École Normale Supérieure — Examen, 18 janvier 2022

Topologie et calcul différentiel — 9h00-12h00 (3 heures)

---

0. The goal of this exercise is to find all solutions $u \in C^1(\mathbb{R}^2,\mathbb{R})$ of the equation

$$2y\frac{\partial u}{\partial x}(x,y) + \frac{\partial u}{\partial y}(x,y) = 0, \quad x,y \in \mathbb{R}. \tag{1}$$

(a) Consider the vector-field $v : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $v(x,y) := (2y,1)$ and let $\varphi^t(x_0,y_0)$ denote the solution of the Cauchy problem for the autonomous differential equation $(x'(t),y'(t)) = v(x(t),y(t))$ with $x(0)=x_0$, $y(0)=y_0$. Prove that $u(\varphi^t(x_0,y_0))$ does not depend on $t\in\mathbb{R}$ if $u\in C^1(\mathbb{R}^2,\mathbb{R})$ satisfies the equation (1).

(b) Find an explicit formula for $\varphi^t(x_0,y_0)$.

(c) Prove that all solutions of the equation (1) can be written as $u(x,y)=g(y^2-x)$, where $g\in C^1(\mathbb{R},\mathbb{R})$.

---

1. (a) Recall the Picard–Lindelöf/Cauchy–Lipschitz theorem on the local existence/uniqueness of solutions of the Cauchy problem $u'(t)=f(t,u(t))$, $u(t_0)=x_0$, where $f:\mathbb{R}\times E\supset\mathcal{O}\to E$ and E is a Banach space.

Let $E=\mathbb{R}^m$, $I :=]-\frac{r}{R},\frac{r}{R}[$ and $f : I\times B(0,r)\to B(0,R)$ be a continuous function. In (b)–(d) we prove **Peano's theorem :** the Cauchy problem $u'(t)=f(t,u(t))$, $u(0)=0$, has at least one solution $u:I\to\mathbb{R}^m$.

(b) Let $n\in\mathbb{N}^*$ and $\tau_n := \frac{r}{nR}$. Define (inductively)

$$u_n(0):=0, \qquad \begin{array}{rcl} u_n((k+1)\tau_n) & := & u_n(k\tau_n)+\tau_n f(k\tau_n,u_n(k\tau_n)), \\ u_n(-(k+1)\tau_n) & := & u_n(-k\tau_n)-\tau_n f(-k\tau_n,u_n(-k\tau_n)), \end{array} \qquad k=0,\dots,n-1.$$

and let the function u_n be linear on each of the segments $[k\tau_n,(k+1)\tau_n]$, where $k=-n,\dots,n-1$. Check the correctness of this definition (i.e., that $\|u_n(\pm k\tau_n)\|<r$ if $k<n$: otherwise, f is not defined).

(c) Recall the Arzelá–Ascoli theorem for the space $C(K,E)$ of continuous functions on a compact K and prove that the functions $u_n\in C([-\frac{r}{R},\frac{r}{R}]\,;\mathbb{R}^m)$ satisfy the assumptions of this theorem.

(d) Prove that each subsequential limit u of functions u_n solves the differential equation $u'(t)=f(t,u(t))$. [*Hint :* as usual, it is useful to rewrite this differential equation in the integral form.]

(e) Find all solutions of the equation $u'(t)=2\sqrt{|u(t)|}$ with $u(0)=0$ (and prove that there are no other ones).

(f) Consider now the equation $u'(t)=2\sqrt{|u(t)|}+b(t)$, where b is a continuous functions and $0<b(t)\le 1$ for all $t\in\mathbb{R}$. Prove that the Cauchy problem for this equation with $u(0)=0$ has a *unique* solution and that this solution is defined for all $t\in\mathbb{R}$.

Now let $E=\ell_0^\infty=\{(x_0,x_1,\dots): x_n\to 0 \text{ as } n\to\infty\}$. Consider a function $f:(x_n)_{n\in\mathbb{N}}\mapsto(2\sqrt{|x_n|}+\frac{1}{n+1})_{n\in\mathbb{N}}$.

(g) Prove that the Cauchy problem $u'(t)=f(u(t))$, $u(0)=0$, does *not* have any local solution. [*Hint :* show that $x_n(t)>t^2$ for all $t>0$ and $n\ge\mathbb{N}$.]

