TEICHMÜLLER GEOMETRY IN THE HIGHEST TEICHMÜLLER SPACE

NICOLAS THOLOZAN

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INTRODUCTION

The goal of these notes is to explain how the space of reparametrizations of the geodesic flow of a hyperbolic surface can be seen both as the Teichmüller space of the weakly unstable foliation and as a subset of the character variety of the fundamental group of the surface into the group of diffeomorphisms of the circle. This work stemmed out of many discussions Bertrand Deroin, who I warmly thank here.

0.1. Higher Teichmüller theory. Our main motivation is to develop a framework in which we hope to study higher Teichmüller theory with the tools of classical Teichmüller theory. Before we get more precise, let us briefly recall what these two terms cover.

Classical Teichmüller theory. Let \( \Sigma \) be a closed oriented surface of genus \( g \geq 2 \). The Teichmüller space \( T(\Sigma) \) of the surface \( \Sigma \) is the space of complex structures on \( \Sigma \) up to isotopy. It carries a properly discontinuous action of the mapping class group of \( \Sigma \), and the quotient under this action is the moduli space of Riemann surfaces of genus \( g \).

Teichmüller theory in a broad sense refers to the study of the geometry of the Teichmüller space, which can be traced back to the XIXth century and continued throughout the XXth century, with the works of Ahlfors, Bers, or Wolpert among many others. In a more restrictive sense it could refer to the work of the Nazi mathematician Oswald Teichmüller, who proved that \( T(\Sigma) \) is homeomorphic to \( \mathbb{R}^{6g-6} \) and constructed its complex structure by studying optimal quasi-conformal maps between Riemann surfaces.

We know since Gauss that a complex structure on \( \Sigma \) is equivalent to a conformal class of Riemannian metrics, and since Poincaré that every such conformal class contains a unique metric of curvature \(-1\). Thus the space \( T(\Sigma) \) is canonically identified with the Fricke space \( \mathcal{F}(\Sigma) \) of hyperbolic metrics on \( \Sigma \) up to isotopy.

Now, a hyperbolic metric on \( \Sigma \) gives an isometry from the universal cover \( \tilde{\Sigma} \) to the Poincaré half plane \( \mathbb{H}^2 \) which is equivariant with respect to some representation of the fundamental group \( \Gamma \) of \( \Sigma \) which is Fuchsian (i.e. discrete and faithful). This gives an identification between \( \mathcal{F}(\Sigma) \) and the space \( \mathcal{X}_{\text{fuchs}}(\Gamma, \text{PSL}(2, \mathbb{R})) \) of Fuchsian representations up to conjugation.

What makes Teichmüller theory so rich is that the three avatars of the Teichmüller space: \( T(\Sigma) \), \( \mathcal{F}(\Sigma) \) and \( \mathcal{X}_{\text{fuchs}}(\Gamma, \text{PSL}(2, \mathbb{R})) \) carry different geometric structures that “miraculously” combine together. A striking example is the fact that Goldman’s symplectic form on \( \mathcal{X}_{\text{fuchs}}(\Gamma, \text{PSL}(2, \mathbb{R})) \) combines with the complex structure of \( T(\Sigma) \) to form the Weil–Petersson metric, which lives naturally on \( \mathcal{F}(\Sigma) \).

Let us point out that some aspects of Teichmüller theory have been generalized to several infinite dimensional contexts, such as Teichmüller theory of surface of infinite type, Teichmüller theory of foliations and universal Teichmüller theory.

Higher Teichmüller theory. In the 90’s, Hitchin discovered that the space \( \mathcal{X}(\Gamma, \text{PSL}(n, \mathbb{R})) \) of representations of \( \Gamma \) into \( \text{PSL}(n, \mathbb{R}) \) had a distinguished connected component, that we will denote \( \mathcal{X}_{\text{bud}}(\Gamma, \text{PSL}(n, \mathbb{R})) \) which coincided with the space \( \mathcal{X}_{\text{fuchs}}(\Gamma, \text{PSL}(2, \mathbb{R})) \) for \( n = 2 \), and which beared some
resemblance with a “higher rank” analog of the Teichmüller space, in particular, it contains a Fuchsian locus isomorphic to $\mathcal{T}(\Sigma)$. This analogy was strengthened by Labourie, who proved that representations in Hitchin’ component have a very powerful dynamical property that he called Anosov property, showing in particular that they are all discrete and faithful. Since then other character varieties $\mathfrak{X}(\Gamma, G)$ have been shown to contain connected components consisting only of Anosov representations. This popularized the term Higher Teichmüller theory to refer, depending on the context, to the study of Hitchin or related components of character varieties, of more generally o the study of Anosov representations of surface groups.

Several authors have been working on extending Teichmüller geometry to these higher Teichmüller spaces. An important work in this direction is that of Bridgeman–Canary–Labourie–Sambarino, who used the thermodynamical formalism to construct a Riemannian metric on $\mathfrak{X}_{hit}(\Gamma, \text{PSL}(n, \mathbb{R}))$, which restricts to the Weil–Petersson metric on the Fuchsian locus. Let us briefly sketch how their construction works: they show that one can embed $\mathfrak{X}_{hit}(\Gamma, \text{PSL}(n, \mathbb{R}))$ into the space of Hölder reparametrizations of the geodesic flow of a hyperbolic metric on $\Sigma$. There lives the pressure metric, which is roughly speaking the second fundamental form of the hypersurface formed by reparametrizations of entropy 1.

What has been missing in higher Teichmüller theory is a higher rank analog of the complex structure of $\mathcal{T}(\Sigma)$. An important open question is for instance whether the pressure metric of Bridgeman–Canary–Labourie–Sambarino is Kähler for some suitable complex structure on $\mathfrak{X}_{hit}(\Gamma, \text{PSL}(n, \mathbb{R}))$.

Here we construct a complex structure on the space of Hölder reparametrizations of the geodesic flow, by identifying it with some foliated Teichmüller space. We also develop a representation theoretic point of view on that space, identifying it with the space of “Anosov representations” of $\Gamma$ into the group of diffeomorphisms of the circle. This that many aspects of the classical Teichmüller theory could be generalized to this “highest Teichmüller space” of reparametrizations.

We hope that some of these results will descend to interesting geometric properties of higher Teichmüller spaces after a careful understanding of how these embed into the space of reparametrizations.

Let us advise the reader that the results presented here are not entirely new. For the most part they seem to be rephrasings of results well-known from hyperbolic dynamists. In particular, the general idea that certain moduli spaces of dynamical systems could be identified with Teichmüller spaces of foliations was introduced by Sullivan in []. In [], Cawley studies introduces the Teichmüller space of Anosov automorphisms of the torus and carries a study which is essentially covers what we do here when replacing the geodesic flow of a hyperbolic surface by the suspension flow of a linear automorphism of the torus. It was however very useful to the author writing the many details of these correspondance that he couldn’t find elsewhere, and we hope it will be useful to some readers too.
0.2. **Three avatars of a highest Teichmüller space.** Let us now state our results with more precision. We first introduce three avatars of what our “highest Teichmüller space” will be.

0.2.1. *Parametrizations of the geodesic foliation.* Let $\Sigma$ be a closed hyperbolic surface of genus at least 2. In the sequel, we denote by $\Gamma$ its fundamental group and by $M_\Gamma$ its unit tangent bundle. The geodesic flow of $\Sigma$ is a flow on $M_\Gamma$. It turns out that the topological manifold $M_\Gamma$ and the foliation $\mathcal{G}$ given by the orbits of the geodesic flow are “independent” of the choice of the hyperbolic structure on $\Sigma$. We call $\mathcal{G}$ the *geodesic foliation*.

Closed leaves of the geodesic foliation correspond to closed geodesics on $\Sigma$ and are in one to one correspondence with conjugacy classes of primitive elements in $\Gamma$. We denote by $[\Gamma]$ this set.

A reparametrization of the geodesic flow is a continuous flow on $M_\Gamma$ which is orbit equivalent to the geodesic flow. To avoid as much as possible references to a background hyperbolic metric, we call such flows *parametrizations of the geodesic foliation*.

Two parametrizations of the geodesic foliation are *conjugate* if they are conjugate by a homeomorphism preserving each geodesic leaf. This equivalence relation is not closed on the space of reparametrizations. We call two parametrizations *weakly conjugate* if one is the uniform limit of conjugates of the other. The space of parametrizations of the geodesic foliation modulo weak conjugation is denoted

$$\text{Par}(\mathcal{G})$$

The subset of equivalence classes of parametrizations which are Hölder regular is denoted by

$$\text{Par}^h(\mathcal{G})$$

If $\varphi$ is a parametrization of the geodesic foliation, the *period map* of $\varphi$ is the function

$$L_\varphi : [\Gamma] \to \mathbb{R}_{>0}$$

associating to a closed geodesic $\gamma$ the time $\varphi$ takes to run through $\gamma$.

There is a natural “scaling” action of $\mathbb{R}_{>0}$ on $\text{Par}(\mathcal{G})$. The *topological entropy* $h_{\text{top}}(\varphi)$ of a flow $\varphi$ provides a way to normalize parametrizations. Indeed, the function $h_{\text{top}}$ is well-defined on $\text{Par}(\mathcal{G})$, continuous, positive and homogeneous of degree $-1$ with respect to the scaling action. Thus every parametrization $\varphi$ admits a unique scaling of entropy 1. We denote by

$$\text{Par}_1(\mathcal{G})$$

the space of parametrizations of entropy 1 up to weak equivalence, and by $\text{Par}_1^h(\mathcal{G})$ its intersection with $\text{Par}^h(\mathcal{G})$.

0.2.2. *Anosov actions on the circle.* Recall that the hyperbolic surface $\Sigma$ is isometric to the quotient $j(\Gamma) \backslash \mathbb{D}$ where $\mathbb{D}$ denotes Poincaré’s hyperbolic disc and $j : \Gamma \to \text{Isom}_+ (\mathbb{D})$ is a Fuchsian representation (i.e. discrete and faithful). The representation $j$ provides an analytic action of $\Gamma$ on the unit circle $\mathbb{S}^1 = \partial \mathbb{D}$. This action has maximal Euler class, i.e. the twisted product bundle $\Sigma \times_j \mathbb{S}^1$ is isomorphic to the unit tangent bundle of $\Sigma$. 
Let $\text{Diff}(S^1)$ denote the group of diffeomorphisms of $S^1$ of class $C^1$ and $\text{Diff}^h(S^1)$ the subgroup of diffeomorphisms with Hölder derivatives. We endow $\text{Diff}(S^1)$ with the $C^1$ topology. By a theorem of Matsumoto, a morphism $\rho : \Gamma \to \text{Diff}(S^1)$ has maximal Euler class if and only if it is semi-conjugate to a Fuchsian representation $j$. We call $\rho$ an Anosov representation into $\text{Diff}(S^1)$ or an Anosov action on $S^1$ if it is conjugate to a Fuchsian representation $j$ via a bi-Hölder homeomorphism. We will see in Section ?? that this is equivalent for the action of $\rho$ to be topologically conjugated to $j$ and expanding (see for instance [1] for the relevance of expanding properties in one dimensional dynamics).

Let $\rho$ be an Anosov action on the circle. For every $\gamma \in \Gamma$, the diffeomorphism $\rho(\gamma)$ has a unique attracting fixed point on $S^1$ that we denote abusively $\gamma_+$. The derivative of $\rho(\gamma)$ at its attracting fixed point is less than 1 and is invariant by conjugation of $\gamma$ by a diffeomorphism. We define the period map of $\rho$ as the map

$$L_\rho : [\Gamma] \to \mathbb{R}_{>0}$$

$$[\gamma] \mapsto -\log(\rho(\gamma)'(\gamma_+)).$$

The space $\text{Hom}(\Gamma, \text{Diff}(S^1))$ of morphisms from $\Gamma$ to $\text{Diff}(S^1)$ inherits a topology from that of $\text{Diff}(S^1)$. The group $\text{Diff}(S^1)$ acts continuously on $\text{Hom}(\Gamma, \text{Diff}(S^1))$ by conjugation. This action may not be proper. We denote by $\mathcal{X}(\Gamma, \text{Diff}(S^1))$ the largest Hausdorff quotient of $\text{Hom}(\Gamma, \text{Diff}(S^1))/\text{Diff}(S^1)$. We also denote by $\mathcal{X}_m(\Gamma, \text{Diff}(S^1))$ the open subset of equivalence classes of Anosov representations, and by $\mathcal{X}_m(\Gamma, \text{Diff}^h(S^1))$ the subset of equivalence classes of Anosov representations with values in $\text{Diff}^h(S^1)$.

0.2.3. Teichmüller space of the weakly stable foliation. The geodesic flow of a hyperbolic surface is a well-known example of an Anosov flow. In particular, it has a weakly unstable foliation $\mathcal{W}^u$ of dimension 2, which contains the geodesic foliation. This foliation is “independent” of the choice of a hyperbolic structure.

A foliated Riemannian metric $g$ on $\mathcal{W}^u$ is the data of a scalar product on each tangent space to $\mathcal{W}^u$ which is of class $C^\infty$ along the leaves and varies transversally continuously for the $C^\infty$ topology. We call it transversally Hölder if there exists $\alpha > 0$ such that the metrics on two $\epsilon$-close leaves are $\epsilon^{\alpha}$-close for the $C^\infty$ topology.

Two metrics $g$ and $h$ are conformally equivalent if there is a continuous function $\sigma$ on $M_{\Gamma}$ such that $h = e^\sigma g$. A foliated conformal structure on $\mathcal{W}^u$ is a conformal equivalence class of Riemannian metrics on $\mathcal{W}^u$. It is transversally Hölder if it admits a transversally Hölder representative.

A foliated homotopy $(h_t)_{0 \leq t \leq 1}$ is a continuous family of continuous self maps of $M_{\Gamma}$ preserving the leaves of $\mathcal{W}^u$ and such that $h_0 = \text{Id}$. Two foliated conformal structures $[g_1]$ and $[g_2]$ are homotopic if there exists a foliated homotopy $(h_t)$ such that $h_t^*[g_2] = g_1$.

Let $[g]$ be a conformal structure on $\mathcal{W}^u$, and let $[\gamma]$ be a closed geodesic. The weakly unstable leaf containing $[\gamma]$ is conformally equivalent (for the conformal structure $[g]$) to $l \setminus \mathbb{D}$ for some hyperbolic isometry $l$. The period map of $[g]$ is the map

$$L_{[g]} : [\Gamma] \to \mathbb{R}_{>0}$$
associating to $\gamma$ the translation length of $l$. The period map is homotopy invariant.

The Teichmüller space of the foliation $W^u$ is the space of foliated conformal structures on $W^u$ modulo homotopy. We denote it by $\mathcal{T}(W^u)$. We denote by $\mathcal{T}^h(W^u)$ the subset of homotopy classes of transversally Hölder conformal structures.

Teichmüller spaces of 2-dimensional foliations were introduced by Sullivan in []. There, he proves that those Teichmüller spaces (in particular $\mathcal{T}(W^u)$) have the geometry of a (possibly infinite dimensional) complex manifold biholomorphic to a bounded domain in a complex Banach space.

Note that we could imagine several variations in the definition of a Teichmüller space. For instance, replacing homotopy equivalence by isotopy equivalence, or weakening the leafwise regularity of our conformal structures. In the classical theory, all these definitions are equivalent thanks to the following classical theorems:

**Theorem 0.1.** Any two smooth structures on a topological surface are isotopic.

**Theorem 0.2.** Two homeomorphisms of a surface which are homotopic are isotopic.

We do not know whether the equivalent results exist for 2-dimensional foliations and try to avoid entering into those details here.

0.3. **Main results.** The purpose of these notes is to clarify the relation between the three spaces described above. This is summarized in the following theorems:

**Theorem 0.3.** There exists a continuous map

$$DF : X_{an}(\Gamma, \text{Diff}(S^1)) \to \text{Par}_1(\mathcal{G})$$

such that

$$L_{DF(\rho)} = L_\rho$$

for all $\rho$. This map is a surjective and restricts to a bijection between $X_{an}(\Gamma, \text{Diff}^h(S^1))$ and $\text{Par}_1^h(\mathcal{G})$.

**Theorem 0.4.** There exists a continuous map

$$CF : \mathcal{T}(W^u) \to \text{Par}_1(\mathcal{G})$$

such that

$$L_{CF([g])} = L_{[g]}$$

for all $[g]$. This map restricts to a bijection between $\mathcal{T}^h(W^u)$ and $\text{Par}_1^h(\mathcal{G})$.

Note that the sets $\text{Par}_1^h(\mathcal{G})$ and $\mathcal{T}^h(W^u)$ are respectively dense in $\text{Par}_1(\mathcal{G})$ and $\mathcal{T}(W^u)$. By Theorem 0.4, one can thus see $\text{Par}_1(\mathcal{G})$ and $\mathcal{T}(W^u)$ as two natural completions of the space of Hölder parametrizations. We do not know whether the two completions coincide (i.e. whether $CF$ is a bijection).
1. Cocycles, cohomology and parametrizations

In this section, we gather some classical results about cocycles along a continuous flow and their relation with reparametrizations of that flow. This leads to the description of the space $\text{Par}(\mathcal{G})$ as a convex cone in an infinite dimensional Banach space. This description seems to date back to Bowen.

In order to stick to our objective, we restrict ourselves to reparametrizations of the geodesic flow of a closed negatively curved surface. However, all the results here could work in the very general setting of parametrizations of a 1-dimensional lamination, except for a few results which use the density of closed orbits and hold for any topologically transitive Anosov flow.

1.1. Geodesic, stable and unstable foliation. In all the paper, $\Sigma$ denotes a closed oriented surface of genus at least 2 and $\Gamma$ its fundamental group. Recall that $\Gamma$ is hyperbolic in the sense of Gromov. Its boundary at infinity $\partial\Sigma \Gamma$ is a topological circle with a canonical Hölder structure. Let $\tilde{\mathcal{M}}_\Gamma$ denote the set of cyclically oriented triples of distinct points of $\partial\Sigma \Gamma$. The group $\Gamma$ acts properly discontinuously and cocompactly on $\tilde{\mathcal{M}}_\Gamma$. We denote the quotient by $\mathcal{M}_\Gamma$.

For every $y \neq z \in \partial\Sigma \Gamma$, let us denote by $\tilde{\mathcal{W}}^s(z)$ the set $\{(x_-, x_t, x_+) \in \tilde{\mathcal{M}}_\Gamma \mid x_+ = z\}$, by $\tilde{\mathcal{W}}^u(y)$ the set $\{(x_-, x_t, x_+) \in \tilde{\mathcal{M}}_\Gamma \mid x_- = y\}$ and by $\tilde{\mathcal{G}}(y, z)$ the set $\tilde{\mathcal{W}}^u(y) \cap \tilde{\mathcal{W}}^s(z)$. Note that the cyclic order on $\partial\Sigma \Gamma$ induces an order on $\tilde{\mathcal{G}}(y, z)$ given by

$$(y, t, z) \leq (y, s, z) \iff (y, t, s, z) \text{ are cyclically ordered.}$$

The sets $\tilde{\mathcal{G}}(y, z)$ are the leaves of a one dimensional Hölder foliation $\tilde{\mathcal{G}}$ of $\tilde{\mathcal{M}}_\Gamma$ which is preserved by the action of $\Gamma$ and thus induces a Hölder foliation $\mathcal{G}$ of $\mathcal{M}_\Gamma$ that we call the geodesic foliation. Similarly, the sets $\tilde{\mathcal{W}}^s(z)$ and $\tilde{\mathcal{W}}^u(y)$ respectively induce Hölder foliations of $\mathcal{M}_\Gamma$ of dimension 2 called the weakly stable and weakly unstable foliations and denoted $\mathcal{W}^s$ and $\mathcal{W}^u$.

**Remark 1.1.** If $x$ is a point in $\mathcal{M}_\Gamma$, we will denote by $\mathcal{G}(x)$, $\mathcal{W}^s(x)$ and $\mathcal{W}^u(x)$ the geodesic, stable and unstable leaves passing through $x$.

