IMPROVED FRACTAL WEYL UPPER BOUND IN OBSTACLE SCATTERING

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ABSTRACT. In this paper, we are interested in the problem of scattering by strictly convex obstacles in the plane. We provide an upper bound for the number $N(r, \gamma)$ of resonances in the box $\{r \leq \operatorname{Re}(\lambda) \leq r + 1; \operatorname{Im}(\lambda) \geq -\gamma\}$. It was proved in the work of [NSZ14] that $N(r, \gamma) = O_\gamma(r^{d_H})$ where $2d_H + 1$ is the Hausdorff dimension of the trapped set of the billiard flow. In this article, we provide an improved upper bound in the band $0 \leq \gamma < \gamma_{cl}/2$, where γ_{cl} is the classical decay rate of the flow. This improved Weyl upper bound is in the spirit of the ones of [Nau12] and [Dya19] in the case of convex co-compact surfaces, and of [DJ17] in the case of open quantum baker's maps.

1. INTRODUCTION

1.1. An improved fractal upper bound.

Scattering by convex obstacles. In this paper, we are interested in the problem of scattering by strictly convex obstacles in the plane. We assume that

$$\mathcal{O} = \bigcup_{j=1}^{J} \mathcal{O}_j$$

where $\mathcal{O}_j$ are open, strictly convex obstacles in $\mathbb{R}^2$ having smooth boundary and satisfying the *Ikawa condition* : for $i \neq j \neq k$, $\overline{\mathcal{O}_i}$ does not intersect the convex hull of $\overline{\mathcal{O}_j} \cup \overline{\mathcal{O}_k}$. Let

$$\Omega = \mathbb{R}^2 \setminus \overline{\mathcal{O}}.$$

It is known that the resolvent of the Dirichlet Laplacian in Ω continues meromorphically to the
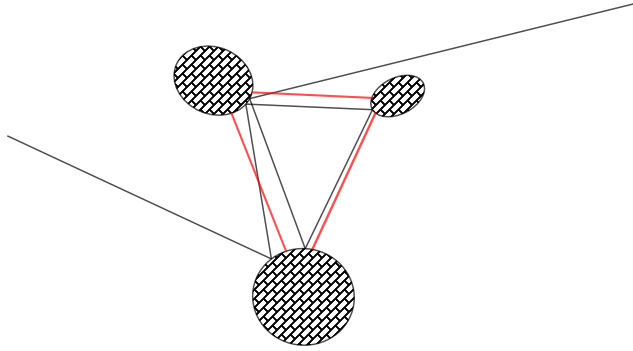

FIGURE 1. Scattering by three obstacles in the plane

logarithmic cover of $\mathbb{C}$ (see for instance [DZ19], Theorem 4.4 in Chapter 4). More precisely, suppose that $\chi \in C_c^\infty(\mathbb{R}^2)$ is equal to one in a neighborhood of $\overline{\mathcal{O}}$.

$$R_\chi(\lambda) := \chi(-\Delta - \lambda^2)^{-1}\chi : L^2(\Omega) \to L^2(\Omega)$$

is holomorphic in the region $\{\operatorname{Im} \lambda > 0\}$ and it continues meromorphically to the logarithmic cover of $\mathbb{C}$. Its poles are the *scattering resonances*. We are interested in the distribution of these scattering resonances in the first sheet of the logarithmic cover (i.e. $\mathbb{C} \setminus i\mathbb{R}^-$), or more precisely, in a conic neighborhood of $\mathbb{R}$.

The multiplicity of a (non-zero) resonance λ_0 is given by

$$m(\lambda_0) = \ \text{rank} \ \frac{1}{2i\pi} \int_\gamma R_\chi(\lambda) d\lambda \ , \ \gamma(t) = \lambda_0 + \epsilon^{2i\pi t} \ , \ 0 \le t \le 1 \ , \ 0 < \epsilon \ll 1$$

We are interested in counting resonances in strips and in this paper we focus on an upper bound for the quantities

$$N(r, \gamma) = \sum_{\substack{r \le \text{Re } \lambda \le r+1 \\ \text{Im } \lambda \ge -\gamma}} m(\lambda)$$

The depth γ of the strip being fixed, we are interested by upper bounds as $r \to +\infty$.

Fractal Weyl bounds. In this regime, it becomes a high-frequency problem and justifies the introduction of a small parameter $h = r^{-1}$. Under this rescaling, it becomes a semiclassical problem. In the semiclassical limit, that is $h \to 0$, the classical dynamics associated with this quantum problem is the billiard flow φ^t in $\Omega \times \mathbb{S}^1$, that is to say, the free motion outside the obstacles with normal reflection on their boundaries. A relevant dynamical object is the trapped set K corresponding to the points $(x, \xi) \in \Omega \times \mathbb{S}^1$ that do not escape to infinity in the backward and forward direction of the flow. In the case of two obstacles, it is a single closed geodesic. As soon as more obstacles are involved, the structure of the trapped set becomes complex and exhibits a fractal structure. This is a consequence of the hyperbolicity of the billiard flow. The structure of the trapped set plays a crucial role in the spectral properties of $-\Delta$. In particular, its Hausdorff dimension appears when estimating $N(r, \gamma)$. In [NSZ14], the authors proved a Fractal Weyl upper bound involving this fractal dimension.

Theorem 1. *Fractal Weyl upper bound* [NSZ14]
Assume that the obstacles $\mathcal{O}_j \subset \mathbb{R}^2$ satisfy the conditions above. Assume that the trapped set of the billiard flow has Hausdorff dimension $2d_H + 1$. Then, for every $\gamma > 0$, there exists $C_\gamma > 0$ such that for all $r \ge 1$,

$$N(r, \gamma) \le C_\gamma r^{d_H}$$

Remark. Their result holds in any dimension, but in dimension $d > 2$, one has to add an extra loss of ε : for every $\varepsilon > 0$, for every $\gamma > 0$, there exists $C_{\varepsilon, \gamma} > 0$ such that for all $r \ge 1$,

$$N(r, \gamma) \le C_{\varepsilon, \gamma} r^{d_H + \varepsilon}$$

This bound is conjectured to be optimal for large values of γ (see [Zwo17], Conjecture 5). However, as soon as a spectral gap exists, the exponent d_H cannot be optimal for any γ. It always exists in dimension 2, as proved in [Vac22] and it holds also in higher dimensions under some pressure condition (see [Ika88]) on the billiard flow. Our Theorem 2 below gives a better bound in dimension 2 for

$$\gamma < \gamma_{cl}/2$$

where γ_{cl} is the classical decay rate of the flow. γ_{cl} is equal to $-P(-\varphi_u)$ where φ_u is defined in (1.9) with the unstable Jacobian and P is the topological pressure for the billiard map on the trapped set (see Definition 3.1). It is also given by the following formula (see [BR75], Proposition 4.4) :

$$(1.1) \qquad\qquad -\gamma_{cl} = \lim_{\varepsilon \to 0} \limsup_{T \to +\infty} \frac{1}{T} \log \text{Leb} \left(\bigcup_{\rho \in K} B_\rho(\varepsilon, T) \right)$$

where

$$B_\rho(\varepsilon, T) = \{(x, \xi) \in \Omega \times \mathbb{S}^1, \forall t \in [0, T], d(\varphi^t(x, \xi), \varphi^t(\rho)) \le \varepsilon\}$$

are Bowen balls.

The theorem we prove in this article is

Theorem 2. *Assume that the obstacles $\mathcal{O}_j \subset \mathbb{R}^2$ satisfy the conditions above. Then, there exists a non increasing function $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying*

- *$\sigma(\gamma) > 0$ for $0 \le \gamma < \gamma_{cl}/2$;*
- *$\sigma(\gamma) = 0$ for $\gamma \ge \gamma_{cl}/2$*

and such that for all $\gamma > 0$ and for all $\varepsilon > 0$ there exists $C_{\gamma,\varepsilon} > 0$ such that

$$\forall r \geq 1, N(r,\gamma) \leq C_{\gamma,\varepsilon} r^{d_H - (\sigma(\gamma) - \varepsilon)_+}$$

Remark. A rather explicit value of σ in term of topological pressure is given by the formula (6.2). Here, we can take $(\sigma(\gamma) - \varepsilon)_+ = \max(\sigma(\gamma) - \varepsilon, 0)$ due to the result of [NSZ14]. When $\gamma \geq \gamma_{cl}/2$, we always have $(\sigma(\gamma) - \varepsilon)_+ = 0$. When $\gamma < \gamma_{cl}/2$, we can find $\varepsilon > 0$ such that the bound given by Theorem 2 improves the one of Theorem 1.

More on obstacle scattering. The problem of wave scattering by obstacles has a long history in the physics and mathematics literature. The case of two obstacles is particularly well-understood (see [Gé88], [Ika82]), so is the diffraction by one convex obstacle (see for instance [BLR87], [HL94]). As soon as 3 or more obstacles are involved, the underlying classical flow - in this case, the billiard flow - becomes highly chaotic. A particularly interesting model is the n-disk system, which has been intensively studied both numerically and experimentally (see for instance [GR89],[BWP$^+$13]) and the fractal upper bound has been successfully tested in [PWB$^+$12] or [LSZ03]. A recent result concerning a spectral gap has been proved in [Vac22], improving the previous result of [Ika88] (see also [NZ09]).

Related results in open hyperbolic systems. The problem of scattering by obstacles falls into the wider class of spectral problems for open hyperbolic systems, that is scattering systems where most trajectories escape to infinity, so that the trapped set has Liouville measure zero, and supports a hyperbolic flow. We refer the reader to the article of review [Non11] for a survey on these open chaotic systems. Among the problems which widely interest mathematicians and physicists, resonance counting and spectral gaps are on the top of the list (see for instance [Zwo17] for results and open problems concerning resonances). An important example is given by the semiclassical scattering by a potential (see 1.2.2), with particular dynamical assumptions on the Hamiltonian flow.

Convex co-compact hyperbolic surfaces. Another class of open hyperbolic systems exhibiting a fractal trapped set consists of the convex co-compact hyperbolic surfaces, which can be obtained as the quotient of the hyperbolic plane $\mathbb{H}^2$ by Schottky groups Γ. The spectral problem concerns the Laplacian on these surfaces and its classical counterpart is the geodesic flow on the cosphere bundle, which is known to be hyperbolic due to the negative curvature of these surfaces. In this context, it is common to write the energy variable $\lambda^2 = s(1-s)$ and study the meromorphic continuation of

$$s \in \mathbb{C} \mapsto (-\Delta - s(1-s))^{-1}$$

The trapped set, and more particularly its dimension, influences the spectrum (see for instance [Bor16] for an introduction to this theory).

Weyl upper bounds. The first Fractal Weyl upper bound for the counting function in strips appeared in the work of Sjöstrand [Sjö90] (see Section 5, Theorem 5.7) for Schrödinger operators $-h^2\Delta + V$ in the analytic case. The author estimated the number of resonances in larger boxes $\{|\operatorname{Re} z| \leq \delta, -\gamma h \leq \operatorname{Im} z \leq 0\}$ in the limit $h \to 0$. More precise upper bounds $O(h^{-d_H})$ for smaller boxes $\{|\operatorname{Re} z| \leq Ch, -\gamma h \leq \operatorname{Im} z \leq 0\}$, which correspond, under the rescaling $r = h^{-1}$ to the boxes we consider, were obtained in different smooth situations : for convex co-compact hyperbolic surfaces ([Zwo99]), in scattering by a potential ([SZ07]), in obstacle scattering ([NSZ14]), for asymptotically hyperbolic manifold ([DD12]). It has been conjectured (see [Zwo17], Conjecture 5) that the bound $N(r,\gamma) = O(r^{d_H})$ is optimal when the strip is sufficiently large. However, numerical experiments (see for instance the appendix of [Dya19] for the case of convex co-compact surfaces) show that it should be possible to improve this bound for strips of width smaller than some threshold. These numerical results lead [Zwo17] to conjecture that

$$\lim_{r \to +\infty} N(r,\gamma) r^{-d_H} = 0 \text{ when } 0 \leq \gamma < \frac{\gamma_{cl}}{2}$$

First results in this direction were obtained in the case of convex co-compact hyperbolic surfaces :

- In [Nau12], the author showed a bound similar to the one in Theorem 2 (without the loss of ε), with a function σ having the same properties ;
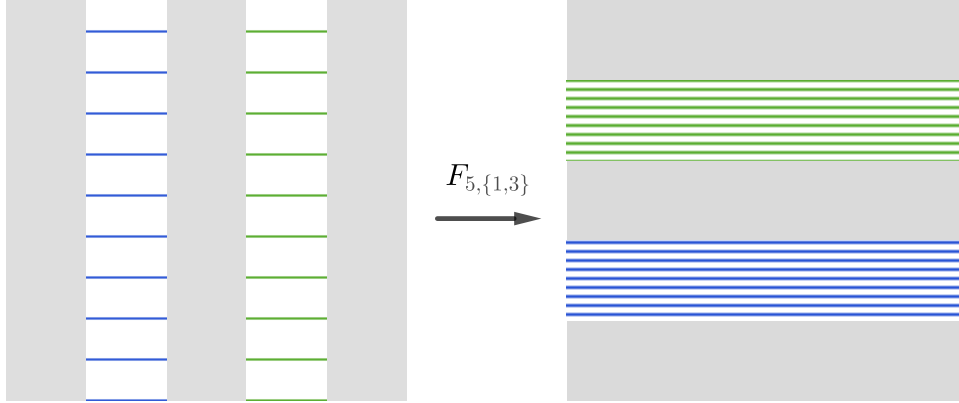
FIGURE 2. Example of an open baker's map.

- In [Dya19], the author obtained the same result with an explicit function σ given by $\sigma(\gamma) = 1 - d_H - 2\gamma$, which satisfies the same properties as the one in Theorem 2 (since in this context $\gamma_{cl} = 1 - d_H$). His result can be generalized to higher dimensional convex co-compact hyperbolic manifold.

Theorem 2 gives a positive answer to this conjecture in obstacle scattering in dimension 2. There is also a stronger conjecture, due to Jakobson-Naud ([JN12]) in the case of convex co-compact surfaces, which states that for every $\gamma < \gamma_{cl}/2$, $N(r, \gamma) = 0$ for $r \gg 1$. Our work is still far from proving this conjecture.

Toy models and open quantum maps. To test these conjectures, it is useful to work on toy models where numerical and theoretical computations are sometimes easier. A very appreciated toy model in the study of open hyperbolic systems is the open baker's map (see for instance [Non11], section 6.1.1). The classical map is a piecewise affine open map $F_{a,\mathcal{A}}$ on the torus $\mathbb{T}^2$, associated with an alphabet $\mathcal{A} \subset \{0, \ldots, a - 1\}$ (a is called the base) (see Figure 2).

It quantization is given by a matrix $M_N \in \mathcal{M}_N(\mathbb{R})$ where N plays the role of $(2\pi h)^{-1}$. In this context, one wants to count the number of eigenvalues of the matrix $M_N \in \mathcal{M}_N(\mathbb{R})$ in the annulus $\{|z| \geq \nu\}$ in the limit $N \to +\infty$.

These toy models are what we call *open quantum maps*. There is a heuristic correspondence between open quantum maps and open quantum systems. These quantized open maps have to be interpreted as propagators at time $t = \log a$ of an open quantum system with constant unstable Jacobian $J^u = a$, so that, to a resonance λ of the open quantum system, it corresponds an eigenvalue $e^{-it\lambda}$ of the open quantum map, with $t = \log a$. In fact, [NSZ11] and [NSZ14] have shown that the study of the resonances in obstacle scattering can be reduced to the study of a family of more general open quantum maps. This reduction is the starting point of the proof of Theorem 2.

Concerning the quantized open baker's map, there are convincing numerical and theoretical results. In [NR07], the authors gave numerical evidence of Weyl upper bounds of the type

$$\# \left(\mathrm{Spec} \, (M_N) \cap \{|z| \geq \nu\} \right) \leq C_\nu N^{d_H}$$

In [DJ17], the author proved an even more precise upper bound, when $N = a^k$:

$$\# \left(\mathrm{Spec} \, (M_{a^k}) \cap \{|z| \geq \nu\} \right) \leq C_\nu (a^k)^{d_H + \varepsilon - \Sigma(\nu)} \, , \, \forall k \in \mathbb{N} \quad ; \quad \Sigma(\nu) = \sigma \left(-\frac{\log \nu}{\log J^u} \right)$$

where $J^u = a$ is the unstable Jacobian of the system and $\sigma(\gamma) = \max(1 - d_H - 2\gamma, 0)$. In particular, σ shares the same properties as the one in Theorem 2, since the classical decay rate of the baker's map is $1 - d_H$. The link between $\Sigma(\nu)$ and $\sigma(\gamma)$ comes from the heuristic interpretation above.

1.2. **Statement of the main theorem.** Our proof of Theorem 2 relies on previous results of [NSZ14]. Their Theorem 5 reduces the study of the scattering resonances $\lambda \in]1/h - R, 1/h +$
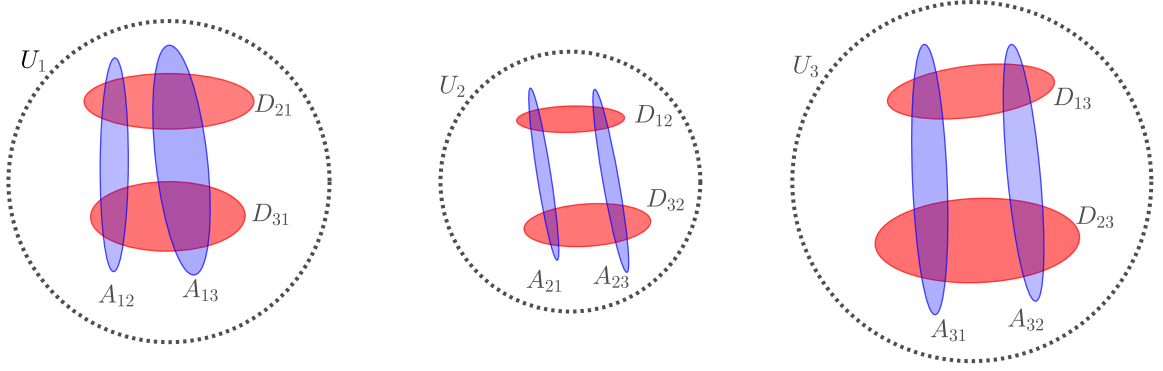
FIGURE 3. A schematic example of open hyperbolic map with $J = 3$ in a case where $D_{ii} = \emptyset$ for $i = 1, 2, 3$.

$R[+i] - R, R[$ to the study of the cancellation of

$$z \in] - R, R[+i] - R, R[\mapsto \det(I - M(z; h))$$

where

(1.2) $$M(z) = M(z; h) : L^2(\partial\mathcal{O}) \to L^2(\partial\mathcal{O})$$

is a family of *open quantum hyperbolic maps* (see below Section 1.2.1). The family $z \mapsto M(z)$ depends holomorphically on $z \in] - R, R[+i] - R, R[$ for some arbitrary $R > 0$ and is sometimes called a *hyperbolic quantum monodromy operator*. The zeros z and the resonances are related by the relation $h\lambda = 1 + z$. The notion of *monodromy* comes from the fact that the outgoing solutions of the equation $-\Delta u = \lambda^2 u$ must satisfy the equation $M(z)u = u$, which dictates the behavior of u on the boundary of the obstacles. We now introduce some definitions required to state the main theorem of this paper. We show how Theorem 3 implies Theorem 2 using the results of [NSZ14] in 6.1.

1.2.1. *Open quantum hyperbolic maps and statement on the main theorem.* The following long definition is based on the definitions in the works of Nonnenmacher, Sjöstrand and Zworski in [NSZ11] and [NSZ14] specialized to the 2-dimensional phase space. Consider open intervals $Y_1, \ldots, Y_J$ of J copies of $\mathbb{R}$ and set :

$$Y = \bigsqcup_{j=1}^{J} Y_j \subset \bigsqcup_{j=1}^{J} \mathbb{R}$$

and consider

$$U = \bigsqcup_{j=1}^{J} U_j \subset \bigsqcup_{j=1}^{J} T^*\mathbb{R} \quad ; \quad \text{where } U_j \Subset T^*Y_j \text{ are open sets}$$

The Hilbert space $L^2(Y)$ is the orthogonal sum $\bigoplus_{i=1}^{J} L^2(Y_i)$.

For $j = 1, \ldots, J$, consider open disjoint subsets $\widetilde{D}_{ij} \Subset U_j$, $1 \leq i \leq J$, *the departure sets*, and similarly, for $i = 1, \ldots, J$ consider open disjoint subsets $\widetilde{A}_{ij} \Subset U_i$, $1 \leq j \leq J$, *the arrival sets* (see Figure 3). We assume that there exists smooth symplectomorphisms, with smooth inverse,

(1.3) $$F_{ij} : \widetilde{D}_{ij} \to F_{ij}\left(\widetilde{D}_{ij}\right) = \widetilde{A}_{ij}$$

We note F for the global smooth map $F : \widetilde{D} \to \widetilde{A}$ where $\widetilde{A}$ and $\widetilde{D}$ are the full arrival and departure sets, defined as

$$\widetilde{A} = \bigsqcup_{i=1}^{J} \bigsqcup_{j=1}^{J} \widetilde{A}_{ij} \subset \bigsqcup_{i=1}^{J} U_i$$

$$\widetilde{D} = \bigsqcup_{j=1}^{J} \bigsqcup_{i=1}^{J} \widetilde{D}_{ij} \subset \bigsqcup_{j=1}^{J} U_j$$

We define the outgoing (resp. incoming) tail by $\mathcal{T}_+ := \{\rho \in U; F^{-n}(\rho) \in U, \forall n \in \mathbb{N}\}$ (resp. $\mathcal{T}_- := \{\rho \in U; F^n(\rho) \in U, \forall n \in \mathbb{N}\}$). We assume that they are closed subsets of U and that the *trapped set*

$$\tag{1.4} \mathcal{T} = \mathcal{T}_+ \cap \mathcal{T}_-$$

is compact. We also assume that

$$\mathcal{T} \text{ is } totally \ disconnected.$$

Remark. It is possible that for some values of i and j, $\widetilde{D}_{ij} = \emptyset$. For instance, when dealing with the billiard map (see subsection 6.1), the sets $\widetilde{D}_{ii}$ are all empty.

We then make the following dynamical assumption.

$$\tag{1.5} \mathcal{T} \text{ is a hyperbolic set for } F$$

Namely, for every $\rho \in \mathcal{T}$, we assume that there exist stable/unstable tangent spaces $E^s(\rho)$ and $E^u(\rho)$ such that :

- $\dim E^s(\rho) = \dim E^u(\rho) = 1$
- $T_\rho U = E^s(\rho) \oplus E^u(\rho)$
- there exists $\lambda > 0$, $C > 0$ such that for every $v \in T_\rho U$ and any $n \in \mathbb{N}$,

$$\tag{1.6} v \in E^s(\rho) \implies ||d_\rho F^n(v)|| \leq C e^{-n\lambda}||v||$$

$$\tag{1.7} v \in E^u(\rho) \implies ||d_\rho F^{-n}(v)|| \leq C e^{-n\lambda}||v||$$

where $||\cdot||$ is a fixed Riemannian metric on U.

The decomposition of $T_\rho U$ into stable and unstable spaces is assumed to be continuous. It allows to define stable and unstable Jacobians $J_n^u(\rho)$ and $J_n^s(\rho)$ (see Definition 3.3 for the precise definition). We define the maximal Lyapounov exponent $\lambda_{\max}$ as

$$\tag{1.8} \lambda_{\max} = \sup_{\rho \in \mathcal{T}} \limsup_{n \to +\infty} \frac{1}{n} \log J_n^u(\rho)$$

We also note

$$\tag{1.9} \varphi_u(\rho) = \log J_1^u(\rho)$$

Remark.

- The definition is valid for any Riemannian metric on U and we can of course suppose that is it the standard Euclidean metric.
- It is a standard fact (See [Mat68]) that there exists a smooth Riemannian metric on U, which is said to be adapted to the dynamic, such that (1.6) and (1.7) hold with $C = 1$.

Here ends the description of the classical map. It encompasses the case of the billiard map, useful when dealing with obstacle scattering (see subsection 6.1). We then associate to F *open quantum hyperbolic maps*, which are its quantum counterpart. The definition of such operators is presented in detail in 2.2.3. An open quantum hyperbolic map $T : L^2(Y) \to L^2(Y)$ is an operator-valued matrices $(T_{ij})_{ij}$ where $T_{ij} : L^2(Y_j) \to L^2(Y_i)$ is a Fourier integral operator associated with F_{ij} (see Definition 2.4 for a precise definition).

We now come to the statement of the main theorem of this paper.

Assumptions of Theorem 3. We consider a family $(M_h(z))_z = (M(z; h))_z$ of open hyperbolic quantum maps, associated with F, as defined in Definition 2.4, and depending holomorphically on a parameter $z \in \Omega = \Omega_R =]-R, R[+i]-R, R[$ with R fixed (but in practice, for applications, it can be chosen arbitrarily large). We suppose that there exists $L > 0$ and $a \in C_c^\infty(T^*Y)$ such that $\mathrm{supp}(a)$ is contained in a compact neighborhood $\mathcal{W}$ of $\mathcal{T}$, $\mathcal{W} \subset \widetilde{D}$, $a = 1$ in a neighborhood of $\mathcal{T}$ and uniformly in Ω,

$$M_h(z)(1 - \mathrm{Op}_h(a)) = O(h^L)_{L^2 \to L^2}$$

Let's note $\alpha_h(z)$ the amplitude of $M_h(z)$ (as defined after definition 2.4). We make the following assumption on α_h : there exists a neighborhood $V \subset U$ of $\mathcal{T}$ and a smooth function $t_{ret} : V \to \mathbb{R}_+$[1] such that $\inf_V t_{ret} > 0$, $\sup_V t_{ret} < +\infty$ and for all $z \in \Omega$ and $\rho \in V$,

$$\alpha_h(z)(\rho) = \exp(-i \operatorname{Im} z t_{ret}(\rho)) + O(h^{1-})_{S_{0+}}$$

that is, there exists $\chi \in C_c^\infty(U)$ supported in a larger neighborhood of $\mathcal{T}$ with $\chi \equiv 1$ on V, such that for every $\eta > 0$, $\chi(\alpha_h(z) - \exp(-izt_{ret}))$ is in $h^{1-\eta}S_\eta$ uniformly for $z \in \Omega$. The definition of the symbol class S_η and S_{0+} are recalled in Section 2.

Remark. In particular, the principal part $\alpha_z(\rho) = \exp(-izt_{ret}(\rho))$ of α_h is independent of h in V. This assumption, which may look strong at first glance, is in fact satisfied in the two applications we consider (see 6.1 and 6.2). In fact, the works of [NSZ11] and [NSZ14] allow to work up to $\operatorname{Im} z = C \log h$. For such z, α is clearly h dependent and lives in the symbol class S_{0+}.

We also assume that $M_h(z)$ is uniformly bounded for $z \in \Omega$ and for all h small enough

$$\|M_h(z)\|_{L^2 \to L^2} \leq C.$$

Let us now define the following quantity :

(1.10)
$$p(\beta) = -\frac{1}{6\lambda_{\max}} P(-\varphi_u + 2\beta t_{ret})$$

where P denotes the topological pressure of $\varphi : \rho \in \mathcal{T} \mapsto -\varphi_u + 2\beta t_{ret}$ with respect to the dynamics of F on $\mathcal{T}$. It is defined as (see also 3.1)

$$P(\varphi) = \lim_{\epsilon \to 0} \limsup_{n \to +\infty} \frac{1}{n} \log P_0(\varphi, n, \epsilon)$$

where

$$P_0(\varphi, n, \epsilon) = \sup \left\{ \sum_{x \in E} \exp\left(\sum_{i=0}^{n-1} \varphi(F^i(x)) \right) \; ; \; E \text{ is } (n, \epsilon) \text{ separated} \right\}$$

(a subset $E \subset \mathcal{T}$ is said to be (n, ϵ) separated if for every $x, y \in E, x \neq y$, there exits $0 \leq i \leq n-1$, $d(F^i(x), F^i(y)) > \epsilon$). The quantity $\sum_{i=0}^{n-1} \varphi(F^i(x))$ is called a Birkhoff sum. The map $\beta \mapsto p(\beta)$ is a non increasing function of β and at $\beta = 0$, we have

$$p(0) = -\frac{1}{6\lambda_{max}} P(-\varphi_u) > 0.$$

For $\Omega' \subset \Omega$, we note $m_M(\Omega') = \sum_{z \in \Omega', f_h(z)=0} m(z)$ where $m(z)$ stands for the multiplicity of z as a zero of $f_h(z) = \det(1 - M_h(z))$. Note that this determinant is well-defined since the operators $M_h(z)$ are constructed trace-class (see the 2.2.3). In this paper, we prove

Theorem 3. *For every $\varepsilon > 0$, $\gamma > 0$ and $0 < R' < R$, there exist $C = C_{\varepsilon, \gamma, R'} > 0$ and $h_0 > 0$ such that*

$$m_M\left(\{|\operatorname{Re} z| < R', \operatorname{Im} z \in [-\gamma, 0]\}\right) \leq Ch^{-d_H + \max(p(\gamma+\varepsilon) - \varepsilon, 0)} \,, \; \forall 0 < h \leq h_0$$

where $2d_H$ is the Hausdorff dimension of $\mathcal{T}$.

1.2.2. *Application in semiclassical scattering by a potential.* The reduction from an open quantum system to an open quantum hyperbolic map, proved in [NSZ14] for the case of obstacle scattering, is also proved in the case of potential scattering in [NSZ11]. As a consequence, we can prove a bound similar to the one given by Theorem 2 in potential scattering. The following theorem is proved in 6.2 using Theorem 3.

Theorem 4. *Let $V \in C_c^\infty(\mathbb{R}^2)$, $E_0 > 0$ and consider the semiclassical pseudodifferential operator $P_h = -h^2\Delta + V - E_0$. Let's note $p(x, \xi) = \xi^2 + V - E_0$ and assume that*

$$dp \neq 0 \text{ on } p^{-1}(0)$$

Let's note H_p the Hamiltonian vector field associated with p and $\Phi_t = \exp(tH_p)$ the corresponding Hamiltonian flow. Let's note K_0 the trapped set of p at energy 0 and let's assume that

  (i) Φ_t is hyperbolic on K_0 ;
  (ii) K_0 is topologically one dimensional.

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[1]In the applications, t_{ret} is a return time function.

Let γ_{cl} be the classical escape rate of the system at energy 0 and $2d_H+1$ be the Hausdorff dimension of K_0. Let $N(R, \gamma; h)$ be the number of resonances of P_h/h in $\{|\operatorname{Re} z| < R, \operatorname{Im} z \in [-\gamma, 0]\}$, counted with multiplicity. Then, there exists a non increasing function $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

- *$\sigma(\gamma) > 0$ for $0 \leq \gamma < \gamma_{cl}/2$;*
- *$\sigma(\gamma) = 0$ for $\gamma \geq \gamma_{cl}/2$*

and such that for all $R, \gamma > 0$ and for all $\varepsilon > 0$ there exists $C_{R,\gamma,\varepsilon} > 0$ and $h_0 > 0$ such that

$$\forall 0 < h \leq h_0 \,, \; N(R, \gamma; h) \leq C_{R,\gamma,\varepsilon} h^{-d_H + \sigma(\gamma) - \varepsilon}.$$

Remark. We are interested in resonances of a Schrödinger operator $P_h = -h^2 \Delta + V - E_0$ in a neighborhood of 0 of size h. To keep notations consistent with the spectral parameter z appearing in Theorem 3, we renormalize to study the resonances of P_h/h in a fixed neighborhood of 0.

Remark. The theorem could be extended to a wider class a perturbations of the Laplacian in manifolds with Euclidean ends. We refer the reader to [NSZ11] (Section 2.1) for more general assumptions

1.2.3. *Sketch of proof of Theorem 3.* In [NSZ14], to prove the Fractal Weyl upper bound, the author modify the monodromy operator $M(z; h)$ and replace it by

$$M_{tG}(z; h) = e^{-tG} M(z; h) e^{tG}$$

where $G = \operatorname{Op}_h(g)$ with g an escape function in the critical symbol class $\tilde{S}_{1/2}$, constructed such that the Fourier integral operator M_{tG} has a small amplitude outside a neighborhood $\mathcal{T}(h^{1/2-})$ of $\mathcal{T}$, and t is a fixed parameter. Here, to avoid the critical symbol class $\tilde{S}_{1/2}$, we will work in the symbol class S_δ for some $\delta = 1/2 - \varepsilon$, so that the interesting neighborhood of $\mathcal{T}$ becomes $\mathcal{T}(h^\delta)$, which has a volume comparable to $h^{\delta(2-2d_H)}$.

Since the zeros of $z \mapsto \det(1 - M_{tG}(z; h))$ coincide (with multiplicity) with the zeros of $\det(1 - M(z; h))$, we wish to count the zeros of $\det(1 - M_{tG}(z; h))$. Jensen's formula and standard spectral inequalities on spectral determinants reduce the estimates on the zeros of $\det(1 - M_{tG}(z; h))$ to a control on the Hilbert-Schmidt norm of $M_{tG}(z; h)$. In [NSZ14], the author show that M_{tG} is close to an operator having a rank comparable to h^{-d_H}, which lead them to a bound of the form

$$||M_{tG}(z; h)||^2_{HS} \leq ||M_{tG}(z; h)||^2_{L^2 \to L^2} \times \; \operatorname{rank} \; \leq C h^{-d_H}$$

To improve the fractal upper bound of [NSZ14] and prove Theorem 3, we start with the simple observation that the zeros of $\det(1 - M_{tG}(z; h))$ are among the zeros of $\det(1 - M_{tG}^n(z; h))$, for any $n \in \mathbb{N}^*$. We use this fact with an exponent $n = n(h)$ depending on h : $n(h) \sim \nu \log 1/h$ for some $\nu > 0$. *A priori*, when $n(h)$ grows logarithmically, M_{tG}^n becomes "nasty" (i.e. no more a Fourier Integral Operator in a suitable class; recall that essentially $g \in S_\delta$), and in particular, it becomes impossible to use Egorov's theorem as soon as $n \geq \varepsilon \log 1/h$, for some small ε (essentially $\frac{1/2-\delta}{\lambda_{\max}}$). However, the action of the operator $M_{tG}(z)$ on coherent states φ_ρ will remain under control for a sufficiently long logarithmic time. We will be able to obtain good estimates up to

$$n(h) \sim \frac{1}{6\lambda_{\max}} \log 1/h$$

To use these estimates, we use the representation of the trace in terms of coherent states :

$$(1.11) \qquad\qquad ||M_{tG}(z; h)^n||^2_{HS} = \frac{1}{2\pi h} \int_U ||M_{tG}(z; h)^n \varphi_\rho||^2 d\rho$$

The main new ingredient in the present paper will consist in controlling precisely the evolved states $M_{tG}^n \varphi_\rho$ for such logarithmic times. The behavior of this state will depend on the initial point ρ (see Proposition 4.2 for a precise and rigorous statement)

- If ρ is not in an h^δ neighborhood of $\mathcal{T}$, we will show that for any $L > 0$, we can find $t = t(L)$ such that the norm of $M_{tG}^n \varphi_\rho$ is $O(h^L)$. As a consequence, the mass in the integral in (1.11) is essentially contained in an h^δ neighborhood of $\mathcal{T}$. In particular, by simply estimating $||M_{tG}(z; h)^n||^2 \leq C$ in a h^δ neighborhood of $\mathcal{T}$, we find that

$$||M_{tG}(z; h)^n||^2_{HS} \leq C h^{-1} h^{\delta(2-2d_H)} \leq C h^{-d_H + O(\varepsilon)}$$

This gives the previous upper bound of [NSZ14]

- For states sufficiently close to $\mathcal{T}$, φ_ρ will evolve into a squeezed coherent state, aligned along the unstable leaves of $\mathcal{T}_+$. This phenomenon can be understood as a delocalization of the coherent state. In the unstable direction, the components of this squeezed state far from $\mathcal{T}_-$ (that it at distance bigger that h^δ) will experience a strong damping due to the escape function. For such state, we are able to control the squared L^2-norm

$$w_z(\rho) := ||M_{tG}(z;h)^n \varphi_\rho||^2$$

by (again, see Proposition 4.2 for the rigorous statement)

$$(1.12) \qquad w_z(\rho) \le C \left(\prod_{i=0}^{n-1} \alpha_z(F^i(\rho)) \right)^2 J_n^u(\rho)^{d_H - 1}$$

where $\alpha_z(\rho) = \exp(-(\operatorname{Im} z) t_{ret}(\rho))$. This is the crucial estimate of this paper.

Plugging this bound into the integral in (1.11), we are able to prove the following upper bound

$$||M_{tG}(z;h)^n||_{HS}^2 \le C_\varepsilon h^{-d_H + \tilde\sigma(z) - O(\varepsilon)} \; ; \; \tilde\sigma(z) = -\frac{1}{6\lambda_{\max}} P(-\varphi_u + 2\log\alpha_z)$$

(see Proposition 4.1). The link between the pressure and (1.12) appears when one writes

$$w_z(\rho) \le C J_n^u(\rho)^{d_H} \exp\left(\sum_{i=0}^{n-1} (-2\operatorname{Im} z t_{ret} - \varphi_u) \circ F^i(\rho) \right)$$

The factor $J_n^u(\rho)^{d_H}$ disappears after integrating (see the proof of Proposition 4.1). It finally gives Theorem 3 (see Section 4).

