d}t \leq \frac{1}{T} norm\alpha(Y)_\infty \underset{T\to+\infty}{\longrightarrow} 0 \ .$$

We conclude that

$$\|\alpha_T\|_\infty \underset{T\to+\infty}{\longrightarrow} \|\Pi([\alpha])\| \ .$$

Therefore, $\|[\alpha]\| = \|\Pi([\alpha])\|$.

□

Proposition 2.13 says in particular that every continuous cocycle is weakly cohomologous to one which is stably Buseman. As a corollary, we obtain the following theorem:

Theorem 2.14.
Every continuous parametrization of the geodesic foliation is weakly conjugate to a parametrization admitting a stable horocyclic foliation.

3. Anosov groups of diffeomorphisms of the circle

In this section, we associate to certain well-behaved circle actions of a surface group a parametrization of the geodesic foliation, unique up to Livsic equivalence. Moreover, these parametrizations admit stable horocycles, and come with a family of measures on unstable leaves with nice properties with respect to the stable horocycles.

3.1. **Expanding actions.** We denote by $\mathbb{S}^1$ the unit circle, by $\mathrm{Diff}(\mathbb{S}^1)$ the group of diffeomorphisms of $\mathbb{S}^1$ of class $\mathcal{C}^1$ and by $\mathrm{Diff}^h(\mathbb{S}^1)$ the subgroup of diffeomorphisms whose derivatives are Hölder regular.

A homomorphism ρ from Γ to $\mathrm{Diff}(\mathbb{S}^1)$ is called a $\mathcal{C}^1$ action of Γ on $\mathbb{S}^1$.

Let ρ be a $\mathcal{C}^1$ action which is topologically conjugate to the action of Γ on $\partial_\infty\Gamma$. In particular, every element $\gamma \in \Gamma$ acts on $\mathbb{S}^1$ with an attracting fixed point and a repelling fixed point. When this does not bring any confusion, we denote these points respectively by γ_+ and γ_- (omitting the dependence in ρ).

Definition 3.1. We define the *period map* of ρ as the function

$$\begin{array}{rccl} L_\rho : & [\Gamma] & \to & \mathbb{R}_+ \\ & [\gamma] & \mapsto & -\log\left(\rho(\gamma)'(\gamma_+)\right) \ . \end{array}$$

This definition is motivated by the proposition:

Proposition 3.2. *Let ρ be a $\mathcal{C}^1$ action of Γ on $\mathbb{S}^1$ topologically conjugate to a Fuchsian action. Let φ be a parametrization of the geodesic foliation. Then there exists a cocycle c_ρ along φ such that*

$$L_\rho = L_{c_\rho} \ .$$

Proof. Let $h : \partial_\infty\Gamma \to \mathbb{S}^1$ be the homeomorphism conjugating the action ρ with the action of Γ on its boundary. Let $\widetilde{E}_\rho$ be the continuous line bundle over $\widetilde{M_\Gamma}$ defined by

$$(\widetilde{E}_\rho)_{(x,y,z)} = T_{h(z)}\mathbb{S}^1 \ ,$$

and let E_ρ be the line bundle over M obtain by quotienting by the action of Γ.

The flow φ_t on M_Γ lifts to a flow $\widehat{\varphi}_t$ on the total space of E_ρ which is linear in the fibers, induced by the transformation

$$((x,y,z),v) \mapsto (\varphi_t(x,y,z),v)$$

on $\widetilde{E}_\rho$.

Let $|\cdot|$ be a continuous norm on L_ρ. We define the cocycle c_ρ by

$$c_\rho(x,t) = \log\frac{|\widehat{\varphi}_t(v)|}{|v|}$$

where v is any vector in $(E_\rho)_x\backslash\{0\}$.

The norm $|\cdot|$ lifts to a Γ-invariant norm on $\widetilde{E}_\rho$ that we still denote by $|\cdot|$. Let γ be an element of Γ and let (γ_-, y, γ_+) be a point on the axis of γ. Let v be a tangent vector to $h(\gamma_+)$. Then, by definition of c_ρ, we have

$$\begin{aligned}
L_{c_\rho}(\gamma) &= \log\frac{|v|_{(\gamma_-,\gamma\cdot y,\gamma_+)}}{|v|_{(\gamma_-,y,\gamma_+)}}\\
&= \log\frac{|\mathrm{d}\rho(\gamma^{-1})(v)|_{(\gamma_-,y,\gamma_+)}}{|v|_{(\gamma_-,y,\gamma_+)}}\\
&= \log\rho(\gamma^{-1})'(h(\gamma_+))\\
&= L_\rho(\gamma) \ .
\end{aligned}$$

□

Remark 3.3. The cocycle c_ρ above is well-defined up to a Livsic coboundary. Indeed, it only depends on the choice of a metric on the line bundle L_ρ, and one easily checks that changing this metric will modify c_ρ by a coboundary.

Definition 3.4. A $\mathcal{C}^1$ action ρ of Γ on $\mathbb{S}^1$ is *expanding* if for every $x \in \mathbb{S}^1$, there exists $\gamma \in \Gamma$

$$|\rho(\gamma)'(x)| > 1 \ .$$

By a straightforward application of Borel–Lebesgue's characterization of compactness, every expanding action satisfies the stronger property:

Proposition 3.5. *Let $\rho : \Gamma \to \mathrm{Diff}(\mathbb{S}^1)$ be an expanding action. Then there exists $g_1, \ldots, g_k \in \Gamma$, a covering of $\mathbb{S}^1$ by open intervals $I_1, \ldots I_k$, and $\varepsilon > 0$ such that*

$$\rho(g_j)'(x) \geq 1 + \varepsilon$$

for all $x \in I_j$.

Theorem 3.6. *Let ρ be a $\mathcal{C}^1$ action of Γ on $\mathbb{S}^1$ topologically conjugate to the action of Γ on $\partial_\infty\Gamma$. Then the following are equivalent:*

(*i*) *The action ρ is expanding,*
(*ii*) *The cocycle c_ρ is expanding,*

(iii) The action ρ is bi-Hölder conjugate to the action of Γ *on* $\partial_\infty \Gamma$.

Proof. $(i) \Rightarrow (iii)$.

We prove more generally that if two dilating $\mathcal{C}^1$ actions ρ_1 and ρ_2 are conjugated by a homeomorphism h, then h is bi-Hölder continuous. In particular, a dilating $\mathcal{C}^1$ action topologically conjugate to the action of Γ on $\partial_\infty \Gamma$ is bi-Hölder conjugate to any Fuchsian action of Γ on $\mathbb{S}^1$.

Let $g_1, \ldots, g_k$ be elements in Γ, $I_1, \ldots, I_k$ be open intervals covering $\mathbb{S}^1$ and ε be positive such that $\rho_1(g_j)' > 1 + \varepsilon$ on I_j. Let $0 < \eta < 1$ be such that $\rho_2(g_j)' > \eta$ on I_j for all $j \in \{1, \ldots, k\}$.

Let $a > 0$ be such that, for all interval $J \in \mathbb{S}^1$, if J has length less or equal to a, then there exists $j \in \{1, \ldots, k\}$ such that $J \subset I_j$.

Let us now fix $x \neq y \in \mathbb{S}^1$. By the expanding property and the definition of a, one can find $n \in \mathbb{N}$ and and $i_1, \ldots, i_n \in \{1, \ldots, k\}$ such that

- For all $0 \leq l < n$,
$$|\rho_1(g_{i_l} \ldots g_{i_1}) \cdot x - \rho_1(g_{i_l} \ldots g_{i_1}) \cdot y| < a \ ,$$
- For all $0 \leq l < n$,
$$g_{i_l} \ldots g_{i_1} \cdot [x, y] \subset I_{i_{l+1}} \ ,$$
-
$$|\rho_1(g_{i_n} \ldots g_{i_1}) \cdot x - \rho_1(g_{i_n} \ldots g_{i_1}) \cdot y| \geq a \ .$$

Note that, since g_j multiplies the length of every interval contained in I_j by at least $1 + \varepsilon$, we have

$$(1 + \varepsilon)^{n-1} |x - y| < a \ . \tag{2}$$

Now, let $h : \mathbb{S}^1 \to \mathbb{S}^1$ be the homeomorphism conjugating ρ_1 and ρ_2 and let $b > 0$ be the infimum of lengths of images by h of an interval of length at least a.

We then have

$$|\rho_2(g_{i_n} \ldots g_{i_1}) \cdot h(x) - \rho_2(g_{i_n} \ldots g_{i_1}) \cdot h(y)| \geq b$$

and therefore

$$|h(x) - h(y)| \geq \eta^n b \ . \tag{3}$$

Putting (2) and (3) together, and setting $\alpha = \frac{\log(1+\varepsilon)}{\log(1/\eta)} > 0$, we get

$$\begin{aligned} |x - y| &< \frac{a}{(1+\varepsilon)^{n-1}} \\ &< a\eta^{(n-1)\alpha} \\ &< \frac{a}{(b\eta)^\alpha} (b\eta^n)^\alpha \\ &< \frac{a}{(b\eta)^\alpha} |h(x) - h(y)|^\alpha \ . \end{aligned}$$

Since this is true for all $x \neq y$, we conclude that h^{-1} is α-Hölder. We obtain simimarly that h is Hölder by switching the roles of ρ_1 and ρ_2.

$(iii) \Rightarrow (ii)$

Let $\rho : \Gamma \to \mathrm{Diff}(\mathbb{S}^1)$ be a $\mathcal{C}^1$ action on $\mathbb{S}^1$ which is bi-Hölder conjugate to a Fuchsian action j. Let $h : \mathbb{S}^1 \to \mathbb{S}^1$ be the homeomorphism conjugating j and ρ and $\alpha > 0$ such that h is α-Hölder.

Fix $\gamma \in \Gamma$. Let γ_+ be the attracting fixed point of $j(\gamma)$ and denote $\lambda = j(\gamma)'(\gamma_+) < 1$. Let y be a point in $\mathbb{S}^1$ distinct from γ_- and γ_+. Then when n goes to $+\infty$,

$$\log |j(\gamma^n) \cdot y - x| \sim \log(\lambda) n \ .$$

Since h is α-Hölder, we thus have

$$\log |\rho(\gamma^n) \cdot h(y) - h(\gamma_+)| = O(\alpha \log(\lambda) n) \ ,$$

which implies that

$$\rho(\gamma)'(h(\gamma_+)) \leq \lambda^\alpha \ .$$

We thus have

$$L_{c_\rho}(\gamma) \geq -\alpha \log(\lambda) = \alpha L_j(\gamma) \ .$$

By proposition 1.21, the cocycle c_ρ is thus expanding.

(ii) $\Rightarrow$ *(i)*

Let $\widetilde{E}_\rho$ be the line bundle over $\widetilde{M_\Gamma}$ defined in the proof of Proposition 3.2. Let us denote by $|\cdot|_0$ the continuous metric on $\widetilde{L}_\rho$ induced by the metric on $\mathbb{S}^1$ and by $|\cdot|_1$ a Γ-invariant continuous metric on $\widetilde{L}_\rho$.

By construction, the metric $|\cdot|_0$ is invariant under the flow $\widehat{\varphi}$. Let K be a compact set in $\widetilde{M_\Gamma}$ such that $\bigcup_{\gamma\in\Gamma} \gamma \cdot K$ covers $\widetilde{M_\Gamma}$ and such that the image of K by the projection $\pi : (x, y, z) \mapsto z$ is the whole circle. Finally, let $A > 1$ be such that for all $x \in K$ and all $u \in (\widetilde{E}_\rho)_x \backslash \{0\}$,

$$\frac{1}{A} < \frac{|u|_1}{|u|_0} < A \ .$$

Let z be any point in $\mathbb{S}^1$. Choose a point $x \in K$ such that $\pi(x) = z$ and a non-zero vector $u \in \widetilde{L}_\rho(x) = T_z\mathbb{S}^1$.

Since the cocycle c_ρ is expanding, there exists a time $t > 0$ such that

$$\frac{|\widehat{\varphi}_t(u)|_1}{|u|_1} > A^2 \ .$$

Let $\gamma \in \Gamma$ be such that $\gamma \cdot \varphi_t(x) \in K$ and let $v \in (\widetilde{L}_\rho)_{\gamma \cdot x}$ be the image of u by γ. We then have

$$\rho(\gamma) \cdot z = \pi(\gamma \cdot x)$$

and

$$\begin{aligned}
\rho(\gamma)'(z) &= \frac{|v|_0}{|\widehat{\varphi}_t(u)|_0} \\
&= \frac{|v|_0}{|u|_0} \quad \text{since } |\cdot|_0 \text{ is } \widehat{\varphi}\text{-invariant} \\
&> \frac{1}{A^2}\frac{|v|_1}{|u_1|} \\
&> \frac{1}{A^2}\frac{|\widehat{\varphi}_t(u)|_1}{|u|_1} \quad \text{since } |\cdot|_1 \text{ is } \Gamma\text{-invariant} \\
&> 1 \ .
\end{aligned}$$

We conclude that the $\mathcal{C}^1$ action ρ is expanding. $\square$

3.2. **The space $\mathfrak{X}_{an}(\Gamma, \mathrm{Diff}(\mathbb{S}^1))$ and the map DF.**

Definition 3.7. Let $\rho : \Gamma \to \mathrm{Diff}(\mathbb{S}^1)$ be a homomorphism. We will say that ρ is an *Anosov action* on $\mathbb{S}^1$ or an *Anosov representation* into $\mathrm{Diff}(\mathbb{S}^1)$ if ρ is Hölder conjugate to a Fuchsian action.