We can now define a (continuous) parametrization of the geodesic foliation to be a continuous flow on $\mathcal{M}_\Gamma$ whose orbits are the leaves of the geodesic foliation. More precisely:

**Definition 1.2.** A continuous parametrization of the geodesic foliation $\mathcal{G}$ on $\mathcal{M}_\Gamma$ is a continuous continuous flow $(\varphi_t)_{t \in \mathbb{R}}$ on $\mathcal{M}_\Gamma$ whose orbits are the leaves of the foliation $\mathcal{G}$ and which respects the orientation of $\mathcal{G}$.

The parametrization $(\varphi_t)$ is Hölder if each $\varphi_t$ is bi-Hölder the map $(t, x) \rightarrow (t, \varphi_t(x))$ is a Hölder homeomorphism of $\mathcal{M}_\Gamma \times \mathbb{R}$.

**Remark 1.3.** A parametrization $\varphi$ lifts to a $\Gamma$-equivariant flow on $\tilde{\mathcal{M}}_\Gamma$ that we still denote $\varphi$. More generally, any object on $\mathcal{M}_\Gamma$ which lifts naturally to $\tilde{\mathcal{M}}_\Gamma$ will called the same when lifted. This will avoid some unnecessarily heavy notations.
Every non-trivial element $\gamma \in \Gamma$ has a unique attracting fixed point $\gamma_+ \in \partial_{\infty} \Gamma$ and a unique repelling repelling fixed point $\gamma_- \in \partial_{\infty} \Gamma \setminus \{\gamma_+\}$.

Let $x \in \partial_{\infty} \Gamma$ be a point on the geodesic $G(\gamma_-, \gamma_+)$. We define $L_{\varphi}(\gamma)$ as the positive number $t$ such that $(\gamma_-, \gamma \cdot x, \gamma_+) = \varphi_t(\gamma_-, x, \gamma_+)$. One easily verifies that $L_{\tilde{\varphi}}(\gamma)$ does not depend on the choice of $x$, is invariant by conjugation of $\gamma$ and verifies $L_{\varphi}(\gamma^n) = nL_{\varphi}(\gamma)$.

**Definition 1.4.** Let $\varphi$ be a continuous parametrization of the geodesic foliation. The map $L_{\varphi} : [\Gamma] \to \mathbb{R}_{>0}$

is called the **period map** of $\varphi$.

**Definition 1.5.** Two continuous parametrizations $(\varphi_t)$ and $(\psi_t)$ of the geodesic foliation are called **conjugate** if there exists a continuous function $f : M_{\Gamma} \to \mathbb{R}$ such that the flows $(\varphi_t)$ and $(\psi_t)$ are conjugated by the homeomorphism $h : x \mapsto \varphi_{f(x)}(x)$.

One can provide the space of all parametrizations of the geodesic foliation with the topology of uniform convergence on compact sets of $M_{\Gamma} \times \mathbb{R}$. Conjugation of flows defines an equivalence relation on this space. We define $\text{Par}(\mathcal{G})$ as the quotient of the space of all parametrizations by the closure of this equivalence relation. The main purpose of this section is to describe the geometry of $\text{Par}(\mathcal{G})$.

1.1.1. **Main example: geodesic flow of a negatively curved surface.** Assume $\Sigma$ is endowed with a Riemannian metric of negative curvature. Then there is a well-known identification of $M_{\Gamma}$ with $T_1 \Sigma$, the unit tangent bundle to $\Sigma$, through which the geodesic flow is a parametrization of the geodesic foliation. The geodesic flow is Anosov and the foliations $\mathcal{W}^s$ and $\mathcal{W}^u$ are precisely the weakly stable and unstable foliations of this Anosov. Though different negatively curved metrics give rise to different flows [2], these flows are orbit equivalent and have the same weakly stable and unstable foliations.

Instead of considering parametrizations of the geodesic foliation, we could have fixed a negatively curved metric on $\Sigma$ and considered **reparametrizations** of its geodesic flow (this approach is more common in the literature). However, for our purpose, we prefer to emphasize that we have no privileged choice of a negatively curved metric or parametrization of the geodesic foliation.

1.2. **Cocycles.** Let us fix a continuous parametrization $\varphi$ of the geodesic foliation. We recall here a few classical facts about continuous cocycles along $\varphi$.

**Definition 1.6.** A continuous cocycle along $\varphi$ is a continuous function $c : M_{\Gamma} \times \mathbb{R} \to \mathbb{R}$

such that $c(x, t + s) = c(x, t) + c(\varphi_t x), s)$.
for all \( x \in M_\Gamma \) and all \( t, s \in \mathbb{R} \).

The following examples relate cocycles to continuous functions on one side and to reparametrizations of \( \varphi \) on the other side.

**Example 1.7.** Let \( f : M_\Gamma \to \mathbb{R} \) be a continuous function. Then the function

\[
c_f : M_\Gamma \times \mathbb{R} \to \mathbb{R}
\begin{align*}
(x, t) & \mapsto \int_0^t f(\varphi(u, x))du
\end{align*}
\]

is a cocycle. We call such a cocycle an *integral cocycle*.

**Example 1.8.** Let \( \psi \) be another parametrization of the geodesic foliation. One can associate to the pair \( (\varphi, \psi) \) the cocycle \( c_{\varphi \to \psi} \) along \( \varphi \) defined by

\[
\psi_{c_{\varphi \to \psi}}(x, t)(x) = \varphi_t(x) .
\]

In other words, \( c_{\varphi \to \psi}(x, t) \) is the time taken by the flow \( \psi \) to move from \( x \) to \( \varphi_t(x) \). We call such a cocycle a *reparametrization cocycle*. When the base flow \( \varphi \) is fixed once and for all, we simply denote this cocycle by \( c_\psi \).

The space \( Z(\varphi) \) of cocycles along \( \varphi \) is a Banach space for the norm

\[
\|c\|_\infty = \sup_{0 \leq t \leq 1} \sup_{x \in M_\Gamma} |c(x, t)| .
\]

**Definition 1.9.** A *Livšic coboundary* is a cocycle \( c \) for which there exists a continuous function \( F \) such that

\[
c(x, t) = F(\varphi_t(x)) - F(x)
\]

for all \( x \in M_\Gamma \) and \( t \in \mathbb{R} \). Two cocycles \( c_1 \) and \( c_2 \) are called *Livšic cohomologous* if \( c_1 - c_2 \) is a Livšic coboundary.

The following propositions reduce the study of cohomology classes of cocycles to that of integral cocycles.

**Proposition 1.10.** Let \( c_f \) be the integral cocycle associated to a continuous function \( f \). Then \( c_f \) is a Livšic coboundary if and only if \( f \) is a derivative along \( \varphi \), i.e. there exists a continuous function \( F : M_\Gamma \to \mathbb{R} \) such that

\[
\frac{1}{\varepsilon}(F(\varphi_\varepsilon(x)) - F(x))
\]

converges uniformly to \( f \).

**Proposition 1.11.** Let \( c \) be a cocycle along \( \varphi \) and let \( T \) be a positive number. Then \( c \) is Livšic cohomologous to the integral cocycle \( c_T \) associated to the function \( x \mapsto \frac{1}{T}c(x, T) \). In particular, every cocycle is Livšic cohomologous to an integral cocycle.

**Proof.** Let \( c \) be a cocycle along \( \varphi \). Define

\[
F : x \mapsto \int_0^T c(x, u)du .
\]
For all \( x \in M_\Gamma \) and all \( t \in \mathbb{R} \), we have
\[
\int_0^t c(\phi_u(x), T) \, dt = \int_0^t c(x, u + T) - c(x, u) \, du \\
= \int_t^{t+T} c(x, u) \, du - \int_0^T c(x, u) \, du \\
= \int_0^T c(x, t + u) \, du - \int_0^T c(x, u) \, du \\
= \int_0^T c(x, t) + c(\phi_t(x), u) \, du - \int_0^T c(x, u) \, du \\
= Tc(x, t) + F(\phi_t(x)) - F(x) .
\]

The integral cocycle associated to the function \( x \mapsto c(x, T) \) is thus cohomologous to \( Tc \).

The space of Livsic coboundaries is not closed in \( Z(\phi) \). We call a uniform limit of Livsic coboundaries a weak coboundary and we say that two cocycles are weakly cohomologous if they differ by a weak coboundary. We denote by \( B(\phi) \) the space of weak coboundaries and by \( H^1(\phi) \) the quotient \( Z(\phi)/B(\phi) \) with the Banach norm
\[
\| [c] \| \overset{\text{def}}{=} \inf_{c' \sim c} \| c' \|_\infty .
\]

Let \( Z'(\phi) \) be the space of continuous functions on \( M_\Gamma \) with the supremum norm and \( B'(\phi) \subset Z'(\phi) \) the closure of the subspace of functions which are derivatives along \( \phi \). Propositions 1.10 and 1.11 imply the following

**Corollary 1.12.** The space \( H^1(\phi) \) is isometric to the quotient \( Z'(\phi)/B'(\phi) \) with the quotient norm
\[
\| [f] \| = \inf_{g \in B'(\phi)} \| f - g \|_\infty .
\]

**Proof.** Consider the linear maps
\[
A : \ Z'(\phi) \to Z(\phi) \\
\quad f \mapsto c_f
\]
and
\[
B : \ Z(\phi) \to Z'(\phi) \\
\quad c \mapsto c(\cdot, 1) .
\]

\( A \) and \( B \) are both continuous of operator norm at most 1, and are inverses up to coboundaries by Propositions 1.10 and 1.11. They thus factor to isometries between \( Z'(\phi)/B'(\phi) \) and \( Z(\phi)/B(\phi) \).

1.3. **The dual space of invariant measures.** Recall that the dual of the space \( Z'(\phi) \) is the space of signed Borel measures on \( M_\Gamma \), i.e. linear forms of the form
\[
\mu : f \mapsto \int_{M_\Gamma} f \, d\mu^+ - \int_{M_\Gamma} f \, d\mu^-
\]
where \( \mu^+ \) and \( \mu^- \) are finite Borel measures on \( M_\Gamma \). It contains the closed subspace of signed Borel measures invariant by the flow \( \phi \), which are characterized by the following proposition:
Proposition 1.13. A signed Borel measure \( \mu \) on \( M_\Gamma \) is \( \varphi \)-invariant if and only if
\[
\mu(f) = 0
\]
for all \( f \in B'(\varphi) \).

The dual statement readily follows from the Hahn–Banach theorem:

Corollary 1.14. Let \( f \) be a continuous function on \( M_\Gamma \). Then \( f \) belongs to \( B'(\varphi) \) if and only if \( \mu(f) = 0 \) for all invariant finite Borel measure \( \mu \).

It follows that the space of invariant signed Borel measures on \( M_\Gamma \) is isomorphic to the dual of \( H^1(\varphi) \). This also gives a characterization of the quotient norm on \( H^1(\varphi) \). For every continuous function \( f \) and every \( T > 0 \), define
\[
I_T f : x \mapsto \frac{1}{T} \int_0^T f(\varphi_t(x)) dt = \frac{1}{T} c_f(x, T).
\]

Note that \( I_T f \) is Livsic cohomologous to \( f \) by Proposition 1.11. Let \( M^1(\varphi) \) denote the space of \( \varphi \)-invariant probability measures on \( M_\Gamma \).

Lemma 1.15. Let \( f \) be a continuous function on \( M_\Gamma \). Then the following equalities hold:
\[
\inf_{g \in B'(\varphi)} \sup_{M_\Gamma} (f - g) = \sup_{\mu \in M^1(\varphi)} \int_{M_\Gamma} f d\mu = \lim_{T \to +\infty} \sup_{M_\Gamma} I_T f .
\]

Corollary 1.16. Let \( c \) be a continuous cocycle. Then we have the equalities
\[
\| [c] \| = \sup_{\mu \in M^1(\varphi)} |\mu(c)| = \lim_{T \to +\infty} \frac{1}{T} \| c(\cdot, T) \|_\infty ,
\]
where \( \mu(c) = \int f d\mu \) for any function \( f \) such that \( c \) is cohomologous to \( c_f \).

Proof of Lemma 1.15. Note first that the function
\[
T \mapsto \sup_{x \in M_\Gamma} \int_0^T f(\varphi_t(x)) dx
\]
is subadditive. Thus \( \sup_{M_\Gamma} I_T f \) converges as \( T \) goes to \(+\infty\).

The inequality
\[
\sup_{\mu \in M^1(\varphi)} \int_{M_\Gamma} f d\mu \leq \inf_{g \in B'(\varphi)} \sup_{M_\Gamma} (f - g)
\]
follows for the inequality \( \int_{M_\Gamma} f d\mu \leq \sup_{M_\Gamma} f \) and the fact that \( \int_{M_\Gamma} g d\mu = 0 \) for \( g \in B'(\varphi) \).

The inequality
\[
\inf_{g \in B'(\varphi)} \sup_{M_\Gamma} (f - g) \leq \lim_{T \to +\infty} \sup_{M_\Gamma} I_T f
\]
follows from the fact that \( I_T f \) is cohomologous to \( f \) for all \( T \).

Finally, let \( x_T \) be a point where \( I_T f \) achieves its supremum and consider the probability measure \( \mu_T \) defined by
\[
\int_{M_\Gamma} g d\mu_T = \frac{1}{T} \int_0^T g(\varphi_t(x_T)) dt .
\]
Let $\nu$ be an accumulation point of $\mu_T$ for the vague topology. Then $\nu$ is $\varphi$-invariant and we have

$$\int_{M_T} f d\nu = \lim_{T \to +\infty} I_T f(x_T) = \lim_{T \to +\infty} \sup_{M_T} I_T f .$$

We conclude that

$$\lim_{T \to +\infty} \sup_{M_T} I_T f \leq \sup_{\mu \in M^1(\varphi)} \int f d\mu .$$

\[\square\]

**Proof of Corollary 1.16.** By Proposition 1.11, we can assume without loss of generality that $c$ is the integral cocycle associated to a continuous function $c_f$.

Set $N(f) = \inf_{g \in B'} \sup_{M_T} (f - g)$. By Lemma 1.15, we have

$$\max(N(f), N(-f)) = \sup_{\mu \in M^1(\varphi)} |\mu(c)| = \lim_{T \to +\infty} \frac{1}{T} \|c(\cdot, T)\|_\infty .$$

On one side, we have

$$\max(N(f), N(-f)) \leq \inf_{g \in B'} \|f - g\|_\infty = ||[c]|| .$$

On the other side, we have

$$\lim_{T \to +\infty} \frac{1}{T} \|c(\cdot, T)\|_\infty \geq ||[c]||$$

since $\frac{1}{T} c(\cdot, T)$ is cohomologous to $f$. Hence $||[c]|| = \max(N(f), N(-f))$. \[\square\]

1.4. **Positive cocycles and reparametrizations.** We still fix a parametrization $\varphi$ of the geodesic foliation. Let $\psi$ be another parametrization. Recall that the **reparametrization cocycle** $c_\psi$ along $\varphi$ is defined by the relation

$$\psi_{c_\psi \psi(\cdot, t)}(x) = \varphi_t(x) .$$

Such a cocycle is **positive**, i.e. $c_\psi(\varphi(x), t) > 0$ for $t > 0$. Conversely, if $c$ is a positive cocycle, then the relation (1) defines a unique parametrization $\psi$ such that $c_{\varphi \rightarrow \psi} = c$.

**Proposition 1.17.** Two parametrizations $\psi_1$ and $\psi_2$ are conjugate if and only if their reparametrization cocycles are Livsic cohomologous.

**Proof.** Let $h$ be a geodesic preserving homeomorphism such that

$$h(\psi_1(t, x)) = \psi_2(t, h(x)) .$$

Define $F(x)$ as the time $s$ such that

$$\psi_2(s, x) = h(x) .$$

We then have

$$\psi_2(c_{\psi_2}(x, t) + F(\varphi(t, x)), x) = \psi_2(F(\varphi(t, x), \varphi(t, x)) \quad \text{by definition of } c_{\psi_2}$$

$$= h(\varphi(t, x)) \quad \text{by definition of } F$$

$$= h(\psi_1(c_{\psi_1}(x, t), x) \quad \text{by definition of } c_{\psi_1}$$

$$= \psi_2(c_{\psi_1}(x, t), h(x))$$

$$= \psi_2(c_{\psi_1}(x, t) + F(x), x) .$$
Therefore,

\[ c_{\psi_1}(x, t) - c_{\psi_2}(x, t) = F(\varphi(x, t)) - F(x) . \]

Conversely, assume there exists a continuous function \( F \) such that

\[ c_{\psi_1}(x, t) - c_{\psi_2}(x, t) = F(\varphi(x, t)) - F(x) . \]

Set \( h(x) = \psi_2(F(x), x) \). Given \( x \in M_\Gamma \) and \( s \in \mathbb{R} \), let \( t \) be such that \( c_{\psi_1}(x, t) = s \). We then have

\[
\psi_2(s, h(x)) = \psi_2(s + F(x), x) = \psi_2(c_{\psi_1}(x, t) + F(x), x) = \psi_2(F(\varphi_1(x)), \varphi_1(x)) \text{ by definition of } F \\
= h(\varphi_1(x)) \text{ by definition of } h = h(\psi_1(s, x)) .
\]

Symmetrically, the map \( g : x \mapsto \psi_1(-F(x), x) \) satisfies

\[ \psi_1 \circ g = g \circ \psi_2 \].

One can verify that \( h \) and \( g \) are inverses of each other. They thus provide the required conjugation. \( \Box \)

**Definition 1.18.** We call two parametrizations \( \psi_1 \) and \( \psi_2 \) weakly conjugate if \( \psi_2 \) is a uniform limit of conjugates of \( \psi_1 \). (Here and elsewhere, uniform limit means limit for the compact-open topology.)

One easily verifies that the map \( \psi \mapsto c_\psi \) is a homeomorphism from the space of parametrizations of the geodesic foliation (with the topology of uniform convergence on compact sets) to the space of positive cocycles. Thus Proposition 1.17 implies the following corollary:

**Corollary 1.19.** Two parametrizations \( \psi_1 \) and \( \psi_2 \) are weakly conjugate if and only if their reparametrization cocycles are weakly cohomologous. Moreover, the map

\[ [\psi] \mapsto [c_\psi] \]

is a homeomorphism from the space \( \text{Par}(\mathcal{G}) \) of weak conjugacy classes of parametrizations to the domain \( H^1_+(\varphi) \) of weak cohomology classes of positive cocycles.

**Remark 1.20.** This corollary asserts in particular that weak conjugacy of flows is indeed an equivalence relation, and that \( \text{Par}(\mathcal{G}) \) is the largest Hausdorff quotient of the space of parametrizations up to (strong) conjugacy.

The domain \( H^1_+(\psi) \) is a convex open cone in \( H^1_+(\psi) \). The following proposition gives further characterizations of it:

**Proposition 1.21.** Let \( c \) be a cocycle along \( \varphi \). The following are equivalent:

1. \( c \) is cohomologous to a positive cocycle,
2. \( c \) is cohomologous to the integral cocycle \( c_f \) associated to a positive function \( f \),
3. There exists \( T > 0 \) such that \( c(x, T) \) is positive for all \( x \),
(4) There exist constants $A, B > 0$ such that 
\[ c(x, t) \geq At - B \]
for all $x \in M$ and all $t \geq 0$.

(5) $\mu(c) > 0$ for all $\mu \in \mathcal{M}^1(\varphi)$.

(6) There exists a constant $A > 0$ such that $\mu(c) > A$ for all $\mu \in \mathcal{M}^1(\varphi)$.

We call a cocycle satisfying these properties and expanding cocycle.

\[ \mu(c) > 0 \text{ for all } \mu \in M_1(\varphi), \]

Proof. The equivalence between (1) and (2) follows from Proposition 1.11. The equivalence between (2), (3) and (6) follows from applying Lemma 1.15 to the cocycle $-c$. The equivalence between (5) and (6) follows from the vague compactness of $M_1(\varphi)$. Finally, the equivalence between (3) and (4) follows from

\[ c(x, nT) = \sum_{k=0}^{n-1} c(\varphi_k T(x), t). \]

Corollary 1.19 gives a homeomorphism

\[ \text{Id}_\varphi : \text{Par}(G) \to H^1_+(\varphi), \]

depending on the choice of a background parametrization $\varphi$. Let us now describe the coordinate changes with respect to different background parametrizations.