The crucial estimate (1.12) is the main novelty of this paper. It relies on propagation of coherent states and a subtle interaction of the evolved state with the escape function (see Figure 4). The proof of (1.12) relies on the following ideas :

- The term

$$(1.13) \qquad \pi_{\alpha,n}(\rho) := \prod_{i=0}^{n-1} \alpha_z(F^i(\rho))$$

  comes from the repeated action of $M(z)$ on φ_ρ.
- The initial state φ_ρ is a wavepacket of size $h^{1/2}$. $M(z)^n \varphi_\rho$ is a squeezed coherent state, microlocalized near $F^n(\rho)$. This is due to the fact that we will work with $n = n(h) \le \frac{1-\eta}{6\lambda_{\max}} \log(1/h)$ for some $\eta > 0$. Nevertheless, it is no more microlocalized in a $h^{1/2}$ neighborhood of this point. It will be more convenient to write it as a Lagrangian state, associated with a local unstable leaf $W_u(\rho_n)$, for some $\rho_n \in \mathcal{T}$ close to $F^n(\rho)$: if $\psi_u(x)$ is a generating function for $W_u(\rho_n)$, that is, if we can write $W_u(\rho_n) = \{(x, \psi_u'(x))\}$, the state will be written

$$a_h(x) e^{\frac{i}{h}\psi_u(x)}$$

  The size of this Lagrangian state along the unstable manifold is controlled by the local Jacobian near ρ and is $O(h^{1/2} J_n^u(\rho))$: we will see that

$$|x| \gg h^{1/2} J_n^u(\rho) \implies a_h(x) = O(h^\infty)$$

- Finally, we need to understand the interaction of the escape function with this evolved state. The action of the escape function damps the part of the state at distance larger that h^δ of $\mathcal{T}$. Since such a state is very close to an unstable manifold, the only relevant damping on this state comes from the components at distance larger that h^δ from $\mathcal{T}_-$. Roughly speaking, to obtain the bound we want, we prove that if $d((x, \psi_u'(x)), \mathcal{T}_-) \le h^\delta$, then
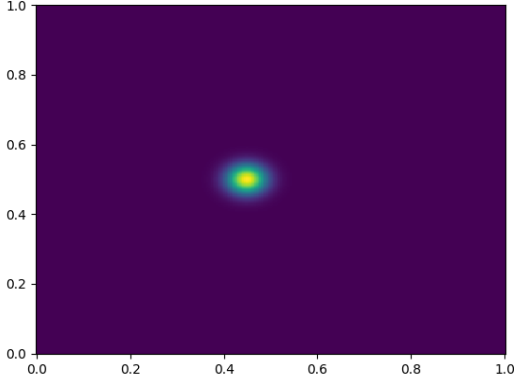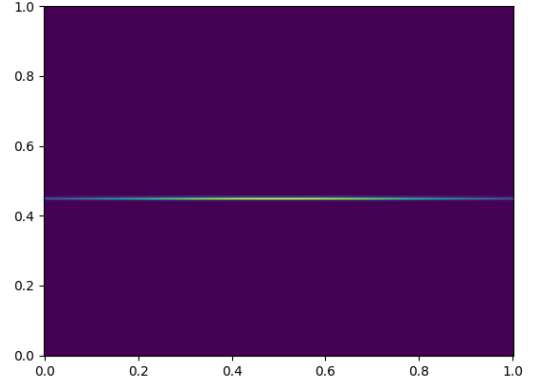
$$(1.14) \qquad a_h(x) \le C\pi_{\alpha,n}(\rho) (J_n^u(\rho))^{-1/2} h^{-1/4}$$

  and we prove that we can neglect the remaining points x such that $d((x, \psi_u'(x)), \mathcal{T}_-) \ge h^\delta$ (see Proposition 5.6). It gives
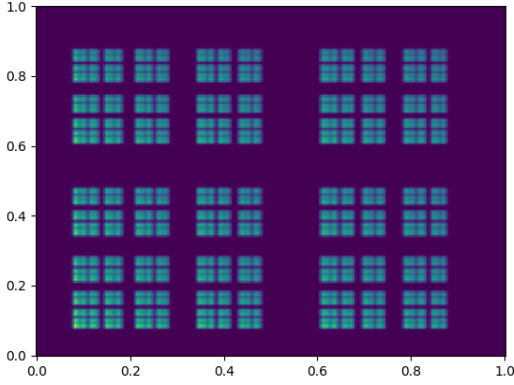
$$||M_{tG}(z)^n \varphi_\rho||_{L^2}^2 \le C\pi_{\alpha,n}(\rho)^2 J_n^u(\rho)^{-1} h^{-1/2} \operatorname{Len}(X_-(\rho, \rho_n))$$
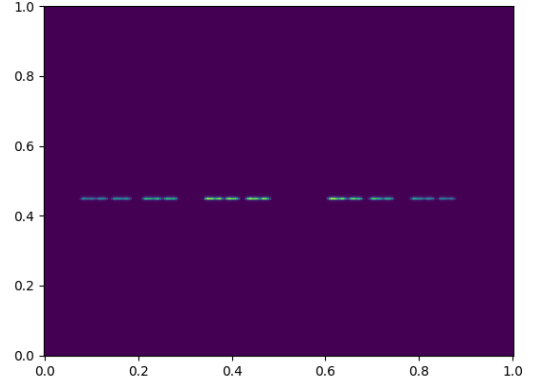
  where

$$X_-(\rho, \rho_n) = \{x \in \mathbb{R}, |x| \le C J_n^u(\rho) h^{1/2}, d((x, \psi_u'(x)), \mathcal{T}_-) \le h^\delta\}$$

(A) A coherent state of size $h^{1/2}$ ...

(B) ...evolved into a squeezed coherent state.

(C) The escape function damps the region far from the trapped set...

(D) ... and is responsible of a damping for the evolved coherent state.

FIGURE 4. We show the evolution in phase space of a coherent state in an open hyperbolic system, associated with an open baker's map. The color is related to the intensity of the Wigner distribution of the state. The damping due to the escape function is shown in (4c). The initial coherent state is shown in (4a), and the evolved state, without damping in (4b). When we apply the damping, the evolved state loses part of its mass (4d)

.

- It remains to control the length of $X_-(\rho, \rho_n)$. We use the fact that $\mathcal{T}_- \cap W_u(\rho_n)$ has box dimension d_H. In fact, we are interested by a piece of $W_u(\rho_n)$ of size $h^{1/2} J_n^u(\rho)$ and we show that such a piece can be covered by N_h balls of radius h^δ with (see Lemma 3.6)

$$N_h \leq C \left(\frac{h^\delta}{h^{1/2} J_n^u(\rho)} \right)^{-d_H}$$

so that

$$\mathrm{Len}(X_-(\rho, \rho_n)) \leq C N_h h^\delta \leq C h^{1/2} J_n^u(\rho)^{d_H} h^{-O(\varepsilon)}$$

- Putting the pieces together, we obtain (1.12).

Plan of the paper. The paper is organized as follows :

- We start with preliminaries in Section 2 and Section 3. Section 2 is devoted to semiclassical results concerning pseudodifferential operators, Fourier integral operators, metaplectic

operators and coherent states. Section 3 focuses on properties of hyperbolic dynamical systems.
- Section 4 reduces the proof of Theorem 3 to the key proposition 4.2, concerning the behavior of the propagated coherent states.
- Section 5 is devoted to the proof of Proposition 4.2.

Notations. Throughout the paper, we will use the same constant C at different places, with different meaning. However, it will always have the same dependence on the dynamical system and the family of operators $M_h(z)$ we work with. That is, we write $f \leq Cg$ instead of : there exists $C > 0$ depending on F and $M_h(z)$ such that $f \leq Cg$. At some point, we will fix a partition of unity of U, associated with local charts, depending on parameters ε and ε_0. The constants C will also depend on these objects. If the constant C has other dependencies, we will make it precise it or write it in subscript if necessary.

Finally, we write $f \sim g$ to mean $C^{-1} f \leq g \leq Cf$.

2. SEMICLASSICAL PRELIMINARIES

2.1. **Pseudodifferential operators and Weyl quantization.** We recall some basic notions and properties of the Weyl quantization on $\mathbb{R}^n$. We refer the reader to [Zwo12] for the proofs of the statements and further considerations on semiclassical analysis and quantizations.

Definitions. We start by defining classes of h-dependent symbols. In the following definitions, m is a positive functions defined on $T^*\mathbb{R}^n$ of the form $\langle \rho \rangle^N$, for some $\mathbb{N} \in \mathbb{Z}$, where $\langle \rho \rangle = \sqrt{1 + |\rho|^2}$ and $\rho = (x, \xi)$ is a point in phase space $T^*\mathbb{R}^n = \mathbb{R}^{2n}$. m is called an order function (in the sense of [Zwo12], 4.4.1)

Definition 2.1. Let $0 \leq \delta < \frac{1}{2}$. We say that an h-dependent family $a := (a(\cdot; h))_{0 < h \leqslant 1} \in C^\infty(\mathbb{R}^{2n})$ is in the class $S_\delta(m)$ (and simply S_δ if $m = 1$) if for every $\alpha \in \mathbb{N}^{2n}$, there exists $C_\alpha > 0$ such that :

$$\forall 0 < h \leq 1, \ \sup_{\rho \in \mathbb{R}^{2n}} |\partial^\alpha a(\rho; h)| \leq C_\alpha h^{-\delta|\alpha|} m(\rho)$$

We will use the notation $S_{0^+}(m) = \bigcap_{\delta > 0} S_\delta(m)$.

We write $a = O(h^N)_{S_\delta(m)}$ to mean that for every $\alpha \in \mathbb{N}^{2n}$, there exists $C_{\alpha,N}$ such that

$$\forall 0 < h \leq 1, \ \sup_{\rho \in \mathbb{R}^{2n}} h^{\delta|\alpha|} \partial^\alpha a(\rho; h)| \leq C_{\alpha,N} h^N m(\rho)$$

If $a = O(h^N)_{S_\delta(m)}$ for all $N \in N$, we'll write $a = O(h^\infty)_{S_\delta(m)}$. For a given symbol $a \in S_\delta$, we say that a has a compact essential support if there exists a compact set K such that[2] :

$$\forall \chi \in C_c^\infty(\mathbb{R}^{2n}), \operatorname{supp} \chi \cap K = \emptyset \implies \chi a = O(h^\infty)_{\mathcal{S}(\mathbb{R}^{2n})}$$

We say that a belongs to the class S_δ^{comp} and its essential support is then the intersection of all such compact K's. We denote it $\operatorname{ess\,supp} a \subset K$. In particular, the class S_δ^{comp} contains all the symbols supported in a h-independent compact set and these symbols correspond, modulo $O(h^\infty)_{\mathcal{S}(T^*\mathbb{R})}$, to all symbols of S_δ^{comp}. For this reason, we will adopt the following notation: if $V \Subset \mathbb{R}^{2n}$ is an open set, we say that $a \in S_\delta^{comp}(V)$ if $a \in S_\delta^{comp}(\mathbb{R}^{2n})$ and $\operatorname{ess\,supp} a \Subset V$.

For a symbol $a = a(\cdot; h) \in S_\delta(m)$, we'll quantize it using Weyl's quantization procedure. It is informally written as :

$$(\operatorname{Op}_h(a)u)(x) = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^{2n}} a\left(\frac{x+y}{2}, \xi; h\right) u(y) e^{i\frac{(x-y)\cdot\xi}{h}} dy d\xi$$

---

[2]Here $\mathcal{S}$ denotes the Schwartz space and the notation $O(h^\infty)_{\mathcal{S}(\mathbb{R}^{2n})}$ means that every semi-norm is $O(h^\infty)$.

We will note $\Psi_\delta(m)$ the corresponding classes of pseudodifferential operators. By definition, the wavefront set of $A = \mathrm{Op}_h(a)$ is $\mathrm{WF}_h(A) = \mathrm{ess\,supp}\, a$.

We say that a family $u = u(h) \in \mathcal{D}'(\mathbb{R}^n)$ is h-tempered if for every $\chi \in C_c^\infty(\mathbb{R}^n)$, there exist $C > 0$ and $N \in \mathbb{N}$ such that $||\chi u||_{H_n^{-N}} \leq Ch^{-N}$. For a h-tempered family u, we say that a point $\rho \in T^*\mathbb{R}^n$ does *not* belong to the wavefront set of u if there exists $a \in S^{comp}$ such that $a(\rho) \neq 0$ and $\mathrm{Op}_h(a)u = O(h^\infty)_{C^\infty}$. We note $\mathrm{WF}_h(u)$ the wavefront set of u.

We say that a family of operators $B = B(h) : C_c^\infty(\mathbb{R}^{n_2}) \to \mathcal{D}'(\mathbb{R}^{n_1})$ is h-tempered if its Schwartz kernel $\mathcal{K}_B \in \mathcal{D}'(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ is h-tempered. We define the twisted wavefront set of B as

$$\mathrm{WF}_h{}'(B) = \{(x, \xi, y, -\eta) \in T^*\mathbb{R}^{n_1} \times T^*\mathbb{R}^{n_2}, (x, \xi, y, \eta) \in \mathrm{WF}_h(\mathcal{K}_B)\}$$

Standard properties. Let us now recall standard results in semiclassical analysis concerning the L^2-boundedness of pseudodifferential operator and their composition. We'll use the following version of Calderon-Vaillancourt Theorem ([Zwo12], Theorem 4.23).

Proposition 2.1. There exists $C_n > 0$ such that the following holds. For every $0 \leq \delta < \frac{1}{2}$, and $a \in S_\delta$, $\mathrm{Op}_h(a)$ is a bounded operator on L^2 and

$$|| \mathrm{Op}_h(a)||_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)} \leq C_n \sum_{|\alpha| \leq 8n} h^{|\alpha|/2} ||\partial^\alpha a||_{L^\infty}$$

As a consequence of the sharp Gärding inequality (see [Zwo12], Theorem 4.32), we also have the precise estimate of L^2 norms of pseudodifferential operator,

Proposition 2.2. Assume that $a \in S_\delta(\mathbb{R}^{2n})$. Then, there exists C_a depending on a finite number of semi-norms of a such that :

$$|| \mathrm{Op}_h(a)||_{L^2 \to L^2} \leq ||a||_\infty + C_a h^{\frac{1}{2} - \delta}$$

We recall that the Weyl quantizations of real symbols are self-adjoint in L^2. If m_1 and m_2 are two order functions of the form $\langle \rho \rangle^{N_i}$, $i = 1, 2$, the composition of two pseudodifferential operators in $\Psi_\delta(m_1)$ and $\Psi_\delta(m_2)$ is a pseudodifferential operator in the class $\Psi_\delta(m_1 m_2)$. More precisely (see [Zwo12], Theorem 4.11 and 4.18), if $(a, b) \in S_\delta(m_1) \times S_\delta(m_2)$, $\mathrm{Op}_h(a) \circ \mathrm{Op}_h(b)$ is given by $\mathrm{Op}_h(a \# b)$, where $a \# b$ is the Moyal product of a and b. It is given by

$$a \# b(\rho) = e^{ihA(D)}(a \otimes b)|_{\rho = \rho_1 = \rho_2}$$

where $a \otimes b(\rho_1, \rho_2) = a(\rho_1)b(\rho_2)$, $e^{ihA(D)}$ is a Fourier multiplier acting on functions on $\mathbb{R}^{4n}$ and, writing $\rho_i = (x_i, \xi_i)$,

$$A(D) = \frac{1}{2}\left(D_{\xi_1} \circ D_{x_2} - D_{x_1} \circ D_{\xi_2}\right)$$

We can estimate the Moyal product by a quadratic stationary phase and get the following expansion which holds in $S_\delta(m_1 m_2)$ for all $N \in \mathbb{N}$,

$$a \# b(\rho) = \sum_{k=0}^{N-1} \frac{i^k h^k}{k!} A(D)^k (a \otimes b)|_{\rho = \rho_1 = \rho_2} + r_N$$

where for all $\alpha \in \mathbb{N}^{2n}$, there exists C_α, independent of a and b, such that

$$||\partial^\alpha r_N||_\infty \leq C_\alpha h^N ||a \otimes b||_{C^{2N+4n+1+|\alpha|}}$$

Weighted Sobolev spaces. We can also define the weighted Sobolev spaces $H_h(m)$. In the case $m = \langle \rho \rangle^N$, we have

$$H_h\left(\langle \rho \rangle^N\right) = \mathrm{Op}_h\left(\langle \rho \rangle^{-N}\right)\left(L^2(\mathbb{R}^n)\right) \subset \mathcal{S}'(\mathbb{R}^n)$$

When $N \geq 0$, $H_h\left(\langle \rho \rangle^N\right)$ coincides with the space of functions $u \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\forall \alpha, \beta \in \mathbb{N}^n \text{ with } \alpha + \beta \leq N \,, \ x^\alpha (h \partial^\beta) u \in L^2(\mathbb{R}^n)$$

and we have the following equivalence of norms :

$$||u||^2_{H_h(\langle \rho \rangle^N)} \sim \sup_{|\alpha| + |\beta| \leq N} ||x^\alpha (h \partial^\beta) u||^2_{L^2}$$

As a consequence of Calderon-Vaillancourt theorem, we have for symbols $a \in S_\delta(m)$:

Proposition 2.3. Let $N \in \mathbb{Z}$. There exists $M \in \mathbb{N}$ and $C > 0$ such that the following holds : For all $a \in S_\delta \left(\langle \rho \rangle^N \right)$, $\operatorname{Op}_h(a) : H_h \left(\langle \rho \rangle^N \right) \to L^2$ is uniformly bounded and

$$\| \operatorname{Op}_h(a) \|_{H_h(\langle \rho \rangle^N) \to L^2} \leq C \sup_{|\alpha| \leq M} h^{|\alpha|/2} \| \langle \rho \rangle^{-N} \partial^\alpha a \|_\infty$$

2.2. **Fourier Integral Operators.** We now review some aspects of the theory of Fourier integral operators. We follow [Zwo12], Chapter 11 and [NSZ14]. We refer the reader to [GS13] for further details on Lagrangian distributions and Fourier integral operators. We also introduce the material needed to understand the definition 2.4 of open hyperbolic quantum maps. We also provide a quantitative version of Egorov's theorem.

2.2.1. *Local symplectomorphisms and their quantization.* We momentarily work in dimension n. Let us note $\mathcal{K}$ the set of symplectomorphisms $\kappa : T^*\mathbb{R}^n \to T^*\mathbb{R}^n$ such that the following holds : there exist continuous and piecewise smooth (in t) families of smooth functions $(\kappa_t)_{t \in [0,1]}, (q_t)_{t \in [0,1]}$ such that :

- $\forall t \in [0,1]$, $\kappa_t : T^*\mathbb{R}^n \to T^*\mathbb{R}^n$ is a symplectomorphism ;
- $\kappa_0 = \operatorname{Id}_{T^*\mathbb{R}^n}, \kappa_1 = \kappa$;
- $\forall t \in [0,1], \kappa_t(0) = 0$;
- there exists $K \Subset T^*\mathbb{R}^n$ compact such that $\forall t \in [0,1], q_t : T^*\mathbb{R}^n \to \mathbb{R}$ and $\operatorname{supp} q_t \subset K$;
- $\frac{d}{dt} \kappa_t = (\kappa_t)^* H_{q_t}$

If $\kappa \in \mathcal{K}$, we note $G_\kappa = Gr'(\kappa) = \{(x, \xi, y, -\eta), (x, \xi) = \kappa(y, \eta)\}$ the twisted graph of κ, which is Lagrangian in $T^*\mathbb{R}^n$. We recall [Zwo12], Lemma 11.4, which asserts that local symplectomorphisms fixing the origin can be seen as elements of $\mathcal{K}$, as soon as we have some geometric freedom.

Lemma 2.1. Let U_0, U_1 be open and precompact subsets of $T^*\mathbb{R}^n$. Assume that $\kappa : U_0 \to U_1$ is a local symplectomorphism fixing 0 and which extends to $V_0 \supseteq U_0$ an open star-shaped neighborhood of 0. Then, there exists $\tilde{\kappa} \in \mathcal{K}$ such that $\tilde{\kappa}|_{U_0} = \kappa$.

If $\kappa \in \mathcal{K}$ and if (q_t) denotes the family of smooth functions associated with κ in its definition, we note $Q(t) = \operatorname{Op}_h(q_t)$. It is a continuous and piecewise smooth family of operators. Then the Cauchy problem

$$(2.1) \qquad \qquad \begin{cases} hD_t U(t) + U(t)Q(t) = 0 \\ \qquad\qquad U(0) = \operatorname{Id} \end{cases}$$

is globally well-posed.

Following [NSZ14], Definition 3.9, we adopt the definition (with $G_\kappa = Gr'(\kappa)$):

Definition 2.2. Fix $\delta \in [0, 1/2)$. We say that $T \in I_\delta(\mathbb{R}^n \times \mathbb{R}^n; G_\kappa)$ if there exist $a \in S_\delta(T^*\mathbb{R}^n)$ and a path (κ_t) from Id to κ satisfying the above assumptions such that $T = \operatorname{Op}_h(a)U(1)$, where $t \mapsto U(t)$ is the solution of the Cauchy problem (2.1).

The class $I_{0+}(\mathbb{R}^n \times \mathbb{R}^n, G_\kappa)$ is by definition $\bigcap_{\delta > 0} I_\delta(\mathbb{R}^n \times \mathbb{R}^n, G_\kappa)$.

It is a standard result, known as Egorov's theorem (see [Zwo12], Theorem 11.1) that if $U(t)$ solves the Cauchy problem (2.1) and if $b_0 \in S_\delta$, then $U(1)^{-1} \operatorname{Op}_h(b_0)U(1)$ is a pseudodifferential operator in Ψ_δ and if $b_1 = b_0 \circ \kappa$, then $U(t)^{-1} \operatorname{Op}_h(b_0)U(t) - \operatorname{Op}_h(b_1) \in h^{1-2\delta}\Psi_\delta$.

Remark. Applying Egorov's theorem and Beal's theorem, it is possible to show that if (κ_t) is a closed path from Id to Id, and $U(t)$ solves (2.1), then $U(1) \in \Psi_0(\mathbb{R}^n)$. In other words, $I_\delta(\mathbb{R} \times \mathbb{R}, Gr'(\operatorname{Id})) \subset \Psi_\delta(\mathbb{R}^n)$. But the other inclusion is trivial. Hence, this in an equality :

$$I_\delta(\mathbb{R}^n \times \mathbb{R}^n, Gr'(\operatorname{Id})) = \Psi_\delta(\mathbb{R}^n)$$

The notation $I(\mathbb{R}^n \times \mathbb{R}^n, G_\kappa)$ comes from the fact that the Schwartz kernels of such operators are Lagrangian distributions associated with G_κ, and in particular have wavefront sets included in C. As a consequence, if $T \in I_\delta(\mathbb{R}^n \times \mathbb{R}^n, G_\kappa)$, $\operatorname{WF}_h'(T) \subset \operatorname{Gr}(\kappa)$.

We also recall that the composition of two Fourier integral operators is still a Fourier integral operator : if $\kappa_1, \kappa_2 \in \mathcal{K}$ and $U_1 \in I_\delta(\mathbb{R}^n \times \mathbb{R}^n, Gr'(\kappa_1)), U_2 \in I_\delta(\mathbb{R}^n \times \mathbb{R}^n, Gr'(\kappa_1))$, then,

$$U_1 \circ U_2 \in I_\delta(\mathbb{R}^n \times \mathbb{R}^n, Gr'(\kappa_1 \circ \kappa_2))$$

2.2.2. *An important example.* Let us focus on a particular case of canonical transformations. Suppose that $\kappa : T^*\mathbb{R}^n \to T^*\mathbb{R}^n$ is a canonical transformation such that

$$(x, \xi, y, \eta) \in \mathrm{Gr}(\kappa) \mapsto (x, \eta)$$

is a local diffeomorphism near $(x_0, \xi_0, y_0, \eta_0)$. Then, there exists a phase function $\psi \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, Ω_x, Ω_η open sets of $\mathbb{R}^n$ and Ω a neighborhood of $(x_0, \xi_0, y_0, -\eta_0)$, such that

$$\mathrm{Gr}'(\kappa) \cap \Omega = \{(x, \partial_x \psi(x, \eta), \partial_\eta \psi(x, \eta), -\eta), x \in \Omega_x, \eta \in \Omega_\eta\}$$

One says that ψ generates $\mathrm{Gr}'(\kappa)$. Suppose that that $a \in S_\delta^{comp}(\Omega_x \times \Omega_\eta)$. Then, the following operator T is an element of $I_\delta(\mathbb{R}^n \times \mathbb{R}^n, \mathrm{Gr}'(\kappa))$:

$$(2.2) \qquad Tu(x) = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^{2n}} e^{\frac{i}{h}(\psi(x,\eta) - y \cdot \eta)} a(x, \eta) u(y) dy d\eta$$

and if $T^*T = \mathrm{Id}$ microlocally near (y_0, η_0) - that is if $(y_0, \eta_0) \notin \mathrm{WF}_h(T^*T - \mathrm{Id})$- then $|a(x, \eta)|^2 = |\det D^2_{x\eta} \psi(x, \eta)| + O(h^{1-2\delta})_{S_\delta}$ near $(x_0, \xi_0, y_0, \eta_0)$. The converse statement holds : microlocally near $(x_0, \xi_0, y_0, \eta_0)$ and modulo a smoothing operator which is $O(h^\infty)$, the elements of $I_\delta(\mathbb{R}^n \times \mathbb{R}^n, \mathrm{Gr}'(\kappa))$ can be written under this form.

2.2.3. *Open quantum hyperbolic maps.* The aim of this part is to provide the precise definition of open quantum hyperbolic maps in 2.4. Let us consider an open hyperbolic map F, as described by the formalism in 1.2.1. We recall that this formalism relies on :

- open intervals $Y_1, \ldots, Y_J$ of $\mathbb{R}$ and $Y = \bigsqcup_{j=1}^J Y_j \subset \bigsqcup_{j=1}^J \mathbb{R}$;
- $U = \bigsqcup_{j=1}^J U_j \subset \bigsqcup_{j=1}^J T^*\mathbb{R}^d$ where $U_j \Subset T^*Y_j$ are open sets;
- For $j = 1, \ldots, J$, open disjoint subsets $\widetilde{D}_{ij} \Subset U_j$, $1 \le i \le J$, *the departure sets*, and for $i = 1, \ldots, J$ open disjoint subsets $\widetilde{A}_{ij} \Subset U_i$, $1 \le j \le J$, *the arrival sets* ;
- Smooth symplectomorphisms $F_{ij} : \widetilde{D}_{ij} \to F_{ij}\left(\widetilde{D}_{ij}\right) = \widetilde{A}_{ij}$

Then, F is the global smooth map $F : \widetilde{D} \to \widetilde{A}$ where $\widetilde{A}$ and $\widetilde{D}$ are the full arrival and departure sets, defined as

$$\widetilde{A} = \bigsqcup_{i=1}^J \bigcup_{j=1}^J \widetilde{A}_{ij} \subset \bigsqcup_{i=1}^J U_i$$

$$\widetilde{D} = \bigsqcup_{j=1}^J \bigcup_{i=1}^J \widetilde{D}_{ij} \subset \bigsqcup_{j=1}^J U_j$$

Finally, we recall that we note $\mathcal{T} \subset U$ the trapped set of F.

Our aim is to define open quantum maps associated with F. We fix a compact set $W \subset \widetilde{A}$ containing some neighborhood of $\mathcal{T}$. Our definition will depend on W. Following [NSZ14] (Section 3.4.2), we now focus on the definition of the elements of $I_\delta(Y \times Y; \mathrm{Gr}(F)')$. An element $T \in I_\delta(Y \times Y; \mathrm{Gr}(F)')$ is a matrix of operators

$$T = (T_{ij})_{1 \le i,j \le J} : \bigoplus_{j=1}^J L^2(Y_j) \to \bigoplus_{i=1}^J L^2(Y_i)$$

Each T_{ij} is an element of $I_\delta(Y_i \times Y_j, \mathrm{Gr}(F_{ij})')$. Let's now describe the recipe to construct elements of $I_\delta(Y_i \times Y_j, \mathrm{Gr}(F_{ij})')$.

We fix $i, j \in \{1, \ldots, J\}$.

- Fix some small $\varepsilon > 0$ and two open covers of U_j, $U_j \subset \bigcup_{l=1}^L \Omega_l$, $\Omega_l \Subset \widetilde{\Omega}_l$, with $\widetilde{\Omega}_l$ star-shaped and having diameter smaller than ε. We note $\mathcal{L}$ the sets of indices l such that $\widetilde{\Omega}_l \subset \widetilde{D}_{ij} \subset U_j$ and we require (this is possible if ε is small enough)

$$F^{-1}(W) \cap U_j \subset \bigcup_{l \in \mathcal{L}} \Omega_l$$

- Introduce a smooth partition of unity associated with the cover (Ω_l), $(\chi_l)_{1 \le l \le L} \in C_c^\infty(\Omega_l, [0,1])$, $\mathrm{supp}\, \chi_l \subset \Omega_l$, $\sum_l \chi_l = 1$ in a neighborhood of $\overline{U_j}$.
- For each $l \in \mathcal{L}$, we denote F_l the restriction to $\widetilde{\Omega}_l$ of F_{ij}. By Lemma 2.1, there exists $\kappa_l \in \mathcal{K}$ which coincides with F_l on Ω_l.

- We consider $T_l = \mathrm{Op}_h(\alpha_i)U_l(1)$ where $U_l(t)$ is the solution of the Cauchy problem (2.1) associated with κ_l and $\alpha_i \in S_\delta^{comp}(T^*\mathbb{R})$.
- We set

(2.3)
$$T^\mathbb{R} = \sum_{l \in \mathcal{L}} T_l \, \mathrm{Op}_h(\chi_l) : L^2(\mathbb{R}) \to L^2(\mathbb{R})$$

$T^\mathbb{R}$ is a globally defined Fourier integral operator. We will note $T^\mathbb{R} \in I_\delta(\mathbb{R} \times \mathbb{R}, \mathrm{Gr}(F_{ij})')$. Its wavefront set is included in $\widetilde{A}_{ij} \times \widetilde{D}_{ij}$.

- Finally, we fix cut-off functions $(\Psi_i, \Psi_j) \in C_c^\infty(Y_i, [0,1]) \times C_c^\infty(Y_j, [0,1])$ such that $\Psi_i \equiv 1$ on $\pi(U_i)$ and $\Psi_j \equiv 1$ on $\pi(U_j)$(here, $\pi : (x,\xi) \in T^*Y \mapsto x \in Y$ is the natural projection) and we adopt the following definitions :

Definition 2.3. We say that $T : \mathcal{D}'(Y_j) \to C^\infty(\overline{Y_i})$ is a Fourier integral operator in the class $I_\delta(Y_i \times Y_j, \mathrm{Gr}(F_{ij})')$ if there exists $T^\mathbb{R} \in I_\delta(\mathbb{R} \times \mathbb{R}, \mathrm{Gr}(F)')$ as constructed above such that

- $T - \Psi_i T \Psi_j = O(h^\infty)_{\mathcal{D}'(Y_j) \to C^\infty(\overline{Y_i})}$;
- $\Psi_i T \Psi_j = \Psi_i T^\mathbb{R} \Psi_j$

For $U_j' \subset U_j$ and $U_i' = F(U_j') \subset U_i$, we say that T (or $T^\mathbb{R}$) is microlocally unitary in $U_i' \times U_j'$ if $TT^* = \mathrm{Id}$ microlocally in U_i' and $T^*T = \mathrm{Id}$ microlocally in U_j'.

Remark. The definition of this class is not canonical since it depends in fact on the compact set W through the partition of unity.

We can now state our definition for open quantum hyperbolic maps associated with F :

Definition 2.4. Fix $\delta \in [0, 1/2[$. We say that $T = T(h)$ is an *open quantum hyperbolic map* associated with F, and we note $T = T(h) \in I_\delta(Y \times Y, \mathrm{Gr}(F)')$ if : for each couple $(i,j) \in \{1, \ldots, J\}^2$, there exists a semi-classical Fourier integral operator $T_{ij} = T_{ij}(h) \in I_\delta(Y_j \times Y_i, \mathrm{Gr}(F_{ij})')$ associated with F_{ij} in the sense of definition 2.3, such that

$$T = (T_{ij})_{1 \le i,j \le J} : \bigoplus_{i=1}^{J} L^2(Y_i) \to \bigoplus_{i=1}^{J} L^2(Y_i)$$

In particular $\mathrm{WF}_h'(T) \subset \widetilde{A} \times \widetilde{D}$. We note $I_{0+}(Y \times Y, \mathrm{Gr}(F)') = \bigcap_{\delta > 0} I_\delta(Y \times Y, \mathrm{Gr}(F)')$.

We will say that $T \in I_{0+}(Y \times Y, \mathrm{Gr}(F)')$ is microlocally invertible near $\mathcal{T}$ if there exists a neighborhood $U' \subset U$ of $\mathcal{T}$ and an operator $T' \in I_{0+}(Y \times Y, \mathrm{Gr}(F^{-1})')$ such that, for every $u = (u_1, \ldots, u_J) \in L^2(Y)$

$\forall j \in \{1, \ldots, J\}, \mathrm{WF}_h(u_j) \subset U' \cap U_j \implies TT'u = u + O(h^\infty)||u||_{L^2}, T'Tu = u + O(h^\infty)||u||_{L^2}$

Suppose that T is microlocally invertible near $\mathcal{T}$ and recall that $T^*T \in \Psi_{0+}(Y)$. Then, we can write

$$T^*T = \mathrm{Op}_h(a_h)$$

where a_h is a smooth symbol in the class $S_{0+}(U)$. We note $\alpha_h = \sqrt{|a_h|}$ and call it the *amplitude* of T. Since T is microlocally invertible near $\mathcal{T}$, $|a_h| > c^2$ near $\mathcal{T}$, for some h-independent constant $c > 0$, showing that α_h is smooth and larger than c in a neighborhood of $\mathcal{T}$.

Remark. If T has amplitude α, at first approximation, T transforms a wave packet u_{ρ_0} of norm 1 centered at a point ρ_0 lying in a small neighborhood of $\mathcal{T}$ into a wave packet of norm $\alpha(\rho_0)$ centered at the point $F(\rho_0)$.

2.2.4. *A precise version of Egorov's theorem.* We will need a more quantitative version of Egorov's theorem, similar to the one in [DJN21] (Lemma A.7). The result does not show that $U(1)^{-1} \mathrm{Op}_h(a)U(1)$ is a pseudodifferential operator (one would need Beal's theorem to say that) but it gives a precise estimate on the remainder, depending on the semi-norms of a. We specialize to the case of dimension 2. The statement is proved in [Vac22] (Proposition 3.3).

Proposition 2.4. Consider $\kappa \in \mathcal{K}$ and note $U(t)$ the solution of (2.1). There exists a family of differential operators $(D_j)_{j \in \mathbb{N}}$ of order j such that for all $a \in S_\delta$ and all $N \in \mathbb{N}$,

(2.4) $\qquad U(1)^{-1} \mathrm{Op}_h(a)U(1) = \mathrm{Op}_h \left(a \circ \kappa + \sum_{j=1}^{N-1} h^j (D_{j+1}a) \circ \kappa \right) + O_\kappa \left(h^N ||a||_{C^{2N+15}} \right)$

By using local charts and composition results (see for instance [DZ19], Proposition E.10), it is possible to build local Fourier integral operators, which, combined with the last proposition, gives

Proposition 2.5. Let $V \subset \mathbb{R}^2 = T^*\mathbb{R}$ an open set and $\kappa : V \to U \subset \mathbb{R}^2$ a symplectic map. Fix $\rho_V \in V$. There exists $W \subset V' \subset V$, neighborhoods of ρ_V and a pair (B, B') of Fourier integral operators in $I_0(\kappa(V') \times V', \mathrm{Gr}'(\kappa)) \times I_0(V' \times \kappa(V'), \mathrm{Gr}'(\kappa^{-1}))$ which satisfy : there exists differential operator $(L_j)_{j \geq 1}$ of order $2j$ and supported in V' such that for all $a \in S_\delta(\mathbb{R}^2)$ with $\mathrm{supp}\, a \subset W$ and for all $N \in \mathbb{N}$,

$$B \operatorname{Op}_h(a) B' = \operatorname{Op}_h(a \circ \kappa^{-1}) + \sum_{j=1}^{N-1} \operatorname{Op}_h\left(L_j a \right) \circ \kappa^{-1}) + O\left(h^N \|a\|_{C^{2N+M}} \right)$$

for some universal integer M.

Proof. It is enough to treat the case $\rho_V = 0 = \kappa(\rho_V)$. It suffices to consider sufficiently small neighborhoods of 0 so that the restriction of κ can be seen as the restriction of an element of $\mathcal{K}$. Then, one uses Proposition 2.4. $\qquad\square$

2.3. **Metaplectic operator.** Among the class of Fourier integral operators acting on $L^2(\mathbb{R})$, metaplectic operators are the one quantizing the linear symplectic transformations of $T^*\mathbb{R} = \mathbb{R}^2$. The main advantage of metaplectic operators compared with general Fourier Integral operators is that the Egorov property is exact (see definition 2.6 below). We recall here a few standard facts on metaplectic operators. We refer the reader to [Zwo12] (Section 11.3) and [CR12] (Chapter 3) for a more precise presentation and other references.

Definition 2.5. For $\rho = (x_0, \xi_0) \in \mathbb{R}^2 = T^*\mathbb{R}$, the phase space translation operator $T_h(\rho)$ is defined as :
$$T_h(\rho)u(x) = e^{-i\frac{x_0 \xi_0}{2h}} e^{i\frac{x \xi_0}{h}} u(x - x_0)$$
It is a unitary on $L^2(\mathbb{R})$ and $T_h(\rho)^* = T_h(-\rho)$. Moreover, $T_h(\rho)^* \operatorname{Op}_h(a) T_h(\rho) = \operatorname{Op}_h(a(\cdot - \rho))$ for any classical observable $a \in \mathcal{S}'(\mathbb{R})$.

Proposition 2.6. (and Definition) Let $\kappa : T^*\mathbb{R} \to T^*\mathbb{R}$ be a symplectic *linear* map. There exists a unitary operator $\mathcal{M}_h(\kappa) : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ such that one of the two following equivalent conditions hold :
   (i) For every $\rho \in T^*\mathbb{R}$, $\mathcal{M}_h(\kappa)T_h(\rho)\mathcal{M}_h(\kappa)^* = T_h(\kappa(\rho))$;
   (ii) For all $a \in \mathcal{S}(\mathbb{R})$, $\mathcal{M}_h(\kappa) \operatorname{Op}_h(a) \mathcal{M}_h(\kappa)^* = \operatorname{Op}_h(a \circ \kappa^{-1})$.
The operator $\mathcal{M}_h(\kappa)$ is unique up to multiplication by an element of $\mathbb{U} = \{z \in \mathbb{C}, |z| = 1\}$.