(h) Check that $f:\ell_0^\infty\to\ell_0^\infty$ is a continuous function. What is wrong with the proof in (b)–(d) if $E=\ell_0^\infty$?

(i) Let $E:=C([0,1],\mathbb{R})$ and $u\in C(I,E)$, where I is an open interval. Prove that $u\in C^1(I,E)$ if and only if the partial derivative $\frac{d}{dt}u(t,x)$ exists for all $(t,x)\in I\times[0,1]$ and is continuous on $I\times[0,1]$.

(j) Let $a,b\in C(I\times[0,1],\mathbb{R})$ and $t_0\in I$. Modify the arguments from items (b)–(d) to prove that there exists $u\in C^1(I,E)$ such that $u(t_0,x)=0$ for all $x\in[0,1]$ and

$$\tfrac{d}{dt}u(t,x)=\textstyle\int_0^x(a(t,y)\sqrt{|u(t,y)|}+b(t,y))dy \quad \text{for all} \quad (t,x)\in I\times[0,1]\,.$$

Turn the page !

2. One says that a compact set $K \subset \mathbb{R}^2 \cong \mathbb{C}$ does not separate 0 from ∞ if there exists a continuous curve $\gamma : [0,1) \to \mathbb{C} \smallsetminus K$ such that $\gamma(0) = 0$ and $\gamma(t) \to \infty$ as $t \to 1$. In this problem, the goal is to prove

> **Janiszewski's theorem :** Let $0 \notin K_1, K_2 \subset \mathbb{R}^2 \cong \mathbb{C}$ be two compact sets. Assume that neither K_1 nor K_2 separate 0 from ∞. If $K_1 \cap K_2$ is connected, then $K_1 \cup K_2$ also does not separate 0 from ∞.

We call $p \subset \mathbb{C}$ an *hv-polyline* if p is a polyline consisting of finitely many vertical and horizontal segments. We say that $P \subset \mathbb{C}$ is a *closed hv-polyline* if it is an hv-polyline whose starting and ending points coincide.

(a) Recall the definition of connected and path-connected topological spaces. Prove that an *open* set $U \subset \mathbb{C}$ is connected if and only if for each $z_1, z_2 \in U$ there exists an hv-polyline $p \subset U$ going from z_1 to z_2.

(b) Recall the definition of a connected component of a topological space. Prove that for each compact $K \subset \mathbb{C}$ the set $\mathbb{C} \smallsetminus K$ has exactly one unbounded connected component.

Below we take for granted the following two facts about *non-self-intersecting* closed hv-polylines $P \subset \mathbb{C}$:

(i) The 'trivial' case of the Jordan curve theorem : the set $\mathbb{C} \smallsetminus P$ has *exactly* two connected components. We say (see (b)) that $z \notin P$ lies *outside* P if it belongs to the unbounded connected component of $\mathbb{C} \smallsetminus P$;
$z \notin P$ lies *inside* P if it belongs to the bounded connected component of $\mathbb{C} \smallsetminus P$.

(ii) if $0 \notin P$, we will denote by $\oint_P z^{-1}dz$ the (Riemann) integral computed along P with the convention that P is oriented counterclockwise. In particular, if P is a *rectangle* with opposite corners (x_1, y_1) and (x_2, y_2), then

$$\oint_P z^{-1}dz = \int_{x_1}^{x_2} \frac{dx}{x+iy_1} + \int_{y_1}^{y_2} \frac{dy}{x_2+iy} + \int_{x_2}^{x_1} \frac{dx}{x+iy_2} + \int_{y_2}^{y_1} \frac{dy}{x_1+iy}$$

provided that $x_1 < x_2$ and $y_1 < y_2$. In this case, a straightforward computation gives

$$\oint_P z^{-1}dz = 2\pi i \ \text{ if } 0 \text{ lies inside } P, \qquad \oint_P z^{-1}dz = 0 \ \text{ if } 0 \text{ lies outside } P. \tag{2}$$

We also take for granted that (2) holds for *all* non-self-intersecting closed hv-polylines P (this can be proven by splitting the inner component of $\mathbb{C} \smallsetminus P$ into two parts with smaller number of edges unless P is a rectangle).