Let $\varphi_1, \varphi_2$ be two parametrizations of the geodesic foliation. Given a cocycle $c$ along $\varphi_1$, define

\[ c \circ c_{\varphi_2 \to \varphi_1} : M \to \mathbb{R} \]

\[ (x, t) \mapsto c(x, c_{\varphi_2 \to \varphi_1}(x, t)). \]

Proposition 1.22. The following holds:

(1) If $c$ is a cocycle along $\varphi_1$, then $c \circ c_{\varphi_2 \to \varphi_1}$ is a cocycle along $\varphi_2$.

(3) If $c$ and $c'$ are cocycles along $\varphi$, then $c \circ c_{\varphi_2 \to \varphi_1}$ and $c' \circ c_{\varphi_2 \to \varphi_1}$ are Livsic (resp. weakly) cohomologous if and only if $c$ and $c'$ are Livsic (resp. weakly) cohomologous.

(4) If $\psi$ is another parametrization, then

\[ c_{\varphi_1 \to \psi} \circ c_{\varphi_2 \to \varphi_1} = c_{\varphi_2 \to \psi}. \]

It follows that the map $c \mapsto c \circ c_{\varphi_2 \to \varphi_1}$ induces a map $I_{\varphi_1 \to \varphi_2}: H^1(\varphi_1) \to H^1(\varphi_2)$. This map is an isomorphism of Banach spaces and maps $H^1_+(\varphi_1)$ to $H^1_+(\varphi_2)$. Finally, we have

\[ \text{Id}_{\varphi_2} \circ \text{Id}_{\varphi_1}^{-1} = I_{\varphi_1 \to \varphi_2}. \]

In conclusion, the space $\text{Par}(G)$ has the structure of a convex Banach cone. More precisely, it is endowed with a family of homeomorphisms to open convex cones in Banach spaces whose transition maps are linear isomorphisms. We call all the $(I_\varphi)$ the affine charts of $\text{Par}(G)$.
1.5. **Period map for cocycles.** We fix again a background parametrization \( \varphi \). Recall that closed leaves in \( \mathcal{G} \) are in bijection with primitive conjugacy classes in \( \Gamma \), and that for \([\gamma] \in [\Gamma] \), the period \( L_\varphi(\gamma) \) is the time taken by the flow \( \varphi \) to go through \( \gamma \).

**Definition 1.23.** Let \( c \) be a cocycle along \( \varphi \). The *period map* of \( c \) is the map

\[
L_c : [\Gamma] \to \mathbb{R} \\
\gamma \mapsto c(x_\gamma, L_\varphi(\gamma))
\]

where \( x_\gamma \) is any point on the closed geodesic \( \gamma \).

**Example 1.24.** If \( c_\psi \) is the reparametrization cocycle associated to \( \psi \), then

\[
L_{c_\psi} = L_\psi .
\]

One easily verifies that \( L_c \) only depends on the weak cohomology class of \( c \). In fact, we have

**Proposition 1.25.** For every \( \gamma \in [\Gamma] \) and every \( c \in Z(\varphi) \),

\[
L_c(\gamma) = L_\varphi(\gamma)\delta_\gamma(c)
\]

where \( \delta_\gamma \) is the \( \varphi \)-invariant probability measure supported on the closed geodesic \( \gamma \).

The closing lemma for the geodesic flow of a hyperbolic surface has the following consequence:

**Lemma 1.26.** The convex hull of the \((\delta_\gamma)_{\gamma \in [\Gamma]}\) is dense in \( M^1(\varphi) \) for the vague topology.

Thus, Proposition 1.21 and 1.16 imply the following

**Corollary 1.27** (Corollary of Proposition 1.21). A cocycle \( c \) is expanding if and only if there exists a constant \( A > 0 \) such that

\[
L_c \geq AL_\varphi .
\]

**Corollary 1.28** (Corollary of Corollary 1.16). Two cocycles \( c \) and \( c' \) along \( \varphi \) are weakly cohomologous if and only if \( L_c \equiv 0 \).

Note that this last corollary is strengthened by Livšic’s theorem under a Hölder regularity hypothesis, see Theorem 6.4.

1.6. **Scaling and the space \( \mathbf{PPar}(\mathcal{G}) \).** Let \( \psi \) be a parametrization of the geodesic foliation and \( \lambda \in \mathbb{R}_{>0} \). The *scaled flow* \( \psi^\lambda \) is defined by

\[
\psi_\lambda^\lambda(x) = \psi_{1/\lambda}(x) .
\]

Scaling defines an action of \( \mathbb{R}_{>0} \) on the space \( \text{Par}(\mathcal{G}) \). We denote by \( \mathbf{PPar}(\mathcal{G}) \) the quotient of \( \text{Par}(\mathcal{G}) \) by this action and call it the projective space of parametrizations.

Let \( \varphi \) be a background parametrization of \( \mathcal{G} \). We then have

\[
c_{\varphi \to \psi^\lambda} = \lambda c_{\varphi \to \psi} .
\]

In other words, the affine chart \( L_\varphi : \text{Par}(\mathcal{G}) \to H^1(\varphi) \) conjugates the scaling action with the scalar multiplication. The space \( \mathbf{PPar}(\mathcal{G}) \) is thus an open domain in the projective space over a Banach space.
Proposition 1.29. The domain $\text{PPar}(G) \subset PH^1(\varphi)$ is a weakly proper convex domain, i.e. it intersects every projective line in a proper interval (and interval which is not the complement of a point).

Remark 1.30. In this infinite dimensional setting, “weakly proper” is weaker than the stronger property of being bounded in some affine chart.

Recall that a (weakly) proper convex domain $\Omega$ carries a natural projectively invariant metric called the Hilbert metric, defined by

$$d(x, y) = \frac{1}{2} \log \left| a, x, b, y \right|,$$

where $a$ and $b$ are the endpoints of the intersection of $\Omega$ with the projective line spanned by $x$ and $y$, and $[a, x, b, y]$ the cross-ratio of those four points. We call the Hilbert metric on $\text{PPar}(G)$ the Hilbert–Thurston distance because, as stated in the next theorem, it is a symmetrization of the distance introduced by Thurston on the Teichmüller space. We denote it by $d_{HT}$.

Theorem 1.31. Let $\varphi$ and $\psi$ be two parametrizations of the geodesic foliation. The Hilbert–Thurston distance between (the projective classes of) $\varphi$ and $\psi$ is given by

$$d_{HT}(\varphi, \psi) = \frac{1}{2} \left( \log \sup_{\gamma \in [\Gamma]} L_\psi(\gamma) + \log \sup_{\gamma \in [\Gamma]} L_\varphi(\gamma) \right).$$

The proofs of both Proposition 1.29 and Theorem 1.31 follow from the following computation:

Lemma 1.32. Let $f$ and $g$ be two points in $H^1_+(\varphi)$. Assume that we don’t have $L_f < L_g$. Then

$$\sup\{\lambda \geq 0 \mid (1 - \lambda)f + \lambda g \in H^1_+(\varphi)\} = \frac{1}{1 - \inf_{\gamma \in [\Gamma]} \frac{L_a(\gamma)}{L_f(\gamma)}}.$$

Proof. For some $\lambda > 0$, we have

$$L_{(1-\lambda)f+\lambda g} = L_f + \lambda(L_g - \lambda L_f)$$

$$= L_f(1 - \lambda(1 - \frac{L_g}{L_f}))$$

Assume first that

$$\lambda \leq \frac{1 - \varepsilon}{1 - \frac{L_a(\gamma)}{L_f(\gamma)}}$$

for all $\gamma \in [\Gamma]$ such that $L_g(\gamma) < L_f(\gamma)$. For some $\gamma \in \Gamma$, if $L_g(\gamma) \geq L_f(\gamma)$, then

$$L_{(1-\lambda)f+\lambda g}(\gamma) \geq L_f(\gamma),$$

and otherwise,

$$L_{(1-\lambda)f+\lambda g}(\gamma) \geq \varepsilon L_f(\gamma).$$

In any case, we get

$$L_{(1-\lambda)f+\lambda g} \geq \varepsilon L_f \geq \varepsilon' L_\varphi,$$

hence $(1 - \lambda) + \lambda g$ belongs to $H^1_+(\varphi)$ by Corollary 1.27.
Assume now that there exists \( \gamma \) such that \( L_g(\gamma) < L_f(\gamma) \) and
\[
\lambda > \frac{1}{1 - \frac{L_g(\gamma)}{L_f(\gamma)}}.
\]
For this \( \gamma \) we then have
\[
L_{(1-\lambda)f+\lambda g}(\gamma) < 0.
\]
Thus \((1-\lambda)f + \lambda g\) does not belong to \(H^1_+(\varphi)\). We conclude that
\[
\sup\{\lambda \mid (1-\lambda)f + \lambda g \in H^1_+(\varphi)\} = \inf_{\gamma \in [\Gamma], L_g(\gamma) < L_f(\gamma)} \frac{1}{1 - \frac{L_g(\gamma)}{L_f(\gamma)}}.
\]
\[
\lambda_+ = \frac{1}{1 - \inf_{\gamma \in [\Gamma]} \frac{L_g(\gamma)}{L_f(\gamma)}}
\]
and
\[
\lambda_- = \frac{1}{1 - \inf_{\gamma \in [\Gamma]} \frac{L_f(\gamma)}{L_g(\gamma)}} - 1.
\]
By definition of the Hilbert distance, we have
\[
d_{HT}(f, g) = \frac{1}{2} \log \left( \frac{\lambda_+ - 1}{\lambda_-} \right)
\]
\[
= \frac{1}{2} \log \left( \frac{1}{\inf_{\gamma \in [\Gamma]} L_g(\gamma)} \inf_{\gamma \in [\Gamma]} \frac{L_f(\gamma)}{L_f(\gamma)} \right)
\]
\[
= \frac{1}{2} \left( \log \left( \sup_{\gamma \in [\Gamma]} \frac{L_f(\gamma)}{L_g(\gamma)} \right) + \log \left( \sup_{\gamma \in [\Gamma]} \frac{L_g(\gamma)}{L_f(\gamma)} \right) \right).
\]

\[\square\]

In Section ?? we will introduce a positive continuous function \( h_{top} \) on \( Par(G) \) which is homogeneous of degree \(-1\) (i.e. \( h_{top}(\varphi^\lambda) = \frac{1}{\lambda} h_{top}(\varphi) \)). Experts will recognize the topological entropy of a flow. We will denote by \( Par_1(G) \) set of parametrization \( \varphi \in Par(G) \) such that \( h_{top}(\varphi) = 1 \). By homogeneity of \( h_{top} \) every parametrization \( \varphi \) admits a unique scaling of entropy 1. Thus \( Par_1(G) \) can be identified with \( PPar(G) \).

2. Buseman cocycles, foliated 1-forms, and horocycles

2.0.1. Buseman cocycles. Let \( \tilde{N}_s^\Gamma \) (resp. \( \tilde{N}_u^\Gamma \)) denote the space of pairs of points \((x, y) \in \tilde{M}_\Gamma^2\) such that \( x \) and \( y \) are contained in the same stable (resp. unstable) leaf.

**Definition 2.1.** A stable Buseman cocycle (resp. unstable Buseman cocycle) on \( M_\Gamma \) is a \( \Gamma \)-invariant continuous function \( B \) on \( \tilde{N}_s^\Gamma \) (resp. \( \tilde{N}_u^\Gamma \)) such that for all \( x, y, z \in \tilde{M}_\Gamma \) belonging to the same weakly stable (resp. weakly unstable) leaf, we have
\[
B(x, z) = B(x, y) + B(y, z).
\]

Let \( \varphi \) be a parametrization of the geodesic flow. Then every Buseman cocycle \( B \) on \( M_\Gamma \) induces a cocycle along \( \varphi \) defined by
\[
c(x, t) = B(x, \varphi_t(x)).
\]

We will say that a cocycle \( c \) is stably (resp. unstably) Buseman if it is associated to a stable (resp. unstable) Buseman cocycle via this construction. We have the following characterization:

**Proposition 2.2.** Let \( c \) be a continuous cocycle along \( \varphi \). The following are equivalent:

- \( c \) is stably (resp. unstably) Buseman,
- the function
  \[
  (x, y) \mapsto c(x, t) - c(y, t)
  \]
  converges uniformly on every compact subset of \( \tilde{N}_s^\Gamma \) (resp. \( \tilde{N}_u^\Gamma \)) when \( t \) goes to \(+\infty\) (resp. \(-\infty\)).

More over, the associated Buseman cocycle is determined by
\[
B(x, y) = \lim_{t \to \pm\infty} c(x, t) - c(y, t).
\]

In particular it is unique.
Corollary 2.3. If $\varphi$ is a Hölder parametrization of the geodesic flow, then every Hölder cocycle along $\varphi$ is stably and unstably Buseman.

We leave the proofs of the previous proposition and corollary as an exercise.

Definition 2.4. A stable (resp. unstable) Buseman cocycle $B$ is a coboundary if there exists a continuous function $F$ on $M_\Gamma$ such that

$$B(x, y) = F(y) - F(x)$$

for all $(x, y) \in \tilde{N}_s^\Gamma$ (resp. $\tilde{N}_u^\Gamma$). Two Buseman cocycles are Livšic cohomologous if their difference is a coboundary.

One easily verifies that two Buseman cocycles are Livsic cohomologous if and only if their associated cocycles along $\varphi$ are Livsic cohomologous.

2.1. Horocyclic foliations.

Definition 2.5. A parametrization $\varphi$ of the geodesic foliation is said to admit a stable (resp. unstable) horocyclic foliation or, for short, stable (resp. unstable) horocycles if there exists a 1-dimensional continuous foliation $H^s$ (resp. $H^u$) of $M_\Gamma$ such that

- Every leaf of $H^s$ (resp. $H^u$) is contained in a leaf of $W^s$ (resp. $W^u$),
- The foliation $H^s$ (resp. $H^u$) is transverse to $G$ in $W^s$ (resp. in $W^u$),
  - i.e. locally, each leaf of $H^s$ and each leaf of $G$ contained in the same weakly stable leaf intersect at exactly one point,
- $H^s$ (resp. $H^u$) is preserved by the flow $\varphi$.

Example 2.6. The geodesic flow of a negatively curved metric on $\Sigma$ is Anosov. It thus admits both stable and unstable horocycles given by the strongly stable and unstable foliations of the flow.

Remark 2.7. The subtlety of this section is that a continuous parametrization of an Anosov flow may not admit strongly stable or unstable foliations.

The relation with Buseman cocycles is given by the following proposition:

Proposition 2.8. Let $\varphi$ and $\psi$ be parametrizations of the geodesic foliation. Assume that $\varphi$ admits stable (resp. unstable) horocycles. Then $\psi$ admits stable (resp. unstable) horocycles if and only if the reparametrization cocycle $c_{\varphi \rightarrow \psi}$ is stably (resp. unstably) Buseman.

Proof. Exercise. □

Definition 2.9. Let $\varphi$ be a parametrization of the geodesic foliation admitting stable (resp. unstable) cocycles. We will say that $\varphi$ admits a horocyclic flow if there exists a flow $h$ on $M_\Gamma$ whose orbit foliation is the stable (resp. unstable) horocyclic foliation, and such that $\varphi$ and $h$ have the following commuting property:

$$\varphi_t \circ h_s \circ \varphi_{-t} = h_{e^{-t} s}$$

(resp. $$\varphi_t \circ h_s \circ \varphi_{-t} = h_{e^t s}$$)

for all $(s, t) \in \mathbb{R}^2$. 
The pair \((\varphi, h)\) gives a locally free action of the affine group on \(M_\Gamma\) whose orbit foliation is \(W^s\) (resp. \(W^u\)). We call it a \textit{stable} (resp. \textit{unstable}) affine action.

\textbf{Example 2.10.} The geodesic and horocyclic flows of a hyperbolic metric on \(\Sigma\) give a stable affine action.

A stable (resp. unstable) affine action \((\varphi, h)\) induces in particular a \textit{foliated smooth structure} (i.e. a transversally continuous family of smooth structures on the leaves of) \(W^s\) (resp. \(W^u\)) as well as two vector fields \(X\) and \(Y\) tangent to \(W^s\) (resp. \(W^u\)) that generate respectively the flow \(\varphi\) and \(h\) and satisfy the commutation relation

\[ [X,Y] = -Y \] (resp. \[ [X,Y] = Y \] )

We will come back extensively on stable affine actions and their relation to foliated hyperbolic structures in Section 4. For now, we only need the existence of a stable affine action and its associated foliated smooth structure.

\subsection*{2.2. stable 1-forms}

In this section, we assume that the weakly stable (resp. unstable) foliation is provided with an affine action \((\varphi, h)\). One can take \(\varphi\) and \(h\) to be respectively the geodesic and horocyclic flow of a hyperbolic metric on \(\Sigma\). In particular, the weakly stable (resp. unstable) leaves carry a smooth structure. Let \(X\) and \(Y\) be the vector fields on \(W^s\) (resp. \(W^u\)) generating the flows \(\varphi\) and \(h\) respectively.

\textbf{Definition 2.11.} A \textit{stable} 1-form (resp. \textit{unstable} 1-form) of class \(C^k\) on \(M_\Gamma\) is a family of 1-forms of class \(C^k\) on the leaves of \(\tilde{W}^s\) (resp. \(\tilde{W}^u\)), preserved by \(\Gamma\), and depending continuously on the leaf for the \(C^k\) topology. It is called \textit{closed} if it is closed on each leaf.

(Recall that a continuous 1-form is closed if it is locally the differential of a \(C^1\) function.)

A closed stable (resp. unstable) 1-form \(\alpha\) gives rise to a stable (resp. unstable) Buseman cocycle \(B_\alpha\) defined by

\[ B_\alpha(x,y) = \int_x^y \alpha . \]

The associated cocycle \(c_\alpha\) along \(\varphi\) is the integral cocycle associated to the function \(\alpha(X)\) (i.e. \(c_\alpha(x,t)\) is the integral of \(\alpha\) from \(x\) to \(\varphi(t,x)\)).

\textbf{Proposition 2.12.} Every (stable or unstable) Buseman cocycle is Livsic cohomologous to the Buseman cocycle associated to a closed (stable or unstable) 1-form of class \(C^\infty\).

\textbf{Proof.} Let \(B\) be a stable Buseman cocycle. We construct \(\alpha\) by “smoothening” \(B\). To do so, we can for instance choose a probability law \(\nu\) on \(\mathbb{R}^2\) with smooth density with respect to Lebesgue and compact support. For \((x,y) \in \tilde{N}^s_\Gamma\), define

\[ \nu_* B(x,y) = \int_{\mathbb{R}^2} B(x, \exp(tX) \circ \exp(sY) \cdot y) d\nu(t,s) . \]
(Watch out that $\nu_* B$ is not a Buseman cocycle.) The function $\nu_* B(x, \cdot)$ is smooth on each stable leaf. Moreover, for $x_1$ and $x_2$ in the same stable leaf, we have

$$\nu_* B(x_1, \cdot) = \nu_* B(x_2, \cdot) + B(x_1, x_2),$$

so the 1-form $\alpha = d\nu_* B(x, \cdot)$ does not depend on $x$. This defines a closed foliated 1-form of class $C^\infty$ on $M_\Gamma$.

Let $B_\alpha$ be the Buseman cocycle associated to $\alpha$. For $x_0, x$ and $y$ in the same stable leaf, we have

$$B_\alpha(x, y) = \nu_* B(x_0, y) - \nu_* B(x_0, x)$$

$$= B(x_0, y) - B(x_0, x) + \nu_* B(y, y) - \nu_* B(x, x)$$

$$= B(x, y) + \nu_* B(y, y) - \nu_* B(x, x).$$

The function $x \mapsto \nu_* B(x, x) = \int_{\mathbb{R}^2} B(x, \exp(tX) \circ \exp(sY) \cdot x) d\nu(t, s)$ is continuous and $\Gamma$-invariant. Therefore, $B$ and $B_\alpha$ are cohomologous.