Proposition 3.8. *Let ρ be an Anosov action on $\mathbb{S}^1$. Then there exists a norm $|\cdot|$ on E_ρ such that*

$$|\widehat{\varphi}_t(u)| > |u|$$

for all $t > 0$ and all $u \neq 0 \in L_\rho$.

Proof. Let $|\cdot|_0$ be a continuous metric on E_ρ. Since the cocycle c_ρ is expanding, we can find constants $K > 1$ and $a > 0$ such that

$$\frac{|\widehat{\varphi}_{-t}(u)|_0}{|u|_0} \leq K e^{-at}$$

for all $u \neq 0 \in E_\rho$. We can thus define a new continuous metric $|\cdot|_1$ on E_ρ by

$$|u|_1 = \int_0^{+\infty} |\widehat{\varphi}_{-s}(u)|_0 \mathrm{d}s \ .$$

We now have

$$\begin{aligned} |\widehat{\varphi}_t(u)|_1 &= \int_0^{+\infty} |\widehat{\varphi}_{t-s}(u)|_0 \mathrm{d}s \\ &= \int_{-t}^{+\infty} |\widehat{\varphi}_{-s}(u)|_0 \mathrm{d}s \\ &> |u|_1 \end{aligned}$$

$\square$

Corollary 3.9. *Let ρ be an Anosov action on $\mathbb{S}^1$. Then there exists a continuous parametrization ψ_ρ of $\mathcal{G}$ and a continuous norm $|\cdot|$ on E_ρ such that*

$$|\widehat{\psi}_\rho(t,u)| = e^t |u|$$

for all $u \in E_\rho$.

Moreover, ψ_ρ is unique up to conjugation and $|\cdot|$ is uniquely determined by ψ_ρ up to a multiplicative constant.

Proof. Let $|\cdot|$ be a continuous metric on E_ρ such that $|\widehat{\varphi}_t(u)| > |u|$. Then the cocycle c_ρ associated to this norm is positive. It is thus the reparametrization cocycle of a flow ψ^ρ. For $x \in M_\Gamma$ and $t \in \mathbb{R}$, let s be such that $c(x,s) = t$. Then for any $u \in L_\rho(x)\backslash\{0\}$, we have

$$\begin{aligned} |\widehat{\psi}^\rho_t(u)| &= |\widehat{\varphi}_s(u)| \\ &= e^{c_\rho(x,s)}|u| \\ &= e^t |u| \ . \end{aligned}$$

This proves the existence of ψ^ρ.

Asume now that there are two norms $|\cdot|_1$ and $|\cdot|_2$ on L_ρ and two reparametrizations ψ_1 and ψ_2 of φ such that

$$|\widehat{\varphi}_i(t,u)|_i = e^t |u|_i$$

for $i = 1, 2$.

Let c be the reparametrization cocycle of ψ_2 along ψ_1. Let x be a point in M_Γ, $u \in (L_\rho)_x \backslash \{0\}$, $t \in \mathbb{R}$ and set $v = \widehat{\psi}_1(t,u)$. We then have

$$t = \log \frac{|v|_1}{|u|_1}$$

and

$$c(x,t) = \log \frac{|v|_2}{|u|_2} \ .$$

It follows that

$$c(x,t) - t = F(\psi_1(t,x)) - F(x) \ ,$$

where

$$F = \frac{|\cdot|_2}{|\cdot|_1} \ .$$

The cocycle c is thus Livsic cohomologous to the tautological cocycle $c_0(x,t) = t$, and ψ_2 is thus equivalent to ψ_1.

Finally, fix ψ_ρ a reparametrization of φ. Assume that $|\cdot|_1$ and $|\cdot|_2$ are two continuous metrics on L_ρ such that

$$|\widehat{\psi}_\rho(t,u)|_i = e^t |u|_i$$

for $i = 1, 2$.

Then the function $\frac{|\cdot|_2}{|\cdot|_1}$ on M_Γ is continuous and invariant by ψ_ρ. It is thus constant by topological transitivity of the geodesic foliation. □

Recall that $\mathrm{Diff}^h(\mathbb{S}^1)$ denotes the set of diffeomorphisms of the circle with Hölder derivatives.

Proposition 3.10. *If ρ takes values in $\mathrm{Diff}^h(\mathbb{S}^1)$, then the associated parametrization ψ_ρ is conjugate to a Hölder continuous reparametrization.*

Proof. Exercise. □

Let us now see how Corollary 3.9 defines a continuous map from the space of Anosov representations to the space $\mathrm{Par}(\mathcal{G})$.

Let us provide the group $\mathrm{Diff}(\mathbb{S}^1)$ with the $\mathcal{C}^1$ topology. Since Γ is finitely generated, the space $\mathrm{Hom}(\Gamma, \mathrm{Diff}(\mathbb{S}^1))$ of homomorphisms from Γ to $\mathrm{Diff}(\mathbb{S}^1)$ embeds in a product of finitely many copies of $\mathrm{Diff}(\mathbb{S}^1)$ and inherits its topology.

The group $\mathrm{Diff}(\mathbb{S}^1)$ acts continuously on $\mathrm{Hom}(\Gamma, \mathrm{Diff}(\mathbb{S}^1))$ by conjugation. Its orbit equivalence relation has a priori no reason to be Hausdorff, so we define

$$\mathfrak{X}(\Gamma, \mathrm{Diff}(\mathbb{S}^1))$$

as the largest Hausdorff quotient of $\mathrm{Hom}(\Gamma, \mathrm{Diff}(\mathbb{S}^1))/\mathrm{Diff}(\mathbb{S}^1)$ (i.e. the quotient of $\mathrm{Hom}(\Gamma, \mathrm{Diff}(\mathbb{S}^1))$ by the smallest closed equivalence relation containing the conjugation).

The set $\mathrm{Hom}_{an}(\Gamma, \mathrm{Diff}(\mathbb{S}^1))$ of Anosov actions is open in $\mathrm{Hom}(\Gamma, \mathrm{Diff}(\mathbb{S}^1))$ and invariant under conjugation. We denote by

$$\mathfrak{X}_{an}(\Gamma, \mathrm{Diff}(\mathbb{S}^1))$$

its image in $\mathfrak{X}(\Gamma, \mathrm{Diff}(\mathbb{S}^1))$.

Corollary 3.9 associates to each Anosov action ρ a parametrization ψ_ρ of the geodesic foliation such that $L_{\psi^\rho} = L_\rho$. This defines a map

$$\widetilde{\mathrm{DF}} : \mathrm{Hom}_{an}(\Gamma, \mathrm{Diff}(\mathbb{S}^1)) \to \mathrm{Par}(\mathcal{G}) \ .$$

Here we prove the following:

Theorem 3.11. *The map* $\widetilde{\mathrm{DF}}$ *factors to a continuous map*

$$\mathrm{DF} : \mathfrak{X}_{an}(\Gamma, \mathrm{Diff}(\mathbb{S}^1)) \to \mathrm{Par}(\mathcal{G}) \ ,$$

which maps $\mathfrak{X}^h_{an}(\Gamma, \mathrm{Diff}(\mathbb{S}^1))$ *into* $\mathrm{Par}^h(\mathcal{G})$.

Let us first see that $\widetilde{\mathrm{DF}}$ factors to $\mathfrak{X}_{an}(\Gamma, \mathrm{Diff}(\mathbb{S}^1))$.

Proposition 3.12. *For every* $\gamma \in \Gamma$*, the function*

$$\begin{array}{rccc} \chi_\gamma : & \mathrm{Hom}_{an}(\Gamma, \mathrm{Diff}(\mathbb{S}^1)) & \to & \mathbb{R} \\ & \rho & \mapsto & L_\rho(\gamma) \end{array}$$

is continuous and invariant by conjugation.

Proof. The conjugation invariance is easy. The continuity follows from the stability of contracting dynamics. □

By universal property of the largest Hausdorff quotient, the functions χ_γ factor to continuous functions on $\mathfrak{X}_{an}(\Gamma, \mathrm{Diff}(\mathbb{S}^1))$. In other words, two Anosov actions in the same equivalence class in $\mathfrak{X}_{an}(\Gamma, \mathrm{Diff}(\mathbb{S}^1))$ have the same period map. Since the map $\widetilde{\mathrm{DF}}$ preserves period maps and since points in $\mathrm{Par}(\mathcal{G})$ are uniquely determined by their period map, we conclude that the map $\widetilde{\mathrm{DF}}$ factors to a map

$$\mathrm{DF} : \mathfrak{X}_{an}(\Gamma, \mathrm{Diff}(\mathbb{S}^1)) \to \mathrm{Par}(\mathcal{G}) \ .$$

Let us now prove the continuity of DF. Note that it is not sufficient to know that $L_\rho(\gamma)$ varies continuously with ρ for each γ: one needs some uniformity in γ.

Lemma 3.13. *Let* (ρ_n) *be a sequence of Anosov actions on the circle converging to* $\rho \in \mathrm{Hom}_{an}(\Gamma, \mathrm{Diff}(\mathbb{S}^1))$*. let* h_n *and* h *denote the homeomorphisms from* $\partial_\infty\Gamma$ *to* $\mathbb{S}^1$ *conjugating the action of* Γ *with* ρ_n *and* ρ *respectively. Then* h_n *converges uniformly to* h.

Proof. For each $\gamma \in \Gamma \backslash \{\mathrm{Id}\}$, the homeomorphism h_n maps γ_+ to the attracting fixed point of $\rho_n(\gamma)$. By stability of contracting dynamics, we deduce that $h_n(\gamma_+)$ converges to $h(\gamma_+)$. Since attracting fixed points of elements in $\Gamma \backslash \{\mathrm{Id}\}$ are dense in $\partial_\infty\Gamma$ we obtain that h_n converges pointwise to h on a dense subset. Now, h_n and h are locally given by continuous and monotonous maps of a compact interval, so Dini's second theorem implies tha h_n converges uniformly to h. □

From here, one could argue that the whole construction of the flat line bundle and corresponding cocycle associated to ρ vary continuously with ρ. Alternatively, one can prove in a more down to earth way the following:

Proposition 3.14. *Let (ρ_n) be a sequence of Anosov actions on the circle converging to $\rho \in \mathrm{Hom}_{an}(\Gamma, \mathrm{Diff}(\mathbb{S}^1))$. Then $\frac{L_{\rho_n}}{L_\rho}$ converges uniformly to* 1.

Since DF preserves period maps, this shows that $d_{HT}(\mathrm{DF}(\rho_n), \mathrm{DF}(\rho))$ converges to 0, concluding the proof of the continuity of DF.

Proof. Fix $\varepsilon > 0$ and choose a finite generating set S of Γ. Let η be such that for all $s \in S$,

$$|\log \rho'(s)(x) - \log \rho'(s)(y)| \leq \varepsilon$$

whenever $|x - y| \leq \eta$. Choose n large enough so that

$$|\log \rho_n(s)'(x) - \log \rho(s)'(x)| \leq \varepsilon$$

for all $x \in \mathbb{S}^1$ and all $s \in S$, and

$$|h_n(x) - h(x)| \leq \eta$$

for all $x \in \partial_\infty \Gamma$.