Most of the time we won't precise that $T_h(\rho)$ and $\mathcal{M}_h(\kappa)$ depend on h and we will simply write $T(\rho)$ and $\mathcal{M}(\kappa)$. We will write the index h (or $h = 1$) when needed. In fact, we can relate $\mathcal{M}_h(\kappa)$ and $\mathcal{M}_1(\kappa)$ by the relation :

$$(2.5) \qquad\qquad\qquad \mathcal{M}_h(\kappa)\Lambda_h = \Lambda_h \mathcal{M}_1(\kappa)$$

where Λ_h is the unitary scaling operator :

$$(2.6) \qquad\qquad\qquad \Lambda_h u(x) = h^{-1/4} u(h^{-1/2} x)$$

A way to obtain metaplectic operators is by solving the Schrödinger equation associated with quadratic Hamiltonians.

Proposition 2.7. Let $S_2(\mathbb{R})$ be the spaces of symmetric matrices of $\mathcal{M}_2(\mathbb{R})$. Let $t \in [0, 1] \mapsto S(t) \in S_2(\mathbb{R})$ be C^1. We note
   • the quadratic time dependent Hamiltonian $H(t, \rho) = \frac{1}{2}(\rho, S(t)\rho)$;
   • $t \in [0, 1] \mapsto \kappa(t)$ the classical flow for the Hamiltonian $H(t)$, which solves the equation
$$\dot{\kappa}(t) = JS(t)\kappa(t)$$
   where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. $\kappa(t)$ is a symplectic linear map for all $t \in [0, 1]$
   • $U(t)$ the propagator of the Schrödinger equation
$$\frac{h}{i}\frac{d}{dt}u(t) + \operatorname{Op}_h(H(t))u = 0$$
   $U(t)$ is a unitary operator on $L^2(\mathbb{R})$ for all $t \in [0, 1]$.

Then, for all $t \in [0,1]$, $U(t)$ is a metaplectic operator associated with the linear symplectic map $\kappa(t)$.

Note that for every $\kappa_1 \in \mathrm{Sp}_2(\mathbb{R})$, there always exists a (non unique) C^1 curve $\kappa : t \in [0,1] \to \mathrm{Sp}_2(\mathbb{R})$ such that $\kappa(0) = I_2$ and $\kappa(1) = \kappa_1$. (see for instance [CR12], Proposition 31 in Chapter 3). So that we can construct $\mathcal{M}(\kappa_1)$ by use of the previous proposition.

Example. The unitary h-Fourier transform $\mathcal{F}_h$, where

$$\mathcal{F}_h u(\xi) = \frac{1}{(2\pi h)^{1/2}} \int_{\mathbb{R}} u(y) e^{-\frac{iy\xi}{h}}$$

is a metaplectic operator associated with J.

Example. Suppose that

$$\kappa = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $a \neq 0$. Then, the following operator is a metaplectic operator associated with κ :

$$(2.7) \qquad \mathcal{M}(\kappa)u(x) = \left(\frac{1}{2\pi h |a|} \right)^{1/2} \int_{\mathbb{R}} e^{\frac{i}{2h}(ca^{-1}x^2 + 2a^{-1}x\xi - a^{-1}b\xi^2)} \mathcal{F}_h u(\xi) d\xi$$

2.4. Coherent states.

2.4.1. *Definitions and notations.* In this subsection, we introduce the notations and definitions we will use for studying coherent states. We refer the reader to [CR12]. The semiclassical coherent state (or Gaussian state) centered at zero will be denoted by

$$(2.8) \qquad \varphi_0(x) = \frac{1}{(\pi h)^{1/4}} e^{-\frac{x^2}{2h}}$$

and the coherent state centered at ρ is simply

$$(2.9) \qquad \varphi_\rho := T(\rho)\varphi_0$$

We also write

$$\varphi_0 = \Lambda_h \Psi_0$$

where Λ_h is defined in (2.6) and Ψ_0 is the renormalized coherent state

$$(2.10) \qquad \Psi_0(x) = \frac{1}{\pi^{1/4}} e^{-\frac{x^2}{2}}$$

We recall that φ_0 (resp. Ψ_0) is the ground sate of the harmonic oscillator $-h^2\partial_x^2 + x^2$ (resp. $-\partial_x^2 + x^2$). The other eigenfunctions of this harmonic oscillator, called excited states, are obtained from φ_0 (resp. Ψ_0) by applying the creation operator $\mathbf{a} = \frac{1}{\sqrt{2h}}(-h\partial_x + x)$ (resp. $\Lambda_h^* \mathbf{a} \Lambda_h = \frac{1}{\sqrt{2}}(-\partial_x + x)$). For $n \in \mathbb{N}$, we can note for instance

$$\varphi_{0,n} = \mathbf{a}^n \varphi_0; \ \Psi_n = \Lambda_h^* \mathbf{a}^n \Lambda_h \Psi_0$$

We recall that $\Psi_n = h_n(x)\Psi_0$ where h_n is a hermite polynomial of degree n. In particular, if $P \in \mathbb{C}[X]$, it is possible to decompose $P(x)\Psi_0(x)$ into a linear combination of excited states up to order $\deg(P)$.

We can also define *squeezed coherent states* :

Definition 2.6. Let $\gamma \in \mathbb{C}$ with $\mathrm{Im}\,\gamma > 0$. The squeezed coherent state, deformed by γ and centered at zero is

$$\varphi_0^{(\gamma)}(x) = (a_\gamma \pi h)^{-1/4} e^{i\gamma \frac{x^2}{2h}}$$

where $a_\gamma = \mathrm{Im}(\gamma)^{-1}$ makes the norm of this state equal to one. We also define the squeezed coherent state centered at $\rho \in T^*\mathbb{R}$ by

$$\varphi_\rho^{(\gamma)} = T(\rho)\varphi_0^{(\gamma)}$$

and the squeezed renormalized coherent state at 0

$$\Psi_0^{(\gamma)}(x) = (a_\gamma \pi)^{-1/4} e^{i\gamma \frac{x^2}{2}}$$

We conclude this section by recalling a useful formula - a resolution of the identity - which is the starting point of our analysis (see [CR12], Proposition 6 in Section 1.2). I

Lemma 2.2. Let $A : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be a trace class operator. Then,

$$\mathrm{tr}(A) = \frac{1}{2\pi h} \int_{T^*\mathbb{R}} < A\varphi_\rho, \varphi_\rho > d\rho$$

where $d\rho$ denotes the Lebesgue measure of $\mathbb{R}^2$.

2.4.2. *Action of metaplectic operators on coherent states.* We recall here how metaplectic operators act on coherent states. We refer the reader to [CR12] (Section 3.2) for a complete proof and a general version in any dimension :

Proposition 2.8. Let $\kappa = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a symplectic linear map. Let $\mathcal{M}(\kappa)$ be a metaplectic operator associated with κ, constructed by use of Proposition 2.7, following a path κ_t from I_2 to κ. Then, we have :

$$\mathcal{M}(\kappa)\varphi_0(x) = (\pi h)^{-1/4}(a+ib)^{-1/2} e^{i\gamma_\kappa \frac{x^2}{2h}}$$

where $\gamma_\kappa = (c+id)(a+ib)^{-1}$.

Remark. The square root $(a+ib)^{1/2}$ is determined by the path κ_t $((a_t + ib_t)^{1/2}$ has to be continuous).
Since $\mathrm{Im}\,\gamma_\kappa = |a+ib|^{-2}$, this proposition shows that for some $\lambda \in \mathbb{U}$,

$$\mathcal{M}(\kappa)\varphi_0 = \lambda \varphi_0^{(\gamma)}$$

Since the metaplectic operators are defined modulo $\mathbb{U}$, in the rest of this article, we will sometimes omit to write the factor λ and and by abuse, we could write $\mathcal{M}(\kappa)\varphi_0 = \varphi_0^{(\gamma)}$. It won't be specified anymore. Anyway, we are concerned by the norm of such states.

We also give the following formula concerning the action of metaplectic operators on excited coherent states (see [Hag98], Section 2) :

Proposition 2.9. Let $\kappa = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a symplectic linear map. Let $\mathcal{M}(\kappa)$ be a metaplectic operator associated with κ, constructed by use of Proposition 2.7, following a path κ_t from I_2 to κ. Then,

$$\mathcal{M}(\kappa)\varphi_{0,n} = (|a+ib|^2 \pi h)^{-1/4} \left(\frac{(a-ib)}{(a+ib)} \right)^{n/2} h_n \left(\frac{x}{h^{1/2}|a+ib|} \right) e^{i\gamma_\kappa \frac{x^2}{2h}}$$

where $\gamma_\kappa = (c+id)(a+ib)^{-1}$.

In the sequel, we will need to estimate the $H_h(\langle\rho\rangle^N)$-norm of squeezed coherent states in terms of the squeezing parameter. Equivalently, we need to control this norm for a state of the form $\mathcal{M}(\kappa)\varphi_\rho$ in terms of κ. To do so, we start by fixing a norm $||\cdot||$ on $\mathcal{M}_2(\mathbb{R})$ For convenience, let's assume that for all linear symplectic map, we have

$$(2.11) \qquad\qquad\qquad\qquad\qquad ||\kappa|| \geq 1$$

For instance, let's say that $||\kappa|| = \sqrt{2}\max(|\kappa|_{11}, |\kappa|_{12}, |\kappa|_{21}, |\kappa|_{22})$. It is not hard to check that this norm satisfies 2.11 since $\det(\kappa) = 1$. The main interest of (2.11) is that $||\kappa||^a \leq ||\kappa||^b$ if $a \leq b$.

We have :

Lemma 2.3. There exists a family of universal constants $(K_{N,k})_{(N,k)\in\mathbb{N}^2}$ such that the following holds : let $N \in \mathbb{N}$, $k \in \mathbb{N}$ and κ be a symplectic linear map. Then, there exists for all $0 < h \leq 1$,

$$||\mathcal{M}(\kappa)(x^k \varphi_0)||_{H_h(\langle\rho\rangle^N)} \leq K_{N+k} \sum_{l=0}^{N} h^{(l+k)/2} ||\kappa||^l$$

Proof. Let's write $\kappa = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. For a state $u \in H_h(\langle\rho\rangle^N)$, we have

$$||u||_{H_h(\langle\rho\rangle^N)}^2 \sim \sum_{\alpha+\beta \leq N} ||\mathrm{Op}_h(x^\alpha \xi^\beta)u||_{L^2}^2$$

Let $\alpha, \beta \in N$ such that $\alpha + \beta \leq N$. We want to estimates $||\mathrm{Op}_h(x^\alpha \xi^\beta)\mathcal{M}(\kappa)(x^k \varphi_0)||_{L^2}^2$. We have

$$\mathrm{Op}_h(x^\alpha \xi^\beta)\mathcal{M}(\kappa)(x^k \varphi_0) = \mathcal{M}(\kappa)\mathrm{Op}_h\left((ax+b\xi)^\alpha (cx+d\xi)^\beta\right)(x^k \varphi_0)$$

Since $\mathcal{M}(\kappa)$ is unitary on L^2, it is enough to estimates the L^2 norm of

$$\mathrm{Op}_h\left((ax+b\xi)^\alpha(cx+d\xi)^\beta\right)(x^k\varphi_0) = \mathrm{Op}_h\left(\sum_{l=(l_1,l_2)\in\mathbb{N}^2, l_1+l_2=\alpha+\beta} B_l(\kappa)x^{l_1}\xi^{l_2}\right)(x^k\varphi_0)$$

where B_l is some l_1+l_2 multilinear form in κ, whose coefficients depend on α and β. In particular, $|B_l(\kappa)| \leq C_l\|\kappa\|^{l_1+l_2}$ for some universal C_l. Finally, we observe that $\|\mathrm{Op}_h(x^{l_1}\xi^{l_2})(x^k\varphi_0)\|_{L^2} := h^{(l_1+l_2+k)/2}C_{(l_1,l_2,k)}$, for some $C_{(l_1,l_2,k)}$ depending only on (l_1,l_2,k), and we find that

$$\|\mathrm{Op}_h\left((ax+b\xi)^\alpha(cx+d\xi)^\beta\right)(x^k\varphi_0)\| \leq C_{(\alpha,\beta,k)}\sum_{p=0}^{\alpha+\beta}\|\kappa\|^p h^{(p+k)/2}$$

we find the required inequality with $K_{N,k}$ depending on the the $C_{\alpha,\beta,k}$ with $\alpha+\beta \leq N$. $\qquad\square$

As a corollary, by specializing at $h=1$, we obtain the following :

Corollary 2.1. There exists a family of constants $K_{N,d}, d\in\mathbb{N}, N\in\mathbb{N}$ such that : for all $P\in\mathbb{C}[X]$, for all symplectic linear map κ and for all $N\in\mathbb{N}$,

$$\|\mathcal{M}_1(\kappa)(P\Psi_0)\|_{H_1(\langle\rho\rangle^N)} \leq K_{N,\deg P}N_\infty(P)\|\kappa\|^N$$

where $N_\infty(P)$ is the sup norm of the coefficients of P.

2.4.3. *Action of pseudodifferential operators on coherent states.* In this subsection, we give precise results for the actions of semiclassical pseudodifferential operators on coherent states, when the symbol of the pseudodifferential operator belong to the class S_δ.

Lemma 2.4. Suppose that $a\in S_\delta(T^*\mathbb{R})$ with $0\leq\delta<1/2$. Assume that $\rho_0=(x_0,\xi_0)\in T^*\mathbb{R}$. Then, for every $N\in\mathbb{N}$, there exists $\rho_N(a,\rho_0)\in L^2$ such that

$$\mathrm{Op}_h(a)\varphi_{\rho_0} = \sum_{k=0}^{N-1}h^{k/2}\psi_k(a,\rho_0) + h^{N/2}\rho_N(a,\rho_0)$$

where

$$\psi_k(a,\rho_0) = T(\rho_0)\Lambda_h\mathrm{Op}_1\left(\sum_{\alpha+\beta=k}\frac{\partial_x^\alpha\partial_\xi^\beta a(\rho_0)}{\alpha!\beta!}x^\alpha\xi^\beta\right)\Psi_0$$

and

$$\|\rho_N(a,\rho_0)\|_{L^2} \leq C_N h^{-\delta N}\sup_{|\gamma|\leq N+M}\|h^{\delta|\gamma|}\partial^\gamma a\|_\infty$$

Remark.   • M is a universal constant.
- The first term of the expansions is $a(\rho_0)\varphi_{\rho_0}$.
- It is effectively an expansion in power of $h^{1/2-\delta}$ since $a\in S_\delta$.
- We could also write $\mathrm{Op}_1(x^\alpha\xi^\beta)\Psi_0$ in the form $P(x)\Psi_0$ where P is a polynomial of degree $\alpha+\beta$, or equivalently, it is a linear combination of the first $|\alpha|+|\beta|$ excited states.

Proof. Let's write $\varphi_{\rho_0} = T(\rho_0)\Lambda_h\Psi_0$. We have

$$\mathrm{Op}_h(a)\varphi_{\rho_0} = \mathrm{Op}_h(a)T(\rho_0)\Lambda_h\Psi_0$$
$$= T(\rho_0)\Lambda_h\mathrm{Op}_1(b_h)\Psi_0$$

where $b_h(\rho) = a(\rho_0+h^{1/2}\rho)$. Let's write the Taylor expansion of a around ρ_0 :

$$b_h(x,\xi) = \sum_{\alpha+\beta\leq N-1}h^{(\alpha+\beta)/2}\frac{\partial_x^\alpha\partial_\xi^\beta a(\rho_0)}{\alpha!\beta!}x^\alpha\xi^\beta + h^{N/2}R_N(x,\xi)$$

where

$$R_N(\rho) = \frac{1}{(N-1)!}\int_0^1\frac{d^N}{dt^N}a(\rho_0+th^{1/2}\rho)(1-t)^{N-1}dt$$

Applying Op_1 to this expansion, we get the required asymptotic with

$$\rho_N(a,\rho_0) = T(\rho_0)\Lambda_h\mathrm{Op}_1(R_N)\Psi_0$$

It remains to estimates the L^2 norm of ρ_N. Since $T(\rho_0)$ is unitary, it is enough to evaluate

$$\Lambda_h\mathrm{Op}_1(R_N)\Psi_0 = \mathrm{Op}_h(\tilde{R}_N)\varphi_0$$

where $\tilde{R}_N(\rho) = R_N(h^{-1/2}\rho) = \frac{1}{(N-1)!}\int_0^1 (1-t)^{N-1} d^N a(\rho_0 + t\rho)(\rho^{\otimes N})dt$ [3]. Using that $a \in S_\delta$, it is not hard to see, after derivation under the integral that, for any $\gamma \in \mathbb{N}^2$ and $\rho \in T^*\mathbb{R}$,

$$|\partial^\gamma \tilde{R}_N(\rho)| \leq C_N \sup_{\gamma_1 \leq N+|\gamma|} ||\partial^{\gamma_1}a||_\infty \langle\rho\rangle^N \leq h^{-\delta(N+|\gamma|)}||h^{\delta|\gamma|}\partial^\gamma a||_\infty \langle\rho\rangle^N$$

This shows that $\tilde{R}_N \in h^{-\delta N}S_\delta(\langle\rho\rangle^N)$ in the sense of [Zwo12] (Definition 4.4.3). Then, we find that

$$h^{\delta N}\operatorname{Op}_h(\tilde{R}_N) : H_h(\langle\rho\rangle^N) \to L^2(\mathbb{R})$$

is a uniformly bounded family of operators, with norm depending on a finite number of semi-norms of $\tilde{R}_N$ in $S_\delta(\langle\rho\rangle^N)$. We conclude by noting that for any $N \in \mathbb{N}$, φ_0 is in $H_h(\langle\rho\rangle^N)$, with a norm bounded uniformly in h. Hence

$$||\rho_N||_{L^2} \leq ||\operatorname{Op}_h(\tilde{R}_N)||_{H_h(\langle\rho\rangle^N)\to L^2(\mathbb{R})} \times ||\varphi_0||_{H_h(\langle\rho\rangle^N)} \leq h^{-\delta N}C_N \sup_{|\gamma|\leq N+M} ||h^{\delta|\gamma|}\partial^\gamma b||_\infty$$

$$\square$$

As a simple corollary, we get :

Corollary 2.2. Assume that a vanishes at order k at ρ_0. Then,

$$\operatorname{Op}_h(a)\varphi_{\rho_0} = O_{L^2}\left(h^{k(1/2-\delta)}\right)$$

In particular, if a vanishes in a neighborhood of ρ_0, we recover that $\operatorname{Op}_h(a)\varphi_{\rho_0} = O(h^\infty)$. This is something well known since $\operatorname{WF}_h(\varphi_{\rho_0}) = \{\rho_0\}$.

2.4.4. *Action of Fourier integral operators on coherent states.* In [CR12] (Chapter 4), the authors study the quantum evolution of coherent states by the propagator of a Schrödinger equation with a time-dependent Hamiltonian. We refer the reader to their work, and in particular to Theorem 21 in this book for this very general version of the evolution of coherent states. Here, we will simply study the action of the particular type of Fourier integral operator of the form given in equation (2.2) on states of the form $T(\rho_0)\mathcal{M}(\kappa)\varphi_0$. In other words, we want to study the action of a Fourier Integral Operator on these squeezed and translated states. More generally, we will consider also squeezed excited states of the form $T(\rho_0)\mathcal{M}(\kappa)\Lambda_h(P(x)\Psi_0(x))$. We will give an asymptotic expansion of these evolved states with a controlled remainder. The dependence of this remainder on κ will be crucial to use recursively the expansion.

Let's describe the framework in which we want to work : we suppose that Ω_x, Ω_η are open intervals of $\mathbb{R}$, $\psi \in C^\infty(\Omega_x \times \Omega_\eta)$ is a phase function that generates the twisted graph of some symplectic map F in some open set $\Omega_0 \subset \mathbb{R}^4$, that is

$$\operatorname{Gr}'(F) \cap \Omega_0 = \left\{(x, \partial_x\psi(x,\eta), \partial_\eta\psi(x,\eta), -\eta), x \in \Omega_x, \eta \in \Omega_\eta\right\}$$

We suppose that $a \in S_{0+}^{comp}(\Omega_x \times \Omega_\eta)$ and we consider the Fourier integral operator :

$$Su(x) = \frac{1}{(2\pi h)}\int_{\mathbb{R}^2} e^{\frac{i}{h}(\psi(x,\eta)-y\cdot\eta)}a(x,\eta)u(y)dyd\eta$$

We do not necessarily assume that S is microlocally unitary, but if it were the case, a would satisfy $|a(x,\eta)|^2 = |\partial_{x\eta}^2\psi(x,\eta)| + O(h^{1-\varepsilon})$ for any $\varepsilon > 0$. More generally, the amplitude α of S as a Fourier integral operator is given, modulo $O(h^{1^-})S_{0+}$, by

$$\alpha_S(y,\eta) = \frac{a(x,\eta)}{|\partial_{x\eta}^2\psi(x,\eta)|^{1/2}} \ , \ F(y,\eta) = (x,\xi)$$

Proposition 2.10. Assume that S satisfies the above assumptions. Let $\kappa \in \mathcal{M}_2(\mathbb{R})$ be a symplectic linear map and $\rho_0 \in T^*\mathbb{R}$. Let's note $\rho_1 = F(\rho_0)$. Let $P \in \mathbb{C}[X]$. Then, there exists a family of polynomials $Q_k(P)_{k\in\mathbb{N}}$ such that
  - $Q_0(P) = \alpha_S(\rho_0)P$ (up to multiplication by an element of $\mathbb{U}$) ;
  - $Q_k(P)$ is a polynomial of degree $\deg P + 3k$ and the map $P \mapsto Q_k(P)$ is linear, with coefficients depending on κ and the derivatives of ψ and a at (x_1, ξ_0) up to the $3k$-th order, and we have
    $$N_\infty(Q_k(P)) \leq C_{3k}(\psi)||a||_{C^k}||\kappa||^{3k}N_\infty(P)$$
  Moreover, if $(x_1, \xi_0) \notin \operatorname{supp} a$, then $Q_k = 0$.

———————————

[3]Here, if $f \in C^N(\mathbb{R}^2, \mathbb{R})$, we note $d^N f(\rho)(h^{\otimes N}) = \frac{d^N}{dt^N}\Big|_{t=0} f(\rho + th)$.

- for every $N \in \mathbb{N}$,

$$(2.12) \qquad S\Big(T(\rho_0)\mathcal{M}(\kappa)\Lambda_h[P\Psi_0]\Big) = T(\rho_1)\mathcal{M}(d_{\rho_0}F \circ \kappa)\Lambda_h \left[\sum_{k=0}^{N-1} h^{k/2}Q_k(P)\Psi_0\right] + R_N$$

with

$$||R_N||_{L^2} \leq h^{N/2}C_{3N+M}(\psi)||a||_{C^{N+M}}||\kappa||^{3N}K_{N,\deg P}N_\infty(P)$$

Here,

- $C_k(\psi)$ depends on the C^k norm of ψ
- M is a universal constant ;
- $N_\infty(P)$ is the sup norm on the coefficients of P ;
- $(K_{N,d})_{(N,d)\in\mathbb{N}^2}$ is a family of universal constants.
- For every $\varepsilon > 0$ and $k \in \mathbb{N}$, there exists $C_{\varepsilon,k}$ such that $||a||_{C^k} \leq C_{\varepsilon,k}h^{-\varepsilon}$.

Remark. This proposition shows that a Fourier Integral operator transforms a wave packet centered at ρ_0 into a wave packet centered at $F(\rho_0)$. However, this transformation squeezes the wave packet according to the linearization of F at ρ_0: this is the effect of $\mathcal{M}(d_{\rho_0}F)$. The control of the error is important if we want to iterate this formula and apply it to squeezed coherent states $\mathcal{M}(\kappa_h)\varphi_0$, with a symplectic linear map κ_h potentially depending of h. As soon as

$$||\kappa_h||^3 \ll h^{-1/2}$$

, the remainder stays smaller than the terms in the expansion. In particular, suppose that $\kappa_h = \kappa_{n(h)}\ldots\kappa_0$ with $||\kappa_i|| \leq e^\lambda$ and $n(h) \sim \nu \log(1/h)$. Then, the approximation is valid as soon as

$$\nu \leq \frac{1-\varepsilon}{6\lambda}$$

Proof. The following computations are performed modulo multiplication by an element of $\mathbb{U}$.

Let's note $\rho_0 = (x_0, \xi_0)$ and $\rho_1 = (x_1, \xi_1)$. Recall that, by definition of ψ,

$$(2.13) \qquad \xi_1 = \partial_x\psi(x_1, \xi_0) \quad ; \quad x_0 = \partial_\eta\psi(x_1, \xi_0)$$

We have, for $u \in L^2(\mathbb{R})$,

$$\left(\Lambda_h^* T(\rho_1)^* S T(\rho_0)\Lambda_h u\right)(x) = h^{1/4}e^{-\frac{ix\xi_1}{h^{1/2}}}\left(ST(\rho_0)\Lambda_h u\right)(h^{1/2}x + x_1)$$

$$= e^{-\frac{ix\xi_1}{h^{1/2}}}\frac{1}{2\pi h}\int_{\mathbb{R}^2} e^{\frac{i}{h}(\psi(h^{1/2}x+x_1,\eta)-y\cdot\eta)}a(h^{1/2}x + x_1, \eta)e^{\frac{iy\xi_0}{h}}u(h^{-1/2}y - x_0)dyd\eta$$

$$= \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{i\tilde\psi_h(x,\eta,y)}a(h^{1/2}x + x_1, h^{1/2}\eta + \xi_0)u(y)dyd\eta$$

after a change of variable, with

$$(2.14) \qquad \tilde\psi_h(x, \eta, y) = \frac{1}{h}\psi(h^{1/2}x + x_1, h^{1/2}\eta + \xi_0) - y\eta - h^{-1/2}(x\xi_1 + x_0\eta)$$

Let us write the Taylor expansion of $\psi(h^{1/2}x + x_1, h^{1/2}\eta + \xi_0)$ at order $N + 1 \in \mathbb{N}$:

$$(2.15) \quad \psi(h^{1/2}x + x_1, h^{1/2}\eta + \xi_0) = \psi(x_1, \xi_0) + h^{1/2}(x\partial_x\psi(x_1, \xi_0) + \eta\partial_\eta\psi(x_1, \xi_0))$$

$$+ \frac{h}{2}(D^2\psi(x_1, \xi_0)(x, \eta), (x, \eta)) + \sum_{k=3}^{N+1} h^{k/2}\psi_k(x, \eta) + h^{(N+2)/2}r_{N+2}^\psi(x, \eta; h)$$

where ψ_k is k-multilinear in (x, η) with coefficients depending on the derivatives of ψ of order k at (x_1, ξ_0) and for $\alpha \in \mathbb{N}^2$,

$$h^{(N+2)/2}r_{N+2}^\psi(x, \eta; h) = \frac{1}{(N+1)!}\int_0^1 (1-t)^{N+1}\frac{d^{N+2}}{dt^{N+2}}\left(\psi(x_1 + th^{1/2}x, \xi_0 + th^{1/2}\eta)\right)dt$$

In particular, we have the estimates

$$(2.16) \qquad |\partial^\alpha r_{N+2}^\psi(x, \eta; h)| \leq C_N \sup_{N+2\leq|\beta|\leq N+2+|\alpha|} h^{(|\beta|-N-2)/2}||\partial^\beta\psi||_\infty\langle(x,\eta)\rangle^{N+2}$$

Recalling (2.13), we can write :

(2.17) $\left(\Lambda_h^* T(\rho_1)^* S T(\rho_0) \Lambda_h u\right)(x) =$

$$\frac{1}{2\pi} \int_{T^*\mathbb{R}} e^{i\left(\frac{1}{2}(D^2\psi(x_1,\xi_0)(x,\eta),(x,\eta)) - y\eta\right)} e^{ih^{1/2} r_3^\psi(x,\eta;h)} a(h^{1/2}x + x_1, h^{1/2}\eta + \xi_0) u(y) dy d\eta$$

Then, we write the Taylor expansion of $e^{ih^{1/2} r_3^\psi(x,\eta;h)}$ at order N:

(2.18)
$$e^{ih^{1/2} r_3^\psi(x,\eta;h)} = \sum_{k=0}^{N-1} \frac{i^k h^{k/2}}{k!} r_3^\psi(x,\eta;h)^k + \underbrace{\frac{i^N h^{N/2}}{(N-1)!} (r_3^\psi(x,\eta;h))^N \int_0^1 e^{ih^{1/2} s r_3^\psi(x,\eta;h)} (1-s)^{N-1} ds}_{\tilde{r}_N}$$

Using (2.15), we write $r_3^\psi = \sum_{j=0}^{N-k-1} h^{j/2} \psi_{3+j} + h^{(N-k)/2} r_{3+N-k}^\psi$ and we can expand

$$\left(r_3^\psi\right)^k = \sum_{\alpha_1 + \cdots + \alpha_k < N-k} h^{(\alpha_1 + \cdots + \alpha_k)/2} \psi_{3+\alpha_1} \cdots \psi_{3+\alpha_k} + h^{(N-k)/2} \text{ Remainder}$$

where the remainder is a linear combination, with universal coefficients, of terms of the form

(2.19) $r_{3+\alpha_1}^\psi \cdots r_{3+\alpha_j}^\psi \psi_{3+\alpha_{j+1}} \cdots \psi_{3+\alpha_k}$; $0 \le j \le k,\, \alpha_1 + \cdots + \alpha_k = N - k$

Gathering all the terms of order $h^{k/2}$ for $k \le N - 1$, together and gathering all the terms of order $h^{N/2}$ in a single remainder term, we have

$$e^{ih^{1/2} r_3^\psi(x,\eta;h)} = \sum_{k=0}^{N-1} h^{k/2} \tilde{P}_k(x,\eta;\psi) + h^{N/2} r_{N,1} + \tilde{r}_N$$

where
   • $\tilde{P}_k(\cdot;\psi)$ is a polynomial of order $3k$ in (x,η) with coefficients of the form $q\left((\partial^\alpha \psi(x_1,\xi_0))_{|\alpha| \le 3+k}\right)$, where q is a universal polynomial of degree k;
   • $r_{N,1}$ is a linear combination of terms of the form (2.19) with $0 \le k \le N-1$, $0 \le j \le k$ and $\alpha_1, \ldots, \alpha_k$, with $\alpha_1 + \cdots + \alpha_k = N - k$;
   • $\tilde{r}_N$ is defined in (2.18).
Similarly, we can Taylor expand $a(h^{1/2}x + x_1, h^{1/2}\eta + \xi_0)$ to find that

(2.20) $e^{ih^{1/2} r_3^\psi(x,\eta;h)} a(h^{1/2}x + x_1, h^{1/2}\eta + \xi_0) = \sum_{k=0}^{N-1} h^{k/2} P_k(x,\eta;\psi,a)$

$$+ \underbrace{h^{N/2} \sum_{k=0}^{N-1} \tilde{P}_k(x,\eta;\psi) r_{N-k}^a(x,\eta;h) + h^{N/2} r_{N,1} \times a(h^{1/2}x + x_1, h^{1/2}\eta + \xi_0)}_{\text{first remainder term}} + \underbrace{\tilde{r}_N \times a(h^{1/2}x + x_1, h^{1/2}\eta + \xi_0)}_{\text{second remainder term}}$$

where $P_k(\cdot;\psi,a)$ is a polynomial of degree $3k$ in (x,η), given by

$$P_k(x,\eta;\psi,a) = \sum_{k_1 + k_2 = k} \tilde{P}_{k_1}(x,\eta;\psi) \times \left(\frac{1}{k_2!} d^{k_2} a(x_1,\xi_0)((x,\eta)^{\otimes k})\right)$$

and for $p \in \mathbb{N}$,

$$r_p^a(x,\eta;h) = \frac{h^{-p/2}}{p!} \int_0^1 (1-t)^{p-1} \frac{d^p}{dt^p} a(x_1 + th^{1/2}x, \xi_0 + th^{1/2}\eta) dt$$

Plugging (2.20) in (2.17) with $u = \mathcal{M}_1(\kappa)(P\Psi_0)$, we find an expansion in power of $h^{1/2}$ for $\Lambda_h^* T(\rho_1)^* S T(\rho_0) \Lambda_h u$.

Identification of the first term. The first term of the expansion is

$$\frac{1}{2\pi} \int_{\mathbb{R}^2} e^{i\left(\frac{1}{2}(D^2\psi(x_1,\xi_0)(x,\eta),(x,\eta))-y\eta\right)} a(x_1,\xi_0) u(y) dy d\eta$$

Differentiating the relation

$$F\left(\partial_\eta \psi(x,\eta), \eta\right) = (x, \partial_x \psi(x,\eta))$$

it not hard to see that

$$d_{\rho_0} F = \frac{1}{\partial_{x\eta}^2 \psi(x_1,\xi_0)} \begin{pmatrix} 1 & -\partial_{\eta\eta}^2 \psi(x_1,\xi_0) \\ \partial_{xx}^2 \psi(x_1,\xi_0) & \left(\partial_{x\eta}^2 \psi(x_1,\xi_0)\right)^2 - \partial_{\eta\eta}^2 \psi(x_1,\xi_0) \partial_{xx}^2 \psi(x_1,\xi_0) \end{pmatrix}$$

As a consequence, comparing with (2.7), we observe that

$$v \mapsto \frac{1}{2\pi} |\partial_{x\eta}\psi(x_1,\xi_0)|^{1/2} \int_{T^*\mathbb{R}} e^{i(\frac{1}{2}D^2\psi(x_1,\xi_0)(x,\eta)-y\eta)} v(y) dy d\eta$$

is a metaplectic operator associated with $d_{\rho_0}F$, that we note $\mathcal{M}_1(d_{\rho_0}F)$, and hence, wee see that

$$ST(\rho_0)\mathcal{M}(\kappa)\Lambda_h[P\Psi_0] = T(\rho_1)\mathcal{M}(d_{\rho_0}F \circ \kappa)\Lambda_h \left[\frac{a(x_1,\xi_0)}{|\partial_{x\eta}\psi(x_1,\xi_0)|^{1/2}} P\Psi_0\right] + \text{(smaller terms)}$$

This gives the required form for $Q_0(P)$.

Identification of higher order terms. For the term of order k in the expansion of (2.17), based on (2.20), we have to understand

$$\frac{1}{2\pi} \int_{\mathbb{R}^2} e^{i\left(\frac{1}{2}(D^2\psi(x_1,\xi_0)(x,\eta),(x,\eta))-y\eta\right)} P_k(x,\eta;\psi,a) u(y) dy d\eta$$

Hence, we focus on terms of the form

$$S_{l,m}(u) = \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{i\left(\frac{1}{2}(D^2\psi(x_1,\xi_0)(x,\eta),(x,\eta))-y\eta\right)} x^l \eta^m u(y) dy d\eta$$

with $l+m \leq 3k$. The term x^l can be put in front of the integral. Concerning, the η term, repeated integrations by part (or equivalently, using the usual properties of the Fourier transform), we find that

$$S_{l,m}(u) = x^l \mathcal{M}_1(d_{\rho_0}F)((i\partial_y)^m u)$$

Now, combining this with the standard commutations properties of metaplectic operators we write

$$S_{l,m}(\mathcal{M}_1(\kappa)[P\Psi_0]) = \mathrm{Op}_1(x^l)\mathcal{M}_1(d_{\rho_0}F)\mathrm{Op}_1((-\xi)^m)\mathcal{M}_1(\kappa)[P\Psi_0]$$
$$= \mathcal{M}_1(d_{\rho_0}F \circ \kappa)\mathrm{Op}_1\left((x \circ (dF(\rho_0) \circ \kappa)^l\right) \mathrm{Op}_1\left((-\xi \circ \kappa)^m\right)[P\Psi_0]$$

Finally, the action of $\mathrm{Op}_1\left((x \circ (d_{\rho_0}F \circ \kappa)^l\right) \mathrm{Op}_1\left((-\xi \circ \kappa)^m\right)$ transforms $P\Psi_0$ into another state of the form $Q\Psi_0$ where Q is of degree $\deg P + l + m$, where the coefficients of Q depend linearly on those of P, with coefficients in the linear combination depending on κ and on $d_{\rho_0}F$. By developing the powers $\left(x \circ (d_{\rho_0}F \circ \kappa)\right)^l$ and $(-\xi \circ \kappa)^m$, we see that the coefficients of Q are bounded by $C_{l,m}||d_{\rho_0}F||^l||\kappa||^{l+m}$ for some constant $C_{l,m}$.

As a consequence, we can write the entire term of order k in the form : $T(\rho_1)\mathcal{M}(d_{\rho_0}F \circ \kappa)\Lambda_h(Q_k(P)\Psi_0))$ where $Q_k(P)$ is a polynomial of order $\deg P + 3k$, the map $P \mapsto Q_k(P)$ is linear and its coefficients depend on κ, the derivatives of ψ and a at (x_1,ξ_0) up to the $3k$-th order. This gives the required polynomial. By putting the terms $||d_{\rho_0}F||^l$ into $C_{3k}(\psi)$ and using the special form of P_k, we obtain the required estimate

$$N_\infty(Q_k(P)) \leq C_{3k}(\psi)||a||_{C^k}||\kappa||^{3k} N_\infty(P).$$

Control of the remainders. The last step of the proof consists in proving that the remainder term has the required bound. As already written with the underbrace in (2.20), this remainder can be decomposed in two terms: they have different properties. Let us start with the first term, and call it $\tilde{r}_{N,1}$.