Let $Z(W^s)$ denote the space of foliated closed 1-forms of class $C^0$. We provide $Z(W^s)$ with the supremum norm:

$$\|\alpha\|_\infty = \sup_M \max \{|\alpha(X)|, |\alpha(Y)|\}.$$ 

Let $B(W^s)$ denote the closure of the subspace of exact 1-forms. Finally, let $H^1(W^s)$ denote the quotient $Z(W^s)/B(W^s)$, provided with the quotient norm.

The map

$$Z(W^s) \rightarrow C^0(M, \mathbb{R})$$

$$\alpha \mapsto \alpha(X)$$

induces a linear map $\Pi : H^1(W^s) \rightarrow H^1(\varphi)$.

**Proposition 2.13.** The map $\Pi$ is an isometric bijection.

**Proof.** Let us prove that $\Pi$ preserves the norm. Since by Proposition 2.12, every Hölder function $f$ is in the image of $\Pi$, we obtain that $\Pi$ has dense image. Since the domain of $\Pi$ is a Banach space, we will conclude that $\Pi$ is an isomorphism.

Let $\alpha$ be a closed foliated 1-form on $W^s$. Then, by definition of the norms, one clearly has

$$\|\alpha\|_\infty \geq \sup_M |\alpha(X)|.$$ 

It follows that

$$\|\Pi([\alpha])\| \leq \|[\alpha]\|.$$ 

To prove the converse inequality, let us define

$$\alpha_T = \frac{1}{T} \int_0^T \varphi_t \alpha dt.$$ 

The form $\alpha_T$ is cohomologous to $\alpha$.

Now, one has

$$\alpha_T(X)(x) = \frac{1}{T} \int_0^T \alpha(X)(\varphi_t(x)).$$
By Corollary 1.16, we thus have
\[ \|\alpha T(X)\|_\infty \rightarrow T \rightarrow +\infty \|\alpha(X)\| = \|\Pi(\alpha)\| . \]

Meanwhile, since \( \varphi_t Y = e^{-t}Y \), one has
\[ |\alpha T(Y)(x)| \leq \frac{1}{T} \int_0^T e^{-t} \|\alpha(Y)\|_\infty \, dt \leq \frac{1}{T} \text{norm}_\alpha(Y) \rightarrow T \rightarrow +\infty 0 . \]

We conclude that
\[ \|\alpha T\| \rightarrow +\infty \|\Pi(\alpha)\| . \]

Therefore, \( \|\alpha\| = \|\Pi(\alpha)\| . \)

Proposition 2.13 says in particular that every continuous cocycle is weakly cohomologous to one which is stably Buseman. As a corollary, we obtain the following theorem:

**Theorem 2.14.**
Every continuous parametrization of the geodesic foliation is weakly conjugate to a parametrization admitting a stable horocyclic foliation.

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3. **Anosov groups of diffeomorphisms of the circle**

In this section, we associate to certain well-behaved circle actions of a surface group a parametrization of the geodesic foliation, unique up to Livsic equivalence. Moreover, these parametrizations admit stable horocycles, and come with a family of measures on unstable leaves with nice properties with respect to the stable horocycles.

3.1. **Expanding actions.** We denote by \( S^1 \) the unit circle, by \( \text{Diff}(S^1) \) the group of diffeomorphisms of \( S^1 \) of class \( C^1 \) and by \( \text{Diff}^h(S^1) \) the subgroup of diffeomorphisms whose derivatives are Hölder regular.

A homomorphism \( \rho \) from \( \Gamma \) to \( \text{Diff}(S^1) \) is called a \( C^1 \) action of \( \Gamma \) on \( S^1 \). Let \( \rho \) be a \( C^1 \) action which is topologically conjugate to the action of \( \Gamma \) on \( \partial_\infty \Gamma \). In particular, every element \( \gamma \in \Gamma \) acts on \( S^1 \) with an attracting fixed point and a repelling fixed point. When this does not bring any confusion, we denote these points respectively by \( \gamma_+ \) and \( \gamma_- \) (omitting the dependence in \( \rho \)).

**Definition 3.1.** We define the period map of \( \rho \) as the function
\[ L_\rho : \begin{array}{c} \Gamma \rightarrow \mathbb{R}_+ \\ \gamma \mapsto -\log(\rho(\gamma)'(\gamma_+)) \end{array} . \]

This definition is motivated by the proposition:

**Proposition 3.2.** Let \( \rho \) be a \( C^1 \) action of \( \Gamma \) on \( S^1 \) topologically conjugate to a Fuchsian action. Let \( \varphi \) be a parametrization of the geodesic foliation. Then there exists a cocycle \( c_\rho \) along \( \varphi \) such that
\[ L_\rho = L_{c_\rho} . \]
Proof. Let \( h : \partial \infty \Gamma \to \mathbb{S}^1 \) be the homeomorphism conjugating the action \( \rho \) with the action of \( \Gamma \) on its boundary. Let \( E_\rho \) be the continuous line bundle over \( M_\Gamma \) defined by

\[
(\tilde{E}_\rho)_{(x,y,z)} = T_{h(z)}\mathbb{S}^1,
\]

and let \( E_\rho \) be the line bundle over \( M \) obtain by quotienting by the action of \( \Gamma \).

The flow \( \varphi_t \) on \( M_\Gamma \) lifts to a flow \( \hat{\varphi}_t \) on the total space of \( E_\rho \) which is linear in the fibers, induced by the transformation

\[
((x, y, z), v) \mapsto (\varphi_t(x, y, z), v)
\]
on \( \tilde{E}_\rho \).

Let \( \cdot \) be a continuous norm on \( L_\rho \). We define the cocycle \( c_\rho \) by

\[
c_\rho(x, t) = \log \frac{|\hat{\varphi}_t(v)|}{|v|}
\]

where \( v \) is any vector in \( (E_\rho)_x \setminus \{0\} \).

The norm \( |\cdot| \) lifts to a \( \Gamma \)-invariant norm on \( \tilde{E}_\rho \) that we still denote by \( |\cdot| \).

Let \( \gamma \) be an element of \( \Gamma \) and let \( (\gamma_-, y, \gamma_+) \) be a point on the axis of \( \gamma \). Let \( v \) be a tangent vector to \( h(\gamma_+) \). Then, by definition of \( c_\rho \), we have

\[
L_{c_\rho}(\gamma) = \log \frac{|v|_{(\gamma_-, y, \gamma_+)} |v|_{(\gamma, y, \gamma_+)}^{-1}}{|v|_{(\gamma_-, y, \gamma_+)}^{-1}} \geq \log \rho(\gamma_+) - \log \rho(\gamma_-) = L_\rho(\gamma).
\]

Remark 3.3. The cocycle \( c_\rho \) above is well-defined up to a Livsic coboundary. Indeed, it only depends on the choice of a metric on the line bundle \( L_\rho \), and one easily checks that changing this metric will modify \( c_\rho \) by a coboundary.

Definition 3.4. A \( C^1 \) action \( \rho \) of \( \Gamma \) on \( \mathbb{S}^1 \) is expanding if for every \( x \in \mathbb{S}^1 \), there exists \( \gamma \in \Gamma \)

\[
|\rho(\gamma)'(x)| > 1.
\]

By a straightforward application of Borel–Lebesgue’s characterization of compactness, every expanding action satisfies the stronger property:

Proposition 3.5. Let \( \rho : \Gamma \to \text{Diff}(\mathbb{S}^1) \) be an expanding action. Then there exists \( y_1, \ldots, y_k \in \Gamma \), a covering of \( \mathbb{S}^1 \) by open intervals \( I_1, \ldots, I_k \), and \( \varepsilon > 0 \) such that

\[
\rho(y_j)'(x) \geq 1 + \varepsilon
\]

for all \( x \in I_j \).

Theorem 3.6. Let \( \rho \) be a \( C^1 \) action of \( \Gamma \) on \( \mathbb{S}^1 \) topologically conjugate to the action of \( \Gamma \) on \( \partial_\infty \Gamma \). Then the following are equivalent:

(i) The action \( \rho \) is expanding,

(ii) The cocycle \( c_\rho \) is expanding.
(iii) The action $\rho$ is bi-Hölder conjugate to the action of $\Gamma$ on $\partial_\infty \Gamma$.

Proof. $(i) \Rightarrow (iii)$. 

We prove more generally that if two dilating $C^1$ actions $\rho_1$ and $\rho_2$ are conjugated by a homeomorphism $h$, then $h$ is bi-Hölder continuous. In particular, a dilating $C^1$ action topologically conjugate to the action of $\Gamma$ on $\partial_\infty \Gamma$ is bi-Hölder conjugate to any Fuchsian action of $\Gamma$ on $S^1$.

Let $g_1, \ldots, g_k$ be elements in $\Gamma$, $I_1, \ldots, I_k$ be open intervals covering $S^1$ and $\varepsilon$ be positive such that $\rho_1(g_j)' > 1 + \varepsilon$ on $I_j$. Let $0 < \eta < 1$ be such that $\rho_2(g_j)' > \eta$ on $I_j$ for all $j \in \{1, \ldots, k\}$.

Let $a > 0$ be such that, for all interval $J \subset S^1$, if $J$ has length less or equal to $a$, then there exists $j \in \{1, \ldots, k\}$ such that $J \subset I_j$.

Let us now fix $x \neq y \in S^1$. By the expanding property and the definition of $a$, one can find $n \in \mathbb{N}$ and $i_1, \ldots, i_n \in \{1, \ldots, k\}$ such that

• For all $0 \leq l < n$,
  $$|\rho_1(g_{i_l} \cdots g_{i_1}) \cdot x - \rho_1(g_{i_l} \cdots g_{i_1}) \cdot y| < a,$$
  • For all $0 \leq l < n$,
  $$g_{i_l} \cdots g_{i_1} \cdot [x, y] \subset I_{i_{l+1}},$$
  • $$|\rho_1(g_{i_n} \cdots g_{i_1}) \cdot x - \rho_1(g_{i_n} \cdots g_{i_1}) \cdot y| \geq a.$$

Note that, since $g_j$ multiplies the length of every interval contained in $I_j$ by at least $1 + \varepsilon$, we have

$$(1 + \varepsilon)^{n-1}|x - y| < a.$$  

Now, let $h : S^1 \to S^1$ be the homeomorphism conjugating $\rho_1$ and $\rho_2$ and let $b > 0$ be the infimum of lengths of images by $h$ of an interval of length at least $a$.

We then have

$$|\rho_2(g_{i_n} \cdots g_{i_1}) \cdot h(x) - \rho_2(g_{i_n} \cdots g_{i_1}) \cdot h(y)| \geq b$$

and therefore

$$(3) \quad |h(x) - h(y)| \geq \eta^n b.$$  

Putting (2) and (3) together, and setting $\alpha = \frac{\log(1+\varepsilon)}{\log(1/\eta)} > 0$, we get

$$|x - y| < \frac{a}{(1+\varepsilon)^{n-1}} \leq a \eta^{(n-1)\alpha}$$

$$< \frac{a}{(b\eta)^{\alpha}} (b\eta)^n \alpha$$

$$< \frac{a}{(b\eta)^{\alpha}} |h(x) - h(y)|^{\alpha}.$$  

Since this is true for all $x \neq y$, we conclude that $h^{-1}$ is $\alpha$-Hölder. We obtain similarly that $h$ is Hölder by switching the roles of $\rho_1$ and $\rho_2$.

$(iii) \Rightarrow (ii)$
Let $\rho: \Gamma \to \text{Diff}(S^1)$ be a $C^1$ action on $S^1$ which is bi-Hölder conjugate to a Fuchsian action $j$. Let $h: S^1 \to S^1$ be the homeomorphism conjugating $j$ and $\rho$ and $\alpha > 0$ such that $h$ is $\alpha$-Hölder.

Fix $\gamma \in \Gamma$. Let $\gamma_+$ be the attracting fixed point of $j(\gamma)$ and denote $\lambda = j(\gamma)'(\gamma_+) < 1$. Let $y$ be a point in $S^1$ distinct from $\gamma_-$ and $\gamma_+$. Then when $n$ goes to $+\infty$,

$$\log |j(\gamma^n) \cdot y - x| \sim \log(\lambda)n.$$ 

Since $h$ is $\alpha$-Hölder, we thus have

$$\log |\rho(\gamma^n) \cdot h(y) - h(\gamma_+)| = O(\alpha \log(\lambda)n),$$

which implies that

$$\rho(\gamma)'(h(\gamma_+)) \leq \lambda^\alpha.$$ 

We thus have

$$L_{c_\rho}(\gamma) \geq -\alpha \log(\lambda) = \alpha L_j(\gamma).$$

By proposition 1.21, the cocycle $c_\rho$ is thus expanding.

$(ii) \Rightarrow (i)$

Let $\tilde{E}_\rho$ be the line bundle over $\tilde{M}_\Gamma$ defined in the proof of Proposition 3.2. Let us denote by $|\cdot|_0$ the continuous metric on $\tilde{L}_\rho$ induced by the metric on $S^1$ and by $|\cdot|_1$ a $\Gamma$-invariant continuous metric on $\tilde{L}_\rho$.

By construction, the metric $|\cdot|_0$ is invariant under the flow $\tilde{\varphi}$. Let $K$ be a compact set in $\tilde{M}_\Gamma$ such that $\bigcup_{\gamma \in \Gamma} \gamma \cdot K$ covers $\tilde{M}_\Gamma$ and such that the image of $K$ by the projection $\pi: (x, y, z) \mapsto z$ is the whole circle. Finally, let $A > 1$ be such that for all $x \in K$ and all $u \in (\tilde{E}_\rho)_x \backslash \{0\}$,

$$\frac{1}{A} < \frac{|u|_1}{|u|_0} < A.$$

Let $z$ be any point in $S^1$. Choose a point $x \in K$ such that $\pi(x) = z$ and a non-zero vector $u \in \tilde{L}_\rho(x) = T_zS^1$.

Since the cocycle $c_\rho$ is expanding, there exists a time $t > 0$ such that

$$\frac{|\tilde{\varphi}_t(u)|_1}{|u|_1} > A^2.$$ 

Let $\gamma \in \Gamma$ be such that $\gamma \cdot \varphi_t(x) \in K$ and let $v \in (\tilde{L}_\rho)_{\gamma \cdot x}$ be the image of $u$ by $\gamma$. We then have

$$\rho(\gamma) \cdot z = \pi(\gamma \cdot x)$$

and

$$\rho(\gamma)'(z) = \frac{|v|_0}{|\tilde{\varphi}_t(u)|_0} = \frac{|v|_0}{|u|_0} \text{ since } |\cdot|_0 \text{ is } \tilde{\varphi}\text{-invariant}$$

$$> \frac{1}{A^2} \frac{|v|_1}{|u|_1}$$

$$> \frac{1}{A^2} \frac{|\tilde{\varphi}_t(u)|_1}{|u|_1} \text{ since } |\cdot|_1 \text{ is } \Gamma\text{-invariant}$$

$$> 1.$$
We conclude that the $C^1$ action $\rho$ is expanding.

3.2. The space $\mathfrak{X}_{an}(\Gamma, \text{Diff}(S^1))$ and the map $DF$.

**Definition 3.7.** Let $\rho: \Gamma \to \text{Diff}(S^1)$ be a homomorphism. We will say that $\rho$ is an *Anosov action* on $S^1$ or an *Anosov representation* into $\text{Diff}(S^1)$ if $\rho$ is Hölder conjugate to a Fuchsian action.

**Proposition 3.8.** Let $\rho$ be an Anosov action on $S^1$. Then there exists a norm $|\cdot|$ on $E_\rho$ such that

$$|\hat{\varphi}_t(u)| > |u|$$

for all $t > 0$ and all $u \neq 0 \in L_\rho$.

**Proof.** Let $|\cdot|_0$ be a continuous metric on $E_\rho$. Since the cocycle $c_\rho$ is expanding, we can find constants $K > 1$ and $a > 0$ such that

$$\frac{|\hat{\varphi}_{-t}(u)|_0}{|u|_0} \leq Ke^{-at}$$

for all $u \neq 0 \in E_\rho$. We can thus define a new continuous metric $|\cdot|_1$ on $E_\rho$ by

$$|u|_1 = \int_0^{+\infty} |\hat{\varphi}_{-s}(u)|_0 ds .$$

We now have

$$|\hat{\varphi}_t(u)|_1 = \int_0^{+\infty} |\hat{\varphi}_{s-t}(u)|_0 ds = \int_{-t}^{+\infty} |\hat{\varphi}_{-s}(u)|_0 ds > |u|_1$$

This proves the existence of $\psi_\rho$.

**Corollary 3.9.** Let $\rho$ be an Anosov action on $S^1$. Then there exists a continuous parametrization $\psi_\rho$ of $G$ and a continuous norm $|\cdot|$ on $E_\rho$ such that

$$|\hat{\psi}_\rho(t, u)| = e^t|u|$$

for all $u \in E_\rho$.

Moreover, $\psi_\rho$ is unique up to conjugation and $|\cdot|$ is uniquely determined by $\psi_\rho$ up to a multiplicative constant.

**Proof.** Let $|\cdot|$ be a continuous metric on $E_\rho$ such that $|\hat{\varphi}_t(u)| > |u|$. Then the cocycle $c_\rho$ associated to this norm is positive. It is thus the reparametrization cocycle of a flow $\psi^\rho$. For $x \in M_\rho$ and $t \in \mathbb{R}$, let $s$ be such that $c(x, s) = t$. Then for any $u \in L_\rho(x) \{0\}$, we have

$$|\hat{\psi}_t^x(u)| = |\hat{\varphi}_s(u)| = e^{c_\rho(x, s)}|u| = e^t|u| .$$

This proves the existence of $\psi^\rho$. 

Asume now that there are two norms $| \cdot |_1$ and $| \cdot |_2$ on $L_\rho$ and two reparametrizations $\psi_1$ and $\psi_2$ of $\varphi$ such that
\[ |\widehat{\psi}_i(t,u)|_i = e^t|u|_i \]
for $i = 1, 2$.

Let $c$ be the reparametrization cocycle of $\psi_2$ along $\psi_1$. Let $x$ be a point in $M_\Gamma$, $u \in (L_\rho)_x \setminus \{0\}$, $t \in \mathbb{R}$ and set $v = \widehat{\psi}_1(t,u)$. We then have
\[ t = \log \frac{|v|_1}{|u|_1} \]
and
\[ c(x,t) = \log \frac{|v|_2}{|u|_2} . \]
It follows that
\[ c(x,t) - t = F(\psi_1(t,x)) - F(x) , \]
where
\[ F = \frac{| \cdot |_2}{| \cdot |_1} . \]

The cocycle $c$ is thus Livsic cohomologous to the tautological cocycle $c_0(x,t) = t$, and $\psi_2$ is thus equivalent to $\psi_1$.

Finally, fix $\psi_\rho$ a reparametrization of $\varphi$. Assume that $| \cdot |_1$ and $| \cdot |_2$ are two continuous metrics on $L_\rho$ such that
\[ |\widehat{\psi}_\rho(t,u)|_i = e^t|u|_i \]
for $i = 1, 2$.

Then the function $\frac{| \cdot |_2}{| \cdot |_1}$ on $M_\Gamma$ is continuous and invariant by $\psi_\rho$. It is thus constant by topological transitivity of the geodesic foliation. \(\square\)

Recall that $\text{Diff}^h(S^1)$ denotes the set of diffeomorphisms of the circle with Hölder derivatives.

**Proposition 3.10.** If $\rho$ takes values in $\text{Diff}^h(S^1)$, then the associated parametrization $\psi_\rho$ is conjugate to a Hölder continuous reparametrization.

**Proof.** Exercise. \(\square\)

Let us now see how Corollary 3.9 defines a continuous map from the space of Anosov representations to the space $\text{Par}(G)$.

Let us provide the group $\text{Diff}(S^1)$ with the $C^1$ topology. Since $\Gamma$ is finitely generated, the space $\text{Hom}(\Gamma, \text{Diff}(S^1))$ of homomorphisms from $\Gamma$ to $\text{Diff}(S^1)$ embeds in a product of finitely many copies of $\text{Diff}(S^1)$ and inherits its topology.