Fix $[\gamma] \in [\Gamma]$ and choose γ a representative of $[\gamma]$ of minimal length k with respect to the generating set S. Write $\gamma = s_1 \dots s_k$ with $s_i \in S$. Since ρ is Anosov, there exists a constant $\lambda > 0$ independent of $[\gamma]$ such that $L_\rho(\gamma) \geq \lambda k$. Let us now compute:

$$\begin{aligned}
\frac{|L_{\rho_n}(\gamma) - L_\rho(\gamma)|}{L_\rho(\gamma)} &= \frac{1}{L_\rho(\gamma)} \left|\log \rho_n(\gamma)'(h_n(\gamma_+)) - \log \rho(\gamma)'(h(\gamma_+))\right| \\
&= \frac{1}{L_\rho(\gamma)} \left| \sum_{i=1}^k \log \rho_p(s_i)'(h_n(s_{i+1} \dots s_k \cdot \gamma_+)) - \sum_{i=1}^k \log \rho(s_i)'(h(s_{i+1} \dots s_k \cdot \gamma_+)) \right| \\
&\leq \frac{1}{\lambda k} \sum_{i=1}^k |\log \rho_n(s_i)'(h_n(s_{i+1} \dots s_k \cdot \gamma_+)) - \rho(s_i)'(h_n(s_{i+1} \dots s_k \cdot \gamma_+))| \\
&\quad + |\rho(s_i)'(h_n(s_{i+1} \dots s_k \cdot \gamma_+)) - \rho(s_i)'(h(s_{i+1} \dots s_k \cdot \gamma_+))| \\
&\leq \frac{2\varepsilon}{\lambda}\,.
\end{aligned}$$

□

3.3. **Anosov actions and stable measures.** Here we investigate further properties of the flow ψ^ρ associated to an Anosov action: we show that ψ^ρ admits stable horocycles, and that the metric $|\cdot|$ on L_ρ which is scaled by φ induces a family of measures on the leaves of $\mathcal{W}^u$ with nice properties with respect to the horocycles.

Let us start with the first point.

Proposition 3.15. *The flow ψ^ρ associated to an Anosov representation into* $\mathrm{Diff}(\mathbb{S}^1)$ *admits stable horocycles.*

Proof. Let $|\cdot|$ be the metric on E_ρ satisfying
$$|\widehat{\psi}_t^\rho(u)| = e^t|u| \ .$$
Let x and $y \in \widetilde{M_\Gamma}$ be two points belonging to the same stable leaf $\widetilde{\mathcal{W}}_s(p)$. Let u be a non-zero vector in $T_p\mathbb{S}^1$. We define
$$B(x,y) = \log \frac{|u|_y}{|u|_x} \ ,$$
where $|u|_x$ and $|u|_y$ denote respectively the norm of u seen as a vector in E_{ρ_x} and in E_{ρ_y}. It is clear that this definition does not depend on u.

One easily verifies that B is a Buseman cocycle. Moreover, for any $x \in \widetilde{M_\Gamma}$ and any $u \in E_{\rho_x}\backslash\{0\}$, we have
$$B(x, \psi_t(x)) = \log \frac{|\widehat{\psi}_t^\rho(u)|}{|u|} = t \ .$$
The sets
$$\mathcal{H}^s(x) = \{y \in \mathcal{W}^s(x) \mid B(x,y) = 0\}$$
thus define stable horocycles for the flow ψ^ρ. □

We now turn to the construction of *unstable measures* which are scaled by the flow. Let us start with some definitions.

Definition 3.16. An *unstable* (resp. *stable*) measure on M_Γ is a collection of Radon measures on the leaves of $\widetilde{\mathcal{W}}^u$ (resp. $\widetilde{\mathcal{W}}^s$) which is preserved by Γ.

Let φ be a parametrization of the geodesic foliation. One can naturally push forward or pull back an unstable measure by φ_t.

Definition 3.17. We say that an unstable measure μ is *scaled* by φ if for every $t \in \mathbb{R}$,
$${\varphi_t}^*\mu = c(t)\mu$$
for some constant $c(t)$. Since φ is a flow, we necessarily have $c(t) = e^{at}$ for some $a \in \mathbb{R}$. We call a the *scaling factor.*

Let us now assume that φ admits a stable horocyclic foliation $\mathcal{H}^s$. Let p and q be two points in $\partial_\infty\Gamma$, labelling two unstable leaves $\widetilde{\mathcal{W}}^u(p)$ and $\widetilde{\mathcal{W}}^u(q)$. Then, for every point $x \in \widetilde{\mathcal{W}}^u(p)\backslash\mathcal{G}(p,q)$, the stable horocycle passing through x intersects $\widetilde{\mathcal{W}}^u(q)$ in a unique point.

Definition 3.18. The map
$$T_{p,q} : \widetilde{\mathcal{W}}^u(p)\backslash\mathcal{G}(p,q) \to \widetilde{\mathcal{W}}^u(q)\backslash\mathcal{G}(q,p)$$
sending x to the unique intersection between $\widetilde{\mathcal{H}}^s(x)$ and $\widetilde{\mathcal{W}}^u(q)$ is called the *holonomy* of the stable horocyclic foliation.

Definition 3.19. An unstable measure μ is called *invariant under horocyclic holonomy* if
$${T_{p,q}}^*\mu_q = \mu_p$$
for all $p, q \in \partial_\infty\Gamma$. An unstable (resp. stable) measure which is invariant under horocyclic holonomy and scaled by the flow φ is called an unstable (resp. stable) *Margulis measure.*

Let now ψ^ρ be the flow associated to an Anosov action ρ. Let $\partial_\infty\Gamma \to \mathbb{S}^1$ be the homeomorphism conjugating the action of Γ on its boundary with ρ. Let $|\cdot|$ be the metric on E_ρ such that

$$|\widehat{\psi}_t^\rho(u)| = e^t|u| \ .$$

We define an unstable measure μ_ρ in the following way: let q be a point $\partial_\infty\Gamma$. Choose $s \mapsto p(s)$ a homeomorphism from $(0,1)$ to $\partial_\infty\Gamma\backslash\{q\}$ such that $h\circ p : (0,1) \to \mathbb{S}^1\backslash\{h(q)\}$ is a diffeomorphism, and choose continuously $x(s)$ in $\widetilde{\mathcal{G}}(q,p(s))$. For a continuous function f on $\widetilde{\mathcal{W}}^u(q)$ with compact support, define

$$\int f \mathrm{d}\mu_q^\rho = \int_0^1 \int_{-\infty}^{+\infty} f(\psi_t^\rho(x(s))e^{-t}|(h\circ p)'(s)|_{x(s)}\mathrm{d}t\mathrm{d}s \ .$$

Proposition 3.20. *The family of measures μ_q^ρ defines an unstable Margulis measure with scale factor* 1*.*

Proof. Let us first remark that the measure μ_q^ρ defined above does not depend on the choice of $x(s)$. Indeed, for another choice $x'(s)$, we can write $x'(s) = \psi_u^\rho(s)(x(s))$ for some $u(s) \in \mathbb{R}$. We then have

$$\begin{aligned}\int_0^1 \int_{-\infty}^{+\infty} f(\psi_t^\rho(x'(s))e^{-t}|(h\circ p)'(s)|_{x'(s)}\mathrm{d}t\mathrm{d}s &= \int_0^1 \int_{-\infty}^{+\infty} f(\psi_{t+u(s)}^\rho(x(s))e^t|(h\circ p)'(s)|_{\psi_u^\rho(s)(x(s))}\mathrm{d}t\mathrm{d}s \\ &= \int_0^1 \int_{-\infty}^{+\infty} f(\psi_{t+u(s)}^\rho(x(s))e^{t+u(s)}|(h\circ p)'(s)|_{x(s)}\mathrm{d}t\mathrm{d}s \\ &= \int_0^1 \int_{-\infty}^{+\infty} f(\psi_t^\rho(x(s))e^t|(h\circ p)'(s)|_{x(s)}\mathrm{d}s\mathrm{d}t \ ,\end{aligned}$$

showing that μ_q^ρ does not depend on the choice of $x(s)$. Now, the fact that μ_q^ρ does not depend on the choice of $p(s)$ either is just the change of variable formula for the integration in s. We conclude that μ_q^ρ is well-defined. Moreover, by Γ-Invariance of the metric $|\cdot|$, we easily verify that $\gamma_*\mu_q^\rho = \mu_{\gamma\cdot q}^\rho$. Hence $(\mu_q^\rho)_{q\in\partial_\infty\Gamma}$ defines an unstable measure on $\mathcal{W}^u$.

From the definition of μ_q^ρ one easily proves that

$$\psi_t^{\rho *}\mu_q^\rho = e^t\mu_q^\rho \ .$$

Thus μ^ρ is scaled by ψ^ρ with scale factor 1.

It remains to prove that μ^ρ is invariant by holonomy along the horocycles of ψ^ρ. We leave that as an exercise to the reader. ☐

4. Foliated affine, hyperbolic and complex structures

In this section, we show that the *foliated affine actions* introduced in Section 2.1 can be recovered from a *foliated conformal structure*. This allows us to construct the map CF.

4.1. Affine actions and affine charts.

Let us first recall the definition of a foliated affine action.

Definition 4.1. A *foliated affine action* on M_Γ is a Γ-equivariant pair of continuous flows $((\varphi_t),(h_s))$ on $\widetilde{M}_\Gamma$ such that

- (φ_t) is a parametrization of $\mathcal{G}$,

- the map $(t,s) \mapsto \varphi_t(h_s(x))$ is a covering from $\mathbb{R}^2$ to the stable leaf $\mathcal{W}^s(x)$,
- $\varphi_{-t} \circ h_s \circ \varphi_t = h_{e^t s}$.

It is called *Hölder continuous* if the flows (φ_t) and (h_s) are Hölder continuous.

Two foliated affine actions (φ_t, h_s) and (φ'_t, h'_s) are *conjugated* if there exists a homeomorphism of M_Γ preserving the leaves of $\mathcal{G}$ and conjugating (φ_t) to (φ'_t) and (h_s) to (h'_s).

As we will see, the data of a foliated affine action is essentially the same as what we call an *equivariant family of affine charts*:

Definition 4.2. An equivariant family of affine charts on $\partial_\infty \Gamma$ is the data, for any $x = (x_-, x_0, x_+) \in \widetilde{M_\Gamma}$, of a homeomorphism $m_x : \partial_\infty \Gamma \backslash \{x_+\} \to \mathbb{R}$ such that:

- $m_x(x_0) = 0$ and $m_x(x_1) = 1$,
- m_x depends continuously on x for the compact open topology
- $m_{\gamma \cdot x} = m_x \circ \gamma^{-1}$ for all $\gamma \in \Gamma$,
- if x and x' belong to the same stable leaf, then $m_{x'} \circ m_x^{-1}$ is an affine transformation of $\mathbb{R}$.

It is called *Hölder continuous* if the homeomorphisms m_x are bi-Hölder continuous and vary Hölder continuously for the compact open topology.

From an equivariant family of affine charts, one gets a foliated affine action by setting

- $\varphi_t(x_-, x_0, x_+) = (x_-, m_x^{-1}(e^t), x_+)$ (where $x = (x_-, x_0, x_+)$,
- $h_s(x_-, x_0, x_+) = (m_x^{-1}(s), m_x^{-1}(s+1), x_+)$.

Note that this affine action has the following property: for every $x = (x_-, x_0, x_+) \in \widetilde{M_\Gamma}$,

$$h_1(x) \in \mathcal{G}(x_0, x_+) \ .$$

We call such a foliated affine action *normalized*.

From a normalized foliated affine action, one gets an equivariant family of affine charts by setting $m_x(y_-) = s$ where s is the unique real number such that the $h_s(x)$ belongs to the geodesic $\mathcal{G}(y_-, x_+)$. One can verify that the construction is inverse of the previous one. There is thus a bijection between normalized foliated affine actions and equivariant families of affine charts. Finally, we have the following:

Proposition 4.3. *Every foliated affine action is conjugated to a unique normalized one.*

Proof. Let (φ_t, h_s) be a foliated affine action. Define $F(x) = (x_-, x_1, x_+)$, where x_1 is such that $h_1(x)$ belongs to $\mathcal{G}(x_1, x_+)$. Then F descends to a homeomorphism of M_Γ preserving the leaves of $\mathcal{G}$.

Set $\varphi'_t = F \circ \varphi_t \circ F^{-1}$ and $h'_s = F \circ h_s \circ F^{-1}$. Then (φ'_t, h'_s) is a foliated affine action. Let (x_-, x_1, x_+) be a point in $\widetilde{M_\Gamma}$. Then

$$\begin{aligned} h'_1(x_-, x_1, x_+) &= F \circ h_1 \circ F^{-1}(x) \\ &\in F(\mathcal{G}(x_1, x_+)) \quad \text{by definition of } F \\ &\in \mathcal{G}(x_1, x_+) \ . \end{aligned}$$

Hence (φ'_t, h'_s) is normalized.

Assume now that two normalized affine actions (φ_t, h_s) and (φ'_t, h'_s) are conjugated via $F : x \mapsto \varphi_{T(x)}(x)$. Then we have

$$\begin{aligned} h'_1(x_-, x_0, x_+) &= F^{-1} \circ h_1 \circ \varphi_{T(x)}(x) \\ &\in F^{-1}(\mathcal{G}(x_{T(x)}, x_+)) = \mathcal{G}(x_{T(x)}, x_+) \ , \end{aligned}$$

where $\varphi_{T(x)}(x) = (x_-, x_{T(x)}, x_+)$. However, since h' is also normalized, $h'(x_-, x_0, x_+)$ belongs to $\mathcal{G}(x_0, x_+)$. We conclude that $F(x) = \varphi_{T(x)}(x) = x$ for all x. Thus F is the identity. □

4.2. Foliated hyperbolic structures.

Definition 4.4. A *foliated hyperbolic structure* on M_Γ is the data, for each $z \in \partial_\infty \Gamma$, of a homeomorphism $m_z : \widetilde{\mathcal{W}}^s(z) \to \mathbb{H}^2$ such that:

- m_z varies continuously with z for the compact open topology,
- For every $\gamma \in \Gamma$,

$$m_{\gamma \cdot z} \circ \gamma \circ m_z^{-1} \in \mathrm{Isom}^+(\mathbb{H}^2) \ .$$

Two foliated hyperbolic structures (m_z) and (m'_z) are *equivalent* if $m'_z \circ m_z^{-1} \in \mathrm{Isom}^+(\mathbb{H}^2)$ for all z.