In the products of the form given by (2.19), gathering the factors $r_{3+\alpha}^{\psi}$ into a single term and the polynomials ψ_k into a single polynomial, we see that the term $r_{N,1}$, appearing in $\tilde{r}_{N,1}$, is a sum of terms of the form $Q_j^{\psi}(x,\eta)R_j^{\psi}(x,\eta;h)$, for $0 \leq j \leq k$, where Q_j^{ψ} is a polynomial of degree $3j$ and $R_j^{\psi}(x,\eta;h)$ satisfies for $\alpha \in \mathbb{N}^2$,

$$|\partial^\alpha R_j^\psi(x,\eta;h)| \leq C_{3N-3j+|\alpha|}(\psi)\langle(x,\eta)\rangle^{3N-3j}$$

where $C_{3N-3j+|\alpha|}(\psi)$ depends on the derivatives of ψ up to the order $3N - 3j + |\alpha|$.[4] Using the same kind of estimates for $r_{N-k}^a(x,\eta;a;h)$, we see that $\tilde{r}_{N,1}$ satisfies :

(2.21) $\qquad \forall \alpha \in \mathbb{N}^2, (x,\eta) \in \mathbb{R}^2, |\partial^\alpha \tilde{r}_{N,1}(x,\eta)| \leq C_{3N+|\alpha|}(\psi)||a||_{C^{N+|\alpha|}}\langle(x,\eta)\rangle^{3N}$

We are now interested in controlling

$$\tilde{R}_{N,1}u(x) := \frac{1}{2\pi}\int_{T^*\mathbb{R}} e^{i\left(\frac{1}{2}(D^2\psi(x_1,\xi_0)(x,\eta),(x,\eta))-y\eta\right)}\tilde{r}_{N,1}(x,\eta)u(y)dyd\eta$$

We will use the following lemma, proved in the appendix A.1 :

Lemma 2.5. Let $\tilde{b}$ be a symbol in $S(\langle\rho\rangle^N)$. Then, there exists a symbol $b \in S(\langle\rho\rangle^N)$ such that for all $0 < h \leq 1$,

$$\frac{1}{2\pi h}\int_{T^*\mathbb{R}} e^{\frac{i}{h}\left(\frac{1}{2}(D^2\psi(x_1,\xi_0)(x,\eta),(x,\eta))-y\eta\right)}\tilde{b}(x,\eta)u(y)dyd\eta = \mathcal{M}(d_{\rho_0}F)\operatorname{Op}_h(b)u(x)$$

Moreover, there exists a universal integer $M' \in \mathbb{N}$ such that b satisfies : for all $\alpha \in \mathbb{N}^2$,

$$\langle\rho\rangle^N|\partial^\alpha b(\rho)| \leq C_\alpha \sup_{|\beta|\leq|\alpha|+M'} \sup_{\rho\in T^*\mathbb{R}} \left(|\partial^\beta\tilde{b}(\rho)|\langle\rho\rangle^N\right)$$

where C_α depends on $d_{\rho_0}F$.

By applying lemma 2.5 (in the case $h = 1$ in the lemma), we can find a symbol $r_{N,h}$ such that

$$\tilde{R}_{N,1} = \mathcal{M}_1(d_{\rho_0}F)\operatorname{Op}_1(r_{N,h})$$

To conclude the treatment of the first part of the remainder, we compute :

$$\begin{aligned}||\tilde{R}_{N,1}\mathcal{M}_1(\kappa)[P\Psi_0]||_{L^2} &= ||\mathcal{M}_1(d_{\rho_0}F)\operatorname{Op}_1(r_{N,h})\mathcal{M}_1(\kappa)[P\Psi_0]||_{L^2}\\ &\leq ||\operatorname{Op}_1(r_{N,h})||_{H_1(\langle\rho\rangle^{3N})\to L^2}\times ||\mathcal{M}_1(\kappa)[P\Psi_0]||_{H_1(\langle\rho\rangle^{3N})}\\ &\leq C_M(r_{N,h})||\kappa||^{3N}K_{N,degP}N_\infty(P)\end{aligned}$$

by using Corollary 2.1, where $C_M(r_{N,h})$ depends on the first M semi-norms of $r_{N,h}$ in $S(\langle\rho\rangle^{3N})$, which, in turn depends on the first $M + M'$ semi-norms of $\tilde{r}_{N,h}$ in $S(\langle\rho\rangle^{3N})$ according to Lemma 2.5. By (2.21), this can be controlled by some constant $C_{3N+M+M'}(\psi)||a||_{C^{N+M+M'}}$.

Let's turn to the second remainder in (2.20). We want to control

$$\frac{1}{2\pi}\int_{\mathbb{R}^2} e^{i\left(\frac{1}{2}(D^2\psi(x_1,\xi_0)(x,\eta),(x,\eta))-y\eta\right)}\tilde{r}_N(x,\eta)u(y)dyd\eta$$

Recalling the precise description of $\tilde{r}_N$ in (2.18), we set, for $s \in [0,1]$:

$$\tilde{R}_s u(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{i\left(\frac{1}{2}(D^2\psi(x_1,\xi_0)(x,\eta),(x,\eta))+ish^{1/2}r_3^\psi(x,\eta;h)-y\eta\right)}r_3^\psi(x,\eta;h)^N a(x_1+h^{1/2}x,\xi_0+h^{1/2}\eta)u(y)dyd\eta$$

and we want to estimate $||\tilde{R}_s\mathcal{M}_1(\kappa)[P\Psi_0]||_{L^2}$ uniformly in $s \in [0,1]$. The symbol

$$b_N(x,\eta) := r_3^\psi(x,\eta;h)^N a(x_1+h^{1/2}x,\xi_0+h^{1/2}\eta)$$

---

[4]These estimates comes from (2.16) and in fact, we can take

$$C_{3N-3j+|\alpha|}(\psi) = \sup_{3N-3j\leq|\beta|\leq3N-3j+|\alpha|} h^{(|\beta|-3N+3j)/2}||\partial^\beta\psi||_\infty$$

lies in the symbol class $S(\langle \rho \rangle^{3N})$, with a control on its semi-norms due to (2.16). Let's admit the following lemma, whose proof is also put in the appendix A.1.

Lemma 2.6. For every $s \in [0,1]$, there exists $B_s(\cdot) \in S\left(\langle \rho \rangle^{6N} \right)$ such that :

- $\tilde{R}_s^* \tilde{R}_s = \mathrm{Op}_1(B_s)$;
- There exists a universal $M \in \mathbb{N}$ such that for all $\alpha \in \mathbb{N}^2$, for all $s \in [0,1]$, with some universal constants C_α,

$$\sup_\rho |\partial^\alpha B_s(\rho)| \leq C_\alpha \left(\sup_{\rho, |\beta| \leq |\alpha|+M} d^\beta b_N(\rho) \langle \rho \rangle^{-3N} \right)^2 \langle \rho \rangle^{6N}$$

This lemma allows us to control

$$
\begin{aligned}
||\tilde{R}_s||^2_{H_1(\langle \rho \rangle^{3N}) \to L^2} &\leq ||\tilde{R}_s^* \tilde{R}_s||_{H_1(\langle \rho \rangle^{3N}) \to H_1(\langle \rho \rangle^{-3N})} \\
&\leq ||\mathrm{Op}_1(B_s)||_{H_1(\langle \rho \rangle^{3N}) \to H_1(\langle \rho \rangle^{-3N})} \\
&\leq C_N \sup_{|\alpha| \leq M} \sup_\rho |\left(\partial^\alpha B_s(\rho) \right) \langle \rho \rangle^{-6N}| \\
&\leq C_N \left(\sup_{|\beta| \leq 2M} \sup_\rho d^\beta b_N(\rho) \langle \rho \rangle^{-3N} \right)^2 \\
&\leq (C_{3N+M'}(\psi) ||a||_{C^{N+M'}})^2
\end{aligned}
$$

We finally conclude as before for $\tilde{R}_{N,1}$ by using Corollary 2.1. This concludes the proof of Proposition 2.10. $\square$

3. DYNAMICAL PRELIMINARIES

3.1. **Hyperbolic dynamics.** We assumed that F is hyperbolic on the trapped set $\mathcal{T}$. As already mentioned, we can fix an adapted Riemannian metric on U such that the following stronger version of the hyperbolic estimates are satisfied for some $\lambda_0 > 0$: for every $\rho \in \mathcal{T}$, $n \in \mathbb{N}$,

$$(3.1) \qquad v \in E_u(\rho) \implies ||d_\rho F^{-n}(v)|| \leq e^{-\lambda_0 n} ||v||$$

$$(3.2) \qquad v \in E_s(\rho) \implies ||d_\rho F^n(v)|| \leq e^{-\lambda_0 n} ||v||$$

Notations. We now use the induced Riemannian distance on U and denote it d.

If $\rho \in \mathcal{T}$, $n \in \mathbb{Z}$, we use this Riemannian metric to define the unstable Jacobian $J_n^u(\rho)$ and stable Jacobian $J_n^s(\rho)$ at ρ by :

$$(3.3) \qquad v \in E_u(\rho) \implies ||d_\rho F^n(v)|| = J_n^u(\rho) ||v||$$

$$(3.4) \qquad v \in E_s(\rho) \implies ||d_\rho F^n(v)|| = J_n^s(\rho) ||v||$$

These Jacobians quantify the local hyperbolicity of the map. Since F is volume preserving, $J_n^u(\rho) J_n^s(\rho) \sim 1$.

Remark. If we define unstable and stable Jacobian $\tilde{J}_n^u$ and $\tilde{J}_n^s$ using another Riemannian metric, then, for every $n \in \mathbb{Z}$ and $\rho \in \mathcal{T}$,

$$\tilde{J}_n^u(\rho) \sim J_n^u(\rho) \quad ; \quad \tilde{J}_n^s(\rho) \sim J_n^s(\rho)$$

From the compactness of $\mathcal{T}$, there exists $\lambda_1 \geq \lambda_0$ which satisfies

$$(3.5) \qquad \forall n \in \mathbb{N}, \forall \rho \in \mathcal{T} \; ; \; e^{n\lambda_0} \leq J_n^u(\rho) \leq e^{n\lambda_1} \text{ and } e^{-n\lambda_1} \leq J_n^s(\rho) \leq e^{-n\lambda_0}$$

In particular, the following Lyapounov exponents are well-defined

$$\lambda_{max} = \sup_{\rho \in \mathcal{T}} \limsup_n \frac{1}{n} \log J_n^u(\rho)$$

$$\lambda_{min} = \inf_{\rho \in \mathcal{T}} \liminf_n \frac{1}{n} \log J_n^u(\rho) > 0$$

We cite here standard facts about the stable and unstable manifolds (see for instance [HK95], Chapter 6).

Lemma 3.1. For any $\rho \in \mathcal{T}$, there exist local stable and unstable manifolds $W_s(\rho), W_u(\rho) \subset U$ satisfying, for some $\varepsilon_1 > 0$ (only depending on F) :

(1) $W_s(\rho), W_u(\rho)$ are C^∞-embedded curves, with the C^∞ norms of the embedding uniformly bounded in ρ.
(2) the boundaries of $W_u(\rho)$ and $W_s(\rho)$ do not intersect $\overline{B(\rho, \varepsilon_1)}$ [5] and $W_{u/s}(\rho) \subset B(\rho, 2\varepsilon_1)$ (these are local unstable/stable manifolds).
(3) $W_s(\rho) \cap W_u(\rho) = \{\rho\}$, $T_\rho W_{u/s}(\rho) = E_{u/s}(\rho)$
(4) $F(W_s(\rho)) \subset W_s(F(\rho))$ and $F^-(W_u(\rho)) \subset W_u(F^{-1}(\rho))$
(5) (a) For each $\rho' \in W_s(\rho), d(F^n(\rho), F^n(\rho')) \to 0$.
    (b) For each $\rho' \in W_u(\rho), d(F^{-n}(\rho), F^{-n}(\rho')) \to 0$.
(6) Let $\theta > 0$ satisfying $e^{-\lambda_0} < \theta < 1$. There exists $C > 0$ (independent of ε_1) such that the following holds :
    (a) If $\rho' \in U$ satisfies $d(F^i(\rho), F^i(\rho')) \leq \varepsilon_1$ for all $i = 0, \ldots, n$ then $d(\rho', W_s(\rho)) \leq C\theta^n \varepsilon_1$ and for $0 \leq i \leq n$, $d(F^i(\rho), F^i(\rho')) \leq C\varepsilon_1 \theta^{\min(i, n-i)}$.
    (b) If $\rho' \in U$ satisfies $d(F^{-i}(\rho), F^{-i}(\rho')) \leq \varepsilon_1$ for all $i = 0, \ldots, n$ then $d(\rho', W_u(\rho)) \leq C\theta^n \varepsilon_1$ and for $0 \leq i \leq n$, $d(F^{-i}(\rho), F^{-i}(\rho')) \leq C\varepsilon_1 \theta^{\min(i, n-i)}$.
(7) If $\rho, \rho' \in \mathcal{T}$ satisfy $d(\rho, \rho') \leq \varepsilon_1$, then $W_u(\rho) \cap W_s(\rho')$ consists of exactly one point of $\mathcal{T}$.

Below, we will require that $C\varepsilon_1 < 1$. Up to making ε_1 smaller, we assume this holds.

For our purpose, we will need a more precise version of these results. The following lemmas are an adaptation of Lemma 2.1 in [DJN21] to our setting, appearing also in [Vac22], where they have been partially proved.

Lemma 3.2. There exist constants $\varepsilon_1 > 0$ and $C > 0$ depending only on (U, F), such that for all $\rho, \rho' \in U$,

(1) if $\rho \in \mathcal{T}$ and $\rho' \in W_s(\rho)$ satisfy $d(\rho, \rho') \leq \varepsilon_1$, then

$$(3.6) \qquad C^{-1} J_n^s(\rho) d(\rho, \rho') \leq d(F^n(\rho), F^n(\rho')) \leq C J_n^s(\rho) d(\rho, \rho') \quad, \quad \forall n \in \mathbb{N}$$

(2) if $\rho \in \mathcal{T}$ and $\rho' \in W_u(\rho)$ satisfy $d(\rho, \rho') \leq \varepsilon_1$, then

$$(3.7) \qquad C^{-1} J_{-n}^u(\rho) d(\rho, \rho') \leq d(F^{-n}(\rho), F^{-n}(\rho')) \leq C J_{-n}^u(\rho) d(\rho, \rho') \quad, \quad \forall n \in \mathbb{N}$$

Proof. We prove (1). (2) is proved in a similar way by inverting the time direction. Let $\rho \in \mathcal{T}, \rho' \in W_s(\rho)$. Since $T_\rho(W_s(\rho)) = E_s(\rho)$ and $d_\rho F(E_s(\rho)) = E_s(F(\rho))$, the Taylor development of F along $W_s(\rho)$ gives :

$$(3.8) \qquad d(F(\rho), F(\rho')) = J_1^s(\rho) d(\rho, \rho') + O\left(d(\rho, \rho')^2\right) = J_1^s(\rho) d(\rho, \rho')\left(1 + O\left(d(\rho, \rho')\right)\right)$$

since $J_1^s \geq e^{-\lambda_1}$. Applying this equality with $F^k(\rho)$ and $F^k(\rho')$ instead of ρ and ρ', and recalling that, by lemma 3.1, $d(F^k(\rho), F^k(\rho')) \leq C\theta^k d(\rho, \rho')$, we can write,

$$(3.9) \qquad d(F^{k+1}(\rho), F^{k+1}(\rho')) = J_1^s(F^k(\rho)) d(F^k(\rho), F^k(\rho'))(1 + O(\theta^k \varepsilon_1))$$

By this last inequality and the chain rule, we have

$$(3.10) \qquad J_n^s(\rho) d(\rho, \rho') \prod_{k=0}^{n-1}(1 - C\theta^k \varepsilon_1) \leq d(F^n(\rho), F^n(\rho')) \leq J_n^s(\rho) d(\rho, \rho') \prod_{k=0}^{n-1}(1 + C\theta^k \varepsilon_1)$$

We conclude by noting that

$$\prod_{k=0}^{n-1}(1 + C\theta^k \varepsilon_1) \leq \prod_{k=0}^{+\infty}(1 + C\theta^k \varepsilon_1) < +\infty \; ; \; \prod_{k=0}^{n-1}(1 - C\theta^k \varepsilon_1) \geq \prod_{k=0}^{\infty}(1 - C\theta^k \varepsilon_1) \geq C^{-1}$$

(note that in the last inequality and in (3.10) , we need to ensure that $\varepsilon_1 C < 1$ so that the product is effectively non zero). $\qquad \square$

The following lemma gives a stronger version of (6) in Lemma 3.1 (it has been proved in [Vac22] - Lemma 3.10-, as the following corollary - Corollary 3.11).

---

[5]in other words, there exists a smooth curve $\gamma : [-\delta, \delta] \to U$ such that $\gamma(0) = \rho$, $\mathrm{ran}(\gamma) \subset W_{u/s}(\rho)$ and $\overline{B(\rho, \varepsilon_1)} \cap W_{i/s}(\rho) = \gamma([-\delta/2, \delta/2])$: it means that the size of the unstable and stable manifolds is bounded from below uniformly.

Lemma 3.3. There exist $C > 0$ and $\varepsilon_1 > 0$, depending only on (U, F), such that for all $\rho, \rho' \in U$ and $n \in \mathbb{N}$: If $\rho \in \mathcal{T}$ and $d\left(F^i(\rho), F^i(\rho')\right) \le \varepsilon_1$ for all $i \in \{0, \dots, n\}$ then

$$(3.11) \qquad d\left(\rho', W_s(\rho)\right) \le \frac{C}{J_n^u(\rho)} d\left(F^n(\rho'), W_s(F^n(\rho))\right)$$

$$(3.12) \qquad d\left(F^n(\rho'), W_u(F^n(\rho))\right) \le \frac{C}{J_n^u(\rho)} d\left(\rho', W_u(\rho)\right)$$

and

$$(3.13) \qquad ||d_{F^n(\rho')} F^{-n}||, ||d_{\rho'} F^n|| \le C J_n^u(\rho)$$

As an immediate consequence of this lemma, we get :

Corollary 3.1. There exists $C > 0$ and $\varepsilon_1 > 0$ (depending only on (U, F)) such that for all $\rho, \rho' \in \mathcal{T}$ and $n \in \mathbb{N}$:

(1) if $d\left(F^i(\rho), F^i(\rho')\right) \le \varepsilon_1$ for all $i \in \{0, \dots, n\}$, then

$$(3.14) \qquad C^{-1} J_n^u(\rho) \le J_n^u(\rho') \le C J_n^u(\rho)$$

(2) if $d\left(F^{-i}(\rho), F^{-i}(\rho')\right) \le \varepsilon_1$ for all $i \in \{0, \dots, n\}$, then

$$(3.15) \qquad C^{-1} J_{-n}^s(\rho) \le J_{-n}^s(\rho') \le C J_{-n}^s(\rho)$$

We also record the following fact (see for instance [DJN21] - Lemma 2.4).

Lemma 3.4. There exist $\varepsilon_1 > 0$ and $C > 0$ such that the following holds : For every $\rho \in \mathcal{T}$, there exists a symplectic coordinate chart $\kappa_\rho : V_\rho \to W_\rho \subset \mathbb{R}^2$ such that

- $B(\rho, \varepsilon_1) \subset V_\rho$;
- $\kappa_\rho(\rho) = (0, 0)$
- $\kappa_\rho\left(W_u(\rho) \cap V_\rho\right) = \{(u, 0), u \in \mathbb{R}\} \cap W_\rho$
- $d\kappa_\rho(E_s(\rho)) = \mathbb{R}(0, 1)$
- For any $N \in \mathbb{N}$, the C^N norm of κ_ρ is bounded uniformly in ρ.

Finally, we conclude by a lemma concerning the linearized dynamics. If $\rho \in \mathcal{T}$ and $\rho' \in W_u(\rho)$, the tangent space $T_{\rho'} W_u(\rho)$ will naturally be denoted $E_u(\rho')$ and if $v \in T_{\rho'} U$, we note $d(v, E_u(\rho'))$ the distance between v and $E_u(\rho')$ using the Riemanniann metric on $T_{\rho'} U$.

Lemma 3.5. There exist $\varepsilon_1 > 0$, $\gamma \in (0, 1)$ and $C > 0$ such that the following holds. Assume that $\rho \in \mathcal{T}$, $\rho' \in W_u(\rho)$, $v_0 \in T_{\rho'} U$ and $n \in \mathbb{N}$ satisfy : $\forall i \in \{0, \dots, n\}, d(F^i(\rho), F^i(\rho')) \le \varepsilon_1$. Assume also that $||v_0|| = 1$ and that $d(v_0, E_u(\rho')) \le \gamma$. Let's note

$$v_n = \frac{d_{\rho'} F^n(v_0)}{||d_{\rho'} F^n(v_0)||} \in T_{F^n(\rho')} U$$

Then (see Figure 5),

$$d\left(v_n, E_u(F^n(\rho'))\right) \le C J_n^u(\rho)^{-2} d(v_0, E_u(\rho'))$$

Remark. This is a form of inclination lemma : the tangent vectors are attracted toward the unstable direction upon the evolution. We provided a quantitative statement. The assumption $d(v_0, E_u(\rho')) \le \gamma$ is a transversality assumption : it says that v_0 has to be sufficiently transverse to the stable direction.

Proof. First note that due to the assumption on ρ and ρ' and Lemma 3.2,

$$d(F^i(\rho), F^i(\rho') \le C\theta^{n-i} d(F^n(\rho), F^n(\rho')) \le C\theta^{n-i} \varepsilon_1$$

for some $0 < \theta < 1$ and for $0 \le i \le n$. We use coordinates charts κ_i centered at $F^i(\rho)$ (for $0 \le i \le n$), given by Lemma 3.4. Let's note (u^i, s^i) the coordinates in κ_i. Since $\kappa_i(W_u(F^i(\rho)) = \{(u^i, 0)\}$, the map F between the charts κ_{i-1} and κ_i is given by :

$$\kappa_i \circ F \circ \kappa_{i-1}^{-1}(u^{i-1}, s^{i-1}) = (\nu_i u^{i-1} + \alpha_i(u^{i-1}, s^{i-1}), \mu_i s^{i-1} + \beta_i(u^{i-1}, s^{i-1}))$$

with $\beta_i(u^{i-1}, 0) = 0$, $d\alpha_i(0, 0) = 0$ and $d\beta_i(0, 0) = 0$. Remark that $\nu_1 \dots \nu_i \sim J_i^u(\rho) \sim (\mu_1 \dots \mu_i)^{-1}$ for $1 \le i \le n$.
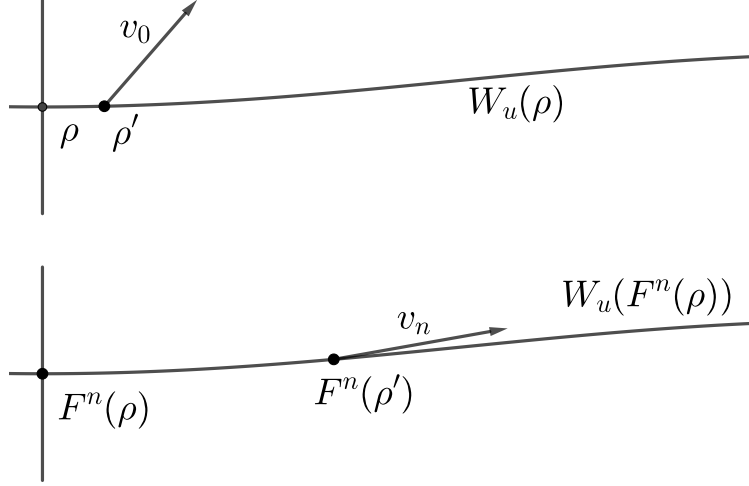
FIGURE 5. The linearized dynamics makes the vector closer and closer to the tangent space of the unstable manifold. See Lemma 3.5. The vertical direction corresponds the to the stable direction, in which the dynamics contracts.

Let us note $w_0 = d_{\rho'}\kappa_0(v_0)$ and $\tilde{w}_n = d_{F^n(\rho')}\kappa_n(v_n)$. Hence, we want to show that $d(\tilde{w}_n, \mathbb{R}e_u) \leq CJ_n^u(\rho)^{-1}d(w_0, \mathbb{R}e_u)$ where $\mathbb{R}e_u = \{(u,0)\}$. Here, to compute the distance, both between points and tangent vectors, we can simply use the usual euclidean distance in $\mathbb{R}^2$. Let us also introduce $w_i = d_{\rho'}(\kappa_i \circ F^i)(v_0)$ and write $w_i = (w_i^u, w_i^s)$. With these notations, we have $w_n = ||d_{\rho'}F^n(v_0)||\tilde{w}_n$ and

$$d(\tilde{w}_n, \mathbb{R}e_u) = \frac{w_n^s}{||d_{\rho'}F^n(v_0)||}$$

Since $||d_{\rho'}F^n(v_0)|| \sim ||w_n||$, we are reduced to prove that

(3.16)
$$\frac{|w_n^s|}{||w_n||} \leq CJ_n^u(\rho)^{-2}|w_0^s|$$

If γ is small enough, we can deduce from the transversality assumption on v_0 that $|w_0^u| \geq \frac{1}{2}|w_0^s|$. In particular, $||w_0||^2 \geq \frac{4}{3}|w_0^s|^2$. Let us note $(u^i, 0)$ the coordinates of $F^i(\rho')$ in the charts κ_i and recall that $|u^i| \leq C\theta^{n-i}\varepsilon_1$. We have the relations

$$w_i^u = \nu_i w_{i-1}^u + d\alpha_i(u^{i-1}, 0) \cdot w_{i-1}$$
$$w_i^s = \mu_i w_{i-1}^s + d\beta_i(u^{i-1}, 0) \cdot w_{i-1}$$

Since $\beta_i(u, 0) = 0$, $d\beta_i(u^{i-1}, 0) \cdot w_{i-1} = \partial_{s^i}\beta_i(u^{i-1}, 0)w_{i-1}^s$. Moreover, $d\beta_i(0, 0) = 0$, and hence, $|\partial_{s^i}\beta_i(u^{i-1}, 0)w_{i-1}^s| \leq C|u^{i-1}||w_{i-1}^s| \leq C\theta^{n+1-i}\varepsilon_1|w_{i-1}^s|$. this gives,

$$|w_i^s| \leq (\mu_i + C\theta^{n+1-i}\varepsilon_1)\ldots(\mu_1 + C\theta^n)|w_0^s|$$

$$|w_i^s| \leq \mu_1\ldots\mu_i \prod_{k=1}^{i}\left(1 + \frac{C\varepsilon_1\theta^{n+1-k}}{\mu_k}\right)|w_0^s|$$

Since $\mu_k \geq c$ for some $c > 0$ and for all $1 \leq k \leq n$, we can estimate

$$\prod_{k=1}^{i}\left(1 + \frac{C\varepsilon_1\theta n + 1 - k}{\mu_k}\right) \leq \prod_{k=0}^{i-1}\left(1 + C\varepsilon_1 c^{-1}\theta^k\right) \leq \prod_{i=0}^{\infty}\left(1 + C\varepsilon_1 c^{-1}\theta^i\right) < +\infty$$

As a consequence, $|w_i^s| \leq CJ_i^u(\rho)^{-1}|w_0^s|$. We now turn to a lower bound for $||w_n||$. From $w_i^u = \nu_i w_{i-1}^u + O(|u^{i-1}|||w_{i-1}||)$, we find that

$$|w_i^u| \geq \nu_i|w_{i-1}^u| - C|u^{i-1}|||w_{i-1}|| \geq \nu_i|w_{i-1}^u| - C\theta^{n+1-i}||w_{i-1}||$$

We observe that $||w_{i-1}|| \leq |w_{i-1}^u| + |w_{i-1}^s| \leq |w_{i-1}^u| + CJ_{i-1}^u(\rho)^{-1}|w_0^s|$, which gives that

$$|w_i^u| \geq |w_{i-1}^u|(\nu_i - C\theta^{n+1-i}) - C\theta^{n+1-i}J_{i-1}^u(\rho)^{-1}|w_0^s|$$

Recall that for $\theta_1 = e^{-\lambda_1}$, we have $J_i^u(\rho)^{-1} \leq \theta_1^i$, so that for some $\theta_2 \in]\max(\theta, \theta_1), 1[$, $\theta^{n+1-i} J_i^u(\rho)^{-1} \leq C\theta_2^n$. Iterating this formula, we find that

$$|w_n^u| \geq (\nu_1 - C\theta^n)\ldots(\nu_n - C\theta)|w_0^u| - \sum_{i=1}^{n-1}(\nu_n - C\theta)\ldots(\nu_i - C\theta^{n+1-i})\theta_2^n|w_0^s|$$

By similar arguments as in the case of $|w_n^s|$, we can show that $(\nu_n - C\theta)\ldots(\nu_1 - C\theta^n) \geq C^{-1}J_n^u(\rho)$ and $|(\nu_n - C\theta)\ldots(\nu_1 - C\theta_n)| \leq \nu_1\ldots\nu_n \leq CJ_n^u(\rho)$. As a consequence, and using the fact that $|w_0^u| \geq 1/4\|w_0\|$ (by the transversality assumption), we find that

$$|w_n^u| \geq C^{-1}(1 - \theta_2^n)J_n^u(\rho)\|w_0\| \geq C^{-1}J_n^u(\rho)\|w_0\|.$$

We conclude that $\|w_n\| \geq |w_n^u| \geq C^{-1}J_n^u(\rho)\|w_0\|$, which gives (3.16). $\qquad\square$

3.2. Topological pressure. Dimensions.

3.2.1. *Topological pressure.* We recall the definition and some formulas for the topological pressure associated with a continuous function $\varphi : \mathcal{T} \to \mathbb{R}$. The dynamical system we consider is the restriction of F on $\mathcal{T}$. We consider a distance function d on $\mathcal{T}$. For $n \in \mathbb{N}$ and $\epsilon > 0$, we say that a subset $E \subset \mathcal{T}$ is (n, ϵ) separated if for every $x, y \in E, x \neq y$, there exists $0 \leq i \leq n - 1$, $d(F^i(x), F^i(y)) > \epsilon$.

Definition 3.1. If φ is a continuous function on $\mathcal{T}$, the topological pressure associated with $\varphi \in C(\mathcal{T}, \mathbb{R})$ is defined as

$$P(\varphi) = \lim_{\epsilon \to 0} \limsup_{n \to +\infty} \frac{1}{n} \log P_0(\varphi, n, \epsilon)$$

where

$$P_0(\varphi, n, \epsilon) = \sup \left\{ \sum_{x \in E} \exp\left(\sum_{i=0}^{n-1} \varphi(f^i(x))\right) \; ; \; E \text{ is } (n, \epsilon) \text{ separated} \right\}$$

In this paper, we will use another formula for the pressure. To state it, let us introduce a few notations : if $\mathcal{Q}$ is a finite open cover of $\mathcal{T}$, we note $\text{diam}\mathcal{Q} = \sup_{A \in \mathcal{Q}} \text{diam}A$ and for $n \in \mathbb{N}$, $\mathcal{Q}^{\wedge n}$ is the open cover of $\mathcal{T}$ by the sets $\bigcap_{i=0}^{n-1} f^{-i}(A_i)$ where $A_0, \ldots, A_{n-1} \in \mathcal{Q}$. For $\varphi : \mathcal{T} \to \mathbb{R}$ continuous, $n \in \mathbb{N}$ and an open cover $\mathcal{Q}$ of $\mathcal{T}$, we define

$$P_1(\varphi, n, \mathcal{Q}) = \inf \left\{ \sum_{A \in \alpha} \sup_{x \in A} \exp\left(\sum_{i=0}^{n-1} \varphi(f^i(x))\right) \; ; \; \alpha \subset \mathcal{Q}^{\wedge n}, \mathcal{T} \subset \bigcup_{A \in \alpha} A \right\}$$

Proposition 3.1. [Wal75] (Theorem 1.6). The following formula holds: for any $\varphi \in C(\mathcal{T}, \mathbb{R})$,

$$(3.17) \qquad P(\varphi) = \lim_{\text{diam}\mathcal{Q} \to 0} \lim_{n \to \infty} \frac{1}{n} \log P_1(\varphi, n, \mathcal{Q})$$

Note that in particular, it asserts that the limit in n exists for all open cover $\mathcal{Q}$.

3.2.2. *Dimensions.* Let us recall the definition of the upper box dimension of a compact metric space (X, d). We denote by $N_X(\varepsilon)$ the minimal number of open balls of radius ε needed to cover X. Then, the upper box dimension of X is defined as :

$$(3.18) \qquad \overline{\dim}X := \limsup_{\varepsilon \to 0} \frac{\log N_X(\varepsilon)}{-\log \varepsilon}$$

In particular, if $\delta > \overline{\dim}_X$, there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \leq \varepsilon_0$, $N_X(\varepsilon) \leq \varepsilon^{-\delta}$.

We recall the following well known result (see for instance [Bar08], Theorem 4.3.2) :

Proposition 3.2. Let s_0 be the unique root of the equation $P(-s\varphi_u) = 0$; $s \in \mathbb{R}$. Then, For every $\rho \in \mathcal{T}$, $\overline{\dim}(\mathcal{T} \cap W_u(\rho)) = \overline{\dim}(\mathcal{T} \cap W_s(\rho)) = s_0$. Moreover, $\overline{\dim}\mathcal{T} = 2s_0$.

Remark. In fact, this holds also for the Hausdorff dimension and the lower-box dimension but we will mainly use the upper-box dimension for practical and technical reasons. In the following, we note $s_0 = d_H$.

We will need the slightly more precise following result, which allows to control $N_{W_{u/s}(\rho) \cap \mathcal{T}}$ uniformly in ρ :

Proposition 3.3. There exists $\varepsilon_1 > 0$ such that the following holds. For every $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that for every $\rho \in \mathcal{T}$, if $X_\rho = W_{u/s}(\rho) \cap \mathcal{T}$,

$$N_{X_\rho}(r) \leq C_\varepsilon r^{-(d_H + \varepsilon)}; \ \forall r \leq \varepsilon_1$$

Proof. Obviously, this holds at every $\rho \in \mathcal{T}$ with C_ε *a priori* depending on ρ. The uniformity is a consequence of the fact that the holonomy maps are Lipschitz, with uniform Lipschitz norm due to the compactness of $\mathcal{T}$ (see for instance [Vac22], Corollary 3.3). Then, due to the compactness of $\mathcal{T}$, one can estimate $N_{W_{u/s}(\rho) \cap \mathcal{T}}(\varepsilon)$ by considering only a finite number of (un)stable leaves as references and apply : *Assume that* (X, d), (Y, d') *are compact metric spaces and* $f : X \to Y$ *is* C-*Lipschitz. Then, for every* $\varepsilon > 0$,

$$N_{f(X)}(\varepsilon) \leq N_X(\varepsilon/C)$$

$\square$

We finish by a lemma estimating the number of balls of size δ needed to cover $\mathcal{T} \cap W_u(\rho_0) \cap J$ where $J \subset W_u(\rho_0)$ is an interval of size l. The difference with Proposition 3.3 is that the size of J can be much small that ε_1.

Lemma 3.6. Let $\rho_O \in \mathcal{T}$. Let $\kappa : U_0 \to V_0 \subset \mathbb{R}$ be a smooth chart such that the image of the local unstable manifold passing through ρ_0 is given by a graph

$$\kappa(W_u(\rho_O) \cap U_0) = \{(x, g(x)), x \in I\}$$

for some open interval I. For $J \subset I$, let's note

$$X(J) = \{x \in J, \kappa^{-1}(x, g(x)) \in \mathcal{T}\}$$

Then, for every $\varepsilon > 0$, there exists $C_\varepsilon > 0$ depending only on ε, F and κ such that : for all $J \subset I$ interval of length l and for all $\delta \in]0, l]$,

$$N_{X(J)}(\delta) \leq C_\varepsilon \left(\frac{\delta}{l}\right)^{-(d_H + \varepsilon)}.$$

Proof. Let's note $N = N_{X(J)}(\delta)$. If $N = 0$, there is obviously nothing to prove. So let's assume that $N \neq 0$ and let's fix a reference point $x_0 \in X(J)$: to x_0 corresponds a point $\rho_0 = \kappa^{-1}(x_0, g(x_0)) \in \mathcal{T}$ and we are interested in a piece of unstable manifold of ρ_0 of size l.

We know that the upper-box dimension of each $\mathcal{T} \cap W_u(\rho)$ is equal to d_H. However, since here we are interested by a small piece of an unstable manifold of size l, we will expand this piece to reach a size of order 1. We note $J_0 = \kappa^{-1}\big(\{(x, g(x)), x \in J\}\big)$ and for $m \in \mathbb{N}$, we note $\rho_m = F^m(\rho_0)$ and introduce

$$T := \max\{m \in \mathbb{N}, F^m(J) \subset W_u(\rho_m) \text{ and } \mathrm{diam} F^m(J_0) \leq \varepsilon_1\}$$

In particular, the definition of T implies that for all $\rho \in J_0$, $F^m(\rho)$ is well-defined for $0 \leq m \leq T$ and satisfies $d(F^m(\rho), F^m(\rho_0)) \leq \varepsilon_1$.

Claim : We first claim that if $J' \subset J$ is a subinterval with $X(J') \neq \emptyset$, then

$$\mathrm{diam}\, J'_T \sim \frac{\mathrm{diam} J'}{J^u_{-T}(\rho_T)}$$

where $J'_T = F^T(J'_0)$ for $J'_0 = \kappa^{-1}\big(\{(x, g(x)), x \in J'\}\big)$. In particular, it holds for $J' = J$.

Proof of the claim : Let's prove this claim and suppose that $J' \subset J$ is an interval of length l' and consider $x' \in X(J')$. Let's note $\rho' = \kappa^{-1}(x', g(x')) \in \mathcal{T}$. If $\hat{x} \in J'$ and $\hat{\rho} = \kappa^{-1}(\hat{x}, g(\hat{x})) \in W_u(\rho')$, we have

$$d(F^T(\rho'), F^T(\hat{\rho})) \sim \frac{d(\rho', \hat{\rho})}{J^u_{-T}(F^T(\rho'))} \sim \frac{|x' - \hat{x}|}{J^u_{-T}(F^T(\rho'))}$$

Since $d(F^m(\rho'), F^m(\rho_0)) \leq \varepsilon_1$ for $0 \leq m \leq T$, we have

$$J^u_{-T}(F^T(\rho')) \sim J^u_{-T}(\rho_T)$$

In particular, if we choose $\hat{x}$ such that $|x' - \hat{x}| \geq \mathrm{diam}\, J'/3$, we have

$$\mathrm{diam}\, J'_T \geq C^{-1} \frac{|x' - \hat{x}|}{J^u_{-T}(\rho_T)} \geq C^{-1} \frac{\mathrm{diam}\, J'}{J^u_{-T}(\rho_T)}$$

For the converse inequality, assume that $\rho_1, \rho_2 \in J_0'$.

$$d(F^T(\rho_1), F^T(\rho_2)) \leq d(F^T(\rho_1), F^T(\rho')) + d(F^T(\rho'), F^T(\rho_2)) \leq C\frac{|x_1 - x'| + |x' - x_2|}{J^u_{-T}(\rho_T)} \leq C\frac{\operatorname{diam} J'}{J^u_{-T}(\rho_T)}$$

which finally gives the required inequality by taking the supremum over ρ_1 and ρ_2.