(c) Let $\gamma : [0,1) \to \mathbb{C}$ be continuous, $\gamma(0) = 0$ and $\gamma(t) \to \infty$ as $t \to 1$; denote $\Gamma := \gamma([0,1)) \subset \mathbb{C}$. Prove that $\oint_P z^{-1}dz = 0$ for all non-self-intersecting closed hv-polylines $P \subset \mathbb{C} \smallsetminus \Gamma$.

(d) Deduce from (c) that $\oint_P z^{-1}dz = 0$ for *all* (possibly, self-intersecting) closed polylines $P \subset \mathbb{C} \smallsetminus \Gamma$.

(e) In the same setup, prove that there exists a function $L : \mathbb{C} \smallsetminus \Gamma \to \mathbb{C}$ such that $\int_p z^{-1}dz = L(z_2) - L(z_1)$ for each hv-polyline $p \subset \mathbb{C} \smallsetminus \Gamma$ going from z_1 to z_2. [*Hint :* choose a reference point $z_0 \in U$ in each of the connected components U of $\mathbb{C} \smallsetminus \Gamma$ and define $L(z) := \int_{p_0(z_0,z)} z^{-1}dz$ for $z \in U$, where $p_0(z_0, z) \subset U$ is an arbitrarily chosen hv-polyline going from z_0 to z. Use (c) to check the required property of L for all $p \subset U$.]

We now come to the setup of Janiszewski's theorem. Let $\Gamma_j := \gamma_j([0,1)) \subset \mathbb{C} \smallsetminus K_j$ and $L_j : \mathbb{C} \smallsetminus \Gamma_j \to \mathbb{C}$ denote the corresponding functions constructed in the item (e).

(f) Argue that there exists a connected component U of the set $\mathbb{C} \smallsetminus (\Gamma_1 \cup \Gamma_2)$ such that $K_1 \cap K_2 \subset U$. Prove that one can choose reference points in the definition of L_1 and L_2 so that $L_1(z) = L_2(z)$ for all $z \in U$.

(g) Prove that there exist open sets V_j, $j = 1, 2$, such that $V_1 \cap V_2 = \emptyset$ and $K_j \smallsetminus U \subset V_j \subset \mathbb{C} \smallsetminus \Gamma_j$. Denote $V := U \cup V_1 \cup V_2 \supset K_1 \cup K_2$ and let $L(z) := L_j(z)$ for $z \in V_j \cup U$, $j = 1, 2$.

(h) Prove that $\int_p z^{-1}dz = L(z_2) - L(z_1)$ for each hv-polyline $p \subset V$ going from z_1 to z_2. In particular, one has $\oint_P z^{-1}dz = 0$ if $P \subset V$ is a closed hv-polyline. [*Hint :* prove that it is always possible to split each segment of $p \subset V$ into finitely many parts, each of which is contained in *one* of the sets $U \cup V_1$ and $U \cup V_2$.]

(i) Arguing by contradiction, assume that the connected component U_0 of $\mathbb{C} \smallsetminus (K_1 \cup K_2)$ that contains 0 is bounded. Prove that there exists a non-self-intersecting closed hv-polyline $P \subset V \cap U_0$ such that 0 lies inside P. [*Hint :* take a small enough $\delta > 0$ and consider the union of all squares $[-\delta+2m\delta, \delta+2m\delta] \times [-\delta+2n\delta, \delta+2n\delta]$, $n, m \in \mathbb{Z}$, that are contained in U_0 .] Conclude the proof of the Janiszewski theorem.

(j) Deduce from the Janiszewski theorem the following fact : if $\gamma : [0,1] \to \mathbb{C} \smallsetminus \{0\}$ is a *Jordan arc* (i.e., γ is a continuous injection; in particular, $\gamma(0) \neq \gamma(1)$), then $\gamma([0,1])$ does not separate 0 from ∞.

SEE THE NEXT PAGE FOR PROBLEM #3 !

3. Stability/asymptotic stability of stationary points of a flow. Consider a differential equation $u'(t) = f(u(t))$, where $f : U \to E$ is a locally Lipschitz function defined on an open set $U \subset E$ in a Banach space E. Recall that $x_0 \in U$ is called a stationary point of this equation if $f(x_0) = 0$ and that we denote by $\varphi^t(x)$ the solution of the Cauchy problem with the initial data $u(0) = x$. Also, recall that x_0 is called

– *stable* if for each $C > 0$ there exists $\varepsilon > 0$ such that $\|\varphi^t(x) - x_0\| \le C$ for all $x \in B(x_0, \varepsilon)$ and all $t \ge 0$;
– *asymptotically stable* if x_0 is stable and there exists $\varepsilon > 0$ such that $x \in B(x_0, \varepsilon) \Rightarrow \varphi^t(x) \to x_0$ as $t \to +\infty$.