A foliated hyperbolic structure is *transversally Hölder* if m_z varies Hölder continuously with z for the compact open topology.

Let (m_z) and (m'_z) be two foliated hyperbolic structures. By compactness of M_Γ, there exists a constant $C > 1$ such that for all $z \in \partial_\infty \Gamma$ and all $x, y \in \widetilde{\mathcal{W}}^s(z)$,

$$d_{\mathbb{H}}(m_z(x), m_z(y)) = 1 \quad \Rightarrow \quad 1/C \leq d_{\mathbb{H}}(m'_z(x), m'_z(y)) \leq C \ .$$

Extending this globally, we get:

Proposition 4.5. *Let (m_z) and (m'_z) be two foliated hyperbolic structures. Then there exist constants $C > 1$ and $K \geq 0$ such that $m'_z \circ m_z^{-1}$ is a (C, K)-quasi-isometry for all z.*

Recall that the affine group is the group of orientation preserving isometries of the upper half-space $\mathbb{H}^2$ fixing infinity. One can thus associate to a foliated affine action the foliated hyperbolic structure that conjugates those affine actions.

To be more precise, let (φ_t, h_s) be a foliated affine action on M_Γ. Choose in a continuous way a point x_z in each stable leaf $\widetilde{\mathcal{W}}^s(z)$.

Proposition 4.6. *The family of maps (m_z) defined by*

$$m_z(\varphi_t(h_s(x_z)) = s + e^t i \ ,$$

is a foliated hyperbolic structure. Moreover, a different choice of (x_z) defines a foliated hyperbolic structure which is equivalent.

The main claim of this section is that, conversely, every foliated hyperbolic structure comes from a foliated affine action.

Theorem 4.7. *Let* (m_z) *be a foliated hyperbolic structure. Then each* m_z *extends continuously to a bi-Hölder homeomorphism* $\bar{m}_z : \partial_\infty \Gamma \to \partial_\infty \mathbb{H}^2$*. For every* $x = (x_-, x_0, z) \in \mathcal{F}(z)$*, define* g_x *as the unique isometry of* $\mathbb{H}^2$ *mapping* $(\bar{m}_z(x_-), \bar{m}_z(x_0), \bar{m}_z(x_+))$ *to* $(0, 1, \infty)$*. Then the family* $(g_x \circ \bar{m}_z)$ *is an equivariant family of affine charts on* $\partial_\infty \Gamma$*.*

Most of the proof of the theorem is straightforward once we know that quasi-isometries of $\mathbb{H}^2$ extend to homeomorphisms of the boundary. The main technical difficulty is to control that this extension varies continuously with z. This is dealt with in the next subsection.

4.3. **boundary extension of quasi-isometries.** Let us first recall the classical Morse lemma for quasi-geodesics in the hyperbolic plane states that every quasi-geodesic ray of $\mathbb{H}^2$ (i.e. every quasi-isometric embedding $f : \mathbb{R}_+ \to \mathbb{H}^2$) is at bounded distance from a geodesic ray.

Proposition 4.8. *Let* $f : \mathbb{R}_+ \to \mathbb{H}^2$ *be a* C*-quasi-geodesic ray. Then* $f(t)$ *converges as* t *goes to* $+\infty$ *to a point* $f(\infty)$ *in* $\partial_\infty \mathbb{H}^2$*, and there exists a constant* D *depending only on* C *such that any* $f(t)$ *is at distance at most* D *from the geodesic ray* $[f(0), f(\infty)]$*.*

Moreover, the point $f(\infty)$ varies continuously with f for the compact open topology. More precisely, we have

Proposition 4.9. *Let* f : *and* $g : \mathbb{R}_+ \to \mathbb{H}^2$ *be two* C*-quasi-geodesics with* $f(0) = g(0)$*. Let* $f(\infty)$ *and* $g(\infty)$ *denote there respective endpoints in* $\partial_\infty \mathbb{H}^2$*. Then there exists a constant* K *depending only on* C *such that, if* $d_{\mathbb{H}}(f(t), g(t)) \leq 1$ *for some* $t \geq 1$*, then*

$$d_\infty^{f(0)}(f(\infty, g(\infty)) \leq K e^{-t/C} \ ,$$

where $d_\infty^{f(0)}$ *denotes the visual distance from* $f(0)$ *on* $\partial_\infty H^2$*.*

As a corollary, one obtains that the boundary map induced by a quasi-isometry varies continuously for the compact-open topology:

Corollary 4.10. *Let* $f : \mathbb{H}^2 \to \mathbb{H}^2$ *be a* C*-quasi-isometric homeomorphism. Then* f *extends to a* $\frac{1}{C}$*-bi-Hölder homeomorphism* $\partial_\infty f : \partial_\infty \mathbb{H}^2 \to \partial_\infty \mathbb{H}^2$*. Moreover, let* $g : \mathbb{H}^2 \to \mathbb{H}^2$ *be another* C*-quasi-isometric homeomorphism. Then there is a constant* K *depending only on* C *such that, if* $R \geq 1$ *and* $d_{\mathbb{H}}(f(x), g(x)) \leq 1$ *for all* x *in* $B(o, R)$*, then*

$$d_\infty^{f(o)}(\partial_\infty f(x), \partial_\infty g(x)) \leq K e^{-R/C}$$

for all $x \in \partial_\infty \mathbb{H}^2$*.*

Proof. Exercise. □

We can now turn to the

Proof of Theorem 4.7. Let us fix a background hyperbolic metric on Σ. This provides us with an identification of $\partial_\infty \Gamma$ with $\partial_\infty \mathbb{H}^2$.

Using the identification of $\widetilde{M}_\Gamma$ with $T_1 \mathbb{H}^2$, one also obtains a reference foliated hyperbolic structure on $\mathcal{W}^s$, which simply projects every leaf $T_1 \mathbb{H}^2$ to $\mathbb{H}^2$. We see this reference foliated structure as an identification of each leaf of $\widetilde{\mathcal{W}^s}$ with $\mathbb{H}^2$.

Let now (m_z) be another foliated hyperbolic structure. Then

$$m_z : \widetilde{\mathcal{W}}^s(z) \simeq \mathbb{H}^2 \to \mathbb{H}^2$$

is a C-quasi-isometry for some C independent of z. It thus extends to a bi-Hölder hoemomorphism $\partial_\infty m_z : \partial_\infty \mathbb{H}^2 \simeq \partial_\infty \Gamma \to \partial_\infty \mathbb{H}^2$. Moreover, for every $x = (x_-, x_0, z) \in \mathcal{F}^s(z)$, there is a unique hyperbolic isometry g_x such that $\bar{m}_x = g_x \circ \partial_\infty m_z$ maps x_- to 0, x_0 to 1 and z to ∞. By restriction, $\bar{m}_x$ defines a bi-Hölder continuous homeomorphism from $\partial_\infty\Gamma\backslash\{z\}$ to $\mathbb{R}$.

By Corollary 4.10, since m_z varies continuously with z for the compact-open topology, so does the family of maps $(\bar{m}_x)$. One easily checks that $(\bar{m}_x)$ also satisfies the other properties of an equivariant family of affine charts.

□

4.4. **Smoothening foliated hyperbolic structures.** So far we associated to every foliated affine action a foliated hyperbolic structure. This hyperbolic structure, however, has rather low regularity (the developping map of each leaf is only continuous, and varies transversally continuously for the $\mathcal{C}^0$ topology. Foliated Teichmüller theory, on the other side, has been developed mainly for leafwise smooth foliated structures. This section is thus devoted to the proof of the following lemma:

Lemma 4.11. *Let $((\varphi_t),(h_s))$ and $((\varphi'_t),(h'_s))$ be two foliated afine actions on M_Γ. Let (m_z) and (n_z) be the foliated hyperbolic structures associated respectively to affine actions (m_z) and (n_z) as in Proposition 4.6. Then (n_z) is isotopic to a foliated hyperbolic structure (n'_z) such that $n'_z \circ m_{z^{-1}}$ is a $\mathcal{C}^\infty$ diffeomorphism that varies continuously with z for the $\mathcal{C}^\infty$ topology.*

Proof. One easily goes from $\mathcal{C}^1$ regularity to $\mathcal{C}^\infty$ regularity by a standard smoothening argument. We focus here on isotoping n_z to a hyperbolic structure with $\mathcal{C}^1$ regularity. Recall first that, by Proposition 1.11, we can assume without loss of generality that (φ'_t) is a reparametrization of (φ_t) which is $\mathcal{C}^1$ along the orbits.

For ε small enough (to be chosen later), define (n_z^ε) by

$$n_z^\varepsilon(x) = \frac{1}{\varepsilon}\int_0^\varepsilon n_z(h_s(x))\mathrm{d}s\ ,$$

where n_z is seen as a map to the upper-half space inside the complex plane. Recall that for every $\gamma \in \Gamma$, we have

$$n_{\gamma\cdot z} = g \circ n_z \circ \gamma^{-1}$$

for some affine transformation g of $\mathbb{H}^2$. This induces the same property for n_z^ε. It remains to see that $n_z^\varepsilon \circ m_z^{-1}$ is a $\mathcal{C}^1$ diffeomorphism for ε small enough.

Since both m_z and n_z map geodesics in $\widetilde{M_\Gamma}$ to vertical geodesics in $\mathbb{H}^2$, we can write $F = n_z \circ m_z^{-1}$ in the form

$$F(x+iy) = f(x) + ig(x,y)\ ,$$

where $f : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism, and where $g(x,\cdot) : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ is a $\mathcal{C}^1$ diffeomorphism varying continuously with x for the $\mathcal{C}^1$ topology. Writing $G^\varepsilon = n_z^\varepsilon \circ m_z^{-1}$, we have

$$G^\varepsilon(x+iy) = \frac{1}{\varepsilon y}\int_0^{\varepsilon y} F(x+s+iy)\mathrm{d}s\ .$$

An elementary computation shows that G^ε is $\mathcal{C}^1$ and that

$$\begin{aligned}
\frac{\partial}{\partial x} G^\varepsilon(x+iy) &= \frac{1}{\varepsilon y}(F(x+\varepsilon y+iy) - F(x+iy)) \\
&= \frac{1}{\varepsilon y}\underbrace{(f(x+\varepsilon y) - f(x))}_{A(\varepsilon)} + i\cdot \frac{1}{\varepsilon y}\underbrace{(g(x+\varepsilon y, y) - g(x,y))}_{B(\varepsilon)} , \\
\frac{\partial}{\partial y} G^\varepsilon(x,y) &= \frac{-1}{y} G^\varepsilon(x+iy) + \frac{1}{y}F(x+\varepsilon y + iy) + \frac{1}{\varepsilon y}\int_0^{\varepsilon y} \frac{\partial}{\partial y} F(x+s+iy)\mathrm{d}s \\
&= \underbrace{\frac{1}{y}\left(f(x+\varepsilon y) - \frac{1}{\varepsilon y}\int_0^{\varepsilon y} f(x+s)\mathrm{d}s\right)}_{C(\varepsilon)} \\
&\quad \underbrace{+i\cdot\frac{1}{y}\left(g(x+\varepsilon y, y) - \frac{1}{\varepsilon y}\int_0^{\varepsilon y} g(x+s,y)\mathrm{d}s\right)}_{D(\varepsilon)} \underbrace{+i\cdot\frac{1}{\varepsilon y}\int_0^{\varepsilon y}\frac{\partial}{\partial y} g(x+s,y)\mathrm{d}s}_{D'(\varepsilon)} .
\end{aligned}$$

Note that, since f is an increasing homeomorphism, we have

$$0 < C(\varepsilon) < A(\varepsilon) .$$

Let us compute the determinent of $\mathrm{d}G^\varepsilon$. We have

$$\begin{aligned}
\text{Jac } G^\varepsilon(x+iy) &= \frac{1}{y\varepsilon}\left(A(\varepsilon)(D(\varepsilon) + D'(\varepsilon)) - B(\varepsilon)C(\varepsilon)\right) \\
&= \frac{A(\varepsilon)}{y\varepsilon}\left(D(\varepsilon) + D'(\varepsilon) - \frac{C(\varepsilon)}{A(\varepsilon)}B(\varepsilon)\right) .
\end{aligned}$$

By continuity of g, the terms $B(\varepsilon)$ and $D(\varepsilon)$ go to 0 as ε goes to 0, while

$$D'(\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} \frac{\partial}{\partial y} g(x,y) > 0 .$$

Thus Jac $G^\varepsilon(x+iy)$ is positive for ε small enough depending only on the (local) module of continuity of F. By compactness of M_Γ and the equivariance of n_z^ε, we deduce the existence of η such that n_z^ε is a diffeomorphism for all z and all for $\varepsilon < \eta$.