End of proof. We have, $J^u_{-T}(\rho_T)\operatorname{diam} J_T \sim \operatorname{diam} J \sim l$. By definition of T,

$$\operatorname{diam} F^{T+1}(J_0) \geq \varepsilon_1$$

so that

$$\operatorname{diam} F^T(J_0) \geq C^{-1}\varepsilon_1$$

and hence, $J^u_{-T}(\rho_T) \leq Cl$ (this C also depends on ε_1, which is not a problem since ε_1 depends only on F). Let us fix $k > 0$, to be determined later. By Proposition 3.1, we can cover $J_T \cap \mathcal{T}$ by N balls of diameter at most $k\delta$ with $N \leq C_\varepsilon(k\delta)^{-d_H - \varepsilon}$. Let's choose $\rho_1, \ldots, \rho_N \in \mathcal{T} \cap J_T$ such that

$$\mathcal{T} \cap J_T \subset \bigcup_{i=1}^N B(\rho_i, k\delta).$$

We note x_i the point in J such that $\rho_i = F^T(\rho_i)$ with $\kappa(\rho_i) = (x_i, g(x_i))$. If $x \in X(J)$, then $\rho := F^T(\kappa^{-1}(x, g(x))) \in \mathcal{T} \cap J_T$ and there exists $i \in \{1, \ldots, N\}$ such that $d(\rho_i, \rho) \leq k\delta$. As a consequence, $|x - x_i| \leq CJ^u_{-T}(\rho_T)d(\rho_i, \rho) \leq C'lk\delta$ for some constant C' depending on F and κ. We now fix $k = (2C'l)^{-1}$, so that $X(J)$ can be covered by N intervals of length δ. As a consequence,

$$N_{X(J)}(\delta) \leq N \leq C_\varepsilon\left(\frac{\delta}{2C'l}\right)^{-d_H - \varepsilon} = C'_\varepsilon\left(\frac{\delta}{l}\right)^{-d_H - \varepsilon}$$

$\square$

3.3. Escape function.
In this subsection, we record the construction of escape functions of [NSZ14], specialized to our open map $F : \widetilde{D} \subset U \to \widetilde{A} \subset U$. We do not give the proof, since it is entirely contained in [NSZ14] (Lemmata 4.1 - 4.4).

Lemma 3.7. Assume that $\mathcal{V}_2$ is a small neighborhood of $\mathcal{T}$ in which F is well defined. Then, there exists $C_0 > 0$ and a neighborhood $\mathcal{V}_1 \subset \mathcal{V}_2$ of $\mathcal{T}$ such that the following holds : For every $\epsilon > 0$, there exist functions $\hat{\varphi}_\pm = \hat{\varphi}_{\pm, \epsilon} \in C^\infty(\mathcal{V}_1 \cup F(\mathcal{V}_1), [\epsilon, +\infty[)$ such that

$$\hat{\varphi}_\pm(\rho) \sim d(\rho, \mathcal{T}_\pm)^2 + \epsilon;$$
$$\pm(\hat{\varphi}_\pm(\rho) - \hat{\varphi}_\pm(F(\rho))) + C_0\epsilon \sim \hat{\varphi}_\pm(\rho);$$
$$\hat{\varphi}_+(\rho) + \hat{\varphi}_-(\rho) \sim \epsilon;$$
$$\partial^\alpha\hat{\varphi}_\pm(\rho) = O\left(\hat{\varphi}_\pm(\rho)^{1-|\alpha|/2}\right).$$

The constants in the $\sim$ and O are independent of $\rho \in \mathcal{V}_1 \cup F(\mathcal{V}_1)$ and ϵ.

Armed with these two functions, we construct the following escape function

(3.19) $$\hat{g}_\epsilon = \log(M\epsilon + \hat{\varphi}_-) - \log(M\epsilon + \hat{\varphi}_+)$$

where $M \gg 1$ is a constant independent of ϵ and sufficiently large so that the following lemma holds :

Lemma 3.8. For M large enough, there exists $C_1 > 0$ such that, uniformly with respect to ϵ,

$$\rho \in \mathcal{V}_1 \cup F(\mathcal{V}_1), d(\rho, \mathcal{T}) \geq C_1\epsilon \implies \hat{g}_\epsilon(F(\rho)) - \hat{g}_\epsilon \geq 1/C_1.$$

Since we will be interested in the dynamics in a neighborhood of $\mathcal{T}$, we fix a smooth cut-off function $\hat{\chi} \in C^\infty_c(\mathcal{V}_1 \cup F(\mathcal{V}_1))$, independent of ϵ, such that $\hat{\chi} = 1$ in a neighborhood of $\mathcal{T}$ and we set

(3.20) $$g_\epsilon = \hat{\chi}\hat{g}_\epsilon$$

As a consequence of the construction of $\hat{\varphi}_\pm$, it is also possible to check that

Lemma 3.9. g_ϵ satisfies the following estimates : there exist $C > 0$, $\mu > 0$ and a family of constants $C_\alpha > 0$, $\alpha \in \mathbb{N}^2$, independent of ϵ such that for all $\rho, \zeta \in U$,

$$|g_\epsilon(\rho)| \leq C |\log \epsilon|$$

$$|\partial^\alpha g_\epsilon(\rho)| \leq C_\alpha \left(\epsilon^{-|\alpha|/2} \right)$$

$$\frac{\exp(g_\epsilon(\rho))}{\exp(g_\epsilon(\zeta))} \leq C \left\langle \frac{\rho - \zeta}{\sqrt{\epsilon}} \right\rangle^\mu$$

This last inequality makes e^{g_ϵ} an order function in the rescaled variable $\rho/\sqrt{\epsilon}$.

We will specialize to $\epsilon = h^{2\delta}$ where $\delta = \frac{1}{2} - \varepsilon$. For this reason, it is important that the constants do not depend on ϵ.

4. Proof of Theorem 3

From now on, $M_h(z) = M(z; h)$ is an open hyperbolic quantum map satisfying the assumptions of Theorem 3. Recall that we note $\alpha_h(z)$ the amplitude of $M_h(z)$. Our aim is to understand the zeros of the Fredholm determinants

$$\det\left(\mathrm{Id} - M_h(z)\right)$$

Since the spectrum of $M_h(z)$ doesn't change by conjugation, we will instead study

$$(4.1) \qquad\qquad M_t(z; h) := e^{-tG} M_h(z) e^{tG}$$

where t will be chosen below and $G = \mathrm{Op}_h(g)$ where $g = g_{h^{2\delta}}$ is the escape function constructed (3.20), specialized in the case $\epsilon = h^{2\delta}$ where $\delta = 1/2 - \varepsilon$, for some fixed ε. To alleviate the notations, we now omit to write that $M_t(z)$ depends on h. The role of this conjugation is to damp the quantum map outside a small neighborhood of the trapped set so that it confers to the new operator nicer microlocal properties. To exploit the hyperbolicity of F and the special structure of the trapped set, we note that the zeros (repeated with multiplicity) of $\det\left(\mathrm{Id} - M_t(z)\right)$ are among the zeros of

$$\det\left(\mathrm{Id} - M_t(z)^{2N}\right)$$

We will use this remark with an exponent $N(h)$ depending on h in a controlled way and we will assume that $N(h) \leq C \log \frac{1}{h}$ for some $C > 0$. A precise value of $N(h)$ will be given later.

4.1. **Application of a Jensen formula.** The proof of Theorem 3 relies on the following Proposition, whose proof will occupy the end of this section. Recall that $\Omega =] - R, R[+i] - R, R[$ with R fixed but large (in particular, $R \geq 4$).

Proposition 4.1. Let $\varepsilon > 0$. Let $g = g_{h^{2\delta}}$ be the escape function defined in (3.20) (with $1/2 - \delta = \varepsilon > 0$)). Let's note $M_t(z) = e^{-t \,\mathrm{Op}_h(g)} M_h(z) e^{t \,\mathrm{Op}_h(g)}$. Let us fix $\beta \in]0, R[$. Then, there exist $t = t_\varepsilon > 0$, $C = C_\varepsilon > 0$, $\nu_\varepsilon > 0$, $\vartheta_\varepsilon > 0$ and $N = N_\varepsilon(h) \in \mathbb{N}$ such that

- When $\varepsilon \to 0$,

$$\nu_\varepsilon = d_H + O(\varepsilon) \; ; \; \vartheta_\varepsilon = \frac{1 - O(\varepsilon)}{6\lambda_{\max}}$$

- at fixed ε, when $h \to 0$, $N_\varepsilon(h) \sim \vartheta_\varepsilon \log(1/h)$
- for all h sufficiently small and for all $z \in \Omega$ with $\mathrm{Im}(z) \in [-\beta, 4]$,

$$(4.2) \qquad\qquad \mathrm{tr}\left(\left(M_t(z)^N\right)^* M_t(z)^N \right) \leq C h^{-\nu_\varepsilon} h^{-\vartheta_\varepsilon P(-2 \,\mathrm{Im}\, z\, t_{ret} - \varphi_u)}$$

Remark. Since $\mathrm{Im}\, z \geq -\beta$ and since the function $s \mapsto P(-2st_{ret} - \varphi_u)$ is non increasing, the right hand side can be estimated by $h^{-\nu_\varepsilon} h^{-\vartheta_\varepsilon P(2\beta t_{ret} - \varphi_u)} = h^{-d_H + p(\beta) - O(\varepsilon)}$. This is where the function $p(\beta) = -\frac{1}{6\lambda_{\max}} P(2\beta t_{ret} - \varphi_u)$ appears.

Armed with this proposition, we can conclude the proof of Theorem 3 by using standard arguments of spectral theory and complex analysis (we mainly borrow the arguments from [DD12], [Dya19]).

Proof of Theorem 3. The exponent d_H is known from [NSZ14] in Theorem 4. We focus on the potential improvement given by $p(\gamma + \varepsilon) - \varepsilon$.

We fix $0 < r < R$ and $\gamma > 0$ and note $\Omega_0 = \{|\mathrm{Re}\, z| \leq r, \mathrm{Im}(z) \in [-\gamma, 2]\}$. For $\eta > 0$, we also note $\Omega_\eta = \{|\mathrm{Re}\, z| < R, \mathrm{Im}\, z \in] - \gamma - \eta, 4[\}$. Since $\det(\mathrm{Id} - M(z; h)) = \det(\mathrm{Id} - M_t(z))$ and due to

the relation: $\mathrm{Id} - A^{2N} = (\mathrm{Id} - A)(\mathrm{Id} + A + \dots A^{2N-1})$, we have (we note $m_T(\Omega_1)$ the numbers of zeros of $\det(I - T)$ in Ω_1, counted with multiplicity),

$$m_M(\Omega_0) \leq m_{M_t}(\Omega_0) \leq m_{M_t^{2N}}(\Omega_0)$$

that is, it is enough to estimates the number of zeros of $f(z) = \det\left(\mathrm{Id} - M_t(z)^{2N}\right)$.

We claim that if H is some Hilbert space and if $A : H \to H$ is a trace-class operator, then $\log|\det(I - A^2)| \leq \|A\|_{HS}^2 = \mathrm{tr}(A^*A)$. Indeed, if we denote $\lambda_j(A)$ (resp. $\sigma_j(A)$) the eigenvalues (resp. singular values) of A repeated with multiplicity, one has,

$$\log|\det(I - A^2)| = \sum_j \log(|1 - \lambda_j(A^2)|) = \sum_j \log|1 - \lambda_j(A)^2| \leq \sum_j \log(1 + \lambda_j(A)^2) \leq \sum_j \lambda_j(A)^2$$

Weyl's inequalities imply that (see for instance [DZ19], Appendix B.5.1)

$$\sum_j \lambda_j(A)^2 \leq \sum_j \sigma_j(A)^2 = \|A\|_{HS}^2 = \mathrm{tr}(A^*A)$$

which gives the desired result. Hence, we have

(4.3) $$\log|\det\left(\mathrm{Id} - M_t(z)^{2N}\right)| \leq \mathrm{tr}\left((M_t^N(z))^* M_t^N(z)\right)$$

which is known to be controlled by Proposition 4.1. Let's note $z_0 = i \in \Omega_0$. By the Riemann mapping theorem, for any $\eta > 0$, there exists a conformal map $c : \Omega_\eta \to \{|z| < 1\}$ such that $c(z_0) = 0$. $c(\Omega_0) \Subset c(\Omega_\eta)$, so that there exists $\delta > 0$ such that $c(\Omega_0) \subset \{|z| < 1 - \delta\}$. We now apply Jensen's formula to the function $f \circ c$. Let $n(t)$ denote the number of zeros of $f \circ c$ (counted with multiplicities), in the disc of radius t. We have, by Jensen's formula,

$$\int_0^{1-\delta/2} \frac{n(t)}{t} dt = \frac{1}{2\pi} \int_0^{2\pi} \log|f \circ c((1 - \delta/2)e^{i\theta})| d\theta - \log|f(z_0)|$$

Therefore,

$$m_M(\Omega_0) \leq m_{M_t^{2N}}(\Omega_0) \leq n(1 - \delta)$$

$$\leq \frac{2}{\delta(1-\delta)} \int_{1-\delta}^{1-\delta/2} \frac{n(t)}{t} dt$$

$$\leq \frac{2}{\delta(1-\delta)} \int_0^{1-\delta/2} \frac{n(t)}{t} dt$$

$$\leq \frac{2}{\delta(1-\delta)} \left(\frac{1}{2\pi} \int_0^{2\pi} \log|f \circ c((1 - \delta/2)e^{i\theta})| d\theta - \log|f(z_0)|\right)$$

$$\leq \frac{2}{\delta(1-\delta)} \left(\sup_{z \in \Omega_\eta} \log|f(z)| - \log|f(z_0)|\right)$$

We apply Proposition 4.1 with a small parameter ε', depending on ε, giving exponents $\nu_{\varepsilon'}, \vartheta_{\varepsilon'}$. Since $\nu_{\varepsilon'} = d_H + O(\varepsilon')$ and $\vartheta_{\varepsilon'} = \frac{1}{6\lambda_{\max}} + O(\varepsilon')$, we can choose ε' small enough so that $\nu_{\varepsilon'} - 6\lambda_{\max}\vartheta_{\varepsilon'} p(\gamma + \eta) \leq d_H - p(\gamma + \eta) + \varepsilon$. Hence, we have

$$\sup_{z \in \Omega_\eta} \log|f(z)| \leq \sup_{z \in \Omega_\eta} h^{-\nu_{\varepsilon'}} h^{-\vartheta_{\varepsilon'} P(-2\,\mathrm{Im}\,z t_{ret} - \varphi_u)} \leq h^{-d_H - \varepsilon + p(\gamma + \eta)}$$

since the map $\beta \mapsto p(\beta)$ is non increasing (recall the definition of $p(\beta)$ in (1.10). To handle the term $-\log|f(z_0)|$, since $\alpha_h(z_0) < 1$ near $\mathcal{T}$, by choosing t large enough, we may ensure that there exists $\rho \in [0, 1[$ such that for h small enough, $\|M_t(z_0)\| \leq \rho$ (see the proof of Lemma 5.3 in [NSZ14]). As a consequence, $\|M_t^{2N(h)}(z_0)\| \leq C_t \rho^{2N}$, so that for h small enough, $\|M_t^{2N}\| < 1/2$. In particular, for such h, $\mathrm{Id} - M_t(z_0)^{2N}$ is invertible and

$$\left\|\left(\mathrm{Id} - M_t(z_0)^{2N}\right)^{-1}\right\| \leq 2.$$

As a consequence, one has

$$
\begin{aligned}
-\log|\det\left(\mathrm{Id}-M_t(z_0)^{2N}\right)| &= \log\left|\det\left(\mathrm{Id}-M_t(z_0)^{2N}\right)^{-1}\right| \\
&= \log\left|\det\left(\mathrm{Id}+M_t(z_0)^{2N}\left(\mathrm{Id}-M_t(z_0)^{2N}\right)^{-1}\right)\right| \\
&\leq \left\|M_t(z_0)^{2N}\left(\mathrm{Id}-M_t(z_0)^{2N}\right)^{-1}\right\|_{tr} \\
&\leq \|M_t(z_0)^{2N}\|_{tr}\left\|\left(\mathrm{Id}-M_t(z)^{2N}\right)^{-1}\right\| \\
&\leq 2\|M_t(z_0)^N\|_{HS} \leq Ch^{-d_H-\varepsilon+p(\gamma+\eta)}
\end{aligned}
$$

This concludes the proof.

$\square$

4.2. **Proof of Proposition 4.1.** We start the proof of Proposition 4.1. We fix some $\varepsilon > 0$ and we froze the complex variable z and note M_h and α_h instead of $M_h(z)$ and $\alpha_h(z)$: we momentarily forget this dependence but keep in mind that $\mathrm{Im}(z) \in [-\beta, 4]$ for some $\beta > 0$. In particular, $\alpha_h(z) = e^{-\mathrm{Im}\,zt_{ret}} + O\left(h^{-1}S_{0+}\right)$ in a neighborhood of $\mathcal{T}$ and the constant in the estimates below can be chosen independent of z.

Reduction to FIO acting on $\mathbb{R}$. We will note $\mathbb{R}_J = \bigsqcup_{j=1}^J \mathbb{R}$ and $L^2(\mathbb{R}_J) = \bigoplus_{j=1}^J L^2(\mathbb{R})$. Recall that by construction (see 2.2.3), M_h is an operator of the form $(M_{ij}(h))$ where $M_{ij}(h) : L^2(Y_j) \to L^2(Y_i)$. It will be more convenient for us to work on $L^2(\mathbb{R})$. For this purpose, recall that, for all i, j, there exists $\tilde{M}_{ij}(h) \in I_{0+}\left(\mathbb{R} \times \mathbb{R}, \mathrm{Gr}(F_{ij})'\right)$ and cut-off functions Ψ_i, Ψ_j such that as operators $L^2(Y_j) \to L^2(Y_i)$

$$
M_{ij}(h) = \Psi_i \tilde{M}_{ij}(h)\Psi_j + O(h^\infty)
$$

and as operator $L^2(\mathbb{R}) \to L^2(\mathbb{R})$,

$$
\tilde{M}_{ij}(h) = \Psi_i \tilde{M}_{ij}(h)\Psi_j + O(h^\infty)
$$

where, in the two equalities above, the $O(h^\infty)$ hold for the trace norm. Let's note $M_\psi(h) = (\Psi_i M_{ij}(h)\Psi_j)_{ij}$. As soon as $N \leq C\log\frac{1}{h}$, $M(h)^N = M_\Psi(h)^N + O(h^\infty)$ as operators $L^2(Y) \to L^2(Y)$ and $M_\Psi(h)^N = \tilde{M}(h)^N + O(h^\infty)$ as operators $L^2(\mathbb{R}_J) \to L^2(\mathbb{R}_J)$. The same holds after conjugation by e^{tG}. In particular, this sows that

$$
\mathrm{tr}_{L^2(Y)}\left((M_t^N)^* M_t^N\right) = \mathrm{tr}_{L^2(\mathbb{R}_J)}\left((\tilde{M}_t^N)^* \tilde{M}_t^N\right) + O(h^\infty)
$$

Since the $O(h^\infty)$ will finally be adsorbed in our required inequality, it is enough to work with $\tilde{M}(h)$ instead of M_h.

From now on, we will write M_h for the operator $\tilde{M}_h : L^2(\mathbb{R}_J) \to L^2(\mathbb{R}_J)$. There exists Ψ_A, Ψ_D such that

$$
\mathrm{supp}\,\Psi_A \Subset \widetilde{A} \quad ; \quad \mathrm{supp}\,\Psi_D \Subset \widetilde{D}
$$

and

$$
\Psi_A M_h = M_h + O(h^\infty); \quad M_h \Psi_D = M_h + O(h^\infty)
$$

Moreover, we will now omit the h-dependence of the semiclassical operators in the notations when this dependence is obvious. In particular, we will simply write M, α or M_t instead of M_h, α_h and $M_t(h)$ respectively.

Notations. A function a on $T^*\mathbb{R}_J = \bigsqcup_{j=1}^J T^*\mathbb{R}$ is a J-uple of functions $(a_1, \ldots, a_J)$. The quantization $\mathrm{Op}_h(a)$ is the diagonal matrix with diagonal entries $\mathrm{Op}_h(a_j)$. The support of a is the disjoint union of the supports of the a_j's, so as the wavefront set of $\mathrm{Op}_h(a)$.

4.2.1. *Refined quantum partition.* In virtue of Proposition 3.1, applied with $\varphi = -2\,\mathrm{Im}\,zt_{ret} - \varphi_u$, there exists $\eta > 0$ such that for any open cover $\mathcal{Q}$ of $\mathcal{T}$ of diameter smaller than η, one has

$$
(4.4) \qquad \left|\lim_{n\to+\infty} \frac{1}{n}\log P_1(\varphi, n, \mathcal{Q}) - P(\varphi)\right| \leq \varepsilon/3
$$

We consider some $\varepsilon_0 > 0$, which is supposed to be small enough to satisfy all the assumptions which will appear in the following and which will follow us throughout the end of the chapter. In particular, we first impose $\varepsilon_0 < \eta$.

Since $\mathcal{T}$ is totally disconnected, there exists an open cover of $\mathcal{T}$ by a finite number of *disjoint* open sets (of U) of diameter smaller than ε_0 :

$$\mathcal{T} \subset \bigcup_{\mathfrak{A} \in \mathcal{Q}} \mathfrak{A}$$

We fix some $\rho_{\mathfrak{A}} \in \mathcal{T} \cap \mathfrak{A}$ and we assume that for all $\mathfrak{A} \in \mathcal{Q}$, there exists $j_{\mathfrak{A}}, l_{\mathfrak{A}}, m_{\mathfrak{A}} \in \{1, \ldots, J\}$ such that

$$\mathfrak{A} \subset B(\rho_{\mathfrak{A}}, 2\varepsilon_0) \subset \widetilde{A}_{j_{\mathfrak{A}} l_{\mathfrak{A}}} \cap \widetilde{D}_{m_{\mathfrak{A}} j_{\mathfrak{A}}} \subset U_{j_{\mathfrak{A}}}$$

ε_0 is supposed to be small enough so that :

- $e^{-\tau_m} \le \alpha_h \le e^{\tau_M}$ in $B(\rho_{\mathfrak{A}}, 2\varepsilon_0)$ for some τ_m, τ_M, for all h small enough.
- If ε_1 denotes the one appearing in Lemma 3.4, $2\varepsilon_0 \le \varepsilon_1$, and then, there exists a chart $\kappa_{\mathfrak{A}} : B(\rho_{\mathfrak{A}}, 2\varepsilon_0) \to W_{\mathfrak{A}} = \kappa_{\mathfrak{A}}(B(\rho_{\mathfrak{A}}, 2\varepsilon_0))$, given by Lemma 3.4, adapted to the dynamics, where $W_{\mathfrak{A}}$ is a subset of $T^*\mathbb{R}$ centered at 0. .
- There exist Fourier integral operators $B_{\mathfrak{A}}, B'_{\mathfrak{A}} \in I_0(\mathbb{R} \times \mathbb{R}, \mathrm{Gr}'(\kappa_{\mathfrak{A}})) \times I_0(\mathbb{R} \times \mathbb{R}, \mathrm{Gr}'(\kappa_{\mathfrak{A}}^{-1}))$, quantizing $\kappa_{\mathfrak{A}}$ in a neighborhood of $\kappa_{\mathfrak{A}}(\overline{\mathfrak{A}}) \times \overline{\mathfrak{A}}$.

Notations. We will still denote $B_{\mathfrak{A}}$ and $B'_{\mathfrak{A}}$ the operators

$$B_{\mathfrak{A}} = \mathrm{Diag}(0, \ldots B_{\mathfrak{A}}, \ldots, 0) : L^2(\mathbb{R}_J) \to L^2(\mathbb{R}_J) \quad ; \quad B'_{\mathfrak{A}} = \mathrm{Diag}(0, \ldots, B'_{\mathfrak{A}}, \ldots, 0) : L^2(\mathbb{R}_J) \to L^2(\mathbb{R}_J)$$

with the non zero entry in position $j_{\mathfrak{A}}$. When we say that $(B_{\mathfrak{A}}, B'_{\mathfrak{A}})$ quantize $\kappa_{\mathfrak{A}}$ in a neighborhood of $\kappa_{\mathfrak{A}}(\overline{\mathfrak{A}}) \times \overline{\mathfrak{A}}$, we mean that $B'_{\mathfrak{A}} B_{\mathfrak{A}} = I + O(h^\infty)$ microlocally in a neighborhood of $\overline{\mathfrak{A}}$ (in the sense that if $\mathrm{supp}(c)$ is included in this neighborhood of $\overline{\mathfrak{A}}$ and if $C = \mathrm{Op}_h(c)$, then $B'_{\mathfrak{A}} B_{\mathfrak{A}} C = C + O(h^\infty)$; $CB'_{\mathfrak{A}} B_{\mathfrak{A}} = C + O(h^\infty)$) and $B_{\mathfrak{A}} B'_{\mathfrak{A}} = I + O(h^\infty)$ microlocally in a neighborhood of $\kappa_{\mathfrak{A}}(\overline{\mathfrak{A}})$.

In virtue of the equation (4.4), there exists $n_0 \in \mathbb{N}$ such that

$$\left| \frac{1}{n_0} \log P_1(\varphi, n_0, \mathcal{Q}) - P(\varphi) \right| \le 2\varepsilon/3$$

As a consequence, there exists a subpartition $(\mathcal{W}_q)_{q \in \mathcal{A}} \subset \mathcal{Q}^{n_0}$ such that $\mathcal{T} \subset \bigcup_{q \in \mathcal{A}} \mathcal{W}_q$ and

$$(4.5) \qquad \sum_{q \in \mathcal{A}} \sup_{\rho \in \mathcal{W}_q \cap \mathcal{T}} \exp\left(\sum_{i=0}^{n-1} \varphi(F^i(\rho)) \right) \le e^{n_0(P(\varphi)+\varepsilon)}$$

For $q \in \mathcal{A}$, we can find an open set $\mathcal{V}_q \Subset \mathcal{W}_q$ such that $\mathcal{T} \cap \mathcal{W}_q \subset \mathcal{V}_q$. $(\mathcal{V}_q)_{q \in \mathcal{A}}$ is still a cover of $\mathcal{T}$. We complete this cover with

$$(4.6) \qquad \mathcal{V}_\infty = \mathbb{R}_J \setminus \bigcup_{q \in \mathcal{A}} \mathcal{V}_q$$

We note $\mathcal{A}_\infty = \mathcal{A} \cup \{\infty\}$. Note also that for $q \in \mathcal{A}$, $\mathcal{W}_q$ is of the form

$$\mathfrak{A}_0 \cap F^{-1}(\mathfrak{A}_1) \cap \cdots \cap F^{-(n-1)}(\mathfrak{A}_{n-1})$$

and in particular $\mathcal{W}_q \subset \mathfrak{A}_0$: we note $j_q, l_q, m_q, \rho_q, \kappa_q, B_q, B'_q, W_q$, instead of $j_{\mathfrak{A}_0}, l_{\mathfrak{A}_0}, m_{\mathfrak{A}_0}, \rho_{\mathfrak{A}_0}, \kappa_{\mathfrak{A}_0}, B_{\mathfrak{A}_0}, B'_{\mathfrak{A}_0}, W_{\mathfrak{A}_0}$. Then, for $q \in \mathcal{A}$, we consider a cut-off function $\chi_q \in C_c^\infty(\mathbb{R}_J, [0,1])$ such that $\mathrm{supp}(\chi_q) \subset \mathcal{W}_q$ and $\chi_q \equiv 1$ in a neighborhood of $\overline{\mathcal{V}_q}$. Finally, we note $\chi_\infty = 1 - \sum_{q \in \mathcal{A}} \chi_q$. We note that χ_q is supported in only one copy of $\mathbb{R}$ in $\mathbb{R}_J$ when $q \in \mathcal{A}$ and χ_∞ has non-zero components in all the copies of $\mathbb{R}$ in $\mathbb{R}_J$. Moreover, $\mathrm{supp}(\chi_\infty) \subset \mathcal{V}_\infty$.

We then quantize the symbols χ_q, $q \in \mathcal{A}_\infty$:

$$(4.7) \qquad A_q = \mathrm{Op}_h(\chi_q)$$

Note that for $q \in \mathcal{A}$, A_q is a diagonal matrix with a single non zero coefficient. The family $(A_q)_{q \in \mathcal{A}_\infty}$ satisfies the following properties :

$$(4.8) \qquad \sum_{q \in \mathcal{A}_\infty} A_q = \mathrm{Id} \quad ; \quad \forall q \in \mathcal{A}_\infty, \|A_q\| \le 1 + O(h)$$

Since $M^{n_0} = \sum_{q \in \mathcal{A}_\infty} M^{n_0} A_q$, we may write

$$M^{nn_0} = \sum_{\mathbf{q} \in \mathcal{A}_\infty^n} M_{\mathbf{q}}$$

where for $\mathbf{q} = q_0 \ldots q_{n-1} \in \mathcal{A}_\infty^n$,

(4.9)
$$M_{\mathbf{q}} := M^{n_0} A_{q_{n-1}} \dots M^{n_0} A_{q_0}$$

For $\mathbf{q} = q_0 \dots q_{n-1} \in \mathcal{A}_\infty^n$, we also define a family of refined neighborhoods, forming a refined cover of $\mathcal{T}$,

(4.10)
$$\mathcal{V}_{\mathbf{q}}^- = \bigcap_{i=0}^{n-1} F^{-in_0}(\mathcal{V}_{q_i}) \quad ; \quad \mathcal{V}_{\mathbf{q}}^+ = F^{nn_0}(\mathcal{V}_{\mathbf{q}}^-) = \bigcap_{i=0}^{n-1} F^{(n-i)n_0}(\mathcal{V}_{q_i})$$

and we adopt the same definitions by changing $\mathcal{V}$ into $\mathcal{W}$. Roughly speaking, we expect that each operator $M_{\mathbf{q}}$ acts from $\mathcal{W}_{\mathbf{q}}^-$ to $\mathcal{W}_{\mathbf{q}}^+$ and is negligible elsewhere. Combining (4.8), the fact that $\alpha_h \leq e^{\tau_M}$ in $B(\rho_{\mathfrak{A}})$ and the bound on M, the following bound is valid :

(4.11)
$$||M_{\mathbf{q}}||_{L^2 \to L^2} \leq \left(e^{\tau_M} + O(h^{1-})\right)^{nn_0}$$

As soon as $|n| \leq C_0 |\log h|$, we have $||M_{\mathbf{q}}||_{L^2 \to L^2} \leq C e^{nn_0 \tau_M}$, for some C depending on C_0 and a finite number of semi-norms of α_h and then

$$||M_{\mathbf{q}}|| \leq C h^{-K}$$

for some $C, K > 0$ depending on C_0 and α_h.

4.2.2. *Local unstable Jacobian.* We want to define unstable Jacobians associated with these refined partition. Let's fix a word $\mathbf{q} = q_0 \dots q_{n-1} \in \mathcal{A}^n$ and assume that $\mathcal{W}_{\mathbf{q}}^- \neq \emptyset$. Fix $\rho \in \mathcal{W}_{\mathbf{q}}^-$. By definition of $\mathcal{W}_{q_i}$, there exists $\mathfrak{A}_{0,i}, \dots, \mathfrak{A}_{n_0-1,i} \in \mathcal{Q}$ such that

$$\mathcal{W}_{q_i} = \bigcap_{j=0}^{n_0-1} F^{-j}(\mathfrak{A}_{j,i})$$

Hence, for $0 \leq l \leq n' = n \times n_0 - 1$, there exists $\rho_l \in \mathcal{T}$ such that $d(\rho_l, F^l(\rho)) \leq 2\varepsilon_0$. Hence,

$$d(F(\rho_l), \rho_{l+1}) \leq d(F(\rho_l), F^{l+1}(\rho)) + d(F^{l+1}(\rho), \rho_{l+1}) \leq C\varepsilon_0$$

That is to say, $(\rho_0, \dots, \rho_{n'})$ is a $C\varepsilon_0$ pseudo orbit. Assume that $\delta_0 > 0$ is a small fixed parameter. In virtue of the shadowing lemma (see [HK95], Section 18.1), if ε_0 is sufficiently small, $(\rho_0, \dots, \rho_{n'})$ is δ_0 shadowed by an orbit of F : there exists $\rho' \in \mathcal{T}$ such that for $i \in \{0, \dots, n'\}$, $d(\rho_i, F^i(\rho')) \leq \delta_0$. Consequently, $d(F^i(\rho), F^i(\rho')) \leq \delta_0 + C\varepsilon_0$. If ρ_2 is another point in $\mathcal{W}_{\mathbf{q}}^-$, for $i = 0, \dots, n'$, $d(F^i(\rho_2), F^i(\rho')) \leq 2\varepsilon_0 + C\varepsilon_0 + \delta_0$. For convenience, set $\varepsilon_2 = 2\varepsilon_0 + \delta_0 + C\varepsilon_0$ and note that ε_2 can be arbitrarily small depending on ε_0. As a consequence, we have proven the following

Lemma 4.1. If $\mathcal{W}_{\mathbf{q}}^- \neq \emptyset$, there exists $\rho' \in \mathcal{T}$ such that $\forall l \in \{0, \dots, nn_0 - 1\}$ and for any $\rho \in \mathcal{W}_{\mathbf{q}}^-$, $d(F^l(\rho), F^l(\rho')) \leq \varepsilon_2$.

We fix any ρ' satisfying the conclusions of this lemma and we arbitrarily set (recall also the definition of $J_n^u(\rho)$ in (3.3) for $\rho \in \mathcal{T}$)

(4.12)
$$J_{\mathbf{q}}^u := J_{nn_0}^u(\rho') = \prod_{j=0}^{n-1} J_{n_0}^u(F^{jn_0}(\rho'))$$

If ρ_1' is another point satisfying this conclusion, we have $d(F^i(\rho'), F^i(\rho_1')) \leq 2\varepsilon_2$ for $i \in \{0, \dots, n'\}$ and in virtue of Corollary (3.1),

$$J_{nn_0}^u(\rho') \sim J_{nn_0}^u(\rho_1')$$

Hence, up to global multiplicative constant, the definition of this unstable Jacobian is independent of the choice of ρ'. Notice that if $\mathcal{W}_{\mathbf{q}}^- \cap \mathcal{T} \neq \emptyset$, any $\rho' \in \mathcal{T} \cap \mathcal{W}_{\mathbf{q}}^-$ satisfies the conclusions of Lemma 4.1 and $J_{\mathbf{q}}^u \sim J_{nn_0}^u(\rho')$.

We have the following facts concerning these local unstable Jacobian :

Lemma 4.2. If ε_0 is small enough, the following holds. There exists $C > 0$ such that for all $\mathbf{q} \in \mathcal{A}^n$ and for all $\rho \in \mathcal{W}_{\mathbf{q}}^-$, we have

- $||d_\rho F^{nn_0}|| \leq C J_{\mathbf{q}}^u$
- $d(F^{nn_0-1}(\rho), \mathcal{T}_+) \leq C \left(J_{\mathbf{q}}^u\right)^{-1} d(\rho, \mathcal{T}_+)$
- $d(\rho, \mathcal{T}_-) \leq C \left(J_{\mathbf{q}}^u\right)^{-1} d(F^{nn_0-1}(\rho), \mathcal{T}_-)$

Proof. The three points are consequences of Lemma 3.3. The first point is an easy one. Concerning the other two, first recall that $\mathcal{T}_+$ (resp. $\mathcal{T}_-$) is, in a neighborhood of $\mathcal{T}$, equal to the union of local unstable (resp. stable manifolds). Let's consider the second inequality. The proof of the third one is similar, by inverting the time direction. We fix $\zeta \in \mathcal{T}$ such that $d(\rho, \mathcal{T}_+) = d(\rho, W_u(\zeta))$ and $d(\zeta, \rho) \leq 2\varepsilon_0$. Recall that by Lemma 4.1, there exists ρ' such that $\forall i \in \{0, \ldots, nn_0 - 1\}$ and $d(F^i(\rho), F^i(\rho')) \leq \varepsilon_2$. We hence consider the unique point $\zeta' \in W_u(\zeta) \cap W_s(\rho')$. Since $\zeta' \in W_s(\rho')$, $d(F^i(\zeta'), F^i(\rho')) \leq C J_i^s(\rho) d(\rho', \zeta')$ for all $0 \leq i \leq nn_0 - 1$. If ε_0 is small enough, we may assume that $C J_i^s(\rho) d(\rho', \zeta') \leq \frac{1}{2}\varepsilon_1$ for $0 \leq i \leq n - 1$ (where ε_1 appears in Lemma 3.3). As a consequence, $J_{nn_0}^u(\zeta') \sim J_{nn_0}^u(\rho') \sim \tilde{J}_{\mathbf{q}}^u$. Moreover, $d(F^i(\rho), F^i(\zeta')) \leq \frac{1}{2}\varepsilon_1 + \varepsilon_2$ for all $0 \leq i \leq nn_0 - 1$. Hence, if $\varepsilon_2 \leq \frac{1}{2}\varepsilon_1$,

$$d(F^{nn_0-1}(\rho), \mathcal{T}_+) \leq d(F^{nn_0-1}(\rho), W_u(F^n(\zeta'))) \leq C J_{nn_0}^u(\zeta') d(\rho, W_u(\zeta')) \leq C \tilde{J}_{\mathbf{q}}^u d(\rho, \mathcal{T}_+)$$

$\square$

4.2.3. *Numerology.* In this subsection, we introduce the parameters we will work with. Recall that ε has been fixed. We set $\delta = 1/2 - \varepsilon$: it is related to the regularity of the escape function g. For technical reasons, we also introduce

$$(4.13) \qquad \delta_0 = \frac{1}{2} - \frac{\varepsilon}{2}; \ \delta_1 = \delta - \frac{\varepsilon}{2} = \frac{1}{2} - \frac{3\varepsilon}{2}$$

satisfying $\delta_1 < \delta < \delta_0 < 1/2$. Recall that n_0 has been chosen in (4.5) and that

$$\lambda_{\max} = \sup_{\rho \in \mathcal{T}} \limsup_{n \to +\infty} \frac{1}{n} \log J_n^u(\rho)$$

We define precisely the parameter ϑ_ε appearing in Proposition 4.1 as

$$(4.14) \qquad \vartheta_\varepsilon = \frac{1 - 4\varepsilon}{6\lambda_{\max}(1 + \varepsilon)^2}$$

The precise value of ϑ_ε will be used in the following : what is important is that $\vartheta_\varepsilon = \frac{1}{6\lambda_{\max}} - O(\varepsilon) < 1/6\lambda_{\max}$. Finally, we set

$$n = n(h) = \left\lfloor \frac{\vartheta_\varepsilon}{n_0} \log \frac{1}{h} \right\rfloor$$

which satisfies

$$e^{\lambda_{\max}(1+\varepsilon)nn_0} \leq h^{-\frac{1-4\varepsilon}{6(1+\varepsilon)}}$$

In particular, we assume that ε is small enough to ensure that

$$h^{\delta_0} h^{-\frac{1-4\varepsilon}{6(1+\varepsilon)}} \leq h^{1/3}$$

This will constraints the width of the evolved coherent states.