The group $\text{Diff}(S^1)$ acts continuously on $\text{Hom}(\Gamma, \text{Diff}(S^1))$ by conjugation. Its orbit equivalence relation has a priori no reason to be Hausdorff, so we define
\[ \mathcal{X}(\Gamma, \text{Diff}(S^1)) \]
as the largest Hausdorff quotient of $\text{Hom}(\Gamma, \text{Diff}(S^1))/\text{Diff}(S^1)$ (i.e. the quotient of $\text{Hom}(\Gamma, \text{Diff}(S^1))$ by the smallest closed equivalence relation containing the conjugation).
The set $\text{Hom}_{an}(\Gamma, \text{Diff}(S^1))$ of Anosov actions is open in $\text{Hom}(\Gamma, \text{Diff}(S^1))$ and invariant under conjugation. We denote by $$X_{an}(\Gamma, \text{Diff}(S^1))$$ its image in $X(\Gamma, \text{Diff}(S^1))$.

Corollary 3.9 associates to each Anosov action $\rho$ a parametrization $\psi_\rho$ of the geodesic foliation such that $L_{\psi_\rho} = L_\rho$. This defines a map $$\tilde{DF} : \text{Hom}_{an}(\Gamma, \text{Diff}(S^1)) \to \text{Par}(G).$$

Here we prove the following:

**Theorem 3.11.** The map $\tilde{DF}$ factors to a continuous map $$DF : X_{an}(\Gamma, \text{Diff}(S^1)) \to \text{Par}(G),$$ which maps $X_{an}^h(\Gamma, \text{Diff}(S^1))$ into $\text{Par}^h(G)$.

Let us first see that $\tilde{DF}$ factors to $X_{an}(\Gamma, \text{Diff}(S^1))$.

**Proposition 3.12.** For every $\gamma \in \Gamma$, the function $$\chi_\gamma : \text{Hom}_{an}(\Gamma, \text{Diff}(S^1)) \to \mathbb{R} \quad \rho \mapsto L_\rho(\gamma)$$ is continuous and invariant by conjugation.

**Proof.** The conjugation invariance is easy. The continuity follows from the stability of contracting dynamics. \hfill \Box

By universal property of the largest Hausdorff quotient, the functions $\chi_\gamma$ factor to continuous functions on $X_{an}(\Gamma, \text{Diff}(S^1))$. In other words, two Anosov actions in the same equivalence class in $X_{an}(\Gamma, \text{Diff}(S^1))$ have the same period map. Since the map $\tilde{DF}$ preserves period maps and since points in $\text{Par}(G)$ are uniquely determined by their period map, we conclude that the map $\tilde{DF}$ factors to a map $$DF : X_{an}(\Gamma, \text{Diff}(S^1)) \to \text{Par}(G).$$

Let us now prove the continuity of $DF$. Note that it is not sufficient to know that $L_\rho(\gamma)$ varies continuously with $\rho$ for each $\gamma$: one needs some uniformity in $\gamma$.

**Lemma 3.13.** Let $(\rho_n)$ be a sequence of Anosov actions on the circle converging to $\rho \in \text{Hom}_{an}(\Gamma, \text{Diff}(S^1))$. Let $h_n$ and $h$ denote the homeomorphisms from $\partial_{\infty}\Gamma$ to $S^1$ conjugating the action of $\Gamma$ with $\rho_n$ and $\rho$ respectively. Then $h_n$ converges uniformly to $h$.

**Proof.** For each $\gamma \in \Gamma \setminus \{\text{Id}\}$, the homeomorphism $h_n$ maps $\gamma_+$ to the attracting fixed point of $\rho_n(\gamma)$. By stability of contracting dynamics, we deduce that $h_n(\gamma_+)$ converges to $h(\gamma_+)$. Since attracting fixed points of elements in $\Gamma \setminus \{\text{Id}\}$ are dense in $\partial_{\infty}\Gamma$ we obtain that $h_n$ converges pointwise to $h$ on a dense subset. Now, $h_n$ and $h$ are locally given by continuous and monotoneous maps of a compact interval, so Dini’s second theorem implies that $h_n$ converges uniformly to $h$. \hfill \Box
From here, one could argue that the whole construction of the flat line bundle and corresponding cocycle associated to $\rho$ vary continuously with $\rho$. Alternatively, one can prove in a more down to earth way the following:

**Proposition 3.14.** Let $(\rho_n)$ be a sequence of Anosov actions on the circle converging to $\rho \in \text{Hom}_\text{an}(\Gamma, \text{Diff}(S^1))$. Then $\frac{\rho_n}{L_{\rho_n}}$ converges uniformly to $1$.

Since $\text{DF}$ preserves period maps, this shows that $d_{HT}(\text{DF}(\rho_n), \text{DF}(\rho))$ converges to $0$, concluding the proof of the continuity of $\text{DF}$.

**Proof.** Fix $\varepsilon > 0$ and choose a finite generating set $S$ of $\Gamma$. Let $\eta$ be such that for all $s \in S$,

$$|\log \rho'(s)(x) - \log \rho'(s)(y)| \leq \varepsilon$$

whenever $|x - y| \leq \eta$. Choose $n$ large enough so that

$$|\log \rho_n(s)'(x) - \log \rho(s)'(x)| \leq \varepsilon$$

for all $x \in S^1$ and all $s \in S$, and

$$|h_n(x) - h(x)| \leq \eta$$

for all $x \in \partial_{\infty} \Gamma$.

Fix $[\gamma] \in [\Gamma]$ and choose $\gamma$ a representative of $[\gamma]$ of minimal length $k$ with respect to the generating set $S$. Write $\gamma = s_1 \ldots s_k$ with $s_i \in S$. Since $\rho$ is Anosov, there exists a constant $\lambda > 0$ independent of $[\gamma]$ such that $L_{\rho}(\gamma) \geq \lambda k$. Let us now compute:

$$\left| \frac{L_{\rho_n}(\gamma) - L_{\rho}(\gamma)}{L_{\rho}(\gamma)} \right| = \frac{1}{L_{\rho}(\gamma)} |\log \rho_n(\gamma)'(h_n(\gamma_+)) - \log \rho(\gamma)'(h(\gamma_+))|$$

$$= \frac{1}{L_{\rho}(\gamma)} \left| \sum_{i=1}^{k} \log \rho_p(s_i)'(h_n(s_{i+1} \ldots s_k \cdot \gamma_+)) - \sum_{i=1}^{k} \log \rho(s_i)'(h(s_{i+1} \ldots s_k \cdot \gamma_+)) \right|$$

$$\leq \frac{1}{\lambda k} \sum_{i=1}^{k} |\log \rho_n(s_i)'(h_n(s_{i+1} \ldots s_k \cdot \gamma_+)) - \rho(s_i)'(h(s_{i+1} \ldots s_k \cdot \gamma_+))|$$

$$+ |\rho(s_i)'(h_n(s_{i+1} \ldots s_k \cdot \gamma_+)) - \rho(s_i)'(h(s_{i+1} \ldots s_k \cdot \gamma_+))|$$

$$\leq \frac{2\varepsilon}{\lambda}.$$ 

\[\Box\]

### 3.3. Anosov actions and stable measures

Here we investigate further properties of the flow $\psi^\rho$ associated to an Anosov action: we show that $\psi^\rho$ admits stable horocycles, and that the metric $|\cdot|$ on $L_{\rho}$ which is scaled by $\varphi$ induces a family of measures on the leaves of $\mathcal{W}^u$ with nice properties with respect to the horocycles.

Let us start with the first point.

**Proposition 3.15.** The flow $\psi^\rho$ associated to an Anosov representation into $\text{Diff}(S^1)$ admits stable horocycles.
Proof. Let $|·|$ be the metric on $E_ρ$ satisfying

$$|\hat{\psi}_t^\rho(u)| = e^t|u|.$$ 

Let $x$ and $y \in \tilde{M}_Γ$ be two points belonging to the same stable leaf $\tilde{W}_s(p)$. Let $u$ be a non-zero vector in $T_pS^1$. We define

$$B(x,y) = \log \frac{|u|_y}{|u|_x},$$

where $|u|_x$ and $|u|_y$ denote respectively the norm of $u$ seen as a vector in $E_ρ_x$ and in $E_ρ_y$. It is clear that this definition does not depend on $u$.

One easily verifies that $B$ is a Buseman cocycle. Moreover, for any $x \in \tilde{M}_Γ$ and any $u \in E_ρ_x \setminus \{0\}$, we have

$$B(x,\psi_t(x)) = \log \frac{|\hat{\psi}_t^\rho(u)|}{|u|} = t.$$ 

The sets

$$H^s(x) = \{y \in W^s(x) \mid B(x,y) = 0\}$$

thus define stable horocycles for the flow $ψ^ρ$. □

We now turn to the construction of unstable measures which are scaled by the flow. Let us start with some definitions.

**Definition 3.16.** An unstable (resp. stable) measure on $M_Γ$ is a collection of Radon measures on the leaves of $\tilde{W}^u$ (resp. $\tilde{W}^s$) which is preserved by $Γ$.

Let $ϕ$ be a parametrization of the geodesic foliation. One can naturally push forward or pull back an unstable measure by $ϕ_t$.

**Definition 3.17.** We say that an unstable measure $µ$ is scaled by $ϕ$ if for every $t \in \mathbb{R}$,

$$ϕ_t^* µ = c(t) µ$$

for some constant $c(t)$. Since $ϕ$ is a flow, we necessarily have $c(t) = e^{at}$ for some $a \in \mathbb{R}$. We call $a$ the scaling factor.

Let us now assume that $ϕ$ admits a stable horocyclic foliation $H^s$. Let $p$ and $q$ be two points in $\partial_∞ Γ$, labelling two unstable leaves $\tilde{W}^u(p)$ and $\tilde{W}^u(q)$. Then, for every point $x \in \tilde{W}^u(p) \setminus G(p,q)$, the stable horocycle passing through $x$ intersects $\tilde{W}^u(q)$ in a unique point.

**Definition 3.18.** The map

$$T_{p,q} : \tilde{W}^u(p) \setminus G(p,q) \rightarrow \tilde{W}^u(q) \setminus G(q,p)$$

sending $x$ to the unique intersection between $H^s(x)$ and $\tilde{W}^u(q)$ is called the holonomy of the stable horocyclic foliation.

**Definition 3.19.** An unstable measure $µ$ is called invariant under horocyclic holonomy if

$$T_{p,q}^* µ_q = µ_p$$

for all $p,q \in \partial_∞ Γ$. An unstable (resp. stable) measure which is invariant under horocyclic holonomy and scaled by the flow $ϕ$ is called an unstable (resp. stable) Margulis measure.
Let now $\psi^t$ be the flow associated to an Anosov action $\rho$. Let $\partial_{\infty} \Gamma \to S^1$ be the homeomorphism conjugating the action of $\Gamma$ on its boundary with $\rho$. Let $| \cdot |$ be the metric on $E_{\rho}$ such that

$$|\tilde{\psi}^t_{p}(u)| = e^t |u| .$$

We define an unstable measure $\mu_{\rho}$ in the following way: let $q$ be a point $\partial_{\infty} \Gamma$. Choose $s \mapsto p(s)$ a homeomorphism from $(0, 1)$ to $\partial_{\infty} \Gamma \setminus \{q\}$ such that $h \circ p: (0, 1) \to S^1 \setminus \{h(q)\}$ is a diffeomorphism, and choose continuously $x(s)$ in $\hat{\mathbb{G}}(q, p(s))$. For a continuous function $f$ on $\mathbb{W}^u(q)$ with compact support, define

$$\int f d\mu_{\rho}(q) = \int_{0}^{1} \int_{-\infty}^{+\infty} f(\psi^t_{p}(x(s)))e^{-t}|(h \circ p)'(s)|_{x(s)} dt ds .$$

**Proposition 3.20.** The family of measures $\mu_{\rho}$ defines an unstable Margulis measure with scale factor $1$.

**Proof.** Let us first remark that the measure $\mu_{\rho}$ defined above does not depend on the choice of $x(s)$. Indeed, for any other choice $x'(s)$, we can write $x'(s) = \psi_{p}(s)(x(s))$ for some $u(s) \in \mathbb{R}$. We then have

$$\int_{0}^{1} \int_{-\infty}^{+\infty} f(\psi^t_{p}(x'(s)))e^{-t}|(h \circ p)'(s)|_{x'(s)} dt ds = \int_{0}^{1} \int_{-\infty}^{+\infty} f(\psi^t_{p+u(s)}(x(s)))e^{-t}|(h \circ p)'(s)|_{\psi_{p}(s)(x(s))} dt ds$$

$$= \int_{0}^{1} \int_{-\infty}^{+\infty} f(\psi^t_{p+u(s)}(x(s)))e^{t+u(s)} |(h \circ p)'(s)|_{x(s)} dt ds$$

$$= \int_{0}^{1} \int_{-\infty}^{+\infty} f(\psi^t_{p}(x(s)))e^{t}|(h \circ p)'(s)|_{x(s)} dt ds ,$$

showing that $\mu_{\rho}$ does not depend on the choice of $x(s)$. Now, the fact that $\mu_{\rho}$ does not depend on the choice of $p(s)$ either is just the change of variable formula for the integration in $s$. We conclude that $\mu_{\rho}$ is well-defined. Moreover, by $\Gamma$-invariance of the metric $| \cdot |$, we easily verify that $\gamma_{*} \mu_{\rho} = \mu_{\gamma_{*} \rho}$. Hence $(\mu_{\rho})_{q \in \partial_{\infty} \Gamma}$ defines an unstable measure on $\mathbb{W}^u$.

From the definition of $\mu_{\rho}$ one easily proves that

$$\psi_{p}^{*} \mu_{\rho}(q) = e^t \mu_{\rho}(q).$$

Thus $\mu_{\rho}$ is scaled by $\psi^t$ with scale factor $1$.

It remains to prove that $\mu_{\rho}$ is invariant by holonomy along the horocycles of $\psi^t$. We leave that as an exercise to the reader. $\square$

4. **Foliated affine, hyperbolic and complex structures**

In this section, we show that the foliated affine actions introduced in Section 2.1 can be recovered from a foliated conformal structure. This allows us to construct the map $\text{CF}$.

4.1. **Affine actions and affine charts.** Let us first recall the definition of a foliated affine action.

**Definition 4.1.** A **foliated affine action** on $M_{\mathcal{F}}$ is a $\Gamma$-equivariant pair of continuous flows $((\varphi_t), (h_s))$ on $M_{\mathcal{F}}$ such that

- $(\varphi_t)$ is a parametrization of $\mathcal{G}$,
• the map $(t, s) \mapsto \varphi_t(h_s(x))$ is a covering from $\mathbb{R}^2$ to the stable leaf $W^s(x)$,
• $\varphi_{-t} \circ h_s \circ \varphi_t = h_{t+s}$.

It is called Hölder continuous if the flows $(\varphi_t)$ and $(h_s)$ are Hölder continuous.

Two foliated affine actions $(\varphi_t, h_s)$ and $(\varphi'_t, h'_s)$ are conjugated if there exists a homeomorphism of $M\Gamma$ preserving the leaves of $\mathcal{G}$ and conjugating $(\varphi_t)$ to $(\varphi'_t)$ and $(h_s)$ to $(h'_s)$.

As we will see, the data of a foliated affine action is essentially the same as what we call an equivariant family of affine charts:

**Definition 4.2.** An equivariant family of affine charts on $\partial_{\infty} \Gamma$ is the data, for any $x = (x_-, x_0, x_+) \in \widetilde{M}\Gamma$, of a homeomorphism $m_x : \partial_{\infty} \Gamma \setminus \{x_+\} \to \mathbb{R}$ such that:

• $m_x(x_0) = 0$ and $m_x(x_1) = 1$,
• $m_x$ depends continuously on $x$ for the compact open topology
• $m_{\gamma x} = m_x \circ \gamma^{-1}$ for all $\gamma \in \Gamma$,
• if $x$ and $x'$ belong to the same stable leaf, then $m_{x'} \circ m^{-1}_x$ is an affine transformation of $\mathbb{R}$.

It is called Hölder continuous if the homeomorphisms $m_x$ are bi-Hölder continuous and vary Hölder continuously for the compact open topology.

From an equivariant family of affine charts, one gets a foliated affine action by setting

• $\varphi_t(x_-, x_0, x_+) = (x_-, m^{-1}_x(e^t), x_+)$ (where $x = (x_-, x_0, x_+)$),
• $h_s(x_-, x_0, x_+) = (m^{-1}_x(s), m^{-1}_x(s+1), x_+)$.

Note that this affine action has the following property: for every $x = (x_-, x_0, x_+) \in \widetilde{M}\Gamma$, 

$$h_1(x) \in \mathcal{G}(x_0, x_+) \ .$$

We call such a foliated affine action normalized.

From a normalized foliated affine action, one gets an equivariant family of affine charts by setting $m_x(y_-) = s$ where $s$ is the unique real number such that the $h_s(x)$ belongs to the geodesic $\mathcal{G}(y_-, x_+)$. One can verify that the construction is inverse of the previous one. There is thus a bijection between normalized foliated affine actions and equivariant families of affine charts. Finally, we have the following:

**Proposition 4.3.** Every foliated affine action is conjugated to a unique normalized one.

*Proof.* Let $(\varphi_t, h_s)$ be a foliated affine action. Define $F(x) = (x_-, x_1, x_+)$, where $x_1$ is such that $h_1(x)$ belongs to $\mathcal{G}(x_1, x_+)$. Then $F$ descends to a homeomorphism of $M\Gamma$ preserving the leaves of $\mathcal{G}$.

Set $\varphi'_t = F \circ \varphi_t \circ F^{-1}$ and $h'_s = F \circ h_s \circ F^{-1}$. Then $(\varphi'_t, h'_s)$ is a foliated affine action. Let $(x_-, x_1, x_+)$ be a point in $\widetilde{M}\Gamma$. Then

$$h'_1(x_-, x_1, x_+) = F \circ h_1 \circ F^{-1}(x)$$

$$\in F(\mathcal{G}(x_1, x_+)) \text{ by definition of } F$$

$$\in \mathcal{G}(x_1, x_+) \ .$$
Proposition 4.6. The family of maps $(m_z)$ defined by
\[ m_z(\varphi_t(h_s(x))) = s + e^{t}i, \]
is a foliated hyperbolic structure. Moreover, a different choice of $(x_z)$ defines a foliated hyperbolic structure which is equivalent.

The main claim of this section is that, conversely, every foliated hyperbolic structure comes from a foliated affine action.
Theorem 4.7. Let \((m_z)\) be a foliated hyperbolic structure. Then each \(m_z\) extends continuously to a bi-Hölder homeomorphism \(\tilde{m}_z : \partial_\infty \Gamma \to \partial_\infty \mathbb{H}^2\). For every \(x = (x_-, x_0, z) \in \mathcal{F}(z)\), define \(g_x\) as the unique isometry of \(\mathbb{H}^2\) mapping \((\tilde{m}_z(x_-), \tilde{m}_z(x_0), \tilde{m}_z(x_+))\) to \((0, 1, \infty)\). Then the family \((g_x \circ \tilde{m}_z)\) is an equivariant family of affine charts on \(\partial_\infty \Gamma\).

Most of the proof of the theorem is straightforward once we know that quasi-isometries of \(\mathbb{H}^2\) extend to homeomorphisms of the boundary. The main technical difficulty is to control that this extension varies continuously with \(z\). This is dealt with in the next subsection.

4.3. Boundary extension of quasi-isometries. Let us first recall the classical Morse lemma for quasi-geodesics in the hyperbolic plane states that every quasi-geodesic ray of \(\mathbb{H}^2\) (i.e. every quasi-isometric embedding \(f : \mathbb{R} \to \mathbb{H}^2\)) is at bounded distance from a geodesic ray.