Hence, for ε small enough, (n_z^ε) is a foliated hyperbolic structure isotopic to (n_z) and such that $n_z^\varepsilon \circ m_z^{-1}$ is $\mathcal{C}^1$ for all z.

□

4.5. **The space $\mathcal{T}(\mathcal{W}^s)$ and the map** CF**.** Teichmüller spaces of 2-dimensional foliations (or, more generally, of 2-dimensional laminations) were introduced by Sullivan in [3]. Building on the works of Ahlfors and Bers, he pointed out that large aspects of classical Teichmüller theory extended to the context of *foliated conformal structures.*

Definition 4.12. A *smooth foliated conformal structure* on $\mathcal{W}^s$ is a family of conformal classes of metrics on the leaves of $\mathcal{W}^s$ which vary continuously with the leaf for the smooth topology. Two foliated conformal structures are *homotopic* if one is the pull-back of the other by a leafwise homotopy of $\mathcal{W}^s$.

The *Teichmüller space of the foliation* $\mathcal{W}^s$, denoted $\mathcal{T}(\mathcal{W}^s)$, is the set of homotopy classes of smooth foliated conformal structures on $\mathcal{W}^s$.

Candel's theorem, the foliated analog of Poincaré's uniformization, shows that one can alternatively see $\mathcal{T}(\mathcal{W}^s)$ as the space of homotopy classes of foliated hyperbolic structures.

Theorem 4.13 (Candel)**.** *Let* $[g]$ *be a smooth foliated conformal structure on* $\mathcal{W}^s$*. Then* $[g]$ *contains a unique foliated Riemannian metric of curvature* -1.

Given $[g_1]$ and $[g_2]$ two foliated conformal classes, define the *conformal dialation* $\mathrm{dil}([g_1],[g_2])$ as the infimum of the constants $K \geq 1$ such that there exists $g_1 \in [g_2]$ and $g_2 \in [g_2]$ with

$$\frac{1}{K} g_1 \leq g_2 \leq K g_1 \ .$$

Definition 4.14. The *Teichmüller distance* between $[g_1]$ and $[g_2]$ is defined as

$$d_{\mathcal{T}}([g_1],[g_2]) = \inf\{\log \mathrm{dil}([g_1'],[g_2']), [g_1'] \text{ homotopic to } [g_1], [g_2'] \text{ homotopic to } [g_2]\} \ .$$

By definition, the Teichmüller distance is well-defined on $\mathcal{T}(\mathcal{W}^s)$. Sullivan proves in [] that it is indeed a distance. Moreover the space $\mathcal{T}(\mathcal{W}^s)$ with the induced topology has a structure of complex Banach manifold:

Theorem 4.15 (Sullivan)**.** *The space* $(\mathcal{T}(\mathcal{W}^s), d_{\mathcal{T}})$ *is homeomorphic to a Banach manifold such that, for every foliated conformal structure* $[g]$*, there is a biholomorphism from* $\mathcal{T}(\mathcal{W}^s)$ *to a bounded open domain in the space* $\mathrm{QD}(\mathcal{W}^s,[g])$ *of* foliated quadratic differentials *which are holomorphic with respect to the conformal structure* $[g]$*.*

Let $[g]$ be a foliated conformal structure on $\mathcal{W}^s$ and let $\gamma \in [\Gamma]$ be a closed leaf of $\mathcal{G}$. This leaf is contained in a unique leaf $\mathcal{W}^s(\gamma)$ which is homeomorphic to a cylinder.

Definition 4.16. The *period map* $L_{[g]}$ of the conformal structure $[g]$ associates to $\gamma \in [\Gamma]$ the translation length of l_γ, where l_γ is an isometry of $\mathbb{H}^2$ such that $(\mathcal{W}^s(\gamma),[g])$ is conformal to $l_\gamma \backslash \mathbb{H}^2$.

We can now turn to the proof of Theorem 0.4. We start with the first part:

Theorem 4.17. *There exists a map* $\mathrm{CF} : \mathcal{T}(\mathcal{W}^s) \to \mathrm{Par}(\mathcal{G})$ *such that*

$$L_{\mathrm{CF}([g])} = L_{[g]} \ .$$

Proof. Let $[g]$ be a foliated conformal structure on $\mathcal{W}^s$. By Candel's theorem, the conformal class $[g]$ contains a unique foliated hyperbolic metric g_{hyp}, which can be seen as a smooth foliated hyperbolic structure. One then associates to g_{hyp} a family of affine charts (m^g) on $\partial_\infty\Gamma$ via Theorem 4.7.

If $[g']$ is homotopic to $[g]$ then g'_{hyp} is homotopic to g_{hyp}. In particular, the developments of a given leaf into $\mathbb{H}^2$ associated respectively to g_{hyp} and g'_{hyp} remain at bounded distance from each other. They thus induce the same boundary maps, and therefore the same families of affine charts on $\partial_\infty \Gamma$. In conclusion, the map

$$[g] \mapsto (m^g)$$

is well defined from $\mathcal{T}(\mathcal{W}^s)$ to the space of equivariant families of affine charts. Finally, the family of affine charts (m^g) defines a foliated affine action (φ_t^g, h_s^g), and we define

$$\begin{array}{rccc} \mathrm{CF}: & \mathcal{T}(\mathcal{W}^s) & \to & \mathrm{Par}(\mathcal{G}) \\ & [g] & \mapsto & [\varphi^g]\ . \end{array}$$

Let us prove that CF preserves the period maps. Let γ be an element in $\Gamma \backslash \mathrm{Id}$. Fix a point $x = (\gamma_-, x_0, \gamma_+)$ and let m be the isometry from $(\widetilde{\mathcal{W}^s}(\gamma_+), g_{hyp})$ to $\mathbb{H}^2$ whose extension to the boundary maps γ_+ to ∞, γ_- to 0 and x_0 to 1.

Since γ acts on $(\widetilde{\mathcal{W}^s}(\gamma_+), g_{hyp})$ as an isometry of translation length $l = L_{[g]}([\gamma])$ and fixes γ_- and γ_+, we have

$$m \circ \gamma \circ m^{-1} : z \mapsto e^l z\ .$$

On the other side, by definition of the affine action associated to m^g, we have $\varphi_t^g(x) = (\gamma_-, x_t, \gamma_+)$, where $m(x_t) = e^t$. We deduce that $\varphi_l^g(x) = \gamma \cdot x$, hence $l = L_{\varphi^g}([\gamma])$. □

It remains to prove the continuity of the map CF. We actually prove a stronger result:

Theorem 4.18. *The map* $\mathrm{CF} : (\mathcal{T}(\mathcal{W}^s), d_{\mathcal{T}}) \to (\mathbf{P}\mathrm{Par}(\mathcal{G}), d_{HT})$ *is Lipschitz continuous.*

Proof. Let $[\gamma]$ be a closed leaf of $\mathcal{G}$ and $[g]$ a foliated conformal structure on $\mathcal{W}^s$. Recall that the hyperbolic length $L_{[g]}(\gamma)$ is proportional to its *extremal length*, defined as

$$EL_{[g]}(\gamma) = \sup_{g'} \inf_{\gamma'} \frac{\mathrm{length}_g(\gamma')^2}{\mathrm{area}(g)}\ ,$$

where the infimum is taken over all curves γ' freely homotopic to γ in $\mathcal{W}^s(\gamma)$ and the suppremum is taken over all metrics g' on $\mathcal{W}^s(\gamma)$ in the conformal class of $[g]$ and of finite area.

Now, one easily verifies that, if $\mathrm{dil}([g_1],[g_2]) = K$, then

$$\frac{1}{K^2} EL_{[g_1]}(\gamma) \leq EL_{[g_2]}(\gamma) \leq K^2 EL_{[g_2]}(\gamma)\ .$$

We deduce that

$$\left|\log\left(\frac{L_{[g_2]}(\gamma)}{L_{[g_1]}(\gamma)}\right)\right| = \left|\log\left(\frac{EL_{[g_2]}(\gamma)}{EL_{[g_1]}(\gamma)}\right)\right| \leq d_{\mathcal{T}}([g_1],[g_2])\ ,$$

and therefore

$$\begin{aligned} d_{HT}(\mathrm{CF}([g_1]), \mathrm{CF}([g_2])) &= \frac{1}{2}\left(\sup_{\gamma\in[\Gamma]} \log\left(\frac{L_{[g_1]}(\gamma)}{L_{[g_2]}(\gamma)}\right) + \sup_{\gamma\in[\Gamma]} \log\left(\frac{L_{[g_2]}(\gamma)}{L_{[g_1]}(\gamma)}\right)\right) \\ &\leq d_{\mathcal{T}}([g_1],[g_2])\ . \end{aligned}$$

□

5. Construction of Margulis measures

In his thesis, Margulis constructed the measure of maximal entropy of an Anosov flow by first constructing what we called *Margulis measures* along stable and unstable leaves. Here we reproduce his argument to prove the following theorem:

Theorem 5.1 (Margulis)**.** *Let φ be a parametrization of the geodesic foliation which admits stable (resp. unstable) horocycles. Then φ admits an unstable (resp. stable) Margulis measure.*

The starting point of Margulis's consruction is a family of unstable measures (ν_p) which are "almost preserved" by the horocycle holonomy. These measures are gven by the volume form associated to some Riemannian metric on unstable leaves. Since we work in low regularity here, one needs an additional argument to find such a family of measures.

5.1. Almost invariant unstable measures.

Definition 5.2. An unstable measure μ is called *almost holonomy invariant* if for every $p, q \in \partial_\infty \Gamma$, we have

$$T_{p,q}{}^* \mu_q = f_{p,q} \mu_p$$

where $f_{p,q}$ is continuous on $\widetilde{\mathcal{W}}^u(p)$, depends continuously on p and q and satisfies

$$|f_{p,q}(x) - 1| \leq \eta(d(x, T_{p,q}(x)))$$

where d is a Γ-invariant distance on $\widetilde{M_\Gamma}$ and $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ is a function such that $\eta(s) \underset{s\to 0}{\longrightarrow} 0$.

Our argument here to construct an almost holonomy invariant unstable measure slightly more elaborate than that of Margulis due to the a priori weak regularity of φ and $\mathcal{H}^s$.

Let us start by fixing a hyperbolic metric on Σ and denote by φ_0 and $\mathcal{H}^s_0$ the associated geodesic flow and stable horocyclic foliation. We provide M_Γ with the smooth structure and the Riemannian metric induced by the identification $M_\Gamma \simeq T_1\Sigma$. We denote by X_0 the vector field generating φ_0.

Each unstable leaf $\widetilde{\mathcal{W}}^u(p)$ is identified with the hyperbolic plane, and we provide it with the hyperbolic area form λ_p. These area forms thus give rise to an unstable measure λ. The following is a good exercise:

Proposition 5.3. *The unstable measure λ is a Margulis measure for φ_0.*

Let now φ be another parametrization of the geodesic foliation which admits a stable horocyclic foliation $\mathcal{H}^s$. We want to find an unstable measure which is almost invariant by the holonomy along $\mathcal{H}^s$.

By Proposition 2.12, without loss of generality, we can assume that the reparametrization cocycle $c_{\varphi_0\to\varphi}$ is the stable Buseman cocycle B associated to a closed stable 1-form α that is smooth on each stable leaf. Let p, q be two points in $\partial_\infty\Gamma$ and let $T_{p,q}$ and $T^0_{p,q}$ denote the holonomies from $\widetilde{\mathcal{W}}^u(p)$ to $\widetilde{\mathcal{W}}^u(q)$ along $\mathcal{H}^s$ and $\mathcal{H}^s_0$ respectively.