4.2.4. *Reduction to L^2-bounds of an evolved coherent state.* We can find a uniform $T_0 \in \mathbb{N}$ such that if $\rho \in \mathcal{V}_\infty$, there exists $k \in \{-T_0, \ldots, T_0\}$ such that $F^k(\rho)$ "falls" in the hole - that is, either there exists $k \in \{1, \ldots, T_0\}$ such that $F^i(\rho) \in \widetilde{D}$ for $1 \leq i \leq k - 1$ and $F^k(\rho) \in U \setminus \widetilde{D}$ or there exists $k \in \{1, \ldots, T_0\}$ such that $F^{-i}(\rho) \in \widetilde{A}$ for $1 \leq i \leq k - 1$ and $F^{-k}(\rho) \in U \setminus \widetilde{A}$. By standard properties of the Fourier integral operators, each component $(M^{T_0})_{ij}$ of M^{T_0} is a Fourier integral operator associated with the component $(F^{T_0})_{ij}$ of F^{T_0}. In particular, $\mathrm{WF}_h'(M^{T_0}) \subset \mathrm{Gr}(F^{T_0})$.

Let us study $M^{2T_0+nn_0} = M^{T_0} M^{nn_0} M^{T_0}$, and let's decompose $M^{nn_0} = \sum_{\mathbf{q} \in \mathcal{A}_\infty^n} M_{\mathbf{q}}$. If $\mathbf{q} = q_0 \ldots q_{n-1} \in \mathcal{A}_\infty^n$ and if there exists an index $i \in \{0, \ldots, n-1\}$ such that $q_i = \infty$, one can isolate this index i and trap A_{q_i} between two Fourier integral operators M_1, M_2, belonging to a finite family of FIO associated to F^{T_0}, so that we can write

$$M^{T_0} M_{\mathbf{q}} M^{T_0} = B_1 M_1 A_\infty M_2 B_2$$

where B_1, B_2 satisfy the L^2-bound :

$$||B_1|| \times ||B_2|| \leq C(||\alpha_h||_\infty)^{nn_0-1} = O(h^{-K})$$

for some integer K, and we have $M_1 A_\infty M_2 = O(h^\infty)$, with constants that can be chosen independent of $\mathbf{q}$. Hence, the same is true for $M^{T_0} M_{\mathbf{q}} M^{T_0}$. So, we can write, keeping in mind that $|\mathcal{A}|^n = O(h^{-K})$ for some $K > 0$:

$$
\begin{aligned}
M^{nn_0+2T_0} &= \sum_{\mathbf{q}\in\mathcal{A}_\infty^n} M^{T_0} M_{\mathbf{q}} M^{T_0} \\
&= \sum_{\mathbf{q}\in\mathcal{A}^n} M^{T_0} M_{\mathbf{q}} M^{T_0} + O(h^\infty) \\
&= M^{T_0}\left(\sum_{\mathbf{q}\in\mathcal{A}^n} M_{\mathbf{q}}\right) M^{T_0} + O(h^\infty)
\end{aligned}
$$

Let us note

$$(4.15) \qquad \mathfrak{M} = M^{n_0}(\mathrm{Id}-A_\infty) = M^{n_0}\sum_{q\in\mathcal{A}} A_q$$

We have shown the following lemma :

Lemma 4.3. There exists $T_0 \in \mathbb{N}$ such that

$$M^{2T_0+nn_0} = M^{T_0}\mathfrak{M}^n M^{T_0} + O(h^\infty)$$

Let us now look at what this equation implies on the trace of $M^{2T_0+nn_0}$. In the following computations, we use : If A is an Hilbert-Schmidt operator and B bounded,

  (i) $\mathrm{tr}(A^*A) = ||A||_{HS}^2$;
  (ii) $||AB||_{HS} \le ||B|| \times ||A||_{HS}$; $||BA||_{HS} \le ||B|| \times ||A||_{HS}$

$$
\begin{aligned}
\mathrm{tr}\left(\left(M_t^{2T_0+nn_0}\right)^* M_t^{2T_0+nn_0}\right) &= ||M_t^{2T_0+nn_0}||_{HS}^2 \\
&= \left|\left|M_t^{T_0}\mathfrak{M}_t^n M_t^{T_0}\right|\right|_{HS}^2 + O(h^\infty) \\
&\le ||M_t^{T_0}||^4 ||\mathfrak{M}_t^n||_{HS}^2 + O(h^\infty) \\
&\le ||M_t^{T_0}||^4 \,\mathrm{tr}\left((\mathfrak{M}_t^n)^* \mathfrak{M}_t^n\right) + O(h^\infty)
\end{aligned}
$$

Hence, is is enough to find the expected upper bound (4.2) for $\mathrm{tr}\left((\mathfrak{M}_t^n)^* \mathfrak{M}_t^n\right)$ to obtain the same kind of upper bounds for $\mathrm{tr}\left(\left(M_t(z)^N\right)^* M_t(z)^N\right)$.

Evolution in local adapted charts. We will be interested in the evolution of coherent states through the action of $\mathfrak{M}$. It will be more convenient to work in the charts κ_q in which the action of F is well adapted to the position-momentum coordinate (x,ξ). For this purpose, we start by writing,

$$\mathfrak{M}_t^n = e^{-tG}\mathfrak{M}^{n-1}\sum_{q\in\mathcal{A}} M^{n_0} A_q e^{tG}$$

Recall that $B'_q B_q = I + O(h^\infty)$ microlocally near $\mathrm{supp}(a_q)$, hence,

$$\mathfrak{M}_t^n = e^{-tG}\mathfrak{M}^{n-1}\sum_{q\in\mathcal{A}} M^{n_0} A_q B'_q B_q e^{tG} B'_q B_q + O(h^\infty)$$

Let's note

$$(4.16) \qquad \tilde{E}_t = B_q e^{tG} B'_q$$

We also fix $\widetilde{A}_q = \mathrm{Op}_h(\tilde{a}_q)$ such that $\mathrm{WF}_h(\widetilde{A}_q) \subset \mathcal{W}_q$ and $\tilde{a}_q = 1$ near $\mathrm{supp}(\chi_q)$. This gives :

$$
\begin{aligned}
\mathrm{tr}\left((\mathfrak{M}_t^n)^* \mathfrak{M}_t^n\right) &= \sum_{q,p \in \mathcal{A}} \mathrm{tr}\left(\left(e^{-tG}\mathfrak{M}^{n-1}M^{n_0}A_p B_p' \tilde{E}_t B_p \widetilde{A}_p\right)^* e^{-tG}\mathfrak{M}^{n-1}M^{n_0}A_q B_q' \tilde{E}_t B_q \widetilde{A}_q\right) + O(h^\infty) \\
&= \sum_{q,p \in \mathcal{A}} \mathrm{tr}\left(B_p^* \left(e^{-tG}\mathfrak{M}^{n-1}M^{n_0}A_p B_p' \tilde{E}_t\right)^* e^{-tG}\mathfrak{M}^{n-1}M^{n_0}A_q B_q' \tilde{E}_t B_q \widetilde{A}_q \widetilde{A}_p^*\right) + O(h^\infty) \\
&= \sum_{q \in \mathcal{A}} \mathrm{tr}\left(B_q^* \left(e^{-tG}\mathfrak{M}^{n-1}M^{n_0}A_q B_q' \tilde{E}_t\right)^* e^{-tG}\mathfrak{M}^{n-1}M^{n_0}A_q B_q' \tilde{E}_t B_q \widetilde{A}_q \widetilde{A}_q^*\right) + O(h^\infty) \\
&\leq CQ \sup_{q \in \mathcal{A}} \mathrm{tr}\left(\left(e^{-tG}\mathfrak{M}^{n-1}M^{n_0}A_q B_q' \tilde{E}_t\right)^* e^{-tG}\mathfrak{M}^{n-1}M^{n_0}A_q B_q' \tilde{E}_t\right) + O(h^\infty)
\end{aligned}
$$

where C is such that $C_0 \|B_q\| \times \|B_q \widetilde{A}_q \widetilde{A}_q^*\| \leq C$ for all $q \in \mathcal{A}$ (and $0 < h \leq 1$) and $Q = |\mathcal{A}|$. The passage from the second to the third line holds since $\widetilde{A}_q \widetilde{A}_p^* = O(h^\infty)$ when $q \neq p$, in virtue of the fact that $\mathcal{W}_p \cap \mathcal{W}_q \neq \emptyset$. This computations show that it is enough to control, uniformly in q, the trace

$$
\tag{4.17}
\mathrm{tr}\left(\left(e^{-tG}\mathfrak{M}^{n-1}M^{n_0}A_q B_q' \tilde{E}_t\right)^* e^{-tG}\mathfrak{M}^{n-1}M^{n_0}A_q B_q' \tilde{E}_t\right)
$$

since we now have :
$$
\tag{4.18}
\mathrm{tr}\left(\left(M_t^{2T_0+nn_0}\right)^* M_t^{2T_0+nn_0}\right) \leq CQ \sup_{q \in \mathcal{A}} \mathrm{tr}\left(\left(e^{-tG}\mathfrak{M}^{n-1}M^{n_0}A_q B_q' \tilde{E}_t\right)^* e^{-tG}\mathfrak{M}^{n-1}M^{n_0}A_q B_q' \tilde{E}_t\right) + O(h^\infty)
$$

From now on, we will note ρ, ζ, etc. points in U and $\hat{\rho}, \hat{\zeta}$, etc. their images in the local charts κ_q. The resolution of identity of Lemma 2.2, valid at the level of operators on $L^2(\mathbb{R})$, extends to the case of matrix operator acting on $L^2(\mathbb{R}_J)$, in the following sense :

$$
\mathrm{tr}(A) = \sum_{j=1}^J \frac{1}{2\pi h} \int_{T^*\mathbb{R}} <A_{jj}\varphi_{\hat{\rho}}, \varphi_{\hat{\rho}}> d\hat{\rho}
$$

Hence, if $K = e^{-tG}\mathfrak{M}^{n-1}M^{n_0}A_q B_q' \tilde{E}_t$, we have

$$
\begin{aligned}
\mathrm{tr}\left(K^*K\right) &= \sum_{j=1}^J \frac{1}{2\pi h} \int_{T^*\mathbb{R}} <(K^*K)_{jj}\varphi_{\hat{\rho}}, \varphi_{\hat{\rho}}> d\hat{\rho} \\
&= \sum_{1 \leq i,j \leq J} \frac{1}{2\pi h} \int_{T^*\mathbb{R}} <K_{ij}\varphi_{\hat{\rho}}, K_{ij}\varphi_{\hat{\rho}}> d\hat{\rho}
\end{aligned}
$$

Since $A_q B_q'$ is diagonal with only one non-zero diagonal entry in position j_q, $B_{ij} = 0$ except when $j = j_q$. We can write :
$$
\tag{4.19}
\mathrm{tr}\left(\left(e^{-tG}\mathfrak{M}^{n-1}MA_q B_q'\right)^* e^{-tG}\mathfrak{M}^{n-1}M^{n_0}A_q B_q' \tilde{E}_t\right) = \frac{1}{2\pi h} \int_{T^*\mathbb{R}} \left\|e^{-tG}\mathfrak{M}^{n-1}M^{n_0}A_q B_q' \tilde{E}_t \tilde{\varphi}_{\hat{\rho}}\right\|^2 d\hat{\rho}
$$

where $\tilde{\varphi}_{\hat{\rho}}$ is the column vector with only one non-zero entry equal to $\varphi_{\hat{\rho}}$ in position j_q.

4.2.5. *End of the proof.* The main ingredient for the proof of the improved fractal Weyl law, which is also the main novelty of this article, is a good control for

$$
\tag{4.20}
w(\hat{\rho}) := \left\|e^{-tG}\mathfrak{M}^{n-1}M^{n_0}A_q B_q' \tilde{E}_t \tilde{\varphi}_{\hat{\rho}}\right\|^2
$$

This weight w depends on the parameter t which governs the weight of the escape function. We omit to write this dependence explicitly : indeed, what is important is that once t is fixed sufficiently large, w will satisfy the expected decay in Proposition 4.2. To state this bound, let's introduce, for $\rho \in \mathcal{W}_{\mathbf{q}}^-$,

$$
\Pi_{\alpha,\mathbf{q}}(\rho) = \prod_{i=0}^{nn_0-1} \alpha\left(F^i(\rho)\right)
$$

where

(4.21) $$\alpha(\rho) = \exp\left(-\operatorname{Im} z t_{ret}(\rho)\right) \; ; \; \rho \in \bigcup_{q \in \mathcal{A}} \mathcal{W}_q$$

so that, for $\rho \in \bigcup_{q \in \mathcal{A}} \mathcal{W}_q$, we have $\alpha_h(\rho) = \alpha(\rho) + O\left(h^{1^-} S_{0+}\right)$. We also introduce the following neighborhood of $\mathcal{T}$

(4.22) $$\mathcal{T}_{\delta,\delta_1} = \left\{ \rho \, ; \, d(\rho, \mathcal{T}_-) \le h^\delta, d(\rho, \mathcal{T}_+) \le h^{\delta_1} \right\} \subset U \subset T^* \mathbb{R}_J$$

Proposition 4.2. For any $L > 0$, there exists $t = t(\varepsilon, L)$ such that the following holds. Let $\hat{\rho} \in \mathbb{R}^2$. If $\hat{\rho} \notin \kappa_q(\mathcal{W}_q)$, then $w(\hat{\rho}) = O\left(\left(\frac{h}{\langle \hat{\rho} \rangle}\right)^\infty\right)$ with uniform constants. Otherwise, assume that $\hat{\rho} = \kappa_q(\rho) \in \kappa_q(\mathcal{W}_q)$. We have

(1) If, for all $\mathbf{q} \in \mathcal{A}^{n+1}$, $\rho \notin \mathcal{W}_{\mathbf{q}}^-$, then $w(\hat{\rho}) = O(h^\infty)$ with uniform constants.
(2) Otherwise, there exists a unique $\mathbf{q} \in \mathcal{A}^{n+1}$ such that $\rho \in \mathcal{W}_{\mathbf{q}}^-$. In that case, for some uniform constants $C > 0$ and $h_0 > 0$, one has, for $0 < h \le h_0$,
    (i) If $\rho \notin \mathcal{T}_{\delta,\delta_1}$, $w(\hat{\rho}) \le h^L$;
    (ii) If $\rho \in \mathcal{T}_{\delta,\delta_1}$,

$$w(\hat{\rho}) \le C \left(\Pi_{\alpha,\mathbf{q}}(\rho)\right)^2 \left(J_{\mathbf{q}}^u\right)^{d_H - 1 + \varepsilon} h^{(\delta_0 - \delta)(d_H + \varepsilon) + \delta - 1/2}.$$

This key proposition is proved in Section 5. We will also require the following proposition :

Proposition 4.3. Let $\mathbf{q} = q_0 \ldots q_n \in \mathcal{A}^{n+1}$ with $n = n(h)$ and assume $\mathcal{W}_{\mathbf{q}}^- \ne \emptyset$. Then, for some uniform constant $C > 0$, and for h small enough, the following estimate holds :

$$\operatorname{Vol}\left(\mathcal{T}_{\delta,\delta_1} \cap \mathcal{W}_{\mathbf{q}}^-\right) \le C h^{2\delta_1} h^{-(\delta+\delta_1)(d_H+\varepsilon)} \left(J_{\mathbf{q}}^u\right)^{-(d_H+\varepsilon)}.$$

Proof. We assume that $\mathcal{W}_{\mathbf{q}}^- \ne \emptyset$. According to Lemma 4.2, there exists $\rho_- \in \mathcal{T}$ such that for all $\rho \in \mathcal{W}_{\mathbf{q}}^-$,

(4.23) $$d(\rho, W_s(\rho_-)) \le C \left(J_{\mathbf{q}}^u\right)^{-1} \varepsilon_0$$

We also consider $\rho_+ \in \mathcal{T}$ such that $d(\rho, \rho_+) \le 2h^{\delta_1}$. In particular, $d(\rho_-, \rho_+) \ll \varepsilon_0$ and we may consider a point $\rho_O \in W_s(\rho_-) \cap W_u(\rho_+)$ and we decide to work in an adapted chart κ centered at ρ_O. We want to estimate the volume of $\kappa\left(\mathcal{T}_{\delta,\delta_1} \cap \mathcal{W}_{\mathbf{q}}^-\right)$. We assume that $\mathcal{W}_q$ is included in the domain of this chart (and so is $\mathcal{W}_{\mathbf{q}}^-$) and we choose this chart such that the image of $W_s(\rho_-) = W_s(\rho_O)$ is given by $\{(0, \xi), \xi \in V\}$: this is possible in virtue of Lemma 3.4 (by considering F^{-1} instead of F to change the unstable manifold into the stable one) [6]. In virtue of (4.23), we have for some uniform constant $C' > 0$,

$$(x, \xi) \in \kappa_q(\mathcal{W}_{\mathbf{q}}^-) \implies |x| \le C' \left(J_{\mathbf{q}}^u\right)^{-1} \varepsilon_0$$

Let's consider $\Xi(\mathcal{T}) = \{\xi \in V, \kappa_q^{-1}(0, \xi) \in \mathcal{T}\}$ and let's cover it by N_s intervals of size $2h^{\delta_1}$ centered at point $\xi_1, \ldots, \xi_{N_s} \in \Xi(\mathcal{T})$. Since $\overline{\dim}\mathcal{T} \cap W_s(\rho_-) = d_H$ and in virtue of Proposition 3.3, we may choose N_s such that

(4.24) $$N_s \le C h^{-\delta_1(d_H+\varepsilon)}$$

for some uniform constant $C > 0$. For $1 \le i \le N_s$, let's note $\sigma_i = \kappa^{-1}(0, \xi_i)$. The local unstable manifold passing through σ_i can be written, in the chart κ, as a graph $\{(x, g_i(x)), x \in U_i\}$. We note

$$X_i(\mathcal{T}) = \{x \in U_i, \kappa^{-1}(x, g_i(x)) \in \mathcal{T}, |x| \le 2C' \left(J_{\mathbf{q}}^u\right)^{-1} \varepsilon_0\}$$

and we cover $X_i(\mathcal{T})$ by $N_{i,u}$ intervals of size $2h^\delta$, centered at points $x_{i,j}$, $1 \le j \le N_{i,u}$. Lemma 3.6 shows that we can take $N_{i,u}$ such that for all $1 \le i \le N_s$,

(4.25) $$N_{i,u} \le C \left(h^\delta J_{\mathbf{q}}^u\right)^{-d_H - \varepsilon}$$

for some uniform constant C.

For $1 \le i \le N_s$ and $1 \le j \le N_{i,u}$, let's also note $\xi_{i,j} = g_i(x_{i,j})$. We claim that there exists a uniform constant $C > 0$ such that

---

[6]In fact, without the assumption on κ being symplectic, we may assume that both $W_s(\rho_O)$ and $W_u(\rho_O)$ are rectified.
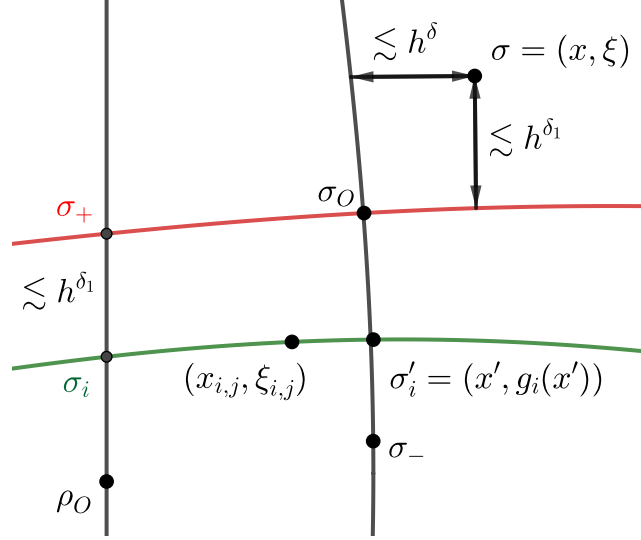
FIGURE 6. The different points introduced in the proof of the claim 4.26. To alleviate the figure, we use the same notations for a point σ and its image trough κ.

$$(4.26) \qquad \kappa\left(\mathcal{T}_{\delta,\delta_1} \cap \mathcal{W}_{\mathbf{q}}^-\right) \subset \bigcup_{i=1}^{N_s} \bigcup_{j=1}^{N_{i,u}} [x_{i,j} - Ch^{\delta_1}, x_{i,j} + Ch^{\delta_1}] \times [\xi_{i,j} - Ch^{\delta_1}, \xi_{i,j} + Ch^{\delta_1}]$$

This claim obviously implies the proposition, by combining it with the bounds on N_s (4.24) and the $N_{i,u}$ (4.25). We now turn to the proof of this claim.

Let's consider $(x,\xi) = \kappa(\sigma) \in \kappa\left(\mathcal{T}_{\delta,\delta_1} \cap \mathcal{W}_{\mathbf{q}}^-\right)$. We introduce different points (and encourage the reader to use Figure 6 to follow the different steps) :

- Since $d(\sigma, \mathcal{T}_+) \leq h^{\delta_1}$, there exists $\sigma_+ \in \mathcal{T}$ such that $d(\sigma, W_u(\sigma_+)) \leq h^{\delta_1}$. We can replace σ_+ by the unique point in the intersection $W_u(\sigma_+) \cap W_s(\rho_O)$ and we can note $\kappa(\sigma_+) = (0, \xi_+)$.
- Since $\xi_+ \in \Xi(\mathcal{T})$, there exists $i \in \{1, \ldots, N_s\}$ such that $|\xi_i - \xi_+| \leq h^{\delta_1}$. In particular, $d(\sigma_i, \sigma_+) \leq Ch^{\delta_1}$.
- Since $d(\sigma, \mathcal{T}_-) \leq h^{\delta}$, there exists $\sigma_- \in \mathcal{T}$ such that $d(\sigma, W_s(\sigma_-)) \leq h^{\delta}$. We note σ_O the unique point in $W_s(\sigma_-) \cap W_u(\sigma_+)$.
- We also note σ_i' the unique point in $W_s(\sigma_-) \cap W_u(\sigma_i)$. Due to the Lipschitzness of the holonomy maps (with uniform Lipschitz constant),

$$d(\sigma_O, \sigma_i') \leq Cd(\sigma_+, \sigma_i) \leq Ch^{\delta_1}$$

- Due to the local product structure near σ_O, we have $d(\sigma, \sigma_O)^2 \sim d(\sigma, W_s(\sigma_O))^2 + d(\sigma, W_u(\sigma_O))^2 \sim h^{2\delta_1} + h^{2\delta}$. It gives $d(\sigma, \sigma_O) \leq Ch^{\delta_1}$ and hence, $d(\sigma_i', \sigma) \leq Ch^{\delta_1}$.
- Let's note $\sigma_i' = (x', g_i(x'))$. Since $x' \in X_i(\mathcal{T})$, there exists $j \in \{1, \ldots, N_{i,u}\}$ such that $|x_{i,j} - x'| \leq h^{\delta}$. Then we have

$$d(\sigma_i', \kappa((x_{i,j}, \xi_{i,j}))) \leq C|x_i' - x_{i,j}| \leq Ch^{\delta}$$

- We conclude that $d(\sigma, \kappa((x_{i,j}, \xi_{i,j}))) \leq Ch^{\delta_1}$, which gives $|x - x_{i,j}| \leq Ch^{\delta_1}, |\xi - \xi_{i,j}| \leq Ch^{\delta_1}$.

$\square$

We can now conclude the proof of the main trace estimate. Set $N = 2T_0 + n(h)$. We want to plug the estimates of Proposition 4.2 into (4.18) and (4.19). For $q \in \mathcal{A}$, let's note

$$O_q = \kappa_q \left(\mathcal{T}_{\delta,\delta_1} \cap \bigcup_{\mathbf{q} \in \mathcal{A}^{n+1}} \mathcal{W}_{\mathbf{q}}^-\right)$$

and let's write

$$
\operatorname{tr}\!\left((M_t^N)^* M_t^N\right) \le C \sup_{q \in \mathcal{A}} \frac{1}{2\pi h} \int_{\mathbb{R}^2} w(\hat{\rho}) d\hat{\rho}
$$

$$
\le \frac{C}{h} \sup_{q \in \mathcal{A}} \left(\int_{O_q} w(\hat{\rho}) d\hat{\rho} + \int_{\mathbb{R}^2 \setminus O_q} w(\hat{\rho}) d\hat{\rho} \right)
$$

$$
\le \sup_{q \in \mathcal{A}} \left(C h^{(\delta - \delta_0)(d_H + \varepsilon) + \delta - 3/2} \sum_{\mathbf{q} \in \mathcal{A}^{n+1}} \int_{\kappa_q \left(\mathcal{T}_{\delta,\delta_1} \cap \mathcal{W}_{\mathbf{q}}^- \right)} \left(\Pi_{\alpha, \mathbf{q}}(\rho) \right)^2 \left(J_{\mathbf{q}}^u \right)^{d_H - 1 + \varepsilon} d\hat{\rho} \right) + O(h^L)
$$

The last inequality holds since the integral outside O_q can be made $O(h^L)$ by choosing correctly t, to make L arbitrarily large. Indeed, using Proposition 4.2, the part outside $\kappa_q(\mathcal{W}_q)$ is $O(h^\infty)$ and the integral on $\kappa_q(\mathcal{W}_q) \setminus O_q$ is $O(h^L)$.

Let $\mathbf{q} = q_0 \ldots q_{n-1} \in \mathcal{A}^n$. We write $\mathcal{W}_{q_i} = \bigcap_{j=0}^{n_0 - 1} F^{-1}(\mathfrak{A}_{i,j})$ with $\mathfrak{A}_{i,j} \in \mathcal{Q}$ and for $\rho \in \mathcal{W}_{\mathbf{q}}^-$. Let's note $\varphi = -2\operatorname{Im} z t_{ret} - \varphi_u$ and recall that $\alpha = \exp(-\operatorname{Im} z t_{ret})$. We have uniformly with respect to $\mathbf{q} \in \mathcal{A}^n$ and $\rho \in \mathcal{W}_{\mathbf{q}}^-$,

$$
\left(\Pi_{\alpha, \mathbf{q}}(\rho) \right)^2 \left(J_{\mathbf{q}}^u \right)^{-1} \le C \left(\Pi_{\alpha, \mathbf{q}}(\rho) \right)^2 \left(J_{nn_0}^u(\rho) \right)^{-1}
$$

$$
\le C \prod_{i=0}^{n-1} \left(\exp \left(\sum_{j=0}^{n_0 - 1} \varphi(F^{i n_0 + j}(\rho)) \right) \right)
$$

$$
\le C \prod_{i=0}^{n-1} \left(\sup_{\rho_i \in \mathcal{W}_{q_i}} \exp \left(\sum_{j=0}^{n_0 - 1} \varphi(F^j(\rho_i)) \right) \right)
$$

$$
\le C \prod_{i=0}^{n-1} \left(C_0 \sup_{\rho_i \in \mathcal{W}_{q_i} \cap \mathcal{T}} \exp \left(\sum_{j=0}^{n_0 - 1} \varphi(F^j(\rho_i)) \right) \right)
$$

The last inequality holds for some $C_0 > 0$ independent of n_0 (and z), since φ is Hölder continuous (with constant uniform with respect to z). Indeed, if ε_0 is small enough, in virtue of Lemma 3.2, there exists $\theta \in [0, 1)$ and $C > 0$ such that if $\rho_1 \in \mathcal{W}_{q_i}$ and if $\rho_2 \in \mathcal{W}_{q_i} \cap \mathcal{T}$ then $d(F^j(\rho_1), F^j(\rho_2)) \le C\theta^{n-j}$. As a consequence, $|\varphi(F^j(\rho_1)) - \varphi(F^j(\rho_2))| \le C\theta_1^{n-j}$ (with $\theta_1 = \theta^\beta$ for some $0 < \beta \le 1$). Since $\sum_{j=0}^{n-1} \theta_1^{n-j} \le \sum_{j=0}^{\infty} \theta_1^j < +\infty$, we find that

$$
\exp \left(\sum_{j=0}^{n_0 - 1} \varphi(F^j(\rho_1)) \right) \sim \exp \left(\sum_{j=0}^{n_0 - 1} \varphi(F^j(\rho_2)) \right)
$$

For $q \in \mathcal{A}$, let's call

$$
p_q = \sup_{\rho \in \mathcal{W}_q \cap \mathcal{T}} \exp \left(\sum_{j=0}^{n_0 - 1} \varphi(F^j(\rho)) \right)
$$

and recall that, due to our special choice of the partition $(\mathcal{W}_q)_q$ (see (4.5)), we have $\sum_{q \in \mathcal{A}} p_q \le e^{n_0(P(\varphi) + \varepsilon)}$. We may assume that n_0 is big enough so that $C_0 \le e^{n_0 \varepsilon}$, and hence, $\sum_{q \in \mathcal{A}} C_0 p_q \le$

$e^{n_0(P(\varphi)+2\varepsilon)}$. As a consequence, we find that

$$\text{tr}\big((M_t^N)^* M_t^N\big) \le Ch^{(\delta-\delta_0)(d_H+\varepsilon)+\delta-3/2} \sum_{\mathbf{q}\in\mathcal{A}^n} \text{Vol}\Big(\mathcal{T}_{\delta,\delta_1} \cap \mathcal{W}_{\mathbf{q}}^-\Big) \big(J_{\mathbf{q}}^u\big)^{d_H+\varepsilon} \prod_{i=0}^{n-1} C_0 p_{q_i}$$

$$\le Ch^{(\delta-\delta_0)(d_H+\varepsilon)+\delta-3/2} \sum_{\mathbf{q}\in\mathcal{A}^n} h^{2\delta_1} h^{-(\delta+\delta_1)(d_H+\varepsilon)} \big(J_{\mathbf{q}}^u\big)^{-(d_H+\varepsilon)} \big(J_{\mathbf{q}}^u\big)^{d_H+\varepsilon} \prod_{i=0}^{n-1} C_0 p_{q_i}$$

$$\le Ch^{-\nu_\varepsilon} \sum_{\mathbf{q}\in\mathcal{A}^n} \prod_{i=0}^{n-1} C_0 p_{q_i} = Ch^{-\nu_\varepsilon} \left(\sum_{q\in\mathcal{A}} C_0 p_q\right)^n \le Ch^{-\nu_\varepsilon} e^{nn_0(P(\varphi)+2\varepsilon)}$$

where

$$\nu_\varepsilon = d_H + (\delta_0 - \delta)(d_H + \varepsilon) + (1/2 - \delta) + (\delta_1 + \delta)\varepsilon + (2\delta_1 - 1) - d_H(\delta + \delta_1 - 1) = d_H + O(\varepsilon)$$

(see the definitions of these exponents in (4.13)). Recalling that $nn_0 \le \vartheta_\varepsilon \log 1/h$, we find that

$$\text{tr}\big((M_t^N)^* M_t^N\big) \le Ch^{-\nu_\varepsilon} h^{-\vartheta_\varepsilon(P(-2\,\text{Im}\,zt_{ret}-\varphi_u)+2\varepsilon)}$$

We can finally insert the term $2\vartheta_\varepsilon\varepsilon$ into the ν_ϵ and we find that

$$\text{tr}\big((M_t^N)^* M_t^N\big) \le Ch^{-\nu_\varepsilon} h^{-\vartheta_\varepsilon P(-2\,\text{Im}\,zt_{ret}-\varphi_u)}$$

This concludes the proof of Proposition 4.1.

5. Proof of Proposition 4.2.

In this section we fix some $q = q_0$ and we aim at proving Proposition 4.2. If $\hat\rho \notin \kappa_q(\mathcal{W}_q)$, as we will explain, the estimate in $O\left(\left(\frac{h}{\langle\hat\rho\rangle}\right)^\infty\right)$ is nothing but a consequence of the fact that $\text{WF}_h(B_q A_q B_q') \Subset \kappa_q(\mathcal{W}_q)$ and one can for instance apply Lemma 15 in [CR12], Chapter 2, Section 3.

The main part of the Proposition 4.2 concerns points $\hat\rho = \kappa_q(\rho) \in \kappa_q(\mathcal{W}_q)$. To prove this proposition, we study separately the actions of the different operators in $e^{-tG}\mathfrak{M}^{n-1} M^{n_0} A_q B_q' \tilde{E}_t$.

- First, we analyze the action of $\tilde{E}_t$. We show that it is essentially given by the multiplication by $e^{tg(\rho)}$.
- We go on studying the propagation of Gaussian coherent state through the iterated actions of $\mathfrak{M}$. The hyperbolicity of the trajectories leads to a deformation of the Gaussian state. The results we obtain are related to the results of [CR12] with Hamiltonian flow. In particular, this is where we use the fact that $\vartheta_\varepsilon < 1/6$. The approximation we use fails for longer logarithmic times.
- Finally, we analyze the action of e^{-tG} on the evolved coherent states. In a way, we treat this evolved state as a Lagrangian state with rapidly oscillating amplitude, of the form $a(x)e^{i\frac{\phi(x)}{h}}$. The scale of oscillation of a is larger than h^δ, scale on which g oscillates. We show that, at leading order, the action of e^{-tG} is well approximated by the multiplication by $e^{-tg(x,\phi'(x))}$.

Notations. In the following, we will be lead to consider states $u \in L^2(\mathbb{R}_J)$ such that all the components of u are $O(h^\infty)$, except one equal to some $v \in L^2(\mathbb{R})$. By abuse, we will note v instead of u as soon as the component where u is non zero is explicit in the context. For instance, we can simply note $\varphi_{\hat\rho}$ instead of $\tilde\varphi_{\hat\rho}$ as soon as we specify that $\hat\rho \in \kappa_q(\mathcal{W}_q)$. Another example : for any $u \in L^2(\mathbb{R}_J)$ and $q \in \mathcal{A}$, $B_q A_q u$ has only one non zero component at j_q and we can use this component to write u. This will be widely used in the sequel since most of the time we will consider this type of elements.

5.1. Preparatory work. Due to standard properties of Fourier integral operators, we can consider a pseudodifferential operator Ξ_q such that $\text{WF}_h(\Xi_q) \subset W_q$, $T^*\mathbb{R} \setminus \text{WF}_h(1 - \Xi_q) \subset \kappa_q(\mathcal{W}_q)$ and $\Xi_q B_q' B_q = \Xi_q + O(h^\infty)$ (recall that $W_q = \kappa_q(B(\rho_q, 2\varepsilon_0))$ and that $\mathcal{W}_q \Subset B(\rho_q, 2\varepsilon_0)$ by construction). With these properties, we have in particular $A_q B_q' \tilde{E}_t = A_q B_q' \tilde{E}_t \Xi_q + O(h^\infty)$. This

allows us to change harmlessly $\tilde{E}_t$ into $E_t := \tilde{E}_t \Xi_q$ in all the computations below. We first write

$$e^{-tG}\mathfrak{M}^{n-1}M^{n_0}A_q B'_q E_t = \sum_{(q_1,\ldots,q_{n-1})\in\mathcal{A}^{n-1}} e^{-tG}M^{n_0}A_{q_{n-1}}\ldots M^{n_0}A_{q_1}M^{n_0}A_q B'_q E_t$$

$$= \sum_{(q_1,\ldots,q_n)\in\mathcal{A}^n} A_{q_n}e^{-tG}M^{n_0}A_{q_{n-1}}\ldots M^{n_0}A_{q_1}M^{n_0}A_q B'_q E_t + O(h^\infty)$$

$$= \sum_{(q_1,\ldots,q_n)\in\mathcal{A}^n} A_{q_n}e^{-tG}B'_{q_n}M_{q_n,q_{n-1}}\ldots M_{q_2,q_1}M_{q_1,q}E_t + O(h^\infty)$$

where

(5.1) $$M_{p,q} = B_p M^{n_0} A_q B'_q$$

We say that a pair (p,q) is admissible if $F^{n_0}(\mathcal{W}_q) \cap \mathcal{W}_p \neq \emptyset$. By standard properties of Fourier integral operators, if (p,q) is not admissible, $M_{p,q} = O(h^\infty)$. We say that a word $(q_1,\ldots,q_n) \in \mathcal{A}^n$ is admissible if all the pairs (q_i, q_{i-1}) are admissible (with $q_0 = q$). Hence, since $n = O\left(\log\frac{1}{h}\right)$, we can restrict the indices in the above sum to the admissible words.