Below we assume that (i) x_0 is a stationary point of the equation $u'(t) = f(u(t))$;
(ii) $\Phi : U \to \mathbb{R}$ is a smooth function such that x_0 is a strict local minimum of Φ;
(iii) there exists $r > 0$ such that $\langle \nabla\Phi(x), f(x)\rangle < 0$ for all $x \in B(x_0, r) \setminus \{x_0\}$.

(a) Check that $\frac{d}{dt}\Phi(\varphi^t(x)) \le 0$ if $\varphi^t(x) \in B(x_0, r)$. Prove that x_0 is a stable stationary point.

(b) Let $E = \mathbb{R}^n$. Prove that in this case x_0 is always asymptotically stable.
[*Hint* : by continuity, one has $\max_{x\in K}\langle \nabla\Phi(x), f(x)\rangle < 0$ for all compact sets $K \not\ni x_0$.]

The next goal is to discuss a possible construction of an equation $u'(t) = f(u(t))$ in the *infinite-dimensional* space $E = \ell^2$ such that the conditions (i)–(iii) hold for $x_0 = 0$ and $\Phi(x) = \|x\|^2$ but the stationary point 0 is *not* asymptotically stable.

Let $Sx := (0, x_0, x_1, x_2, \ldots)$ and ${}^tSx := (x_1, x_2, x_3, \ldots)$ be the shift operators in the (real) Banach space ℓ^2.

We start with considering the linear equation $v'(t) = A\,v(t)$, where $A := S - {}^tS$.

(c) Prove that $x \mapsto Ax$ is a Lipschitz function. Write $\nabla\Phi(x)$ explicitly. Prove that $\Phi(x) = \|x\|^2$ is a conserved quantity for this equation (i.e., that $\Phi(v(t))$ does not depend on t if $v(t)$ satisfies $v'(t) = A\,v(t)$).

(d) Denote $B(x) := \|x\|^2 - \langle x, Sx\rangle$. Prove that $B(x) > 0$ if $x \ne 0$ and that $\inf_{x:\|x\|=1} B(x) = 0$.

(e) Write $\nabla B(x)$ explicitly and prove that $\frac{d}{dt}B(v(t)) \le 0$ if $v'(t) = A\,v(t)$.

Now let $f(x) := Ax - (B(x))^k x$, where $k \gg 1$ and consider the differential equation $u'(t) = f(u(t))$.

(f) Check that $\langle \nabla\Phi(x), f(x)\rangle < 0$ if $x \ne 0$.

(g) For $u(t) \ne 0$, write $u(t) = r(t)v(t)$, where $r(t) := \|u(t)\| \in \mathbb{R}_+$ and $v(t) := u(t)/\|u(t)\|$. Check that the equation $u'(t) = f(u(t))$ implies that $v'(t) = A\,v(t)$ and $r'(t) = -(r(t))^{2k+1}(B(v(t)))^k$.

(h) Deduce from (e) that

$$\frac{1}{\|u(t)\|^{2k}} - \frac{1}{\|u(0)\|^{2k}} \;=\; 2k\int_0^t (B(v(t)))^k dt\,. \tag{3}$$

and conclude that the stable stationary point 0 of the equation $u'(t) = f(u(t))$ *cannot* be asymptotically stable if the equation $v'(t) = A\,v(t)$ admits a trajectory $v(t) \ne 0$ such that $\int_0^{+\infty}(B(v(t)))^k dt < +\infty$.

[! !] The proof is **<u>not</u>** complete. Though we know from the item (d) that $B(v(t))$ decays along all trajectories of the equation $v'(t) = A\,v(t)$ and that $\inf_{\|x\|=1} B(x) = 0$, this does not directly imply even the existence of a trajectory $v(t)$ such that $B(v(t)) \to 0$ as $t \to +\infty$, not to speak about the rate of convergence. To proceed further one needs more involved tools and this would be (by far) too much for the exam.

Bon courage !