Proposition 5.4. *There is a constant* C *(independent of* p *and* q*) such that for all* $x \in \widetilde{\mathcal{W}}^u(p)$*,*

$$T^0_{q,p} \circ T_{p,q}(x) = \varphi_0(x, s(x)) \ ,$$

where

- s *is continuous and*
$$|s(x)| \leq Cd(x, T_{p,q}(x)) \ ,$$
- s *is differentiable along* φ_0*,* $X_0 \cdot s$ *is continuous and*
$$|X_0 \cdot s(x)| \leq Cd(x, T_{p,q}(x)) \ .$$

Define X_0

Corollary 5.5. *The stable measure* λ *is almost invariant under the holonomy of* $\mathcal{H}^s$*.*

Proof of Proposition 5.4. By construction, $T_{p,q}$ and $T^0_{q,p}$ map a geodesic to a geodesic. Indeed, if x and y belong to a geodesic contained in $\widetilde{\mathcal{W}}^u(p)$, then $T_{p,q}(x)$ and $T_{p,q}(y)$ belong to the same unstable leaf $\widetilde{\mathcal{W}}^u(q)$ and to the same stable leaf $\widetilde{\mathcal{W}}^s(x) = \widetilde{\mathcal{W}}^s(y)$ (since $T_{p,q}$ "follows" stable horocycles). The same holds for $T^0_{q,p}$. Therefore, for x and $T^0_{q,p} \circ T_{p,q}(x)$ belong to the same geodesic.

Let $s(x)$ be such that $T^0_{q,p} \circ T_{p,q}(x) = \varphi_0(x, s(x))$. Let α_0 be the smooth closed stable 1-form such that $\int_y^{\varphi_0(y,t)} = t$. By definition of $T^0_{q,p}$, we have $\int_{T_{p,q}(x)}^{T^0_{q,p} \circ T_{p,q}(x)} \alpha_0 = 0$ and thus

$$\begin{aligned} s(x) &= \int_x^{\varphi_0(x,s(x))} \alpha_0 \\ &= \int_x^{T_{p,q}(x)} \alpha_0 \ . \end{aligned}$$

By continuity of α_0 and $T_{p,q}$, we have

$$|s(x)| \leq Cd(x, T_{p,q}(x)) \ ,$$

where C is a uniform bound on α_0.

Let us now prove the derivability of s along φ_0. Recall that φ_0 and φ are respectively generated by the vector fields X_0 and X, tangent to the geodesic foliations, such that $\alpha_0(X_0) = \alpha(X) = 1$.

By construction, we have $T_{p,q}(\varphi(x, \varepsilon)) = \varphi(T_{p,q}(x), \varepsilon)$. Thus

$$\begin{aligned} s(\varphi(x,\varepsilon)) - s(x) &= \int_{\varphi(x,\varepsilon)}^{T_{p,q}(\varphi(x,\varepsilon))} \alpha_0 - \int_x^{T_{p,q}(x)} \alpha_0 \\ &= \int_x^{\varphi(x,\varepsilon)} \alpha_0 - \int_{T_{p,q}(x)}^{\varphi(T_{p,q}(x),\varepsilon)} \alpha_0 \ . \end{aligned}$$

It follows that s is derivable along φ and

$$X \cdot s(x) = \alpha_0(X)_x - \alpha_0(X)_{T_{p,q}(x)} \ .$$

Since $\alpha(X) = \alpha_0(X_0) = 1$, we have $X_0 = fX$ where $f = \alpha(X_0) = \frac{1}{\alpha_0(X)}$. Hence s is derivable along φ_0 and

$$X_0 \cdot s(x) = 1 - \frac{\alpha_x(X_0)}{\alpha_{T_{p,q}(x)}(X_0)} \ .$$

Since $\alpha(X_0)$ is continuous, positive and smooth in restriction to weakly stable leaves, we deduce the existence of a constant C such that

$$|X_0 \cdot s(x)| \leq C d(x, T_{p,q}(x)) \ .$$

□

Let us now deduce Corollary 5.5. Set $\sigma(x) = T^0_{q,p} \circ T_{p,q}(x) = \varphi_0(x, s(x))$. Since ${T^0_{p,q}}^* \lambda_p = \lambda_q$, we have

$$T^*_{p,q} \lambda_q = \sigma^* \lambda_p \ .$$

Now, there are coordinates (u, v) on $\widetilde{\mathcal{W}}^u(p)$ with respect to which $\varphi_0((u, v), t) = (u + t, v)$ and such that $\lambda_p = e^{-u} \mathrm{d}u\mathrm{d}v$. Corollary 5.5 thus follows from the following computation:

Lemma 5.6. *Let $\sigma : \mathbb{R}^2 \to \mathbb{R}^2$ be a homeomorphism given by*

$$\sigma(u, v) = (u + s(u, v), v) \ ,$$

where s is continuous and differentiable with respect to u with continuous partial derivative. Then

$$\sigma^*(e^{-u} \mathrm{d}u \ dv) = e^{-s}(1 + \frac{\partial s}{\partial u}) \mathrm{d}u\mathrm{d}v \ .$$

Proof. Exercise. □

5.2. **Margulis measures on unstable leaves.** To construct an unstable measure which is holonomy invariant and scaled by φ, Margulis's approach is roughly to "pull back" the measure λ by φ^t for large t and suitably rescale it. The crucial point of this approach is the following lemma, which gives a "uniform way" to rescale $\varphi_t^* \lambda$. Let us first set some terminology.

We call a subset K of M_Γ a *compact subset of* $\mathcal{W}^u$ if it is a finite union of subsets K_i which are each contained in a single unstable leaf and compact for the topology of the leaf.

Given a compact subset K of $\mathcal{W}^u$, we call a function $f : M_\Gamma \to \mathbb{R}$ a *continuous function on* $\mathcal{W}^u$ *with support in* K if f is continuous in restriction to each leaf and vanishes outside K. We denote by $\mathcal{C}(K)$ the space of continuous functions on $\mathcal{W}^u$ with support in K, and by denote by $\mathcal{C}_c(\mathcal{W}^u)$ the vector space of continuous functions on $\mathcal{W}^u$ with compact support. We endow this space with the norm

$$\|f\|_\infty = \sup_{M_\Gamma} |f| \ .$$

Lemma 5.7. *There exists a non-negative function $f_0 \in \mathcal{C}_c(\mathcal{W}^u)$ such that, for every compact subset K of $\mathcal{W}^u$, there is a constant $C = C(K, f_0)$ such that for every $f \in \mathcal{C}(K)$ and all $t \geq 0$, we have*

$$\left| \int f \circ \varphi_{-t} \mathrm{d}\lambda \right| \leq C \, \|f\|_\infty \int f_0 \circ \varphi_{-t} \mathrm{d}\lambda \ .$$

Proof. Given an open subset U of $\mathcal{W}^u$, denote by $\mathcal{H}^s(U)$ the union of all the leaves of $\mathcal{H}^s$ intersecting U. We first claim that we can find U with compact closure and large enough so that $\mathcal{H}^s(U) = M_\Gamma$.[1] Indeed,

$$\mathcal{H}^s\left(\bigcup_{t\in\mathbb{R}} \varphi_t(U)\right) = \bigcup_{t\in\mathbb{R}} \mathcal{H}^s(\varphi_t(U))$$

is a non empty open set saturated by the leaves of $\mathcal{W}^u$. It is thus equal to the whole M_Γ by minimality of the weakly unstable foliation. By compactness of M_Γ, there is $T > 0$ such that

$$\mathcal{H}^s\left(\bigcup_{-T\leq t\leq T} \varphi_t(U)\right) = M_\Gamma\ .$$

Let us hence fix an open subset U_0 of $\mathcal{W}^u$ with compact closure such that every horocycle intersects U_0, and let f_0 be a continuous non-negative function on $\mathcal{W}^u$ with compact support such that $f_0 = 1$ on U_0. Let now K be a compact subset of $\mathcal{W}^u$. Let us lift K and U_0 to compact and open sets $\widetilde{K}$ and $\widetilde{U_0}$ respectively contained in $\widetilde{\mathcal{W}^u}(p)$ and $\widetilde{\mathcal{W}^u}(q)$ for some $p, q \in \partial_\infty\Gamma$, and lift f_0 to a continuous Γ-invariant function on $\widetilde{\mathcal{W}^u}$ (that we still denote f_0). By construction of U_0, we can find a covering of K by finitely many open subsets $(V_i)_{1\leq i\leq k}$ and $\gamma_1, \ldots, \gamma_k \in \Gamma$ such that V_i is contained in $T_{\gamma_i\cdot q,p}(\gamma_i \cdot U_0)$. Let f be a continuous function with support in K. Using partitions of unity, we can assume without loss of generality that f has support in one of the V_i's, say V_1. We can also assume that $\gamma_1 = \mathrm{Id}$.

For all $t \geq 0$, we have

$$\begin{aligned}
\left|\int f\circ\varphi_{-t}\mathrm{d}\lambda_p\right| &\leq \|f\|_\infty\, \lambda(\varphi_t(V_1)) \\
&= \|f\|_\infty \int_{T_{p,q}(\varphi_t(V_1))} T_{q,p*}\lambda_p \\
&\leq Cst\, \|f\|_\infty \sup_{x\in\varphi_t(V_1)} d(x, T_{q,p}(x))\lambda_q(T_{p,q}(\varphi_t(V_1))) \\
&\leq Cst\, \|f\|_\infty \sup_{x\in\varphi_t(V_1)} d(x, T_{q,p}(x)) \int f_0\circ\varphi_{-t}\mathrm{d}\lambda_q \quad \text{since } f_0\circ\varphi_{-t} \text{ is positive and equ}
\end{aligned}$$

Finally, since $T_{q,p}$ is the holonomy along the stable horocycles of φ, we have that $d(\varphi_t(x), T_{p,q}(\varphi_t(x))$ goes to 0 as t goes to $+\infty$, from which we deduce that $\sup_{x\in\varphi_t(V_1)} d(x, T_{q,p}(x))$ is bounded uniformly in t, giving the inequality

$$\left|\int f\circ\varphi_{-t}\mathrm{d}\lambda\right| \leq Cst'\, \|f\|_\infty \int f_0\circ\varphi_{-t}\mathrm{d}\lambda\ .$$

□

Let $L = \mathbb{R}^{\mathcal{C}_c(\mathcal{W}^u)}$ denote the space of all functions on $\mathcal{C}_c(\mathcal{W})^u$, provided with the product topology (i.e. the topology of pointwise convergence). We

[1]Note that any U would suit if we knew that the horocycle foliation was minimal. We believe it is true but could not easily adapt Hedlund's theorem in our setting.

see an unstable measure μ as an element of L by setting

$$\mu(f) = \int f \mathrm{d}\mu \ .$$

For $t \in \mathbb{R}_+$, define $\lambda_t(f) = \frac{\int f\circ\varphi_{-t}\mathrm{d}\lambda}{\int f_0\circ\varphi_{-t}\mathrm{d}\lambda}$. For each $T \geq 0$, let Ω_T denote the convex hull of $\{\lambda_t, t \geq T\}$ in $L = \mathbb{R}^{\mathcal{C}_c(\mathcal{W}^u)}$.

By Lemma 5.7, for every $f \in \mathcal{C}_c(\mathcal{W}^u)$, there is a constant C_f such that $|\lambda_t(f)| \leq C_f$ for all $t \in \mathbb{R}_+$. Thus

$$\Omega_T \subset \{\alpha \in L \mid \alpha(f) \leq C_f \text{ for all } f \in \mathcal{C}_c(\mathcal{W})^u\}$$

and the closure of Ω_T is compact for all T.

For each $s \geq 0$, there is a constant A_s such that

$$\lambda_t(f_0 \circ \varphi_{-s}) \geq A_s \ .$$

thus, $\alpha \to \alpha(f_0 \circ \varphi_{-s})$ is positive on $\overline{\Omega}_0$. Since it is continuous, we conclude that the operator

$$\begin{array}{rccl} \widehat{\varphi}_s^* : & \overline{\Omega}_0 & \to & \overline{\Omega}_s \subset \overline{\Omega}_0 \\ & \alpha & \mapsto & \widehat{\varphi}_s^*\alpha : f \mapsto \frac{\alpha(f\circ\varphi_{-s})}{\alpha(f_0\circ\varphi_{-s})} \end{array}$$

is continuous. The following proposition concludes the proof of Theorem 5.1.

Proposition 5.8. *There exists a point in $\overline{\Omega}_0$ which is fixed by $\widehat{\varphi}_s^*$ for all s. This point is associated to an unstable Margulis measure for φ.*

Proof. The Tychonoff fixed point theorem implies that each $\widehat{\varphi}^*_{\frac{1}{2^n}}$ has a fixed point μ_n in $\overline{\Omega}_0$. Let μ be an accumulation point of (μ_n). Then μ is fixed by $\widehat{\varphi}_s^*$ for every diadic s.

Let K be a compact subset of $\mathcal{W}^u$. By Lemma 5.7, for every $f \in \mathcal{C}(K)$, we have $\lambda_t(f) \leq C(K, f_0) \|f\|_\infty$ for all $t \geq 0$. Passing to the convex hull and then to the limit, we deduce that μ is linear on $\mathcal{C}(K)$, continuous, and non negative on positive functions. Riesz's representation theorem then implies that μ is an unstable measure. We also get that μ is fixed by $\widehat{\varphi}_s^*$ for all s by continuity. Thus μ is scaled by φ.