Proposition 4.8. Let \(f : \mathbb{R} \to \mathbb{H}^2\) be a \(C\)-quasi-geodesic ray. Then \(f(t)\) converges as \(t\) goes to \(+\infty\) to a point \(f(\infty)\) in \(\partial_\infty \mathbb{H}^2\), and there exists a constant \(D\) depending only on \(C\) such that any \(f(t)\) is at distance at most \(D\) from the geodesic ray \([f(0), f(\infty)]\).

Moreover, the point \(f(\infty)\) varies continuously with \(f\) for the compact open topology. More precisely, we have

Proposition 4.9. Let \(f : \mathbb{R} \to \mathbb{H}^2\) and \(g : \mathbb{R} \to \mathbb{H}^2\) be two \(C\)-quasi-geodesics with \(f(0) = g(0)\). Let \(f(\infty)\) and \(g(\infty)\) denote there respective endpoints in \(\partial_\infty \mathbb{H}^2\). Then there exists a constant \(K\) depending only on \(C\) such that, if \(d_{\mathbb{H}^2}(f(t), g(t)) \leq 1\) for some \(t \geq 1\), then

\[
\frac{d_{\mathbb{H}^2}^f(0)}{d_{\mathbb{H}^2}^g(0)}(f(\infty), g(\infty)) \leq Ke^{-t/C},
\]

where \(d_{\mathbb{H}^2}^f(0)\) denotes the visual distance from \(f(0)\) on \(\partial_\infty \mathbb{H}^2\).

As a corollary, one obtains that the boundary map induced by a quasi-isometry varies continuously for the compact-open topology:

Corollary 4.10. Let \(f : \mathbb{H}^2 \to \mathbb{H}^2\) be a \(C\)-quasi-isometric homeomorphism. Then \(f\) extends to a \(\frac{1}{C}\)-bi-Hölder homeomorphism \(\partial_\infty f : \partial_\infty \mathbb{H}^2 \to \partial_\infty \mathbb{H}^2\). Moreover, let \(g : \mathbb{H}^2 \to \mathbb{H}^2\) be another \(C\)-quasi-isometric homeomorphism. Then there is a constant \(K\) depending only on \(C\) such that, if \(R \geq 1\) and \(d_{\mathbb{H}^2}(f(x), g(x)) \leq 1\) for all \(x\) in \(B(o, R)\), then

\[
d_{\mathbb{H}^2}^f(\partial_\infty f(x), \partial_\infty g(x)) \leq Ke^{-R/C}
\]

for all \(x \in \partial_\infty \mathbb{H}^2\).

Proof. Exercise. \(\square\)

We can now turn to the

Proof of Theorem 4.7. Let us fix a background hyperbolic metric on \(\Sigma\). This provides us with an identification of \(\partial_\infty \Gamma\) with \(\partial_\infty \mathbb{H}^2\).

Using the identification of \(\mathcal{M}_T\) with \(T_1 \mathbb{H}^2\), one also obtains a reference foliated hyperbolic structure on \(\mathcal{W}^s\), which simply projects every leaf \(T_1 \mathbb{H}^2\) to \(\mathbb{H}^2\). We see this reference foliated structure as an identification of each leaf of \(\mathcal{W}^s\) with \(\mathbb{H}^2\).
Let now \((m_z)\) be another foliated hyperbolic structure. Then
\[
m_z : \tilde{W}^s(z) \simeq \mathbb{H}^2 \to \mathbb{H}^2
\]
is a C-quasi-isometry for some \(C\) independent of \(z\). It thus extends to a bi-Hölder homomorphism \(\partial_{\infty}m_z : \partial_{\infty}\mathbb{H}^2 \simeq \partial_{\infty}\Gamma \to \partial_{\infty}\mathbb{H}^2\). Moreover, for every \(x = (x_-,x_0,z) \in \mathcal{F}^s(z)\), there is a unique hyperbolic isometry \(g_x\) such that \(\tilde{m}_x = g_x \circ \partial_{\infty}m_z\) maps \(x_-\) to 0, \(x_0\) to 1 and \(z\) to \(\infty\). By restriction, \(\tilde{m}_x\) defines a bi-Hölder continuous homeomorphism from \(\partial_{\infty}\Gamma \setminus \{z\}\) to \(\mathbb{R}\).

By Corollary 4.10, since \(m_z\) varies continuously with \(z\) for the compact-open topology, so does the family of maps \((\tilde{m}_x)\). One easily checks that \((\tilde{m}_x)\) also satisfies the other properties of an equivariant family of affine charts.

\[\square\]

4.4. Smoothening foliated hyperbolic structures. So far we associated to every foliated affine action a foliated hyperbolic structure. This hyperbolic structure, however, has rather low regularity (the developing map of each leaf is only continuous, and varies transversally continuously for the \(C^0\) topology. Foliated Teichmüller theory, on the other side, has been developed mainly for leafwise smooth foliated structures. This section is thus devoted to the proof of the following lemma:

**Lemma 4.11.** Let \(((\varphi_t), (h_s))\) and \(((\varphi'_t), (h'_s))\) be two foliated affine actions on \(M_\Gamma\). Let \((m_z)\) and \((n_z)\) be the foliated hyperbolic structures associated respectively to affine actions \((m_z)\) and \((n_z)\) as in Proposition 4.6. Then \((n_z)\) is isotopic to a foliated hyperbolic structure \((n'_z)\) such that \(n'_z \circ m_{z^{-1}}\) is a \(C^\infty\) diffeomorphism that varies continuously with \(z\) for the \(C^\infty\) topology.

**Proof.** One easily goes from \(C^1\) regularity to \(C^\infty\) regularity by a standard smoothening argument. We focus here on isotoping \(n_z\) to a hyperbolic structure with \(C^1\) regularity. Recall first that, by Proposition 1.11, we can assume without loss of generality that \((\varphi'_t)\) is a reparametrization of \((\varphi_t)\) which is \(C^1\) along the orbits.

For \(\varepsilon\) small enough (to be chosen later), define \((n^\varepsilon_z)\) by
\[
n^\varepsilon_z(x) = \frac{1}{\varepsilon} \int_{0}^{\varepsilon} n_z(h_s(x))ds ,
\]
where \(n_z\) is seen as a map to the upper-half space inside the complex plane. Recall that for every \(\gamma \in \Gamma\), we have
\[
n^\varepsilon z \Gamma = g \circ n_z \circ \gamma^{-1}
\]
for some affine transformation \(g\) of \(\mathbb{H}^2\). This induces the same property for \(n^\varepsilon_z\). It remains to see that \(n^\varepsilon_z \circ m_{z^{-1}}\) is a \(C^1\) diffeomorphism for \(\varepsilon\) small enough.

Since both \(m_z\) and \(n_z\) map geodesics in \(\tilde{M}_\Gamma\) to vertical geodesics in \(\mathbb{H}^2\), we can write \(F = n_z \circ m_{z^{-1}}\) in the form
\[
F(x + iy) = f(x) + ig(x,y) ,
\]
where \(f : \mathbb{R} \to \mathbb{R}\) is an increasing homeomorphism, and where \(g(x,\cdot) : \mathbb{R}_{>0} \to \mathbb{R}_{>0}\) is a \(C^1\) diffeomorphism varying continuously with \(x\) for the \(C^1\) topology. Writing \(G^\varepsilon = n^\varepsilon_z \circ m_{z^{-1}}\), we have
\[
G^\varepsilon(x + iy) = \frac{1}{\varepsilon y} \int_{0}^{\varepsilon y} F(x + s + iy)ds .
\]
An elementary computation shows that $G^\varepsilon$ is $C^1$ and that
\[
\frac{\partial}{\partial x} G^\varepsilon(x + iy) = \frac{1}{\varepsilon y} (F(x + \varepsilon y + iy) - F(x + iy))
\] 
\[
= \frac{1}{\varepsilon y} \left( f(x + \varepsilon y) - f(x) \right) + i \cdot \frac{1}{\varepsilon y} \left( g(x + \varepsilon y, y) - g(x, y) \right),
\]
\[
\frac{\partial}{\partial y} G^\varepsilon(x, y) = -\frac{1}{y} G^\varepsilon(x + iy) + \frac{1}{y} F(x + \varepsilon y + iy) + \frac{1}{\varepsilon y} \int_0^{\varepsilon y} \frac{\partial}{\partial y} F(x + s + iy) ds
\] 
\[
= \frac{1}{y} \left( f(x + \varepsilon y) - \frac{1}{\varepsilon y} \int_0^{\varepsilon y} f(x + s) ds \right) + \frac{1}{\varepsilon y} \int_0^{\varepsilon y} \frac{\partial}{\partial y} f(x + s) ds.
\]

Note that, since $f$ is an increasing homeomorphism, we have 
\[0 < C(\varepsilon) < A(\varepsilon).\]

Let us compute the determinant of $dG^\varepsilon$. We have
\[
\text{Jac } G^\varepsilon(x + iy) = \frac{1}{\varepsilon y} \left( A(\varepsilon)(D(\varepsilon) + D'(\varepsilon)) - B(\varepsilon)C(\varepsilon) \right)
\] 
\[
= \frac{A(\varepsilon)}{\varepsilon y} \left( D(\varepsilon) + D'(\varepsilon) - \frac{C(\varepsilon)}{A(\varepsilon)} B(\varepsilon) \right).
\]

By continuity of $g$, the terms $B(\varepsilon)$ and $D(\varepsilon)$ go to $0$ as $\varepsilon$ goes to $0$, while
\[D'(\varepsilon) \to \frac{\partial}{\partial y} g(x, y) > 0.\]

Thus $\text{Jac } G^\varepsilon(x + iy)$ is positive for $\varepsilon$ small enough depending only on the (local) module of continuity of $F$. By compactness of $M_F$ and the equivariance of $n_\varepsilon^x$, we deduce the existence of $\eta$ such that $n_\varepsilon^x$ is a diffeomorphism for all $z$ and all for $\varepsilon < \eta$.

Hence, for $\varepsilon$ small enough, $(n_\varepsilon^x)$ is a foliated hyperbolic structure isotopic to $(n_z)$ and such that $n_\varepsilon^x \circ m_z^{-1}$ is $C^1$ for all $z$. 

\[\square\]

4.5. The space $\mathcal{T}(\mathcal{W}^s)$ and the map $\text{CF}$. Teichmüller spaces of 2-dimensional foliations (or, more generally, of 2-dimensional laminations) were introduced by Sullivan in [3]. Building on the works of Ahlfors and Bers, he pointed out that large aspects of classical Teichmüller theory extended to the context of foliated conformal structures.

**Definition 4.12.** A smooth foliated conformal structure on $\mathcal{W}^s$ is a family of conformal classes of metrics on the leaves of $\mathcal{W}^s$ which vary continuously with the leaf for the smooth topology. Two foliated conformal structures are homotopic if one is the pull-back of the other by a leafwise homotopy of $\mathcal{W}^s$. 

The Teichmüller space of the foliation \( \mathcal{W}^s \), denoted \( \mathcal{T}(\mathcal{W}^s) \), is the set of homotopy classes of smooth foliated conformal structures on \( \mathcal{W}^s \).

Candel’s theorem, the foliated analog of Poincaré’s uniformization, shows that one can alternatively see \( \mathcal{T}(\mathcal{W}^s) \) as the space of homotopy classes of foliated hyperbolic structures.

**Theorem 4.13** (Candel). Let \([g]\) be a smooth foliated conformal structure on \( \mathcal{W}^s \). Then \([g]\) contains a unique foliated Riemannian metric of curvature \(-1\).

Given \([g_1]\) and \([g_2]\) two foliated conformal classes, define the **conformal dilatation** \( \text{dil}([g_1], [g_2]) \) as the infimum of the constants \( K \geq 1 \) such that there exists \( g_1 \in [g_2] \) and \( g_2 \in [g_2] \) with

\[
\frac{1}{K} g_1 \leq g_2 \leq Kg_1.
\]

**Definition 4.14.** The **Teichmüller distance** between \([g_1]\) and \([g_2]\) is defined as

\[
d_T([g_1], [g_2]) = \inf\{ \log \text{dil}([g_1], [g_2]), [g_1'] \text{ homotopic to } [g_1], [g_2'] \text{ homotopic to } [g_2] \}.
\]

By definition, the Teichmüller distance is well-defined on \( \mathcal{T}(\mathcal{W}^s) \). Sullivan proves in [I] that it is indeed a distance. Moreover the space \( \mathcal{T}(\mathcal{W}^s) \) with the induced topology has a structure of complex Banach manifold:

**Theorem 4.15** (Sullivan). The space \( (\mathcal{T}(\mathcal{W}^s), d_T) \) is homeomorphic to a Banach manifold such that, for every foliated conformal structure \([g]\), there is a biholomorphism from \( \mathcal{T}(\mathcal{W}^s) \) to a bounded open domain in the space \( \text{QD}(\mathcal{W}^s, [g]) \) of foliated quadratic differentials which are holomorphic with respect to the conformal structure \([g]\).

Let \([g]\) be a foliated conformal structure on \( \mathcal{W}^s \) and let \( \gamma \in [\Gamma] \) be a closed leaf of \( \mathcal{G} \). This leaf is contained in a unique leaf \( \mathcal{W}^s(\gamma) \) which is homeomorphic to a cylinder.

**Definition 4.16.** The **period map** \( L_{[g]} \) of the conformal structure \([g]\) associates to \( \gamma \in [\Gamma] \) the translation length of \( l_\gamma \), where \( l_\gamma \) is an isometry of \( \mathbb{H}^2 \) such that \( (\mathcal{W}^s(\gamma), [g]) \) is conformal to \( l_\gamma \backslash \mathbb{H}^2 \).

We can now turn to the proof of Theorem 0.4. We start with the first part:

**Theorem 4.17.** There exists a map \( \text{CF} : \mathcal{T}(\mathcal{W}^s) \rightarrow \text{Par}(\mathcal{G}) \) such that

\[
L_{\text{CF}([g])} = L_{[g]}.
\]

**Proof.** Let \([g]\) be a foliated conformal structure on \( \mathcal{W}^s \). By Candel’s theorem, the conformal class \([g]\) contains a unique foliated hyperbolic metric \( g_{\text{hyp}} \), which can be seen as a smooth foliated hyperbolic structure. One then associates to \( g_{\text{hyp}} \) a family of affine charts \((m^g)\) on \( \partial_{\infty} \Gamma \) via Theorem 4.7.

If \([g']\) is homotopic to \([g]\) then \( g'_{\text{hyp}} \) is homotopic to \( g_{\text{hyp}} \). In particular, the developments of a given leaf into \( \mathbb{H}^2 \) associated respectively to \( g_{\text{hyp}} \) and \( g'_{\text{hyp}} \) remain at bounded distance from each other. They thus induce the same
boundary maps, and therefore the same families of affine charts on $\partial_\infty \Gamma$. In conclusion, the map

$$[g] \mapsto (m^g)$$

is well defined from $\mathcal{T}(\mathcal{W}^s)$ to the space of equivariant families of affine charts. Finally, the family of affine charts $(m^g)$ defines a foliated affine action $(\varphi^g_t, h^g_t)$, and we define

$$\text{CF} : \mathcal{T}(\mathcal{W}^s) \to \text{Par}(\mathcal{G})$$

$$[g] \mapsto [\varphi^g].$$

Let us prove that CF preserves the period maps. Let $\gamma$ be an element in $\Gamma \setminus \text{Id}$. Fix a point $x = (\gamma_-, x_0, \gamma_+)$ and let $m$ be the isometry from $(\tilde{\mathcal{W}}^s(\gamma_+), g_{hyp})$ to $\mathbb{H}^2$ whose extension to the boundary maps $\gamma_+$ to $\infty$, $\gamma_-$ to 0 and $x_0$ to 1.

Since $\gamma$ acts on $(\tilde{\mathcal{W}}^s(\gamma_+), g_{hyp})$ as an isometry of translation length $l = L_{[g]}(\gamma)$ and fixes $\gamma_-$ and $\gamma_+$, we have

$$m \circ \gamma \circ m^{-1} : z \mapsto e^l z.$$

On the other side, by definition of the affine action associated to $m^g$, we have $\varphi^g_t(x) = (\gamma_-, x_1, \gamma_+)$, where $m(x_1) = e^t$. We deduce that $\varphi^g_t(x) = \gamma \cdot x$, hence $l = L_{\varphi^g}([\gamma])$. □

It remains to prove the continuity of the map CF. We actually prove a stronger result:

**Theorem 4.18.** The map $\text{CF} : (\mathcal{T}(\mathcal{W}^s), d_T) \to (\text{Par}(\mathcal{G}), d_{HT})$ is Lipschitz continuous.

**Proof.** Let $[\gamma]$ be a closed leaf of $\mathcal{G}$ and $[g]$ a foliated conformal structure on $\mathcal{W}^s$. Recall that the hyperbolic length $L_{[g]}(\gamma)$ is proportional to its extremal length, defined as

$$E\text{L}_{[g]}(\gamma) = \sup_{g' \gamma'} \frac{\text{length}_g(\gamma')^2}{\text{area}(g)},$$

where the infimum is taken over all curves $\gamma'$ freely homotopic to $\gamma$ in $\mathcal{W}^s(\gamma)$ and the supremum is taken over all metrics $g'$ on $\mathcal{W}^s(\gamma)$ in the conformal class of $[g]$ and of finite area.

Now, one easily verifies that, if $\text{dil}([g_1], [g_2]) = K$, then

$$\frac{1}{K^2}E\text{L}_{[g_1]}(\gamma) \leq E\text{L}_{[g_2]}(\gamma) \leq K^2E\text{L}_{[g_2]}(\gamma).$$

We deduce that

$$\left| \log \left( \frac{L_{[g_2]}(\gamma)}{L_{[g_1]}(\gamma)} \right) \right| = \left| \log \left( \frac{E\text{L}_{[g_2]}(\gamma)}{E\text{L}_{[g_1]}(\gamma)} \right) \right| \leq d_T([g_1], [g_2]),$$

and therefore

$$d_{HT}(\text{CF}([g_1]), \text{CF}([g_2])) = \frac{1}{2} \left( \sup_{\gamma \in \Gamma} \log \left( \frac{L_{[g_1]}(\gamma)}{L_{[g_2]}(\gamma)} \right) + \sup_{\gamma \in \Gamma} \log \left( \frac{L_{[g_2]}(\gamma)}{L_{[g_1]}(\gamma)} \right) \right)$$

$$\leq d_T([g_1], [g_2]).$$

□
5. Construction of Margulis measures

In his thesis, Margulis constructed the measure of maximal entropy of an Anosov flow by first constructing what we called Margulis measures along stable and unstable leaves. Here we reproduce his argument to prove the following theorem:

**Theorem 5.1 (Margulis).** Let $\varphi$ be a parametrization of the geodesic foliation which admits stable (resp. unstable) horocycles. Then $\varphi$ admits an unstable (resp. stable) Margulis measure.

The starting point of Margulis’s construction is a family of unstable measures $(\nu_p)$ which are “almost preserved” by the horocycle holonomy. These measures are given by the volume form associated to some Riemannian metric on unstable leaves. Since we work in low regularity here, one needs an additional argument to find such a family of measures.

5.1. Almost invariant unstable measures.

**Definition 5.2.** An unstable measure $\mu$ is called *almost holonomy invariant* if for every $p, q \in \partial_{\infty} \Gamma$, we have

$$T_{p,q}^* \mu_q = f_{p,q} \mu_p$$

where $f_{p,q}$ is continuous on $\mathcal{W}^u(p)$, depends continuously on $p$ and $q$ and satisfies

$$|f_{p,q}(x) - 1| \leq \eta(d(x, T_{p,q}(x)))$$

where $d$ is a $\Gamma$-invariant distance on $\widetilde{M}_\Gamma$ and $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ is a function such that $\eta(s) \to 0$ as $s \to 0$.

Our argument here to construct an almost holonomy invariant unstable measure slightly more elaborate than that of Margulis due to the a priori weak regularity of $\varphi$ and $\mathcal{H}^s$.

Let us start by fixing a hyperbolic metric on $\Sigma$ and denote by $\varphi_0$ and $\mathcal{H}_0^s$ the associated geodesic flow and stable horocyclic foliation. We provide $M_{\Gamma}$ with the smooth structure and the Riemannian metric induced by the identification $M_{\Gamma} \simeq T_1 \Sigma$. We denote by $X_0$ the vector field generating $\varphi_0$.