Suppose that (p,q) is an admissible pair. By composition of Fourier integral operators, $M_{p,q}$ is a Fourier integral operator associated with the symplectic map $F_{p,q} := \kappa_p \circ F^{n_0} \circ \kappa_q^{-1}$. Since $\text{diam}(\mathcal{W}_q) \leq \varepsilon_0$, by taking ε_0 sufficiently small, we can assume that $F^{n_0}(\mathcal{W}_q)$ is included in the domain of κ_p. Indeed, there exists $\tilde{\rho}_q \in \mathcal{W}_q$ such that $F^{n_0}(\tilde{\rho}_q) \in \mathcal{W}_p$ and hence if $\rho \in \mathcal{W}_q$,

$$d(F^{n_0}(\rho), \rho_p) \leq d(F^{n_0}(\rho), F^{n_0}(\tilde{\rho}_q)) + d(F^{n_0}(\tilde{\rho}_q), \rho_p) \leq C\varepsilon_0$$

We note (y,η) the variables in the charts and $(\partial_y, \partial_\eta)$ the canonical basis of $\mathbb{R}^2$ and we have

- $F_{p,q}(0) = \kappa_p \circ F^{n_0}(\rho_q) = O(\varepsilon_0)$;
- $d_0 F_{p,q} = d_{F(\rho_q)}\kappa_p \circ d_{\rho_q}F^{n_0} \circ \left[d_{\rho_q}\kappa_q\right]^{-1}$;
- $d_{\rho_q}F^{n_0}(E_u(\rho_q)) = E_u(F^{n_0}(\rho_q))$ and $\rho \mapsto E_u(\rho)$ is Lipschitz. Hence, if we note $e_u(\rho_q) = \left(d_{\rho_q}\kappa_q\right)^{-1}(\partial_y) \in E_u(\rho_q)$, due to the definitions of the adapted charts in Lemma 3.4, there exists $\lambda_{p,q} \in \mathbb{R}^*$ such that

$$d_{\rho_q}(\kappa_p \circ F^{n_0})(e_u(\rho_q)) = \lambda_{p,q}\partial_y + O(\varepsilon_0)$$

- Similarly, $d_0 F_{p,q}(\partial_\eta) = \mu_{p,q}\partial_\eta + O(\varepsilon_0)$ for some $\mu_{p,q} \in \mathbb{R}^*$

Eventually, we use the fact that $F_{p,q} - F_{p,q}(0) - d_0 F_{p,q} = O(\varepsilon_0)_{C^1(W_q)}$ and we get that

(5.2) $$F_{p,q}(y,\eta) = (\lambda_{p,q}y + y_r(y,\eta), \mu_{p,q}\eta + \eta_r(y,\eta)), (y,\eta) \in W_q$$

where $y_r(y,\eta)$ and $\eta_r(y,\eta)$ are $O(\varepsilon_0)_{C^1}$. In particular, if ε_0 is small enough, $(x,\xi,y,\eta) \in \text{Gr}(F_{p,q}) \mapsto (x,\eta)$ is a local diffeomorphism near $(0,0,0,0)$. Then, there exists a phase function $\psi_{p,q}$ which generates $F_{p,q}$ in a neighborhood Ω of $(0,0,0,0)$. Assuming ε_0 small enough, we can assume that $F_{p,q}(W_q) \times W_q \subset \Omega$.

As a consequence (see for instance [Ale08], [Zwo12] Chapter 10), the Fourier integral operator $M_{p,q}$ can be written under the form (2.2), up to $O(h^\infty)$, that is,

(5.3) $$M_{p,q}u(x) = \frac{1}{2\pi h}\int_{\mathbb{R}^2} e^{\frac{i}{h}(\psi_{p,q}(x,\eta)-y\eta)}\alpha_{p,q}(x,\eta)u(y)dyd\eta$$

where $\alpha_{p,q}$ is a symbol in $S_{0+}(\mathbb{R}^2)$. It has an asymptotic expansion

(5.4) $$\alpha_{p,q} \sim \sum_{j\geq 0} h^j \alpha_{p,q}^{(j)}$$

where $\alpha_{p,q}^{(j)} \in h^{0^-}S_{0+}$, for all $j \geq 1$ (that is, $\alpha_{p,q}^{(j)} \in \bigcap_{\eta>0} h^{-\eta}S_{0+}$) and we have
(5.5)

$$|\alpha_{p,q}^{(0)}(x,\eta)| = |\partial_{x,\eta}^2\psi(x,\eta)|^{1/2}\chi_q(\rho) \times \left(\prod_{i=0}^{n_0-1}\alpha\circ F^i(\rho)\right) \quad ; \quad \rho = \kappa_q^{-1}(y,\eta) \; ; \; (x,\xi) = F_{p,q}(y,\eta)$$

Here, we use the fact that in W_q, $\alpha_h = \alpha + O\left(h^{1^-}S_{0+}\right)$ to put the $O\left(h^{1^-}S_{0+}\right)$ in $\alpha_{p,q}^{(1)}$. Moreover, we have the following support properties : for $j \in \mathbb{N}$,

(5.6) $$(x,\eta) \in \text{supp}(\alpha_{p,q}^{(j)}) \implies (y,\eta) \in \kappa_q\left(\text{supp}(\chi_q)\right) \quad ; \quad (x,\xi) = F_{p,q}(y,\eta)$$

We now pick an admissible word $(q_1, \ldots, q_n)$ and for $\hat{\rho} \in \mathbb{R}^2$, we aim at studying

$$||A_{q_n} e^{-tG} B'_{q_n} M_{q_n, q_{n-1}} \ldots M_{q_2, q_1} M_{q_1, q} E_t \tilde{\varphi}_{\hat{\rho}}||$$

We have $M_{q_1, q} E_t = B_{q_1} M^{n_0} A_q B'_q B_q e^{tG} B'_q$. Since $\mathrm{WF}_h \left(A_q B'_q B_q e^{tG} \right) \subset \mathrm{supp}(\chi_q)$ and B'_q is a Fourier integral operator associated with κ_q^{-1}, we can find $\tilde{\chi}_q$ such that $\mathrm{supp}(\tilde{\chi}_q) \subset \kappa_q(\mathcal{W}_q) \subset W_q$ and

$$A B'_q B_q e^{tG} B'_q = A B'_q B_q e^{tG} B'_q \mathrm{Op}_h(\tilde{\chi}_q) + O(h^\infty)$$

To prove the estimate in $O\left(\left(\frac{h}{\langle \hat{\rho} \rangle} \right)^\infty \right)$, we invoke Lemma 15 in [CR12], Chapter 2, Section 3, which allows us to say that, if $\hat{\rho} \notin \kappa_q(\mathcal{W}_q)$,

$$\mathrm{Op}_h(\tilde{\chi}_q) \varphi_{\hat{\rho}} = O\left(\left(\frac{h}{\langle \hat{\rho} \rangle} \right)^\infty \right).$$

Since both $M_{q_n, q_{n-1}} \ldots M_{q_1, q}$ and the number of terms in the sum are $O(h^{-K})$ for some $K > 0$, we deduce the first part of Proposition 4.2 :

Lemma 5.1. Uniformly for $\hat{\rho} \notin \kappa_q(\mathcal{W}_q)$, $w(\hat{\rho}) = O\left(\left(\frac{h}{\langle \hat{\rho} \rangle} \right)^\infty \right)$.

We now focus on $\hat{\rho} \in \kappa_q(\mathcal{W}_q) \Subset W_q$, for which $F_{p,q}(\hat{\rho})$ is well defined. We finish this preparatory subsection with an important computation. First note that the neighborhood $\bigcup_{q \in \mathcal{A}} \mathcal{W}_q$ has been fixed by dynamical considerations. We may assume that the cut-off function $\hat{\chi}$ used in (3.20) for the construction of g, is chosen such that $\mathrm{supp} \, \hat{\chi} \Subset \bigcup_{q \in \mathcal{A}} \mathcal{W}_q$. As a consequence, we can apply Proposition 2.5 and we have :

Lemma 5.2. For all $q \in \mathcal{A}$, there exists $(g_{j,q})_{j \geq 1} \in S_\delta$ such that for all $N \in \mathbb{N}$, the following holds:

$$B_q G B'_q = \mathrm{Op}_h \left(g \circ \kappa_q^{-1} \right) + \sum_{j=1}^{N-1} h^{j(1-2\delta)} \mathrm{Op}_h(g_{j,q}) + R_N$$

where $||R_N||_{L^2 \to L^2} \leq C_{2N+M,g} h^{N(1-2\delta)}$, for some constants $C_{2N+M,g}$ depending on semi-norms of g in S_δ up to order $2N + M$.

Remark. Even if $g \notin S_\delta$, it still satisfies $|\partial^\alpha g| \leq C_{|\alpha|,g} h^{-\delta|\alpha|}$ as soon as $\alpha \neq 0$. This allows us to fairly define these semi-norms.

Proof. Let's note that $g \in \log \frac{1}{h} S_\delta$ so that we can apply Proposition 2.5 to $\left(\log \frac{1}{h} \right)^{-1} g$. We can find differential operators $L_{j,q}$ such that

$$B_q G B'_q = \mathrm{Op}_h(g \circ \kappa_q^{-1}) + \sum_{j=1}^{N-1} h^j \mathrm{Op}_h \left((L_{j,q} g) \circ \kappa_q^{-1} \right) + O_{L^2 \to L^2} \left(h^N ||g||_{C^{2N+M}} \right)$$

In fact, due to the properties of $\hat{g}$, $\partial^\alpha g \in h^{-\delta|\alpha|} S_\delta$ as soon as $\alpha \neq 0$ and the terms $g_{j,q} := h^{2\delta j}(L_{j,q} g) \circ \kappa_q^{-1} \in S_\delta$ for $j \geq 1$. Moreover, the O is in fact an $O(C_{2N+M,g} h^{N(1-2\delta)})$, where $C_{2N+M,g}$ depends on semi-norms of g in S_δ up to order $2N + M$. □

5.2. **Action of E_t.** We begin with the action of E_t. Recall the definition of $E_t = B_q e^{tG} B'_q \Xi_q$ in (4.16).

Lemma 5.3. For any $N \in \mathbb{N}$ and $\hat{\rho} = \kappa_q(\rho) \in \kappa_q(\mathcal{W}_q)$, there exists $\psi_0, \ldots, \psi_{2N-1}$ and r_N such that

$$E_t \varphi_{\hat{\rho}} = \sum_{j=0}^{2N-1} h^{j(1/2-\delta)} \psi_j + r_N$$

satisfying :

- $\psi_0 = e^{tg(\rho)} \varphi_{\hat{\rho}}$;
- For $1 \leq j \leq 2N - 1$, ψ_j is of the form

$$\psi_j = e^{tg(\rho)} T(\hat{\rho}) \Lambda_h \left(P_{t,h}^{(j)} \Psi_0 \right)$$

where $P_{t,h}^{(j)}$ is a polynomial of degree at most $2j$, with coefficients depending on t, g (and hence, h) and $\hat{\rho}$. It satisfies, $N_\infty(P_{t,h}^{(j)}) \leq C_{j,t}$ with h-independent constants, depending on derivatives of g.

- $||r_N||_{L^2} \leq C_N(1+|t|)^{2N+1}h^{N(1-2\delta)}h^{-K_0 t}$ for some $K_0 > 0$ depending on g.

Proof. Let's fix $\hat{\rho} = \kappa_q(\rho) \in \kappa_q(\mathcal{W}_q)$, $N \in \mathbb{N}$ and set $\phi(t) = E_t \varphi_{\hat{\rho}}$. ϕ solves the equation

$$\phi'(t) = B_q G e^{tG} B_q' \Xi_q \varphi_{\hat{\rho}}$$

Since $B_q B_q' = I$ microlocally near $\mathrm{WF}_h(\Xi_q)$, we have $e^{tG} B_q' \Xi_q = B_q' B_q e^{tG} B_q' \Xi_q + O(h^\infty)$. Hence, up to $O(h^\infty)$, $\psi(t)$ solves $\phi'(t) = \tilde{G}\phi(t)$ with $\tilde{G} = B_q G B_q'$. It is enough to find an expansion for the solution of this equation. By Lemma 5.2, there exists C_N (depending on g) such that, with $G_j = \mathrm{Op}_h(g_{j,q})$ and $G_0 = \mathrm{Op}_h(g \circ \kappa_q^{-1})$,

$$\left\|\tilde{G} - \sum_{j=0}^{N-1} h^{j(1-2\delta)} G_j\right\| \leq C_N h^{N(1-2\delta)}$$

Set $\psi(t) = T(\hat{\rho})^* \phi(t)$. It solves : $\psi'(t) = T(\hat{\rho})^* \tilde{G} T(\hat{\rho})\psi(t)$. We also set $u(t) = \Lambda_h^* \psi(t)$, which solves $u'(t) = Au(t)$ where $A = \Lambda_h^* T(\hat{\rho})^* \tilde{G} T(\hat{\rho}) \Lambda_h$. Let's also note $\tilde{A}_j = \Lambda_h^* T(\hat{\rho})^* G_j T(\hat{\rho}) \Lambda_h = \mathrm{Op}_1(a_j)$ where $a_j(\hat{\zeta}) = g_{j,q}(\hat{\rho} + h^{1/2}\hat{\zeta})$. We wish to apply the formalism of Appendix A.2 with $H = L^2(\mathbb{R})$, the operator $A : H \to H$, $\mathcal{C} = \{P\Psi_0, P \in \mathbb{C}[X]\}$ with initial state $u(0) = \Psi_0$. The parameter h in Appendix A.2 is replaced by $\tilde{h} = h^{1/2-\delta}$. If $P \in \mathbb{C}[X]$, we approximate the action of A by

$$A(P\Psi_0) = \sum_{j=0}^{N-1} \tilde{h}^{2j}\tilde{A}_j(P\Psi_0) + O_N\left(\tilde{h}^{2N}||P\Psi_0||\right)$$

$$= \sum_{j=0}^{N-1} \tilde{h}^{2j}\left(\sum_{k=0}^{2N-1-2j} \tilde{h}^k A_{j,k}(P\Psi_0) + R_{N,j}(P\Psi_0)\right) + O_N\left(\tilde{h}^{2N}||P\Psi_0||\right)$$

where, according to Lemma 2.4,

$$A_{j,k} = \mathrm{Op}_1\left(\sum_{\alpha+\beta=k} h^{\delta k}\frac{\partial_x^\alpha \partial_\xi^\beta h_j}{\alpha!\beta!}(0)x^\alpha\xi^\beta\right) \quad ; \quad R_{N,j}(P\Psi_0) = O_{N,j,\deg P}\left(\tilde{h}^{2N-2j}N_\infty(P)\right)$$

where the constant in $O_{N,j,\deg P}$ depend on g trough its semi-norms, but are h-independent. Gathering the term of same order together, we can write

$$A(P\Psi_0) = \sum_{l=0}^{2N-1} \tilde{h}^l A_l(P\Psi_0) + O_{N,\deg P}\left(\tilde{h}^{2N}N_\infty(P)\right)$$

here $A_l = \sum_{2j+k=l} A_{j,k}$. It is not hard to see that $A_l(P\Psi_0) = P_l\Psi_0$ where $P \mapsto P_l$ is linear and $\deg P_l \leq \deg P + 2l$. Since $g_{j,q} \in S_\delta$ if $j \geq 1$ and since $h^{|\gamma|\delta}\partial^\gamma g = O(1)$, we observe that as soon as $l \geq 1$, there exists C_l depending on g (trough a finite number of semi-norms), but independent of h, such that

$$N_\infty(P_l) \leq C_l N_\infty(P)$$

Concerning A_0, it is clear that it is in fact $g(\rho)\,\mathrm{Id}$. We now apply the formulas given in Appendix A.2 and use the notations introduced in this appendix, that is

$$R_{2N-1}(t) = e^{tA}\Psi_0 - e^{tA_0}\sum_{l=0}^{2N-1} \tilde{h}^l v_l(t)$$

with v_l constructed inductively by (A.3) and $v_0 = \Psi_0$. Since A_0 is a multiplication, $A_k(s) = A_k$ for all $s \in \mathbb{R}$ and we see by induction that $v_k(t)$ is of the form

$$v_k(t) = \sum_{l=0}^k t^l P_{l,k}\Psi_0 = P_k(t)\Psi_0$$

where $P_{l,k} \in \mathbb{C}[X]$ has degree at most $2k$. In particular, $N_\infty(P_k(t)) \le c_k(1+|t|)^k$ for some h-independent c_k depending on g. Concerning the remainder, we have

$$||\tilde{r}_{k,2N}(t)|| = e^{tg(\rho)} \left|\left| \left(A - \sum_{j=0}^{2N-k-1} \tilde{h}^j A_j \right) v_k(t) \right|\right|$$

$$= O_{N,k} \left(e^{tg(\rho)} \tilde{h}^{2N-k} N_\infty(P_k(t)) \right)$$

$$\le C_{N,k}(1+|t|)^k e^{tg(\rho)} \tilde{h}^{2N-k}$$

Finally, we recall that $R'_{2N-1}(t) = AR_{2N-1}(t) + \sum_{j=0}^{2N-1} \tilde{h}^j \tilde{r}_{j,2N}(t)$. Hence, integrating this inequality, we find that

$$||R_{2N-1}(t)|| \le \int_0^t ||A|| \times ||R_{2N-1}(s)||ds + C_N \tilde{h}^N e^{tg(\rho)}(1+|t|)^{2N-1}$$

By a version of Gronwall's lemma, we can find a constant C_N such that

$$||R_{2N-1}(t)|| \le C_N \tilde{h}^{2N} e^{t \max(|g(\rho)|,||A||)} t^{2N+1}$$

(where C_N depends on finitely many semi-norms of g). Since $g \in \log(1/h)S_\delta$, we can find $K_0 > 0$ such that $\max(|g(\rho)|,||A||) \le K_0 \log(1/h)$. Going back to $\phi(t)$, we have proved the Lemma. $\square$

Remark. t is supposed to be fixed, so that the only meaningful term involving t is h^{-Bt}. The other mentions of t can be put into the constants C_N. All the polynomials depend also on h, we will omit to mention it in the subscripts, but we keep in mind that in the following, all the polynomials potentially depend on h. Nevertheless, their N_∞-norm can be controlled in an h-independent way.

5.3. **Repeated actions of $M_{q_i,q_{i-1}}$.** We fix some $\mathbf{q} = qq_1 \dots q_n \in \mathcal{A}^{n+1}$. Each term in the development of $E_t\varphi_{\hat{\rho}}$ is a sum of term of the form

$$e^{tg(\rho)}T(\hat{\rho})\Lambda_h(P_0\Psi_0)$$

with some $P_0 \in \mathbb{C}[X]$ depending on h. We now focus on the evolution of each of these terms under the repeated actions of $M_{q_i,q_{i-1}}$. We recall that this operator has the form

(5.7) $$M_{p,q}u(x) = \frac{1}{2\pi h} \int_{\mathbb{R}^2} e^{\frac{i}{h}(\psi_{p,q}(x,\eta)-y\eta)} \alpha_{p,q}(x,\eta)u(y)dyd\eta$$

with

(5.8) $$\alpha_{p,q} \sim \sum_{j\ge0} h^j \alpha_{p,q}^{(j)}$$

This will allow us to use Proposition 2.10, but we will have to deal with two different scales of asymptotic expansion : h and $h^{1/2}$. To simplify the notations in this context, we note for $1 \le i \le n$,

$$M_{q_i,q_{i-1}} = M_i$$
$$\psi_{q_i,q_{i-1}} = \psi_i$$
$$F_{q_i,q_{i-1}} = F_i$$
$$F^{(i)} = F_i \circ \dots \circ F_1 = \kappa_{q_i} \circ F^{n_0 i} \circ \kappa_q^{-1}$$
$$\alpha_{q_i,q_{i-1}}^{(j)} = \alpha_i^{(j)}$$

For $0 \le i \le n$, we also note $\hat{\rho}_i = F_i \circ \dots \circ F_1(\hat{\rho})$ (with $\hat{\rho}_0 = \hat{\rho}$) and set $\hat{\rho}_i = (x_i, \xi_i)$.

We fix a parameter N and we start with an initial state

(5.9) $$u_0 = T(\hat{\rho}_0)\Lambda_h(P_0\Psi_0)$$

with P_0 a polynomial of degree d_0. Our aim is to show that we have an asymptotic expansion for $u_i = M_i \dots M_1 u_0$ of the form

$$u_i = \sum_{2j+k<2N} h^j h^{k/2} u_i^{(j,k)} + r_i^{(N)}$$

where $u_i^{(j,k)}$ has the form

$$T(\hat{\rho}_i)\mathcal{M}\left(d_{\hat{\rho}}F^{(i)}\right)\Lambda_h\left(P_i^{(j,k)}\Psi_0\right)$$

with $P_i^{(j,k)}$ polynomial and with a good control on $r_i^{(N)}$. For $1 \le i \le n$ and $0 \le j \le N-1$, we apply Proposition 2.10 to the operator

$$(M_i^{(j)}u)(x) = \frac{1}{2\pi h}\int_{\mathbb{R}^2}e^{\frac{i}{h}(\psi_i(x,\eta)-y\eta)}\alpha_i^{(j)}(x,\eta)u(y)dyd\eta$$

and for a state of the form

$$u = T(\hat{\rho}_{i-1})\mathcal{M}\left(d_{\hat{\rho}}F^{(i-1)}\right)\Lambda_h(P\Psi_0)$$

For each such polynomial P, we can find a family $Q_i^{(j,k)}(P)$ of polynomials such that

- $Q_i^{(j,0)}(P) = \frac{\alpha_i^{(j)}(x_i,\xi_{i-1})}{|\partial_{x\eta}^2\psi_i(x_i,\xi_{i-1})|^{1/2}}P$ (up to a multiplicative factor of norm 1 that we omit in the proof) ;
- $Q_i^{(j,k)}(P)$ is a polynomial of degree $\deg P + 3k$ and the map $P \mapsto Q_i^{(j,k)}$ is linear, with coefficients depending on $F^{(i)}$ and the derivatives of ψ_i and $\alpha_i^{(j)}$ at (x_i,ξ_{i-1}) up to the $3k$-th order and we have

$$N_\infty(Q_i^{(j,k)}) \le C_{3k}(\psi_i)\|\alpha_i^{(j)}\|_{C^k}\|d_{\hat{\rho}}F^{(i)}\|^{3k}N_\infty(P)$$

  Moreover, if $(x_i,\xi_{i-1}) \notin \operatorname{supp}\alpha_i^{(j)}$, then $Q_i^{(j,k)} = 0$.
- for every $N \in \mathbb{N}$,

(5.10)

$$M_i^{(j)}\left(T(\hat{\rho}_{i-1})\mathcal{M}(d_{\hat{\rho}}F^{(i-1)})\Lambda_h[P\Psi_0]\right) = T(\hat{\rho}_i)\mathcal{M}(d_{\hat{\rho}}F^{(i)})\Lambda_h\left[\sum_{k=0}^{N-1}h^{k/2}Q_i^{(j,k)}(P)\Psi_0\right] + R_i^{(j,N)}(P)$$

with

$$\|R_i^{(j,N)}(P)\|_{L^2} \le h^{N/2}C_{3N+M}(\psi_i)\|\alpha_i^{(j)}\|_{C^{N+M}}\|d_{\hat{\rho}}F^{(i)}\|^{3N}K_{N,\deg P}N_\infty(P)$$

Remark. In virtue of the properties of $\alpha_i^{(j)}$, the condition $(x_i,\xi_{i-1}) \in \operatorname{supp}\alpha_i^{(j)} \iff F^{in_0}(\rho) \in \operatorname{supp}(\chi_q\alpha)$.

We also write the expansion of M_i in the form, for every N,

(5.11) $$M_i = \sum_{j=0}^{N-1}h^jM_i^{(j)} + \widetilde{S}_i^{(N)}$$

with

$$\|\widetilde{S}_i^{(N)}\| \le \widetilde{C}_{i,N,\varepsilon}h^{N(1-\varepsilon)}$$

Since M_i belongs to the finite family $(M_{p,q})$, we can replace $\widetilde{C}_{i,N,\varepsilon}$ by $\widetilde{C}_{N,\varepsilon} = \sup_i \widetilde{C}_{i,N,\varepsilon}$.

We now give the iteration formulas for the required expansion. We state $P_0^{(0,0)} = P_0$ and $P_0^{(j,k)} = 0$ for the other values of (j,k). For $2j + k < 2N$, we define inductively $P_i^{(j,k)} = P_{i,P_0}^{(j,k)}$ by the formula (to alleviate the notations, we will omit to specify the dependence in P_0 when this is not necessary) :

(5.12) $$P_i^{(j,k)} = \sum_{j_1+j_2=j}\sum_{k_1+k_2=k}Q_i^{(j_2,k_2)}\left(P_{i-1}^{(j_1,k_1)}\right)$$

Concerning the remainder term, we set

(5.13) $$r_i^{(N)} = r_{i,P_0}^{(N)} = M_i\left(r_{i-1}^{(N)}\right) + \sum_{2j+k<2N}h^{j+k/2}\widetilde{S}_i^{(N-j-\lceil k/2\rceil)}\left(u_{i-1}^{(j,k)}\right)$$

$$+ \sum_{2j_1+2j_2+k_1<2N}h^{N-j_1-j_2-k_1/2}R_i^{(j_1,2(N-j_1-j_2)-k_1)}\left(P_{i-1}^{(j_1,k_1)}\right)$$

Lemma 5.4. With the above notations, we have for $1 \le i \le n$,

$$u_i = \sum_{2j+k<2N}h^jh^{k/2}u_{i,P_0}^{(j,k)} + r_{i,P_0}^{(N)} \quad ; \quad u_i^{(j,k)} = T(\hat{\rho}_i)\mathcal{M}\left(d_{\hat{\rho}}F^{(i)}\right)\Lambda_h\left(P_{i,P_0}^{(j,k)}\Psi_0\right)$$

We now analyze these formulas to understand more precisely these terms and obtain a good control of the remainder. In particular, concerning the polynomial $P_i^{(j,k)}$, we want to control their degree and the norms of their coefficients.

Leading term. First note that the leading term (that is the term $(0,0)$) has a nice form. Indeed, up to a factor of norm 1, it is given by

$$P_i^{(0,0)} = P_0 \times \prod_{l=1}^{i} \frac{\alpha_l^{(0)}(x_l, \xi_{l-1})}{|\partial_{x\eta}^2 \psi_l(x_l, \xi_{l-1})|^{1/2}}$$

$$= P_0 \times \prod_{l=0}^{i-1} \left[\left(\chi_{q_l} \prod_{j=0}^{n_0-1} \alpha \circ F^j \right) (F^{ln_0}(\rho)) \right]$$

The product on the right plays a crucial role in the analysis. Let's note

$$p_{\alpha,l}(\rho) = \left(\chi_{q_l} \prod_{j=0}^{n_0-1} \alpha \circ F^j \right) (F^{ln_0}(\rho)) \quad ; \quad \pi_{\alpha,i}(\rho) = \prod_{l=0}^{i-1} p_{\alpha,l}(\rho)$$

We remark that

$$\pi_{\alpha,i}(\rho) \leq \Pi_{\alpha,q_0\dots q_{i-1}}(\rho)$$

Recall that $\Pi_{\alpha,\mathbf{q}}(\rho) = \prod_{i=0}^{nn_0-1} \alpha\left(F^i(\rho)\right)$. To simplify the notations, let's note $\Pi_{\alpha,i} = \Pi_{\alpha,q_0\dots q_{i-1}}$.

Moreover, combining the support property (5.6) of $\alpha_i^{(j)}$, Remark 5.3 and the properties of $Q_i^{(j,k)}$ given by Proposition 2.10, we see that for $\mathbf{q} = q_0\dots q_{i-1}$,

(5.14) $$\rho \notin \mathcal{W}_{\mathbf{q}}^- \implies \forall j,k \in \mathbb{N}, \quad Q_i^{(j,k)} = 0$$

Analysis of the polynomial $P_{i,P_0}^{(j,k)}$. According to (5.14), we assume that $\rho = \kappa_q^{-1}(\hat\rho) \in \mathcal{W}_{q_0\dots q_{i-1}}^-$. Otherwise, there is nothing more to say. We start by the easiest part of the analysis :

Lemma 5.5. For all $0 \leq i \leq n$ and all (j,k) with $2j + k < 2N$, $P_{i,P_0}(j,k)$ is of degree at most $3k + \deg P_0$.

Proof. We argue by induction on i. This is obvious for the case $i = 0$. To pass from $i-1$ to i, we use (5.12) which shows that

$$\deg P_i^{(j,k)} \leq \max_{j_1+j_2=j, k_1+k_2=k} \deg Q_i^{(j_2,k_2)}\left(P_{i-1}^{(j_1,k_1)}\right)$$

$$\leq \max_{j_1+j_2=j, k_1+k_2=k} 3k_2 + \deg P_{i-1}^{(j_1,k_1)}$$

$$\leq \max_{j_1+j_2=j, k_1+k_2=k} 3k_2 + 3k_1 + d_0$$

$$\leq 3k + d_0$$

$\square$

The analysis of $N_\infty\left(P_i^{(j,k)}\right)$ is a bit more tedious.

Lemma 5.6. For every $\varepsilon > 0$, there exists a family of constants $C_{j,k,\varepsilon}$ depending on the dynamical system and on M_h such that: For all $0 \leq i \leq n$ and all (j,k) with $2j + k < 2N$, we have

$$N_\infty\left(P_{i,P_0}^{(j,k)}\right) \leq C_{j,k,\varepsilon} h^{-k\varepsilon} i^{2j+k} \Pi_{\alpha,i}(\rho) \left(J_{q_0\dots q_{i-1}}^u\right)^{3k} N_\infty(P_0)$$

Remark. The dependence on i is of major importance. Here, $i \leq n = O(\log 1/h)$. Hence, the term i^{2j+k} is essentially harmless compared to the second part $\Pi_{\alpha,i}(\rho)\left(J_{q_0\dots q_{i-1}}^u\right)^{3k} h^{-k\varepsilon}$. The factor $\Pi_{\alpha,i}(\rho)$ does not depend on k and is common to all the terms. It can be put in front of the all expansion. On the contrary, the growth of $J_{q_0\dots q_{i-1}}^u$ influences the precision and the validity of the expansion. So that the expansion holds, we need to require

$$\left(J_{q_0\dots q_{i-1}}^u\right)^3 h^{-\varepsilon} \ll h^{-1/2}$$

As a consequence, this is where the assumption

$$\vartheta_\varepsilon < \frac{1 - 4\varepsilon}{6\lambda_{\max}}$$

(see its definition in (4.14)) is important and lead to a valid expansion.

Remark. The constant $C_{j,k,\varepsilon}$ depends on M_h through its amplitude α_h as a Fourier integral operator in a class $I_\eta(\mathbb{R} \times \mathbb{R}, \mathrm{Gr}(F)')$ (for some $\eta = \eta(\varepsilon)$) and it depends only a finite number $N_{j,k}$ of derivatives.

Proof. To alleviate the notations, we renormalize P_0 so that $N_\infty(P_0) = 1$. We fix (j,k) such that $2j + k < 2N$. By iterating (5.12), we find that

$$P_i^{(j,k)} = \sum_{\substack{j_1 + \cdots + j_i = j \\ k_1 + \cdots + k_i = k}} Q_i^{(j_i, k_i)} \circ \cdots \circ Q_1^{(j_1, k_1)}(P_0)$$

We now use the simple following idea : when i is large that is when $i \gg 2j + k$, and when $j_1 + \cdots + j_i = j$ and $k_1 + \cdots + k_i = k$, most of the couples (j_l, k_l) are equal to $(0,0)$. From a more quantitative point of view, we have

$$\#\{1 \le l \le i, (j_l, k_l) \ne (0,0)\} \le 2j + k$$

Indeed,

$$2j + k = 2(j_1 + \cdots + j_i) + (k_1 + \cdots + k_i) \ge \#\{1 \le l \le i, (j_l, k_l) \ne (0,0)\}$$

Let's note $\mathcal{P}(i, 2j + k)$ the set of subsets of $\{1, \ldots, i\}$ of cardinals smaller than $2j + k$. For $\mathcal{L} \in \mathcal{P}(i, 2j + k)$ we define the set of indices $\mathcal{I}_\mathcal{L} \subset \mathbb{N}^i \times \mathbb{N}^i$ by

$$(\overrightarrow{j}, \overrightarrow{k}) = ((j_1, \ldots, j_i), (k_1, \ldots, k_i)) \in \mathcal{I}_\mathcal{L} \iff \begin{cases} j_1 + \cdots + j_i = j \\ k_1 + \cdots + k_i = k \\ \forall 1 \le l \le i, (j_l, k_l) \ne (0,0) \iff l \in \mathcal{L} \end{cases}$$

With these notations, we have

$$P_i^{(j,k)} = \sum_{\mathcal{L} \in \mathcal{P}(i, 2j+k)} \sum_{(\overrightarrow{j}, \overrightarrow{k}) \in \mathcal{I}_\mathcal{L}} Q_i^{(j_i, k_i)} \circ \cdots \circ Q_1^{(j_1, k_1)}(P_0)$$

Let's fix $\mathcal{L} \in \mathcal{P}(i, 2j + k)$ and $(\overrightarrow{j}, \overrightarrow{k}) \in \mathcal{I}_\mathcal{L}$. Let's write $\mathcal{L} = \{l_1 < \cdots < l_m\}$. Since $Q_l^{(0,0)}$ is simply a multiplication by $p_{\alpha,l}(\rho)$, we have :

$$Q_i^{(j_i, k_i)} \circ \cdots \circ Q_1^{(j_1, k_1)}(P_0) = \left(\prod_{l \notin \mathcal{L}} p_{\alpha,l} \right) \times Q_{l_m}^{(j_{l_m}, k_{l_m})} \circ \cdots \circ Q_{l_1}^{(j_{l_1}, k_{l_1})}(P_0)$$

Using Proposition 2.10, we can estimate

$$N_\infty \left(Q_{l_m}^{(j_{l_m}, k_{l_m})} \circ \cdots \circ Q_{l_1}^{(j_{l_1}, k_{l_1})}(P_0) \right) \le N_\infty(P_0) \times \prod_{p=1}^m C_{3k_{l_p}}(\psi_{l_p}) \|\alpha_{l_p}^{(j_{l_p})}\|_{C^{k_{l_p}}} \|d_{\hat{\rho}} F^{(l_p)}\|^{3k_{l_p}}$$

For $1 \le l \le i$, ψ_l (resp. $\alpha_l^{(\cdot)}$) belongs to a finite family of functions (corresponding to the finite number of admissible transitions). Hence, recalling that $\alpha_l^{(j)} \in S_{0^+}$ if $j = 0$ and $h^{0^-} S_{0^+}$ if $j \ge 1$, we can find a global uniform constant depending on the dynamical system, and on a certain number $N_{j,k}$ of derivatives of α such that for all $j' \le j$, $k' \le k$ and for all l,

$$C_{3k'}(\psi_l)\|\alpha_l^{(j')}\|_{C^{k'}} \le C_{j,k,\varepsilon} \begin{cases} h^{-k'\varepsilon/2} \text{ if } j' = 0 \\ h^{-k'\varepsilon/2} h^{-\eta_{k,j}} \text{ if } j' \ge 1 \end{cases}.$$

where we artificially choose $\eta_{k,j} = \frac{k\varepsilon}{2j}$ and use the fact that $\alpha_l^{(j)} \in S_{\varepsilon/2}$ (resp. $h^{-\eta_{j,k}} S_{\varepsilon/2}$) if $j' = 0$ (resp. $j' \ge 1$). As a consequence, since $j_{l_1} + \cdots + j_{l_m} = j$ and $k_{l_1} + \cdots + k_{l_p} = k$, we have

$$N_\infty \left(Q_{l_m}^{(j_{l_m}, k_{l_m})} \circ \cdots \circ Q_{l_1}^{(j_{l_1}, k_{l_1})}(P_0) \right) \le C_{j,k,\varepsilon}^m N_\infty(P_0) h^{-k\varepsilon/2} h^{-j\eta_{k,j}} \left(\sup_{1 \le l \le i} \|d_{\hat{\rho}} F^{(l)}\| \right)^{3k}$$

Since $m \le 2j + k$, there exists a global constant, still denoted $C_{j,k,\varepsilon}$, such that, uniformly in $\mathcal{I}_\mathcal{L}$,

$$N_\infty \left(Q_{l_m}^{(j_{l_m}, k_{l_m})} \circ \cdots \circ Q_{l_1}^{(j_{l_1}, k_{l_1})}(P_0) \right) \le C_{j,k,\varepsilon} N_\infty(P_0) h^{-k\varepsilon} \left(\sup_{1 \le l \le i} \|d_{\hat{\rho}} F^{(l)}\| \right)^{3k}$$

We remark that for $1 \leq l \leq n$, $||d_{\hat{\rho}}F^{(l)}|| \leq C||d_{\hat{\rho}}F^{n_0 l}|| \leq C J^u_{q_0 \ldots q_{l-1}}$. Finally, since $|\alpha| \geq e^{-\tau_m}$ in the neighborhood $\bigcup_{q \in \mathcal{A}} \mathcal{W}_q$ of $\mathcal{T}$, we see that for every $(\overrightarrow{j}, \overrightarrow{k}) \in \mathcal{I}_{\mathcal{L}}$ we have

$$N_\infty \left(Q_i^{(j_i, k_i)} \circ \cdots \circ Q_1^{(j_1, k_1)}(P_0) \right) \leq C_{j,k,\varepsilon} N_\infty(P_0) h^{-k\varepsilon} \Pi_{\alpha, i}(\rho) \left(J^u_{q_0 \ldots q_{i-1}} \right)^{3k}$$

We can now conclude the proof. Indeed, we have

$$N_\infty \left(P_i^{(j,k)} \right) \leq \sum_{\mathcal{L} \in \mathcal{P}(i, 2j+k)} \#\mathcal{I}_{\mathcal{L}} \times \left(C_{j,k,\varepsilon} N_\infty(P_0) h^{-k\varepsilon} \Pi_{\alpha, i}(\rho) \left(J^u_{q_0 \ldots q_{i-1}} \right)^{3k} \right)$$

If $\mathcal{L} \in \mathcal{P}(i, 2j+k)$, we estimate (crudely) the cardinal of $\mathcal{I}_{\mathcal{L}}$ by

$$\#\mathcal{I}_{\mathcal{L}} \leq (j+1)^{\#\mathcal{L}}(k+1)^{\#\mathcal{L}} \leq (j+1)^{2j+k}(k+1)^{2j+k}$$

Finally,

$$i \mapsto \sum_{\mathcal{L} \in \mathcal{P}(i, 2j+k)} 1 = \sum_{m=0}^{2j+k} \binom{i}{m}$$

is a polynomial function of i, of degree $2j+k$. Hence, thee exists $C_{j,k}$ such that

$$\left| \sum_{\mathcal{L} \in \mathcal{P}(i, 2j+k)} \binom{i}{\#\mathcal{L}} \right| \leq C_{j,k} i^{2j+k}$$

This concludes the proof. $\qquad\square$

Control of the remainder. Armed with Lemma 5.6 and the iterative formula (5.13), we can deduce a control for the remainder term. Let's consider $B \geq 1$ such that $||M_{q,p}|| \leq B$ for all admissible pair (q,p) (it is possible to take $B \leq (1+\varepsilon)||\alpha||_\infty$, or even with ε going to 0 as $h \to 0$, but the precise value of B is not relevant for this term). For this reason, we will also get rid of the precise value of $\Pi_{\alpha, i}$ and assume that $||\alpha||_\infty \leq B$ so that $||\Pi_{\alpha, i}|| \leq B^i$.