It remains to prove that μ is holonomy invariant. Let f be a continuous function on $\widetilde{\mathcal{W}}^u$ with compact support $K \subset \widetilde{\mathcal{W}}^u(p)$ for some $p \in \partial_\infty\Gamma$, and let q be another point in $\partial_\infty\Gamma$. We have

$$\begin{aligned} |\lambda_t(f\circ T_{p,q}) - \lambda_t(f)| &= \left| \frac{\int f\circ\varphi_{-t}\circ T_{p,q}\mathrm{d}\lambda_q - \int f\circ\varphi_{-t}\mathrm{d}\lambda_p}{\int f_0\circ\varphi_{-t}\mathrm{d}\lambda} \right| \\ &= \left| \frac{\int f\circ\varphi_{-t}\mathrm{d}T^*_{q,p}\lambda_q - \int f\circ\varphi_{-t}\mathrm{d}\lambda_p}{\int f_0\circ\varphi_{-t}\mathrm{d}\lambda} \right| \\ &\leq Cst \sup_{x\in\varphi_t(K)} d(x, T_{p,q}(x)) \frac{\int |f\circ\varphi_{-t}|\mathrm{d}\lambda}{\int f_0\circ\varphi_{-t}\mathrm{d}\lambda} \\ &\leq Cst'(f) \sup_{x\in\varphi_t(K)} d(x, T_{p,q}(x)) \ . \end{aligned}$$

Since $T_{p,q}$ is the holonomy along the horocyclic foliation of φ, we have that $d(\varphi_t(x), T_{p,q}(\varphi_t(x))) \underset{t\to+\infty}{\longrightarrow} 0$ uniformly on K. Passing to the convex hull and to the limit, we deduce that

$$\int f \circ T_{p,q} \mathrm{d}\mu = \int f \mathrm{d}\mu \ .$$

Thus μ is invariant under horocyclic holonomy. □

5.3. **Scaling factor and entropy.** In general, we don't know whether unstable Marguils measure is unique (see Section 6.2). Nonetheless, we prove here that the scaling factor of any such measure is the same, and that this scaling factor defines in fact a continuous function h_{top} on $\mathrm{Par}(\mathcal{G})$. Experts will have recognized the *topological entropy*. More precisely we could prove the following:

Theorem 5.9. *Let φ be a parametrization of $\mathcal{G}$ with stable horocycles, and let μ be an unstable Margulis measure for φ with scale factor a. Then:*

- *a is the topological entropy of φ,*
- *There exists a φ-invariant probability measure ν on M_Γ which disintegrates to μ along stable horocycles,*
- *The measure ν has metric entropy equal to a (equivalently, ν is a measure of maximal entropy).*

Since we try to avoid introducing the entropy here, we content ourselves with the following theorem:

Theorem 5.10. *Let exists a function*

$$h_{top} : \mathrm{Par}(\mathcal{G}) \to \mathbb{R}_{>0}$$

such that if φ is a parametrization of $\mathcal{G}$ with stable (resp. unstable) horocycles and μ is an unstable (resp. stable) Margulis measure for φ, then the scale factor of μ equals $h_{top}([\varphi])$ (resp. $-h_{top}([\varphi])$).

To prove this, we describe a standard procedure to combine a stable and an unstable measure into a measure on M_Γ.

Let μ be a stable measure on $\mathcal{W}^s$ and let c be a continuous curve contained in a leaf of $\mathcal{W}^u$ and transverse to $\mathcal{G}$ inside that leaf. We define the projection of μ to c as the measure μ_c defined by

$$\mu_c(I) = \mu \left(\bigcup_{t \geq 0} \varphi_t(I) \right) \ ,$$

where I is some interval in c and φ is any parametrization of $\mathcal{G}$.

Let now φ be a parametrization of $\mathcal{G}$ admitting a stable horocyclic foliation $\mathcal{H}$. We call a family of measures on the leaves of $\mathcal{H}$ a *horocyclic measure*. Given a stable measure μ, we get an horocyclic measure $\mu_{\mathcal{H}}$ by projecting μ onto the leaves of $\mathcal{H}$.

Proposition 5.11. *Let φ and ψ be two parametrizations of $\mathcal{G}$ such that*

- *φ admits a stable horocyclic foliation $\mathcal{H}$,*
- *there is a locally finite stable measure μ which is scaled by ψ with scale factor b.*

Then the horocyclic projection $\mu_{\mathcal{H}}$ is locally finite and stafisfies

$$\varphi_t^* \mu_{\mathcal{H}} = e^{-bc(t,\cdot)} \mu_{\mathcal{H}} \ ,$$

where c is the reparametrization cocycle of ψ with respect to φ.

Proof. Let $\mathcal{H}(x)$ be some horocycle of φ and identify the weakly stable leave $\widetilde{\mathcal{W}}^s(x)$ containing $\mathcal{H}(x)$ with $Hh(x) \times \mathbb{R}$ via the map

$$(y,t) \mapsto \psi_t(y) \ .$$

Since the measure μ is scaled by ψ, it can be written in those coordinates in the form

$$\mathrm{d}\mu = \mathrm{d}\nu \otimes e^{-bt} \mathrm{d}t \ ,$$

where ν is a locally finite measure on $\mathcal{H}(x)$. An easy computation then shows that

$$\mu_{\mathcal{H}(x)} = \frac{1}{b} \nu \ .$$

Let I be an interval in $\mathcal{H}(x)$. Withe respect to the coordinates above, we have

$$\varphi_t(I) = \{(y, c(y,t)), y \in I\} \ .$$

We thus have

$$\begin{aligned} \mu_{\mathcal{H}}(\varphi_t(I)) &= \int_{y\in I} \int_{s=c(y,t)}^{+} \infty e^{-bs} \mathrm{d}s \mathrm{d}\nu(y) \\ &= \int_{y \in I} e^{-bc(y,t)} \mathrm{d}\mu_{\mathcal{H}}(y) \ . \end{aligned}$$

This concludes the proof. □

Now, if φ is a parametrization of $\mathcal{G}$ with stable horocycles, ν a locally finite horocyclic measure and μ an unstable measure which is invariant under horocycle holonomy, one obtains a finite measure $\mu \rtimes \nu$ on M_Γ by setting locally

$$\int f \mathrm{d}\mu \rtimes \nu = \int_{y \in \mathcal{W}^u(x)} \int_{z \in \mathcal{H}(y)} f(z) \mathrm{d}\nu(z) \mathrm{d}\mu(y) \ ,$$

where x is any point on M_Γ and f a continuous function supported in a neighbourhood of x. (The holonomy invariance of μ guaranties that the measure is well-defined independently of x.)

Using the fact that the total mass of μ must be preserved by the flow φ, we prove the following:

Lemma 5.12. *Let φ^1 and φ^2 be a parametrizations of $\mathcal{G}$ admitting respectively stable and unstable horocycles. Let μ^1 (resp. μ^2) be an unstable (resp. stable) Margulis measure for φ^1 (resp. φ^2) with scale factor a (resp. $-b$). Then*

$$\inf_{\gamma \in [\Gamma]} \frac{L_{\varphi^2}(\gamma)}{L_{\varphi^1}(\gamma)} \leq \frac{a}{b} \leq \sup_{\gamma \in [\Gamma]} \frac{L_{\varphi^2}(\gamma)}{L_{\varphi^1}(\gamma)} \ .$$

Proof. Let $\mathcal{H}$ be the stable horocyclic foliation of φ^1 and consider the finite measure

$$\nu = \mu^1 \rtimes \mu^2_{\mathcal{H}}$$

on M_Γ. By Proposition **??** and since μ^1 is scaled by φ^1, we have

$$\varphi_t^{1*}\nu = e^{at-bc(\cdot,t)} \ .$$

Since the total mass of the measure μ must be preserved, we must have

$$\inf_{x\in M_\Gamma} e^{at-bc(\cdot,t)} \leq 1 \leq \sup_{x\in M_\Gamma} e^{at-bc(\cdot,t)} \ ,$$

which rewrites

$$\inf_{x\in M_\Gamma} \frac{c(x,t)}{t} \leq \frac{a}{b} \leq \sup_{x\in M_\Gamma} \frac{c(x,t)}{t} \ .$$

Taking the limit as t goes to $+\infty$ and applying Lemma 1.15, we get the conclusion. □

We can now deduce Theorem 5.10 from Lemma 5.12.

Proof of Theorem 5.10. Let $[\varphi]$ be a point in $\mathrm{Par}(\mathcal{G})$. By Theorem 2.14, there exists φ^1 in $[\varphi]$ which admits stable horocycles. By Theorem 5.1 there exists an unstable Margulis measure μ^1 for φ^1 with scale factor a_1. We want to set

$$h_{top}([\varphi]) = a_1 \ .$$

To see that this is well-defined, let (φ^2,μ^2) be another such pair, with $\varphi^2 \in [\varphi]$ and μ^2 scaled by φ^2 with scale factor a_2. There also exists $\psi \in [\varphi]$ which admits unstable horocycles, and a stable Margulis measure ν for ψ with scale factor $-b$. Since φ^1, φ^2 and ψ all have the same period map, applying Lemma 5.12 gives

$$\frac{a_1}{b} = \frac{a_2}{b} = 1 \ ,$$

hence $a_1 = a_2 = b$. Thus h_{top} is well-defined.

Similarly, the continuity of h_{top} follows from Lemma 5.12 and the continuity of

$$([\varphi],[\psi]) \mapsto \sup \frac{L_\varphi}{L_\psi} \ .$$

□

Note that the function h_{top} satisfies

$$h_{top}(\varphi^\lambda) = \frac{1}{\lambda} h_{top}(\varphi) \ .$$

Therefore, every flow φ admits a unique scaling of entropy 1. This gives an isomorphism between $\mathbf{P}\mathrm{Par}(\mathcal{G})$ and the hypersurface

$$\mathrm{Par}_1(\mathcal{G}) = \{[\varphi] \in \mathrm{Par}(\mathcal{G}) \mid h_{top}([\varphi]) = 1 \ .$$

Finally, by construction of the maps DF and CF, we have

Proposition 5.13. *The maps* DF *and* CF *take values into the set* $\mathrm{Par}_1(\mathcal{G})$.

5.4. **Surjectivity of the map** DF. In Section **??**, we associated to an Anosov representation of Γ into $\mathrm{Diff}(\mathbb{S}^1)$ a parametrization φ of $\mathcal{G}$ with stable horocycles and a unstable Margulis measure with scale factor 1. Here we explain how to recover a $\mathcal{C}^1$ action from the data of the flow and its Margulis measure.

Let φ be a parametrization of $\mathcal{G}$ with stable horocycles and μ^u an unstable Margulis measure for φ. We want to integrate the projections of the Margulis measure on unstable paths transverse to $\mathcal{G}$. We first need the following lemma:

Lemma 5.14. *Let φ be a parametrization of $\mathcal{G}$ with stable horocycles, let μ^u be an unstable Margulis measure, and c an unstable path transverse to $\mathcal{G}$. Then the projection μ^u_c of μ^u to c has full support and no atom.*

Proof. Let $\mathrm{Supp}(\mu^u)$ denote the union of the supports of μ^u on each leaf. Since μ^u is scaled by φ and holonomy invariant, $\mathrm{Supp}(\mu^u)$ is a union of weakly stable leaves. Moreover, the complement of $\mathrm{Supp}(\mu^u)$ intersects each unstable leaf in an open set. The holonomy invariance thus implies that $\mathrm{Supp}(\mu^u)$ is closed. Therefore $\mathrm{Supp}(\mu^u) = M_\Gamma$ by minimality of the weakly unstable foliation (which follows from instance from the minimality of the action of Γ on $\partial_\infty\Gamma$). We easily deduce that the projections of μ^u have full support.

Assume now that some projections have atoms. Let us first bound the size of these atomes. For this, let U be a relatively compact unstable open domain such that every stable horocycle intersects U, and take $V = \bigcup_{t\geq 0}\varphi_{-t}(U)$. By local finiteness and scaling property of μ^u, we have $\mu^u(V) = A < +\infty$. Therefore, for every $x \in U$,

$$\mu^u(\{\varphi_{-t}(x), t \geq 0\}) \leq A \ .$$

Since μ^u is holonomy invariant and since every stable horocycle intersects U, the same conclusion holds for every $x \in M_\Gamma$.