Each unstable leaf $\mathcal{W}^u(p)$ is identified with the hyperbolic plane, and we provide it with the hyperbolic area form $\lambda_p$. These area forms thus give rise to an unstable measure $\lambda$. The following is a good exercise:

**Proposition 5.3.** The unstable measure $\lambda$ is a Margulis measure for $\varphi_0$.

Let now $\varphi$ be another parametrization of the geodesic foliation which admits a stable horocyclic foliation $\mathcal{H}^s$. We want to find an unstable measure which is almost invariant by the holonomy along $\mathcal{H}^s$.

By Proposition 2.12, without loss of generality, we can assume that the reparametrization cocycle $c_{\varphi_0 \to \varphi}$ is the stable Buseman cocycle $B$ associated to a closed stable 1-form $\alpha$ that is smooth on each stable leaf. Let $p, q$ be two points in $\partial_{\infty} \Gamma$ and let $T_{p,q}$ and $T_{p,q}^0$ denote the holonomies from $\widetilde{\mathcal{W}}^u(p)$ to $\widetilde{\mathcal{W}}^u(q)$ along $\mathcal{H}^u$ and $\mathcal{H}_0^s$ respectively.
Proposition 5.4. There is a constant $C$ (independent of $p$ and $q$) such that for all $x \in \tilde{W}^{u}(p)$,

$$T_{q,p}^0 \circ T_{p,q}(x) = \varphi_0(x, s(x)),$$

where

- $s$ is continuous and

$$|s(x)| \leq Cd(x, T_{p,q}(x)),$$

- $s$ is differentiable along $\varphi_0$, $X_0 \cdot s$ is continuous and

$$|X_0 \cdot s(x)| \leq Cd(x, T_{p,q}(x))$$

Define $X_0$.

Corollary 5.5. The stable measure $\lambda$ is almost invariant under the holonomy of $\mathcal{H}^s$.

Proof of Proposition 5.4. By construction, $T_{p,q}$ and $T_{q,p}^0$ map a geodesic to a geodesic. Indeed, if $x$ and $y$ belong to a geodesic contained in $\tilde{W}^{u}(p)$, then $T_{p,q}(x)$ and $T_{p,q}(y)$ belong to the same unstable leaf $\tilde{W}^{u}(q)$ and to the same stable leaf $\tilde{W}^{s}(x) = \tilde{W}^{s}(y)$ (since $T_{p,q}$ “follows” stable horocycles). The same holds for $T_{q,p}^0$. Therefore, for $x$ and $T_{q,p}^0 \circ T_{p,q}(x)$ belong to the same geodesic.

Let $s(x)$ be such that $T_{q,p}^0 \circ T_{p,q}(x) = \varphi_0(x, s(x))$. Let $\alpha_0$ be the smooth closed stable 1-form such that $\int_{y}^{\varphi_0(y,t)} = t$. By definition of $T_{q,p}^0$, we have

$$\int_{T_{p,q}(x)}^{T_{q,p}^0 \circ T_{p,q}(x)} \alpha_0 = 0$$

and thus

$$s(x) = \int_{x}^{\varphi_0(x,s(x))} \alpha_0 = \int_{x}^{T_{p,q}(x)} \alpha_0.$$

By continuity of $\alpha_0$ and $T_{p,q}$, we have

$$|s(x)| \leq Cd(x, T_{p,q}(x)),$$

where $C$ is a uniform bound on $\alpha_0$.

Let us now prove the derivability of $s$ along $\varphi_0$. Recall that $\varphi_0$ and $\varphi$ are respectively generated by the vector fields $X_0$ and $X$, tangent to the geodesic foliations, such that $\alpha_0(X_0) = \alpha(X) = 1$.

By construction, we have $T_{p,q}(\varphi(x, \varepsilon)) = \varphi(T_{p,q}(x), \varepsilon)$. Thus

$$s(\varphi(x, \varepsilon)) - s(x) = \int_{\varphi(x,\varepsilon)}^{\varphi(T_{p,q}(x),\varepsilon)} \alpha_0 - \int_{x}^{T_{p,q}(x)} \alpha_0 = \int_{x}^{\varphi(T_{p,q}(x),\varepsilon)} \alpha_0 - \int_{T_{p,q}(x)}^{\varphi(x,\varepsilon)} \alpha_0.$$

It follows that $s$ is derivable along $\varphi$ and

$$X \cdot s(x) = \alpha_0(X)_x - \alpha_0(X)_{T_{p,q}(x)}.$$
Since \( \alpha(X) = \alpha_0(X_0) = 1 \), we have \( X_0 = fX \) where \( f = \alpha(X_0) = \frac{1}{\alpha_0(X)} \).

Hence \( s \) is derivable along \( \varphi_0 \) and

\[
X_0 \cdot s(x) = 1 - \frac{\alpha_x(X_0)}{\alpha_{T_{p,q}(x)}(X_0)} .
\]

Since \( \alpha(X_0) \) is continuous, positive and smooth in restriction to weakly stable leaves, we deduce the existence of a constant \( C \) such that

\[
|X_0 \cdot s(x)| \leq C d(x, T_{p,q}(x)) .
\]

Hence \( s \) is derivable along \( \varphi_0 \) and

\[
X_0 \cdot s(x) = 1 - \alpha_x(X_0) - \alpha_0(X) .
\]

Since \( \alpha_0(X) \) is continuous, positive and smooth in restriction to weakly stable leaves, we deduce the existence of a constant \( C \) such that

\[
|X_0 \cdot s(x)| \leq C d(x, T_{p,q}(x)) .
\]

□

Let us now deduce Corollary 5.5. Set \( \sigma(x) = T_{q,p}^0 \circ T_{p,q}(x) = \varphi_0(x, s(x)) \).

Since \( T_{p,q}^0 \lambda_p = \lambda_q \), we have

\[
T_{p,q}^* \lambda_q = \sigma^* \lambda_p .
\]

Now, there are coordinates \( (u, v) \) on \( \tilde{W}_u(p) \) with respect to which \( \varphi_0((u, v), t) = (u + t, v) \) and such that \( \lambda_p = e^{-u} du dv \). Corollary 5.5 thus follows from the following computation:

**Lemma 5.6.** Let \( \sigma : \mathbb{R}^2 \to \mathbb{R}^2 \) be a homeomorphism given by \( \sigma(u, v) = (u + s(u, v), v) \), where \( s \) is continuous and differentiable with respect to \( u \) with continuous partial derivative. Then

\[
\sigma^*(e^{-u} du dv) = e^{-s(1 + \frac{\partial s}{\partial u})} du dv .
\]

**Proof.** Exercise. □

**5.2. Margulis measures on unstable leaves.** To construct an unstable measure which is holonomy invariant and scaled by \( \varphi \), Margulis’s approach is roughly to “pull back” the measure \( \lambda \) by \( \varphi_t \) for large \( t \) and suitably rescale it. The crucial point of this approach is the following lemma, which gives a “uniform way” to rescale \( \varphi_t^* \lambda \). Let us first set some terminology.

We call a subset \( K \) of \( M_T \) a compact subset of \( W_u \) if it is a finite union of subsets \( K_i \) which are each contained in a single unstable leaf and compact for the topology of the leaf.

Given a compact subset \( K \) of \( W_u \), we call a function \( f : M_T \to \mathbb{R} \) a continuous function on \( W_u \) with support in \( K \) if \( f \) is continuous in restriction to each leaf and vanishes outside \( K \). We denote by \( C(K) \) the space of continuous functions on \( W_u \) with support in \( K \), and by denote by \( C_c(W_u) \) the vector space of continuous functions on \( W_u \) with compact support. We endow this space with the norm

\[
\|f\|_\infty = \sup_{M_T} |f| .
\]

**Lemma 5.7.** There exists a non-negative function \( f_0 \in C_c(W_u) \) such that, for every compact subset \( K \) of \( W_u \), there is a constant \( C = C(K, f_0) \) such that for every \( f \in C(K) \) and all \( t \geq 0 \), we have

\[
\left| \int f \circ \varphi_{-t} d\lambda \right| \leq C \|f\|_\infty \int f_0 \circ \varphi_{-t} d\lambda .
\]
Proof. Given an open subset $U$ of $\mathcal{W}^u$, denote by $\mathcal{H}^s(U)$ the union of all the leaves of $\mathcal{H}^s$ intersecting $U$. We first claim that we can find $U$ with compact closure and large enough so that $\mathcal{H}^s(U) = M^\Gamma$. Indeed,

$$\mathcal{H}^s \left( \bigcup_{t \in \mathbb{R}} \varphi_t(U) \right) = \bigcup_{t \in \mathbb{R}} \mathcal{H}^s(\varphi_t(U))$$

is a non empty open set saturated by the leaves of $\mathcal{W}^u$. It is thus equal to the whole $M^\Gamma$ by minimality of the weakly unstable foliation. By compactness of $M^\Gamma$, there is $T > 0$ such that

$$\mathcal{H}^s \left( \bigcup_{-T \leq t \leq T} \varphi_t(U) \right) = M^\Gamma.$$

Let us hence fix an open subset $U_0$ of $\mathcal{W}^u$ with compact closure such that every horocycle intersects $U_0$, and let $f_0$ be a continuous non-negative function on $\mathcal{W}^u$ with compact support such that $f_0 = 1$ on $U_0$. Let now $K$ be a compact subset of $\mathcal{W}^u$. Let us lift $K$ and $U_0$ to compact and open sets $\tilde{K}$ and $\tilde{U}_0$ respectively contained in $\tilde{\mathcal{W}}^u(p)$ and $\tilde{\mathcal{W}}^u(q)$ for some $p, q \in \partial_x \Gamma$, and lift $f_0$ to a continuous $\Gamma$-invariant function on $\tilde{\mathcal{W}}^u$ (that we still denote $f_0$). By construction of $\tilde{U}_0$, we can find a covering of $K$ by finitely many open subsets $(V_i)_{1 \leq i \leq k}$ and $\gamma_1, \ldots, \gamma_k \in \Gamma$ such that $V_i$ is contained in $T_{\gamma_i, q, p}(V_i, U_0)$. Let $f$ be a continuous function with support in $K$. Using partitions of unity, we can assume without loss of generality that $f$ has support in one of the $V_i$’s, say $V_1$. We can also assume that $\gamma_1 = \text{Id}$.

For all $t \geq 0$, we have

$$\left| \int f \circ \varphi_{-t} d\lambda_p \right| \leq \| f \|_\infty \lambda(\varphi_t(V_1))$$

$$= \| f \|_\infty \int_{T_{p, q}\varphi_t(V_1)} T_{q, p, x} \lambda_p$$

$$\leq Cst \| f \|_\infty \sup_{x \in \varphi_t(V_1)} d(x, T_{q, p}(x)) \lambda_q(T_{p, q}(\varphi_t(V_1))))$$

$$\leq Cst \| f \|_\infty \sup_{x \in \varphi_t(V_1)} d(x, T_{q, p}(x)) \int f_0 \circ \varphi_{-t} d\lambda_q \quad \text{since } f_0 \circ \varphi_{-t} \text{ is positive and equi-}$$

Finally, since $T_{q, p}$ is the holonomy along the stable horocycles of $\varphi$, we have that $d(\varphi_t(x), T_{p, q}(\varphi_t(x)))$ goes to $0$ as $t$ goes to $+\infty$, from which we deduce that $\sup_{x \in \varphi_t(V_1)} d(x, T_{q, p}(x))$ is bounded uniformly in $t$, giving the inequality

$$\left| \int f \circ \varphi_{-t} d\lambda \right| \leq Cst' \| f \|_\infty \int f_0 \circ \varphi_{-t} d\lambda.$$

Let $L = \mathbb{R}^C_c(\mathcal{W}^u)$ denote the space of all functions on $C_c(\mathcal{W}^u)$, provided with the product topology (i.e. the topology of pointwise convergence). We

\footnote{Note that any $U$ would suit if we knew that the horocycle foliation was minimal. We believe it is true but could not easily adapt Hedlund’s theorem in our setting.}
see an unstable measure $\mu$ as an element of $L$ by setting

$$\mu(f) = \int f \, d\mu.$$  

For $t \in \mathbb{R}_+$, define $\lambda_t(f) = \frac{\int f \circ \varphi_{-t} \, d\lambda}{\int f \circ \varphi_{-t} \, d\lambda}$. For each $T \geq 0$, let $\Omega_T$ denote the convex hull of $\{\lambda_t, t \geq T\}$ in $L = \mathbb{R}C_0(\mathcal{W}^u)$.

By Lemma 5.7, for every $f \in C_c(\mathcal{W}^u)$, there is a constant $C_f$ such that $|\lambda_t(f)| \leq C_f$ for all $t \in \mathbb{R}_+$. Thus

$$\Omega_T \subset \{\alpha \in L \mid \alpha(f) \leq C_f \text{ for all } f \in C_c(\mathcal{W}^u)\}$$  

and the closure of $\Omega_T$ is compact for all $T$.

For each $s \geq 0$, there is a constant $A_s$ such that

$$\lambda_t(f_0 \circ \varphi_{-s}) \geq A_s.$$  

thus, $\alpha \to \alpha(f_0 \circ \varphi_{-s})$ is positive on $\overline{\Omega}_0$. Since it is continuous, we conclude that the operator

$$\hat{\varphi}_s^*: \overline{\Omega}_0 \to \overline{\Omega}_0 \subset \overline{\Omega}_0$$  

$$\alpha \mapsto \hat{\varphi}_s^* \alpha : f \mapsto \frac{\alpha(f \circ \varphi_{-s})}{\alpha(f_0 \circ \varphi_{-s})}$$

is continuous. The following proposition concludes the proof of Theorem 5.1.

**Proposition 5.8.** There exists a point in $\overline{\Omega}_0$ which is fixed by $\hat{\varphi}_s^*$ for all $s$. This point is associated to an unstable Margulis measure for $\varphi$.

**Proof.** The Tychonoff fixed point theorem implies that each $\hat{\varphi}_s^*$ has a fixed point $\mu_n$ in $\overline{\Omega}_0$. Let $\mu$ be an accumulation point of $(\mu_n)$. Then $\mu$ is fixed by $\hat{\varphi}_s^*$ for every diadic $s$.

Let $K$ be a compact subset of $\mathcal{W}^u$. By Lemma 5.7, for every $f \in C(K)$, we have $\lambda_t(f) \leq C(K, f_0) \|f\|_\infty$ for all $t \geq 0$. Passing to the convex hull and then to the limit, we deduce that $\mu$ is linear on $C(K)$, continuous, and non negative on positive functions. Riesz’s representation theorem then implies that $\mu$ is an unstable measure. We also get that $\mu$ is fixed by $\hat{\varphi}_s^*$ for all $s$ by continuity. Thus $\mu$ is scaled by $\varphi$.

It remains to prove that $\mu$ is holonomy invariant. Let $f$ be a continuous function on $\mathcal{W}^u$ with compact support $K \subset \mathcal{W}^u(p)$ for some $p \in \partial_\infty \Gamma$, and let $q$ be another point in $\partial_\infty \Gamma$. We have

$$|\lambda_t(f \circ T_{p,q}) - \lambda_t(f)| = \left| \frac{\int f \circ \varphi_{-t} \circ T_{p,q} \, d\lambda_q - \int f \circ \varphi_{-t} \, d\lambda}{\int f_0 \circ \varphi_{-t} \, d\lambda} \right|$$  

$$= \left| \frac{\int f \circ \varphi_{-t} \, d\lambda_q - \int f \circ \varphi_{-t} \, d\lambda_p}{\int f_0 \circ \varphi_{-t} \, d\lambda} \right|$$  

$$\leq Cst \sup_{x \in \varphi(t)(K)} d(x, T_{p,q}(x)) \frac{\int |f \circ \varphi_{-t}| \, d\lambda}{\int f_0 \circ \varphi_{-t} \, d\lambda}$$  

$$\leq Cst^t(f) \sup_{x \in \varphi(t)(K)} d(x, T_{p,q}(x)).$$

Since $T_{p,q}$ is the holonomy along the horocyclic foliation of $\varphi$, we have that $d(\varphi_t(x), T_{p,q}(\varphi_t(x))) \underset{t \to +\infty}{\to} 0$ uniformly on $K$. Passing to the convex hull
and to the limit, we deduce that
\[ \int f \circ T_{p,q} d\mu = \int f d\mu . \]
Thus \( \mu \) is invariant under horocyclic holonomy. \( \square \)

5.3. Scaling factor and entropy. In general, we don’t know whether unstable Margulis measure is unique (see Section 6.2). Nonetheless, we prove here that the scaling factor of any such measure is the same, and that this scaling factor defines in fact a continuous function \( h_{\text{top}} \) on \( \text{Par}(G) \). Experts will have recognized the topological entropy. More precisely we could prove the following:

**Theorem 5.9.** Let \( \varphi \) be a parametrization of \( G \) with stable horocycles, and let \( \mu \) be an unstable Margulis measure for \( \varphi \) with scale factor \( a \). Then:

- \( a \) is the topological entropy of \( \varphi \),
- There exists a \( \varphi \)-invariant probability measure \( \nu \) on \( M_\Gamma \) which disintegrates to \( \mu \) along stable horocycles,
- The measure \( \nu \) has metric entropy equal to \( a \) (equivalently, \( \nu \) is a measure of maximal entropy).

Since we try to avoid introducing the entropy here, we content ourselves with the following theorem:

**Theorem 5.10.** Let exists a function
\[ h_{\text{top}} : \text{Par}(G) \to \mathbb{R}_{>0} \]
such that if \( \varphi \) is a parametrization of \( G \) with stable (resp. unstable) horocycles and \( \mu \) is an unstable (resp. stable) Margulis measure for \( \varphi \), then the scale factor of \( \mu \) equals \( h_{\text{top}}([\varphi]) \) (resp. \( -h_{\text{top}}([\varphi]) \)).

To prove this, we describe a standard procedure to combine a stable and an unstable measure into a measure on \( M_\Gamma \).

Let \( \mu \) be a stable measure on \( W^s \) and let \( c \) be a continuous curve contained in a leaf of \( W^u \) and transverse to \( G \) inside that leaf. We define the projection of \( \mu \) to \( c \) as the measure \( \mu_c \) defined by
\[ \mu_c(I) = \mu \left( \bigcup_{t \geq 0} \varphi_t(I) \right) , \]
where \( I \) is some interval in \( c \) and \( \varphi \) is any parametrization of \( G \).

Let now \( \varphi \) be a parametrization of \( G \) admitting a stable horocyclic foliation \( H \). We call a family of measures on the leaves of \( H \) a horocyclic measure. Given a stable measure \( \mu \), we get an horocyclic measure \( \mu_H \) by projecting \( \mu \) onto the leaves of \( H \).

**Proposition 5.11.** Let \( \varphi \) and \( \psi \) be two parametrizations of \( G \) such that

- \( \varphi \) admits a stable horocyclic foliation \( H \),
- there is a locally finite stable measure \( \mu \) which is scaled by \( \psi \) with scale factor \( b \).
Then the horocyclic projection \( \mu_H \) is locally finite and satisfies
\[
\varphi_t^* \mu_H = e^{-bc(t, \cdot)} \mu_H ,
\]
where \( c \) is the reparametrization cocycle of \( \psi \) with respect to \( \varphi \).

**Proof.** Let \( \mathcal{H}(x) \) be some horocycle of \( \varphi \) and identify the weakly stable leave \( \bar{W}^s(x) \) containing \( \mathcal{H}(x) \) with \( Hh(x) \times \mathbb{R} \) via the map
\[
(y, t) \rightarrow \psi_t(y) .
\]
Since the measure \( \mu \) is scaled by \( \psi \), it can be written in those coordinates in the form
\[
d\mu = d\nu \otimes e^{-bt}dt ,
\]
where \( \nu \) is a locally finite measure on \( H(x) \). An easy computation then shows that
\[
\mu_H(y) = \frac{1}{b\nu} .
\]
Let \( I \) be an interval in \( \mathcal{H}(x) \). With respect to the coordinates above, we have
\[
\varphi_t(I) = \{(y, c(y,t)), y \in I\} .
\]
We thus have
\[
\mu_H(\varphi_t(I)) = \int_{y \in I} \int_{s=c(y,t)}^\infty e^{-bs}ds d\nu(y) = \int_{y \in I} e^{-bc(y,t)}d\mu_H(y) .
\]
This concludes the proof. \( \square \)

Now, if \( \varphi \) is a parametrization of \( G \) with stable horocycles, \( \nu \) a locally finite horocyclic measure and \( \mu \) an unstable measure which is invariant under horocycle holonomy, one obtains a finite measure \( \mu \times \nu \) on \( M_\Gamma \) by setting locally
\[
\int f d\mu \times \nu = \int y \in W^u(x) \int z \in \mathcal{H}(y) f(z)d\nu(z)d\mu(y) ,
\]
where \( x \) is any point on \( M_\Gamma \) and \( f \) a continuous function supported in a neighbourhood of \( x \). (The holonomy invariance of \( \mu \) guaranties that the measure is well-defined independently of \( x \).)

Using the fact that the total mass of \( \mu \) must be preserved by the flow \( \varphi \), we prove the following:

**Lemma 5.12.** Let \( \varphi^1 \) and \( \varphi^2 \) be a parametrizations of \( G \) admitting respectively stable and unstable horocycles. Let \( \mu^1 \) (resp. \( \mu^2 \)) be an unstable (resp. stable) Margulis measure for \( \varphi^1 \) (resp. \( \varphi^2 \)) with scale factor \( a \) (resp. \( -b \)). Then
\[
\inf_{\gamma \in [\Gamma]} \frac{L_{\varphi^2}(\gamma)}{L_{\varphi^1}(\gamma)} \leq a \leq \sup_{\gamma \in [\Gamma]} \frac{L_{\varphi^2}(\gamma)}{L_{\varphi^1}(\gamma)} .
\]

**Proof.** Let \( \mathcal{H} \) be the stable horocyclic foliation of \( \varphi^1 \) and consider the finite measure
\[
\nu = \mu^1 \times \mu^2_\mathcal{H} .
\]
on $M_\Gamma$. By Proposition ?? and since $\mu^1$ is scaled by $\varphi^1$, we have

$$\varphi^1_*\nu = e^{at - bc(t)}.$$

Since the total mass of the measure $\mu$ must be preserved, we must have

$$\inf_{x \in M_\Gamma} e^{at - bc(t)} \leq 1 \leq \sup_{x \in M_\Gamma} e^{at - bc(t)},$$

which rewrites

$$\inf_{x \in M_\Gamma} \frac{c(x, t)}{t} \leq \frac{a}{b} \leq \sup_{x \in M_\Gamma} \frac{c(x, t)}{t}.$$

Taking the limit as $t$ goes to $+\infty$ and applying Lemma 1.15, we get the conclusion. $\square$

We can now deduce Theorem 5.10 from Lemma 5.12.

**Proof of Theorem 5.10.** Let $[\varphi]$ be a point in $\text{Par}(G)$. By Theorem 2.14, there exists $\varphi^1$ in $[\varphi]$ which admits stable horocycles. By Theorem 5.1 there exists an unstable Margulis measure $\mu^1$ for $\varphi^1$ with scale factor $a_1$. We want to set

$$h_{\text{top}}([\varphi]) = a_1.$$

To see that this is well-defined, let $(\varphi^2, \mu^2)$ be another such pair, with $\varphi^2$ in $[\varphi]$ and $\mu^2$ scaled by $\varphi^2$ with scale factor $a_2$. There also exists $\psi \in [\varphi]$ which admits unstable horocycles, and a stable Margulis measure $\nu$ for $\psi$ with scale factor $-b$. Since $\varphi^1$, $\varphi^2$ and $\psi$ all have the same period map, applying Lemma 5.12 gives

$$\frac{a_1}{b} = \frac{a_2}{b} = 1,$$

hence $a_1 = a_2 = b$. Thus $h_{\text{top}}$ is well-defined.

Similarly, the continuity of $h_{\text{top}}$ follows from Lemma 5.12 and the continuity of

$$(|[\varphi], [\psi]) \mapsto \sup \frac{L_{\varphi}}{L_{\psi}}.$$

$\square$

Note that the function $h_{\text{top}}$ satisfies

$$h_{\text{top}}(\varphi^\lambda) = \frac{1}{\lambda} h_{\text{top}}(\varphi).$$

Therefore, every flow $\varphi$ admits a unique scaling of entropy 1. This gives an isomorphism between $\text{PPar}(G)$ and the hypersurface

$$\text{Par}_1(G) = \{[\varphi] \in \text{Par}(G) \mid h_{\text{top}}([\varphi]) = 1\}.$$

Finally, by construction of the maps $\text{DF}$ and $\text{CF}$, we have

**Proposition 5.13.** The maps $\text{DF}$ and $\text{CF}$ take values into the set $\text{Par}_1(G)$. 
5.4. **Surjectivity of the map DF.** In Section ??, we associated to an Anosov representation of $\Gamma$ into $\text{Diff}(\mathbb{S}^1)$ a parametrization $\varphi$ of $\mathcal{G}$ with stable horocycles and an unstable Margulis measure with scale factor 1. Here we explain how to recover a $C^1$ action from the data of the flow and its Margulis measure.

Let $\varphi$ be a parametrization of $\mathcal{G}$ with stable horocycles and $\mu^u$ an unstable Margulis measure for $\varphi$. We want to integrate the projections of the Margulis measure on unstable paths transverse to $\mathcal{G}$. We first need the following lemma:

**Lemma 5.14.** Let $\varphi$ be a parametrization of $\mathcal{G}$ with stable horocycles, let $\mu^u$ be an unstable Margulis measure, and $c$ an unstable path transverse to $\mathcal{G}$. Then the projection $\mu^u_c$ of $\mu^u$ to $c$ has full support and no atom.

**Proof.** Let $\text{Supp}(\mu^u)$ denote the union of the supports of $\mu^u$ on each leaf. Since $\mu^u$ is scaled by $\varphi$ and holonomy invariant, $\text{Supp}(\mu^u)$ is a union of weakly stable leaves. Moreover, the complement of $\text{Supp}(\mu^u)$ intersects each unstable leaf in an open set. The holonomy invariance thus implies that $\text{Supp}(\mu^u)$ is closed. Therefore $\text{Supp}(\mu^u) = M_\Gamma$ by minimality of the weakly unstable foliation (which follows from instance from the minimality of the action of $\Gamma$ on $\partial_\infty \Gamma$). We easily deduce that the projections of $\mu^u$ have full support.

Assume now that some projections have atoms. Let us first bound the size of these atoms. For this, let $U$ be a relatively compact unstable open domain such that every stable horocycle intersects $U$, and take $V = \bigcup_{t \geq 0} \varphi^{-t}(U)$. By local finiteness and scaling property of $\mu^u$, we have $\mu^u(V) = A < +\infty$. Therefore, for every $x \in U$,

$$\mu^u(\{\varphi^{-t}(x), t \geq 0\}) \leq A.$$ 

Since $\mu^u$ is holonomy invariant and since every stable horocycle intersects $U$, the same conclusion holds for every $x \in M_\Gamma$.

Applying this to $\varphi_s(x)$ with large $s$, one gets

$$\mu^u(\{\varphi_{s-t}(x), t \geq 0\}) = e^s \mu^u(\{\varphi_{-t}(x), t \geq 0\}) \leq A,$$

which implies that $\mu^u(\{\varphi_{-t}(x), t \geq 0\})$ has no atom. Thus, the projections of $\mu^u$ have no atom. \hfill \Box

Let us now use these projections to form a $C^1$ atlas on $\partial_\infty \Gamma$. Let $I = [q_1, q_2]$ be an interval in $\Gamma$, $p$ a point in $\partial_\infty \Gamma$ that does not belong to $I$, and $c$ a continuous map from $I$ to the unstable leaf $\mathcal{W}^u(p)$ such that $c(q)$ belongs to the geodesic $\mathcal{G}(p,q)$ for all $q \in [q_1, q_2]$. Define

$$h_{p,c} : I \rightarrow \mathbb{R} \quad q \mapsto \mu^u(c([q_1, q])).$$

By Lemma 5.14, $h_{p,c}$ is a homeomorphism from $I$ to $[0, h_{p,c}(q_2)]$.

**Proposition 5.15.** Let $h_{p,c}$ and $h_{p',c'}$ be two homeomorphisms constructed as above. Then $h_{p',c'} \circ h_{p,c}^{-1}$ is a $C^1$ diffeomorphism.
Proof. Let \((p', c')\) be another choice of \((p, c)\). The holonomy invariance of \(\mu^u\) implies that
\[ h_{p', c'} = h_{p, c''} \]
where \(c'' = T_{p', p} \circ c\). Now, \(c(q)\) and \(c'(q)\) belong to the same geodesic, so we can write
\[ c''(q) = \varphi_t(q)(c(q)) \]
for some continuous function \(t\). The scaling property of \(\mu^u\) gives
\[ h_{p, c''}(q) = \int_0^{h_{p, c}(q)} e^{t(h_{p, c}^{-1}(s))} ds. \]
(See the proof of Proposition 5.11.)
It follows that \(h_{p', c'} \circ h_{p, c}^{-1} = h_{p, c''} \circ h_{p, c}^{-1}\) is a \(C^1\) diffeomorphism.

By Proposition 5.15, the family of charts \((I, h_{p, c})\) define a \(C^1\) atlas on \(\partial_\infty \Gamma\). Since this family is globally \(\Gamma\)-invariant, the associated \(C^1\) structure on \(\partial_\infty \Gamma\) is \(\Gamma\)-invariant.
Let \(h\) be a homeomorphism from \(\partial_\infty \Gamma\) to \(S^1\) which is a diffeomorphism in the local charts \(h_{p, c}\). Then \(h\) conjugates the action of \(\Gamma\) on \(\partial_\infty \Gamma\) to a \(C^1\) action \(\rho\) on \(S^1\).

**Proposition 5.16.** We have
\[ L_\rho = h_{\text{top}}(\varphi)L_\varphi. \]

**Proof.** Exercise.

**Corollary 5.17.** The map \(DF : \mathcal{X}_m(\Gamma, \text{Diff}(S^1)) \to \text{Par}_1(G)\) is surjective.

**Proof.** Let \([\varphi]\) be a point in \(\text{Par}_1(G)\). By Theorem 2.14, there exists a flow \(\varphi\) in \([\varphi]\) which admits stable horocycles. By Theorem 5.1, the exists an unstable Margulis measure \(\mu^u\) for \(\varphi\) with scale factor 1. Let \(\rho : \Gamma \to \text{Diff}(S^1)\) be constructed as above. Then \(\rho\) is topologically conjugate to the action of \(\Gamma\) on \(\partial_\infty \Gamma\). The equality \(L_\rho = L_\varphi\) implies that \(\rho\) is Anosov by Proposition 1.21, and gives
\[ DF([\rho]) = [\varphi]. \]

**Remark 5.18.** Though we did not explicitly proved it, the reader can convince himself that the construction of \(\rho\) from \((\varphi, \mu^u)\) is inverse of the construction of \((\varphi, \mu^u)\) from \(\rho\). There is thus a bijection \(DF\) between the set of conjugacy classes of Anosov representations into \(\text{Diff}(S^1)\) and the set of pairs \((\varphi, \mu^u)\) up to conjugacy. This bijection factors to the map \(DF\) when forgetting the second coordinate and passing to Hausdorff quotients.

The question of whether \(DF\) is injective is thus deeply related to the question of whether the unstable Margulis measure of a flow is unique. We prove uniqueness Hölder parametrizations in the next section.

6. Constructing inverses of DF and CF

In this section, we construct inverses of DF and CF in restriction to the set of Hölder parametrizations of entropy 1. The Hölder assumption is necessary to guaranty that the flow considered admits both stable and unstable horocycles.
6.1. **Anosov parametrizations.** Let us first gather the various results presented above and specialize them to the case of Anosov parametrization.

**Definition 6.1.** A parametrization of the geodesic foliation is **Anosov** if it admits both stable and unstable horocycles.

This is verified for instance by the geodesic flow $\varphi^0$ of a hyperbolic metric.

**Proposition 6.2.** Let $c$ be a Hölder cocycle along $\varphi^0$. Then $c$ is both stably and unstably Buseman.

**Proof.** Up to a coboundary, we can assume $c$ is the integral cocycle associated to a Hölder function $f$. If $x$ and $y$ belong to the same stable leaf, then $\varphi^0_t(x)$ and $\varphi^0_t(y)$ get exponentially close for $t \to +\infty$, and so do $f(\varphi^0_t(x))$ and $f(\varphi^0_t(y))$ since $f$ is Hölder. We deduce that

$$\int_0^t f(\varphi^0_s(x)) - f(\varphi^0_s(y)) \, ds$$

converges as $s$ goes to $+\infty$. One proves with a little extra care that the convergence is uniform on every compact. Thus $c_f$ is stably Buseman. The same argument with $x$ and $y$ in the same unstable leaf and $t \to -\infty$ shows that $c_f$ is also unstably Buseman. \qed

Now, if $\varphi$ is a Hölder parametrization of $\mathcal{G}$, then the reparametrization cocycle of $\varphi$ with respect to $\varphi^0$ is Hölder. One thus obtains the following (well-known) corollary:

**Corollary 6.3.** Every Hölder parametrization of $\mathcal{G}$ is Anosov.

A variant of Livšic theorem asserts that Anosov parametrizations are characterized up to conjugacy by their periods.

**Theorem 6.4** (Livšic). Let $\varphi$ and $\psi$ be two Anosov parametrizations of $\mathcal{G}$. If $L_\psi \equiv L_\psi$, then $\varphi$ and $\psi$ are conjugate.

**Proof.** Exercise. \qed

Let us denote by $\text{Par}^{an}(\mathcal{G})$ the of Anosov parametrizations of $\mathcal{G}$. It is a dense convex subcone of $\text{Par}(\mathcal{G})$ which contains $\text{Par}^{h}(\mathcal{G})$. We denote $\text{Par}^{an}_1(\mathcal{G})$ its intersection with $\text{Par}_1(\mathcal{G})$ and by $\text{PPar}^{an}(\mathcal{G})$ its projection to $\text{PPar}(\mathcal{G})$.

Let $\varphi$ be an Anosov parametrization of the geodesic foliation. Let $\mathcal{H}^s$ and $\mathcal{H}^u$ denote respectively the stable and unstable horocyclic foliations of $\varphi$. Let $\mu^s$ and $\mu^u$ denote respectively some stable and unstable measures scaled by $\varphi$ and invariant under horocyclic holonomy and denote by $\nu^s$ and $\nu^u$ their respective projections to $\mathcal{H}^s$ and $\mathcal{H}^u$. In the next sections, we explain how to recover from these data an Anosov action on the circle and a foliated affine action.

6.2. **Inverses of $DF$ and $CF$.** This inverse has essentially been constructed in Section 5.4 if we are given an unstable Margulis measure. Similarly, we construct the inverse of $CF$ via projections of a stable Margulis measure.
Let $\varphi$ be an Anosov parametrization of $\mathcal{G}$ with $h_{\text{top}}(\mu) = 1$, and let $\mu^s$ be a stable Margulis measure. Recall that $\mu^s_H$ denotes its projection onto the horocyclic foliation of $\varphi$.

**Proposition 6.5.** There exists a stable horocyclic flow $(h_s)$ for $\varphi$ such that $\mu^s_H([x, h_s(x)]) = s$ for all $s \geq 0$.

**Proof.** Given $x \in M_\Gamma$, let $f_x$ be the primitive of $\mu^s$ on $\mathcal{H}^s(x)$ vanishing at $x$. By Proposition ??, the measure $\mu^s$ on $\mathcal{H}^s(x)$ has full support and is atome free, by compactness of $M_\Gamma$, one can fine some uniform $\varepsilon > 0$ such that the total mass $\mu^s(\mathcal{H}^s(x))$ is at least $\varepsilon$. Applying this to $\varphi_t(x)$, one obtains that $\mu^s(\mathcal{H}^s(x)) = e^t \mu^s(\mathcal{H}^s(\varphi_t(x))) \geq e^t \varepsilon$.

Thus $\mu^s(\mathcal{H}^s(x))$ is infinite and $f_x$ is a global homeomorphism.

We can now set $h_s(x) = f_x^{-1}(s)$. It is clear that $h_s$ is a flow. The holonomy invariance of $\mu^s$ give the continuity of $h_s$ when moving in directions transverse to the stable leaves, and the scaling property gives the relation $\varphi_t \circ h_s = h_{e^{-t}s}$.

□

To conclude the construction of the inverses of $DF$ and $CF$ we need the following:

**Proposition 6.6.** Let $\varphi$ be parametrization of the geodesic foliation with stable (resp. unstable) horocycles. Assume there exist $(h_s)$ and $(h'_s)$ two stable (resp. unstable) horocyclic flows. Then there is a constant $\lambda \neq 0$ such that $h'_s = h_{\lambda s}$.

**Proof.** Exercise. □

**Corollary 6.7.** Let $\varphi$ be an Anosov parametrization of $\mathcal{G}$. Then the stable and unstable Margulis measures of $\varphi$ are unique up to a multiplicative constant.

**Proof.** Let $\mu^1_s$ and $\mu^2_s$ be two stable Margulis measures. By Proposition ??, the projections of $\mu^1_s$ and $\mu^2_s$ on $\mathcal{H}^s$ differ by a multiplicative constant. Hense so do $\mu^1_s$ and $\mu^2_s$. □

We can now finally define a map

$$DF^{-1} : \text{Par}^a_1(\mathcal{G}) \rightarrow \mathcal{X}_{an}(\Gamma, \text{Diff}(S^1))$$

in the following way: for $[\varphi] \in \text{Par}^a_1(\mathcal{G})$, let $\varphi \in [\varphi]$ be the parametrization of $\mathcal{G}$ which admits both stable and unstable horocycles ($\varphi$ is unique by Livsic’s theorem). Let $\mu^u$ be the stable Margulis measure of $\varphi$ (which is unique by Corollary 6.7) and define $DF^{-1}([\varphi])$ to be the class of the Anosov $C^1$ action associated to $(\varphi, \mu^u)$ in Section 5.4. It follows from the results of Section 5.4 that

$$DF \circ DF^{-1} = \text{Id}_{\text{Par}^a_1(\mathcal{G})} .$$

Similarly, we can construct a map

$$CF^{-1} : \text{Par}^a_1(\mathcal{G}) \rightarrow \mathcal{T}(\mathcal{W}^s)$$
in the following way: for \([\varphi] \in \Par_{an}(G)\), let \(\varphi \in [\varphi]\) be the parametrization of \(G\) which admits both stable and unstable horocycles (\(\varphi\) is unique by Livsic’s theorem). Let \((h_s)\) be the horocycle flow of \(\varphi\) constructed in Proposition 6.5. By Proposition 6.6, the flow \((h_s)\) is well-defined up to a scaling. Note however that \(\varphi_a\) centralizes \(\varphi\) and conjugates \(h_s\) to \(h_{e^{-a}}\). Thus the foliated affine action \((\varphi,h)\) is well-defined up to conjugation, and thus defines a point \(\CF^{-1}([\varphi]) \in \mathcal{T}(W^s)\) by the results of Section 4, which satisfies

\[
\CF \circ \CF^{-1}([\varphi]) = [\varphi].
\]

6.3. Hölder regularity. Recall that, by Corollary 6.3, \(\Par_h(G)\) is contained in \(\Par_{an}(G)\). To conclude the proof of Theorems 0.4 and 0.3, we just need the following Proposition:

**Proposition 6.8.** The maps \(\DF^{-1}\) and \(\CF^{-1}\) map \(\Par_{1}(G)\) respectively to \(X_{an}(\Gamma, \Diff^{h}(S^1))\) and to \(T^{h}(W^s)\).

**Proof.** Exercise. \(\square\)

**References**


CNRS – École Normale Supérieure, 75230 Paris Cedex 5, France

E-mail address: nicolas.tholozan@ens.fr