Applying this to $\varphi_s(x)$ with large s, one gets

$$\mu^u(\{\varphi_{s-t}(x), t \geq 0\}) = e^s \mu^u(\{\varphi_{-t}(x), t \geq 0\}) \leq A \ ,$$

which implies that $\mu^u(\{\varphi_{-t}(x), t \geq 0\})$ has no atom. Thus, the projections of μ^u have no atom. □

Let us now use these projections to form a $\mathcal{C}^1$ atlas on $\partial_\infty\Gamma$. Let $I = [q_1, q_2]$ be an interval in Γ, p a point in $\partial_\infty\Gamma$ that does not belong to I, an c a continuous map from I to the unstable leaf $\widetilde{\mathcal{W}}^u(p)$ such that $c(q)$ belongs to the geodesic $\mathcal{G}(p,q)$ for all $q \in [q_1, q_2]$. Define

$$\begin{array}{rccl} h_{p,c} : & I & \to & \mathbb{R} \\ & q & \mapsto & \mu^u_c(I)(c([q_1, q])) \ . \end{array}$$

By Lemma 5.14, $h_{p,c}$ is a homeomorphism from I to $[0, h_{p,c}(q_2)]$.

Proposition 5.15. *Let $h_{p,c}$ and $h_{p',c'}$ be two homeomorphisms constructed as above. Then $h_{p',c'} \circ h_{p,c}^{-1}$ is a $\mathcal{C}^1$ diffeomorphism.*

Proof. Let (p', c') be another choice of (p, c). The holonomy invariance of μ^u implies that

$$h_{p',c'} = h_{p,c''} \ ,$$

where $c'' = T_{p',p} \circ c$. Now, $c(q)$ and $c''(q)$ belong to the same geodesic, so we can write

$$c''(q) = \varphi_{t(q)}(c(q))$$

for some continuous function t. The scaling property of μ^u gives

$$h_{p,c''}(q) = \int_0^{h_{p,c}(q)} e^{t(h_{p,c}^{-1}(s)} \mathrm{d}s \ .$$

(See the proof of Proposition 5.11.)

It follows that $h_{p',c'} \circ h_{p,c}^{-1} = h_{p,c''} \circ h_{p,c}^{-1}$ is a $\mathcal{C}^1$ diffeomorphism. □

By Proposition 5.15, the family of charts $(I, h_{p,c})$ define a $\mathcal{C}^1$ atlas on $\partial_\infty\Gamma$. Since this family is globally Γ-invariant, the associated $\mathcal{C}^1$ structure on $\partial_\infty\Gamma$ is Γ-invariant.

Let h be a homeomorphism from $\partial_\infty\Gamma$ to $\mathbb{S}^1$ which is a diffeomorphism in the local charts $h_{p,c}$. Then h conjugates the action of Γ on $\partial_\infty\Gamma$ to a $\mathcal{C}^1$ action ρ on $\mathbb{S}^1$.

Proposition 5.16. *We have*

$$L_\rho = h_{top}(\varphi) L_\varphi \ .$$

Proof. Exercise. □

Corollary 5.17. *The map* $\mathrm{DF} : \mathfrak{X}_{an}(\Gamma, \mathrm{Diff}(\mathbb{S}^1)) \to \mathrm{Par}_1(\mathcal{G})$ *is surjective.*

Proof. Let $[\varphi]$ be a point in $\mathrm{Par}_1(\mathcal{G})$. By Theorem 2.14, there exists a flow φ in $[\varphi]$ which admits stable horocycles. By Theorem 5.1, the exists an unstable Margulis measure μ^u for φ with scale factor 1. Let $\rho : \Gamma \to \mathrm{Diff}(\mathbb{S}^1)$ be constructed as above. Then ρ is topologically conjugate to the action of Γ on $\partial_\infty\Gamma$. The equality $L_\rho = L_\varphi$ implies that ρ is Anosov by Proposition 1.21, and gives

$$\mathrm{DF}([\rho]) = [\varphi] \ .$$

□

Remark 5.18. Though we did not explicitly proved it, the reader can convince himself that the construction of ρ from (φ, μ^u) is inverse of the construction of (φ, μ^u) from ρ. There is thus a bijection $\widehat{\mathrm{DF}}$ between the set of conjugacy classes of Anosov representations into $\mathrm{Diff}(\mathbb{S}^1)$ and the set of pairs (φ, μ^u) up to conjugacy. This bijection factors to the map DF when forgetting the second coordinate and passing to Hausdorff quotients.

The question of whether DF is injective is thus deeply related to the question of whether the instable Margulis measure of a flow is unique. We prove uniqueness Hölder parametrizations in the next section.

6. Constructing inverses of DF and CF

In this section, we construct inverses of DF and CF in restriction to the set of Hölder parametrizations of entropy 1. The Hölder assumption is necessary to guaranty that the flow considered admits both stable and unstable horocycles.

6.1. **Anosov parametrizations.** Let us first gather the various results presented above and specialize them to the case of *Anosov parametrization.*

Definition 6.1. A parametrization of the geodesic foliation is *Anosov* if it admits both stable and unstable horocycles.

This is verified for instance by the geodesic flow φ^0 of a hyperbolic metric.

Proposition 6.2. *Let c be a Hölder cocycle along φ^0. Then c is both stably and unstably Buseman.*

Proof. Up to a coboundary, we can assume c is the integral cocycle associated to a Hölder function f. If x and y belong to the same stable leaf, then $\varphi_t^0(x)$ and $\varphi_(^t y)$ get exponentially close for $t \to +\infty$, and so do $f(\varphi_t^0(x))$ and $f(\varphi_t^0(y))$ since f is Hölder. We deduce that

$$\int_0^t f(\varphi_s^0(x)) - f(\varphi_s^0(y))\mathrm{d}s$$

converges as s goes to $+\infty$. One proves with a little extra care that the convergence is uniform on every compact. Thus c_f is stably Buseman. The same argument with x and y in the same unstable leaf and $t \to -\infty$ shows that c_f is also unstably Buseman. □

Now, if φ is a Hölder parametrization of $\mathcal{G}$, then the reparametrization cocycle of φ with respect to φ^0 is Hölder. One thus obtains the following (well-known) corollary:

Corollary 6.3. *Every Hölder parametrization of $\mathcal{G}$ is Anosov.*

A variant of Livsic theorem asserts that Anosov parametrizations are characterized up to conjugacy by their periods.

Theorem 6.4 (Livšic)**.** *Let φ and ψ be two Anosov parametrizations of $\mathcal{G}$. If $L_\psi \equiv L_\psi$, then φ and ψ are conjugate.*

Proof. Exercise. □

Let us denote by $\mathrm{Par}^{an}(\mathcal{G})$ the of Anosov parametrizations of $\mathcal{G}$. It is a dense convex subcone of $\mathrm{Par}(\mathcal{G})$ which contains $\mathrm{Par}^h(\mathcal{G})$. We denote $\mathrm{Par}_1^{an}(\mathcal{G})$ its intersection with $\mathrm{Par}_1(\mathcal{G})$ and by $\mathbf{P}\mathrm{Par}^{an}(\mathcal{G})$ its projection to $\mathbf{P}\mathrm{Par}(\mathcal{G})$.

Let φ be an Anosov parametrization of the geodesic foliation. Let $\mathcal{H}^s$ and $\mathcal{H}^u$ denote respectively the stable and unstable horocyclic foliations of φ. Let μ^s and μ^u denote respectively some stable and unstable measures scaled by φ and invariant under horocyclic holonomy and denote by ν^s and ν^u their respective projections to $\mathcal{H}^s$ and $\mathcal{H}^u$. In the next sections, we explain how to recover from these data an Anosov action on the circle and a foliated affine action.

6.2. **Inverses of** DF **and** CF**.** This inverse has essentially been constructed in Section 5.4 if we are given an unstable Margulis measure. Similarly, we construct the inverse of CF via projections of a stable Margulis measure.

Let φ be an Anosov parametrization of $\mathcal{G}$ with $h_{top}(\mu) = 1$, and let μ^s be a stable Margulis measure. recall that $\mu^s_{\mathcal{H}}$ denotes its projection onto the horocyclic foliation of φ.

Proposition 6.5. *There exists a stable horocyclic flow (h_s) for φ such that*

$$\mu^s_{\mathcal{H}}([x, h_s(x)]) = s$$

for all $s \geq 0$.

Proof. Given $x \in M_\Gamma$, let f_x be the primitive of μ^s on $\mathcal{H}^s(x)$ vanishing at x. By Proposition **??**, the measure μ^s on $\mathcal{H}^s(x)$ has full support and is atome free, by compactness of M_Γ, one can fine some uniform $\varepsilon > 0$ such that the total mass $\mu^s(\mathcal{H}^s(x))$ is at least ε. Applying this to $\varphi_t(x)$, one obtains that

$$\mu^s(\mathcal{H}^s(x)) = e^t \mu^s(\mathcal{H}^s(\varphi_t(x))) \geq e^t \varepsilon \ .$$

Thus $\mu^s(\mathcal{H}^s(x))$ is infinite and f_x is a global homeomorphism.

We can now set $h_s(x) = f_x^{-1}(s)$. It is clear that h_s is a flow. The holonomy invariance of μ^s give the continuity of h_s when moving in directions transverse to the stable leaves, and the scaling property gives the relation

$$\varphi_t \circ h_s \circ \varphi_{-t} = h_{e^{-t}s} \ .$$

□

To conclude the construction of the inverses of DF and CF we need the following:

Proposition 6.6. *Let φ be parametrization of the geodesic foliation with stable (resp. unstable) horocycles. Assume there exist (h_s) and (h'_s) two stable (resp. unstable) horocyclic flows. Then there is a constant $\lambda \neq 0$ such that*

$$h'_s = h_{\lambda s} \ .$$

Proof. Exercise. □

Corollary 6.7. *Let φ be an Anosov parametrization of $\mathcal{G}$. Then the stable and unstable Margulis measures of φ are unique up to a multiplicative constant.*

Proof. Let μ^s_1 and μ^2_s be two stable Margulis measures. By Proposition **??**, the projections of μ^s_1 and μ^s_2 on $\mathcal{H}^s$ differ by a multiplicative constant. Hense so do μ^s_1 and μ^s_2. □

We can now finally define a map

$$\mathrm{DF}^{-1} : \mathrm{Par}_1^{an}(\mathcal{G}) \to \mathfrak{X}_{an}(\Gamma, \mathrm{Diff}(\mathbb{S}^1))$$

in the following way: for $[\varphi] \in \mathrm{Par}_1^{an}(\mathcal{G})$, let $\varphi \in [\varphi]$ be the parametrization of $\mathcal{G}$ which admits both stable and unstable horocycles (φ is unique by Livsic's theorem). Let μ^u be the stable Margulis measure of φ (which is unique by Corollary 6.7) and define $\mathrm{DF}^{-1}([\varphi])$ to be the class of the Anosov $\mathcal{C}^1$ action associated to (φ, μ^u) in Section 5.4. It follows from the results of Section 5.4 that

$$\mathrm{DF} \circ \mathrm{DF}^{-1} = \mathrm{Id}_{\mathrm{Par}_1^{an}(\mathcal{G})} \ .$$

Similarly, we can construct a map

$$\mathrm{CF}^{-1} : \mathrm{Par}_1^{an}(\mathcal{G}) \to \mathcal{T}(\mathcal{W}^s)$$

in the following way: for $[\varphi] \in \mathrm{Par}_1^{an}(\mathcal{G})$, let $\varphi \in [\varphi]$ be the parametrization of $\mathcal{G}$ which admits both stable and unstable horocycles (φ is unique by Livsic's theorem). Let (h_s) be the horocycle flow of φ constructed in Proposition 6.5. By Proposition 6.6, the flow (h_s) is well-defined up to a scaling. Note however that φ_a centralizes φ and conjugates h_s to $h_{e^{-as}}$. Thus the foliated affine action (φ, h) is well-defined up to conjugation, and thus defines a point $\mathrm{CF}^{-1}([\varphi]) \in \mathcal{T}(\mathcal{W}^s)$ by the results of Section 4, which satisfies

$$\mathrm{CF} \circ \mathrm{CF}^{-1}([\varphi]) = [\varphi] \ .$$

6.3. **Hölder regularity.** Recall that, by Corollary 6.3, $\mathrm{Par}_1^h(\mathcal{G})$ is contained in $\mathrm{Par}_1^{an}(\mathcal{G})$. To conclude the proof of Theorems 0.4 and 0.3, we just need the following Proposition:

Proposition 6.8. *The maps* DF^{-1} *and* CF^{-1} *map* $\mathrm{Par}_1^h(\mathcal{G})$ *respectively to* $\mathfrak{X}_{an}(\Gamma, \mathrm{Diff}^h(\mathbb{S}^1))$ *and to* $\mathcal{T}^h(\mathcal{W}^s)$.

Proof. Exercise. □

References

[1] Andrés Navas. *Groups of circle diffeomorphisms*. University of Chicago Press, 2011.

[2] Jean-Pierre Otal. Le spectre marqué des longueurs des surfaces à courbure négative. *Annals of Mathematics*, 131(1):151–162, 1990.

[3] Dennis Sullivan. Linking the universalities of Milnor–Thurston, Feigenbaum and Ahlfors–Bers. *Topological methods in modern mathematics*, pages 543–564, 1993.

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