Plugging the previous estimates into (5.13), we get

$$||r_i^{(N)}|| \leq B||r_{i-1}^{(N)}|| + \sum_{2j+k<2N} h^{j+k/2} \widetilde{C}_{N-j-\lceil k/2 \rceil, \varepsilon} h^{(N-j-\lceil k/2 \rceil)(1-\varepsilon)} ||u_{i-1}^{(j,k)}||$$

$$+ \sum_{2j_1+2j_2+k_1<2N} h^N C_{3(2N-2j_1-2j_2-k_1)+M}(\psi_{i-1}) h^{-\varepsilon M} \left(h^{-\varepsilon}||d_{\hat{\rho}}F^{(i)}||^3 \right)^{2N-2j_1-2j_2-k_1} K_{3k_1, 3k_1 + \deg P_0} N_\infty(P_{i-1}^{(j_1, k_1)})$$

Recall that $||P\Psi_0||_{L^2} \leq K_{\deg P} N_\infty(P)$ for some family of constants K_n depending only on n. By the expression of $u_i^{(j,k)}$, we have

$$||u_i^{(j,k)}||_{L^2} = ||P_i^{(j,k)} \Psi_0||_{L^2} \leq K_{3k + \deg P_0} N_\infty \left(P_i^{(j,k)} \right)$$

We also recall that we can bound $||d_{\hat{\rho}}F^{(i)}||$ by $C J_{q_0 \ldots q_{i-1}}$ for some global constant C.

$$||r_i^{(N)}|| \leq B||r_{i-1}^{(N)}|| + \sum_{2j+k<2N} h^{N(1-\varepsilon)} \widetilde{C}_{N-j-\lceil k/2 \rceil, \varepsilon} K_{3k} C_{j,k,\varepsilon} B^i i^{2j+k} \left(J^u_{q_0 \ldots q_{i-1}} \right)^{3k}$$

$$+ \sum_{2j_1+2j_2+k_1<2N} h^N C_{N,j,\varepsilon} h^{-M\varepsilon} K_{3k_1} C_{j_1, k_1, \varepsilon} B^i i^{2j_1+k_1} \left(h^{-\varepsilon} \left(J^u_{q_0 \ldots q_{i-1}} \right)^3 \right)^{2(N-j_1-j_2)}$$

Finally, we plug the bound $J^u_{q_0 \ldots q_{i-1}} \leq C_\varepsilon e^{i\lambda_{\max}(1+\varepsilon)}$ into the previous inequality. We can find a constant $C_{N, \deg P_0, \varepsilon}$ such that

$$||r_i^{(N)}|| \leq B||r_{i-1}^{(N)}|| + C_{N, \deg P_0, \varepsilon} B^i i^{2N} e^{6Ni\lambda_{\max}(1+\varepsilon)} h^{N(1-2\varepsilon)} h^{-M\varepsilon}$$

This being valid for all $1 \leq i \leq n$, by induction on i, we find that

$$||r_i^{(N)}|| \leq \sum_{l=0}^{i} B^{i-l} \times C_{N, \deg P_0, \varepsilon} B^l l^N e^{6Nl\lambda_{\max}(1+\varepsilon)} h^{N(1-2\varepsilon)} h^{-M\varepsilon}$$

$$\leq C_{N, \deg P_0, \varepsilon} B^i \sum_{l=0}^{i} l^N e^{6Nl\lambda_{\max}(1+\varepsilon)} h^{N(1-2\varepsilon)} h^{-M\varepsilon}$$

Let $c_{N,\varepsilon} > 0$ be such that $\sum_{l=0}^{i} l^N e^{6Nl\lambda_{\max}(1+\varepsilon)} \leq c_{N,\varepsilon} e^{6Ni\lambda_{\max}(1+\varepsilon)^2}$ for all $i \in \mathbb{N}$. This gives, for a constant $C_{N,\deg P_0,\varepsilon}$,

$$||r_i^{(N)}|| \leq C_{N,\deg P_0,\varepsilon} B^i e^{6Ni\lambda_{\max}(1+\varepsilon)^2} h^{N(1-2\varepsilon)} h^{-M\varepsilon}$$

To conclude, recall that $n(h) \leq \vartheta_\varepsilon \log 1/h$ with $\vartheta_\varepsilon = \frac{1-4\varepsilon}{6\lambda(1+\varepsilon)^2}$. Hence, as soon as $i \leq n(h)$, $e^{6Ni\lambda_{\max}(1+\varepsilon)^2} \leq h^{-N(1-4\varepsilon)}$ and this shows the following lemma

Lemma 5.7. There exists constants $C_{N,d,\varepsilon}$ such that for all $N \in \mathbb{N}$ and for all $P_0 \in \mathbb{C}[X]$, we have for all $1 \leq i \leq n(h)$,

$$||r_{i,P_0}^{(N)}|| \leq C_{N,\deg P_0,\varepsilon} h^{2N\varepsilon} h^{-K} N_\infty(P_0)$$

with $K = \vartheta_\varepsilon \log B + M\varepsilon$.

First consequences. Since N can be taken arbitrarily large, we recover the known fact that a wave packet centered at $\hat{\rho}$ is changed after n steps into an excited squeezed state centered at $F^{(n)}(\hat{\rho})$. The squeezing is governed by the unstable Jacobian along the orbit of ρ. In particular, we obtain the expected following corollary, which gives the first point in Proposition 4.2.

Corollary 5.1. Let's note $\mathbf{q} = q_0 \ldots q_n \in \mathcal{A}^{n+1}$. Let $\hat{\rho} \in \kappa_q(\mathcal{W}_q)$ and let us note $\rho = \kappa_q^{-1}(\hat{\rho})$.

- If $\rho \notin \mathcal{W}_{\mathbf{q}}^-$, then $A_{q_n} e^{-tG} B'_{q_n} M_{q_n,q_{n-1}} \ldots M_{q_1,q} E_t \varphi_{\hat{\rho}} = O(h^\infty)$.
- If $\rho \in \mathcal{W}_{\mathbf{q}}^-$,

$$e^{-tG}\mathfrak{M}^{n-1} M^{n_0} A_q B'_q E_t \varphi_\rho = A_{q_n} e^{-tG} B'_{q_n} M_{q_n,q_{n-1}} \ldots M_{q_1,q_0} E_t \varphi_\rho + O(h^\infty)$$

with constants independent of $\mathbf{q}$ and ρ.

Proof. This is a consequence of the previous results and the fact that $\mathrm{WF}_h(B_{q_n} A_{q_n} e^{-tG} B'_{q_n}) \Subset \kappa_{q_n}(\mathcal{W}_{q_n})$. $\square$

Moreover, we can combine Lemma 5.3 (the running index in the formula of Lemma 5.3 was j, it becomes l in the sum below) and Lemma 5.4 to get :

Proposition 5.1. Assume that $\hat{\rho} = \kappa_q(\rho) \in \kappa_q(\mathcal{W}_q)$ with $\rho \in \mathcal{W}_{q_0 \ldots q_n}^-$. Then, for any $N \in \mathbb{N}$, we have the following expansion (with $n = n(h)$)

$$M_{q_n,q_{n-1}} \ldots M_{q_1,q_0} E_t \varphi_{\hat{\rho}} = \sum_{2j+k+l<2N} h^{j+k/2} h^{l(1/2-\delta)} u_n^{(j,k,l)} + R_n^{(N)}$$

where

$$u_n^{(j,k,l)} = e^{tg(\rho)} T(\hat{\rho}_n) \mathcal{M}(d_{\hat{\rho}} F^{(n)}) \Lambda_h \left(P_n^{(j,k,l)} \Psi_0 \right)$$

$P_n^{(j,k,l)}$ is a polynomial of degree at most $3k + 2l$ and

$$N_\infty \left(P_n^{(j,k,l)} \right) \leq C_{j,k,l,\varepsilon} n^{2j+k} \Pi_{\alpha,n}(\rho) \left(J_{q_0 \ldots q_{n-1}}^u \right)^{3k} h^{-k\varepsilon}$$

Concerning the leading term, $P_n^{(0,0,0)} = \pi_{\alpha,n}(\rho)$. Concerning the remainder $R_n^{(N)}$ we have

$$||R_n^{(N)}||_{L^2} \leq C_{N,\varepsilon} h^{-(K+K_0 t)} h^{2N\varepsilon}$$

Proof. We simply state $P_n^{(j,k,l)} = P_{n,P_{t,h}^{(l)}}^{(j,k)}$ which satisfies the required bound for the degree and N_∞. Here, $P_{t,h}^{(l)}$ appears in the expansion of Lemma 5.3. Lemma 5.3 and Lemma 5.4 show that

$$M_{q_n,q_{n-1}} \ldots M_{q_1,q_0} E_t \varphi_\rho = \sum_{2j+k+l<2N} h^{j+k/2} h^{l(1/2-\delta)} u_n^{(j,k,l)} + R_n^{(N)}$$

with $u_n^{(j,k,l)}$ given by the required formula and

$$R_n^{(N)} = M_{q_n,q_{n-1}} \ldots M_{q_1,q_0} r_{2N} + \sum_{l=0}^{2N-1} h^{l(1/2-\delta)} r_{n,P_{t,h}^{(l)}}^{(N-\lfloor l/2 \rfloor)}$$

We can use the bound $||r_{2N}|| \leq C_N h^{N(1-2\delta)}$ and the bound for $r_{n,P}^{(N)}$ in Lemma 5.7. Since the degrees of the polynomial $P_{t,h}^{(l)}$ are bounded by $4N$, we can forget the depence in $\deg P$ in the estimates of Lemma 5.7, so that we find

$$||R_n^{(N)}|| \leq C_{N,\varepsilon} h^{-K} h^{-K_0 t} \sum_0^{2N} h^{l(1/2-\delta)} h^{2(N-l/2)\varepsilon} \leq C_{N,\varepsilon} h^{-(K+K_0 t)} h^{2N\varepsilon}$$

where the last inequality follows from $\varepsilon = 1/2 - \delta$. $\qquad\square$

Remark. This expansion mixes up different scales :
- the scale $h^{1-2\delta} = h^{2\varepsilon}$: it comes from the symbol class in which g lives ;
- a second scale which is the scale $h^{1/2}$ when n is independent of h. In our context, it is better to think this second scale to be $h^{1/2}(J_{q_0...q_n}^u)^3 h^{-\varepsilon}$. This scale depends on the starting point ρ. The definition of ϑ_ε ensures that the higher order terms are smaller than the leading term, .

Since we can choose N as large as we want, we can ensure that the remainder decays in h and that the leading term controls the whole expansion. Note also that the constants $C_{j,k,l,\varepsilon}$ and $C_{N,\varepsilon}$ depends on g and $M = M_h(z)$ and they can be chosen uniform in $z \in \Omega(h) \cap \{\operatorname{Im} z \in [-\beta, 4]\}$.

5.4. **Final action of $A_{q_n} e^{-tG} B'_{q_n}$.** From now on, and until the end of the section, we assume that $\rho \in \mathcal{W}_{\mathbf{q}}^-$ and we prove the missing items of Proposition 4.2.

We need to understand the action of $e^{-tG} B'_{q_n}$ on the terms $u_n^{(j,k,l)}$ of the last expansion. Since all these terms have the same form, we consider a general polynomial P of degree d and want to understand

$$e^{-tG} B'_{q_n} \left(T(\hat{\rho}_n) \mathcal{M}\left(d_{\hat{\rho}} F^{(n)} \right) \Lambda_h(P\Psi_0) \right)$$

It is no more possible to reuse the strategy of Lemma 5.3. Indeed, if g still oscillates on scale h^δ, $\mathcal{M}\left(d_{\hat{\rho}} F^{(n)} \right) \Lambda_h(P\Psi_0)$ is no more a wave packet in a box of size $h^{1/2}$. To see that in a model case, assume that $d_{\hat{\rho}} F^{(n)}$ is given by the diagonal matrix

$$\begin{pmatrix} \lambda_h & 0 \\ 0 & \lambda_h^{-1} \end{pmatrix}$$

with $\lambda_h \sim J_{q_0...q_{n-1}}^u \sim h^{-\alpha}$ where

$$\lambda_{\min}\vartheta_\varepsilon \leq \alpha \leq \lambda_{\max}\vartheta_\varepsilon = \frac{1-4\varepsilon}{6(1+\varepsilon)^2}$$

Then $\mathcal{M}\left(d_{\hat{\rho}} F^{(n)} \right)$ is nothing but $\Lambda_{\lambda_h^2}$ and hence, $\mathcal{M}\left(d_{\hat{\rho}} F^{(n)} \right) \Lambda_h(P\Psi_0) = \Lambda_{h\lambda_h^2}(P\Psi_0)$. This states oscillate in the x-direction on a scale $h^{1/2-\alpha} \gg h^\delta$.

5.4.1. *Precise description of $d_{\hat{\rho}} F^{(n)}$.* It is not possible to write $d_{\hat{\rho}} F^{(n)}$ exactly as a diagonal matrix in the standard position/momentum variable. However, the following lemma shows that $d_{\hat{\rho}} F^{(n)}$ stays close to a diagonal matrix :

Lemma 5.8. There exists ε_2 which can be made arbitrarily small depending on ε_0 such that the following holds. There exists $\lambda_{n,\mathbf{q}}, \mu_{n,\mathbf{q}} \in \mathbb{R}^+$ such that for all n, $\mathbf{q} = q_0 \ldots q_n$ and $\hat{\rho} \in \kappa_q \left(\mathcal{W}_{\mathbf{q}}^-, \right)$, we have for some global constant $C > 0$:
- $C^{-1} J_{\mathbf{q}}^u \leq \lambda_{n,\mathbf{q}} \leq C J_{\mathbf{q}}^u$;
- $C^{-1} \leq \mu_{n,\mathbf{q}} \lambda_{n,\mathbf{q}} \leq C$;
- $d_{\hat{\rho} F^{(n)}}$ is close to a diagonal matrix :

$$\left\|d_{\hat{\rho}} F^{(n)} - \begin{pmatrix} \lambda_{n,\mathbf{q}} & 0 \\ 0 & \mu_{n,\mathbf{q}} \end{pmatrix}\right\| \leq \varepsilon_2 J_{\mathbf{q}}^u$$

Proof. We note $\rho_i = F^{in_0}(\rho) = \kappa_{q_i}^{-1} \circ F^{(i)}(\hat{\rho})$. Recall also that $F^{(i)} = \kappa_{q_i} \circ F^{in_0} \circ \kappa_{q_0}^{-1}$.

Step 1 : Reduction to $\rho \in \mathcal{T}$**.** By definition of $\mathcal{W}_{q_i}$, for $i \in \{0, \ldots, n\}$, we have $d(\rho_{q_i}, \rho_i) \leq 2\varepsilon_0$. Hence,

$$d(F^{n_0}(\rho_{q_i}), \rho_{i+1}) \leq d(F^{n_0}(\rho_{q_i}), F^{n_0(i+1)}(\rho)) + d(F^{n_0(i+1)}(\rho), \rho_{i+1}) \leq C\varepsilon_0$$

for a constant C only depending on F. That is to say, $(\rho_0, \ldots, \rho_n)$ is a $C\varepsilon_0$ pseudo orbit for F^{n_0}. Assume that $\delta_0 > 0$ is a small fixed parameter. In virtue of the Shadowing Lemma ([HK95] , Section 18.1), if ε_0 is sufficiently small, $(\rho_0, \ldots, \rho_n)$ is δ_0 shadowed by an orbit of F^{n_0} i.e. there exists $\rho' \in \mathcal{T}$ such that for $i \in \{0, \ldots, n\}$, $d(\rho_i, F^{in_0}(\rho')) \leq \delta_0$. Consequently, $d(F^{in_0}(\rho), F^{in_0}(\rho')) \leq \delta_0 + C\varepsilon_0$. For convenience, set $\varepsilon_2 = \delta_0 + C\varepsilon_0$ and note that ε_2 can be arbitrarily small depending on ε_0. By Lemma 3.3, for $1 \leq i \leq n$,

$$||d_\rho F^{in_0}|| \leq C J^u_{in_0}(\rho') \quad ; \quad C^{-1} J^u_{q_0 \ldots q_{i-1}} \leq J^u_{in_0}(\rho') \leq C J^u_{q_0 \ldots q_{i-1}}$$

Hence, using the relation

$$d_\rho F^{nn_0} - d_{\rho'} F^{nn_0} = \sum_{k=0}^{n-1} d_{F^{n_0(k+1)}(\rho')} F^{n_0(n-k-1)} \circ \left(d_{F^{n_0 k}(\rho)} F^{n_0} - d_{F^{n_0 k}(\rho')} F^{n_0} \right) \circ d_\rho F^{n_0 k}$$

we find that

$$
\begin{aligned}
||d_\rho F^{nn_0} - d_{\rho'} F^{nn_0}|| &\leq C \sum_{k=0}^{n-1} J^u_{n_0(n-k-1)} \left(F^{n_0(k+1)}(\rho') \right) \left||d_{F^{n_0 k}(\rho)} F^{n_0} - d_{F^{n_0 k}(\rho')} F^{n_0} \right|| J^u_{n_0 k}(\rho') \\
&\leq C \sum_{k=0}^{n-1} d \left(F^{n_0 k}(\rho), F^{n_0 k}(\rho') \right) J^u_{nn_0}(\rho') \\
&\leq C J^u_{\mathbf{q}} \sum_{k=0}^{n-1} \theta^{\min(k,n-k)} \varepsilon_0 \\
&\leq C J^u_{\mathbf{q}} \varepsilon_0
\end{aligned}
$$

where we use the Lemma 3.1 in the third equality and the last one follows from $\sum_{k=0}^{n-1} \theta^{\min(k,n-k)} \leq 2 \sum_{k=0, \lceil n/2 \rceil} \theta^k \leq 2 \sum_{k=0}^{\infty} \theta^k < +\infty$. It is not hard to deduce from this that

$$||d_{\hat{\rho}} F^{(n)} - d_{\kappa_q(\rho')} F^{(n)}|| \leq C J^u_{\mathbf{q}} \varepsilon_0$$

Hence, it is enough to prove the Lemma for $d_{\kappa_q(\rho')} F^{(n)}$.

Step 2 : The case $\rho \in \mathcal{T}$**.** We assume that $\rho \in \mathcal{T}$. The spaces $E_u(\rho), E_s(\rho), E_u(F^{nn_0}(\rho))$ and $E_s(F^{nn_0}(\rho))$ are well-defined. For $q \in \mathcal{A}$ and $\bullet = s, u$, the maps $\zeta \in \mathcal{W}_q \cap \mathcal{T} \mapsto d_\zeta \kappa_q(E_\bullet(\zeta))$ are Lipschitz. Since $d_\rho \kappa_q(E_u(\rho_q)) = \mathbb{R}\partial_y$, $d_\rho \kappa_q(E_s(\rho_q)) = \mathbb{R}\partial_\eta$ and $d(\rho_{q_0}, \rho) \leq C\varepsilon_2$, $d(\rho_{q_n}, F^{nn_0}(\rho)) \leq C\varepsilon_2$, we can fix unit vectors

$$e_0^u \in d_\rho \kappa_{q_0}(E_u(\rho)) \ , \ e_0^s \in d_\rho \kappa_{q_0}(E_s(\rho))$$

$$e_n^u \in d_{F^{nn_0}(\rho)} \kappa_{q_n}(E_u(F^{nn_0}(\rho))) \quad e_n^s \in d_{F^{nn_0}(\rho)} \kappa_{q_n}(E_s(F^{nn_0}(\rho)))$$

such that $e_0^u, e_n^u = \partial_y + O(\varepsilon_2)$ and $e_0^s, e_n^s = \partial_\eta + O(\varepsilon_2)$. If we note P_0 (resp. P_n) the change-of-basis matrix from the natural basis of $\mathbb{R}^2$ to (e_0^u, e_0^s) (resp. (e_n^u, e_n^s)), then $P_0, P_n = I_2 + O(\varepsilon_2)$ (with global constants in O not depending on n). Moreover, since $d_{\hat{\rho}} F^{(n)}(e_0^u) \in \mathbb{R}e_n^u$ and $d_{\hat{\rho}} F^{(n)}(e_0^s) \in \mathbb{R}e_n^s$, the matrix $P_n^{-1} d_{\hat{\rho}} F^{(n)} P_0$ is diagonal. Let's write it

$$\begin{pmatrix} \lambda_{n,\mathbf{q}} & 0 \\ 0 & \mu_{n,\mathbf{q}} \end{pmatrix}$$

$\lambda_{n,\mathbf{q}}$ (resp. $\mu_{n,\mathbf{q}}$) is nothing but *an* unstable (resp. stable) Jacobian for ρ, and hence $\lambda_{n,\mathbf{q}} \sim J^u_{\mathbf{q}}$. Since $\det d_\rho F^{nn_0} = 1$, $\lambda_{n,\mathbf{q}} \mu_{n,\mathbf{q}} = \det(P_0)^{-1} \det(P_n) = 1 + O(\varepsilon_2)$. Finally,

$$P_n^{-1} d_{\hat{\rho}} F^{(n)} P_0 = (I_2 + O(\varepsilon_2)) d_{\hat{\rho}} F^{(n)} (I_2 + O(\varepsilon_2)) = d_{\hat{\rho}} F^{(n)} + O\left(||d_{\hat{\rho}} F^{(n)}|| \varepsilon_2 \right) = d_{\hat{\rho}} F^{(n)} + O\left(\varepsilon_2 J^u_{\mathbf{q}} \right)$$

This concludes the proof. $\square$

As a consequence of this lemma, in the standard position/momentum coordinates, we can write

$$(5.15) \qquad d_{\hat{\rho}}F^{(n)} = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \; ; \; a_n \sim J_{\mathbf{q}}^u \; ; \; b_n, c_n, d_n = O\left(\varepsilon_2 J_{\mathbf{q}}^u\right)$$

Here, a_n, b_n, c_n, d_n depend on ρ, but we won't make this dependence precise since ρ is fixed until the end of the section. Since we want to understand the action of $\mathcal{M}\left(d_{\hat{\rho}}F^{(n)}\right)$ on excited coherent states, we also introduce

$$(5.16) \qquad \gamma_n = (c_n + id_n)(a_n + ib_n)^{-1} \; ; \; \beta_n = \mathrm{Re}(\gamma_n) \; ; \; \alpha_n = \mathrm{Im}(\gamma_n)^{-1} = |a_n + ib_n|^2$$

We've got the basic estimates

$$(5.17) \qquad \alpha_n \sim \left(J_{\mathbf{q}}^u\right)^2; \quad \beta_n = O(\varepsilon_2)$$

Now assume that $P \in \mathbb{C}[X]$ and decompose P into the basis of the renormalized hermite polynomials $(h_n) : P = \sum_{k=0}^{\deg P} a_k(P)h_k$. By Proposition 2.9,

$$\mathcal{M}\left(d_{\hat{\rho}}F^{(n)}\right)\Lambda_h\left(h_k\Psi_0\right)(x) = (\alpha_n \pi h)^{-1/4}\left(\frac{a_n - ib_n}{a_n + ib_n}\right)^{k/2} h_k\left(\frac{x}{(\alpha_n h)^{1/2}}\right)e^{i\gamma_n \frac{x^2}{2h}} = c_{n,k}\Lambda_{\alpha_n h}\left(h_k\Psi_0\right)(x)e^{i\beta_n \frac{x^2}{2h}}$$

with $|c_{n,k}| = 1$. As a consequence, there exist linear maps $\Phi_n : \mathbb{C}[X] \to \mathbb{C}[X]$ such that for all $n \in \mathbb{N}$ and $P \in \mathbb{C}[X]$,

- $\deg \Phi_n(P) = \deg P$ for all $P \in \mathbb{C}[X]$;
- $N_\infty(\Phi_n(P)) \le K_{\deg P} N_\infty(P)$ where $K_{\deg P}$ depends only on $\deg P$;
- and the following relation holds

$$(5.18) \qquad \mathcal{M}\left(d_{\hat{\rho}}F^{(n)}\right)\Lambda_h\left(P\Psi_0\right) = \Lambda_{\alpha_n h}\left(\Phi_n(P)\Psi_0\right)e^{i\beta_n \frac{x^2}{2h}}$$

Remark. We can interpret this state as a (highly-oscillating) Lagrangian state associated with the Lagrangian manifold $\{(x, \beta_n x)\}$, with amplitude $a(x) = \Lambda_{\alpha_n h}\left(\Phi_n(P)\Psi_0\right)(x)$. Since $\alpha_n \sim \left(J_{\mathbf{q}}^u\right)^2$, $\alpha_n \sim h^{-\alpha}$ for some $\alpha \ge 2\lambda_{\min}\vartheta_\varepsilon$, the amplitude a oscillates on a scale $h^{1/2-\alpha/2}$. Compared with the initial state φ_0, localized in position in an interval of size $h^{1/2}$, this expression shows a stretching in position. Moreover, this scale is larger than the scale h^δ on which the symbol g oscillates.

5.4.2. *Asymptotic expansion for the exponential.* We now aim at understanding the state $A_{q_n}e^{-tG}B'_{q_n}u$ where u is of the form

$$u(x) = T(\hat{\rho}_n)\left(\Lambda_{\alpha_n h}f\right)(x)e^{i\beta_n \frac{x^2}{2h}}$$

where $f = P\Psi$ for some $P \in \mathbb{C}[X]$. We first claim that

$$A_{q_n}e^{-tG}B'_{q_n} = A_{q_n}B'_{q_n}e^{-tB_{q_n}GB'_{q_n}} + O(h^\infty)$$

Proof. Set $A(t) = A_{q_n}e^{-tG}B'_{q_n}e^{tB_{q_n}GB'_{q_n}}$. At $t = 0$, $A(0) = A_{q_n}B'_{q_n}$. We differentiate:

$$\dot{A}(t) = A_{q_n}e^{-tG}\left[B'_{q_n}B_{q_n}GB'_{q_n} - GB'_{q_n}\right]e^{tB_{q_n}GB'_{q_n}}$$

The operator $A_{q_n}e^{-tG}$ is bounded on L^2 and has its semiclassical wavefront set included in $\mathrm{supp}\,\chi_{q_n}$. In particular, $A_{q_n}e^{-tG}\left(B'_{q_n}B_{q_n} - \mathrm{Id}\right) = O(h^\infty)$ (uniformly for t in a bounded interval). This shows that $A'(t) = O(h^\infty)$. We conclude that $A(t) = A_{q_n}B'_{q_n} + O(h^\infty)$. $\qquad\square$

Hence we aim at understanding the action of $e^{-tB_{q_n}GB'_{q_n}}$. We make use of Lemma 5.2 and we write for all $N \in \mathbb{N}$,

$$G_{q_n} := B_{q_n}GB'_{q_n} = \mathrm{Op}_h\left(g \circ \kappa_{q_n}^{-1}\right) + \sum_{j=1}^{N-1} h^{j(1-2\delta)}\,\mathrm{Op}_h(g_{j,q_n}) + R_N$$

with $\|R_N\| \le C_N h^{N(1-2\delta)}$. Let's write $g_{0,q_n} = g \circ \kappa_{q_n}^{-1}$. Similarly, we have

$$A_{q_n}B'_{q_n}e^{-tG_{q_n}}T(\hat{\rho}_n) = A_{q_n}B'_{q_n}T(\hat{\rho}_n)e^{-tT(\hat{\rho}_n)^*G_{q_n}T(\hat{\rho}_n)}$$

and we recall that $T(\hat{\rho}_n)^*\mathrm{Op}_h(a)T(\hat{\rho}_n) = \mathrm{Op}_h(a(\cdot + \hat{\rho}_n))$ for any $a \in \mathcal{S}'$. Let's note $h_j(\hat{\zeta}) = g_{j,q_n}(\hat{\rho}_n + \hat{\zeta})$, so that

$$A := T(\hat{\rho}_n)^*G_{q_n}T(\hat{\rho}_n) = \sum_{j=0}^\infty h^{j(1-2\delta)}\,\mathrm{Op}_h(h_j) + O_N(h^{N(1-2\delta)})$$

Recall that in virtue of Lemma 5.2, $h_0 \in \log(1/h)S_\delta$ and $h_j \in S_\delta$ for $j \geq 1$.

Finally, we need to understand the action of e^{-tA} on states $u(x) = \Lambda_{\alpha_n h} f(x) e^{i\beta_n \frac{x^2}{2h}}$. We want to apply the formalism of Appendix A.2 with $H = L^2(\mathbb{R})$ and A. The class of elements which will interest us is defined as follows : we say that a h-dependent family of states $u = u_h \in L^2(\mathbb{R})$ belongs to the class $\mathcal{C}$ if u has the form :

$$u(x) = a(x)e^{i\beta_n \frac{x^2}{2h}}$$

where $a = a_h \in C^\infty(\mathbb{R})$ satisfies : for all $p \in \mathbb{N}$, there exists C_p such that

$$(5.19) \qquad |a^{(p)}(x)| \leq C_p h^{-\delta p}(\alpha_n h)^{-1/4}\left(1 + \frac{x^2}{\alpha_n h}\right)^{-1}$$

This class depends on h (and n, which himself depends on h). For such a state u, we define the natural semi-norms on $\mathcal{C}$:

$$(5.20) \qquad q_p(u) = \sup_{k \leq p} \sup_{x \in \mathbb{R}} \left(|a^{(k)}(x)|h^{\delta k}(\alpha_n h)^{1/4}\left(1 + \frac{x^2}{\alpha_n h}\right)\right)$$

In particular, one has $||u|| \leq Cq_0(u)$.

Remark. In fact, the introduction of the semi-norms q_j with the factor $(1 + \frac{x^2}{\alpha_n h})^{-1}$ is purely technical : it allows to work in a symbol class depending on this order function (see the proof of Lemma 5.10 in the appendix A.1.3). In the end, we will simply need to estimate the semi-norm q_0 of each term of the expansion of an evolved state $e^{-tA}u$, but this will require to control (a finite number of) semi-norms q_j of the initial state u. This reason has motivated the introduction of the q_j's. We will mainly consider states u with exponential decay and what is important is that $\partial^k \Psi_0 \leq C_{k,p}(1 + x^2)^{-p/2}$ for all $k, p \in \mathbb{N}$.

The following lemma ensures that the states we work with are indeed in $\mathcal{C}$, as soon as $h^{2\delta} \ll \alpha_n h$. Recall that $\alpha_n \geq Ch^{-\alpha_{\min}}$ where $\alpha_{\min} = 2\lambda_{\min}\vartheta_\varepsilon$. Then, it suffices to require

$$\varepsilon = 1/2 - \delta \leq \alpha_{\min}/2.$$

This is clearly not a problem since we want to work with $\delta = 1/2 - \varepsilon$ very close to $1/2$ and we assume that this is true, that is, we assume that

$$\varepsilon \leq \alpha_{\min}/2.$$

Lemma 5.9. Assume that $u(x) = \Lambda_{\alpha_n h}(P\Psi_0)e^{i\beta_n \frac{x^2}{2h}}$ where $P \in \mathbb{C}[X]$ has degree d. Then $u \in \mathcal{C}$ and for all $j \in \mathbb{N}$, there exists constants $C_{d,j}$ depending only on d and j such that $q_j(u) \leq C_{d,j}N_\infty(P)$

Proof.

$$\left|(\Lambda_{\alpha_n h}(P\Psi_0))^{(j)}(x)\right| = \left|(\alpha_n h)^{-j/2}\Lambda_{\alpha_n h}((P\Psi_0)^{(j)})(x)\right|$$

$$\leq h^{-\delta j}(\pi h \alpha_n)^{-1/4}\left|(P_j\Psi_0)((\alpha_n h)^{-1/2}x)\right|$$

Here, we use that $\alpha_n h \gg h^{2\delta}$. and P_j is a polynomial which depends linearly on P, with $\deg P_j = \deg P + j$ and $N_\infty(P_j) \leq C_{d,j}N_\infty(P)$. Hence, we have

$$q_k(u) \leq \sup_{j \leq k} \sup_{x \in \mathbb{R}} \left|(P_j\Psi_0)((\alpha_n h)^{-1/2}x)\right|\left(1 + \frac{x^2}{\alpha_n h}\right)$$

$$\leq \sup_{j \leq k} \sup_{x \in \mathbb{R}} |(P_j\Psi_0)(x)|\left(1 + x^2\right)$$

$$\leq \sup_{j \leq k} C_{d,j}N_\infty(P_j) \leq C_{d,k}N_\infty(P)$$

$\square$

To apply the formalism of Appendix A.2, we will require the following lemma. This a more or less direct application of the stationary phase theorem in the quadratic case. We write its proof in appendix A.1.3. This lemma explains how to compute $\mathrm{Op}_h(m)u$ for $u \in \mathcal{C}$ and $m \in S_\delta$.

Lemma 5.10. There exists $M > 0$ such that the following holds. Assume that $m \in S_\delta$ or $m = h_0$. Then, for all $k \in \mathbb{N}$, there exists $A_k(m) : \mathcal{C} \to \mathcal{C}$ such that for $u \in \mathcal{C}$, written under the form $u(x) = a(x)e^{i\beta_n \frac{x^2}{2h}}$, we have

- $A_0(m)u(x) = m(x, \beta_n x)u(x)$;
- For $k \geq 1$, $A_k(m)$ is of the form

$$(5.21) \qquad A_k(m)u(x) = \sum_{l \leq 2k} c_l(x)\partial_x^l a(x) e^{i\beta_n \frac{x^2}{2h}}$$

  where $|c_l^{(p)}(x)| \leq C_{l,k,p} h^{(l-p)\delta}$.
- For all $(j,k) \in \mathbb{N}^2 \setminus \{(0,0)\}$, there exists $c_{j,k} > 0$ such that for all $u \in \mathcal{C}$, $q_j(A_k(m)u) \leq c_{j,k} q_{2k+j}(u)$;
- For all $N \in \mathbb{N}^*$ and for all $j \in \mathbb{N}$, there exists $C_{j,N} > 0$ such that

$$q_j\left(\mathrm{Op}_h(m)u - \sum_{k=0}^{N-1} h^{k(1-2\delta)} A_k(m)u\right) \leq C_{j,N} q_{j+2N+M}(u) h^{N(1-2\delta)}$$

Remark. We need to distinguish the cases $m = h_0$ and $m \in S_\delta$ because h_0 is not in S_δ (recall that we only have $h_0 = O(\log(1/h))$). However, h_0 satisfies $|\partial^\alpha h_0| \leq C_\alpha h^{-|\alpha|\delta}$ as soon as $|\alpha| \geq 1$. This explains why we restrict on $(j,k) \neq (0,0)$ in the third item but in the case $m \in S_\delta$, the expression given in the first item shows that it also holds for $(j,k) = (0,0)$.

Gathering the terms of same order in the expansions of each $\mathrm{Op}_h(h_k)$ given by Lemma 5.10, we can build the family of operators

$$A_k : \mathcal{C} \to \mathcal{C} \; ; \; A_k = \sum_{j+l=k} A_j(h_l).$$

Each A_k has the same form as (5.21) and they satisfy, for all $u \in \mathcal{C}$,

- $A_0 u(x) = h_0(x, \beta_n x)u(x)$.
- For all $(j,k) \in \mathbb{N}^2 \setminus \{(0,0)\}$, there exists $c_{j,k} > 0$ such that for all $u \in \mathcal{C}$, $q_j(A_k u) \leq c_{j,k} q_{2k+j}(u)$;
- For all $N \in \mathbb{N}^*$ and for all $j \in \mathbb{N}$, there exists $C_{j,N} > 0$ such that

$$q_j\left(Au - \sum_{k=0}^{N-1} h^{k(1-2\delta)} A_k u\right) \leq C_{j,N} q_{j+2N+M}(u) h^{N(1-2\delta)}$$

We now use the formulas and notations of Appendix A.2 to show :

Proposition 5.2. Assume that $P \in \mathbb{C}[X]$ is of degree d and consider the state $u = \Lambda_{\alpha_n h}(P\Psi_0)e^{\frac{i\beta_n x^2}{2h}}$. Then, t being fixed, there exists a family of functions (f_k) and $K_1 > 0$ such that,

- $v_0(x) = u(x)$;
- For all $N \in \mathbb{N}^*$, there exists $C_{N,d}$ such that

$$\left\|A_{q_n}\left(e^{-tG}B'_{q_n}T(\hat{\rho}_n)u - \sum_{k=0}^{N-1} h^{k(1-2\delta)} B'_{q_n} T(\hat{\rho}_n)u_k\right)\right\| \leq C_{N,d} h^{N(1-2\delta)} h^{-tK_1} N_\infty(P)$$

  where

$$u_k(x) = \exp\left(-th_0(x, \beta_n x)\right) v_k(x) \; ; \; v_k(x) = f_k(x)\left(\Lambda_{\alpha_n h}\Psi_0\right)(x)e^{\frac{i\beta_n x^2}{2h}}$$

- For all $k \in \mathbb{N}$, there exists $c_{k,d} > 0$ such that for all $x \in \mathbb{R}$,

$$|f_k(x)| \leq c_{k,d}\left(1 + \frac{x^2}{\alpha_n h}\right)^{k/2} N_\infty(P)$$

Remark. In particular, these last estimates imply that $v_k \in \mathcal{C}$.

Proof. We use the notations and formulas of Appendix A.2, with parameter $\tilde{h} = h^{1-2\delta}$. We define a family $(v_k(t))$ by the iterative formula (A.3). The operator A_0 is nothing but the multiplication by

$$a_0(x) = h_0(x, \beta_n x)$$

and hence, e^{sA_0} is the multiplication by $\exp(sa_0)$.

Let us note $A_k(s) = e^{-sA_0} A_k e^{sA_0}$ and let us show that $A_k(s)u(x)$ has the same form as (5.21), with the functions $c_l(x)$ replaced by functions $\tilde{c}_l(s, x)$. We have

$$e^{-sa_0} c_l(x)\partial_x^l \left(e^{sa_0}a(x)\right) = c_l(x) \sum_{m=0}^{l} \binom{l}{m} a_m(s, x)a^{(l-m)}(x)$$

$$= \sum_{m=0}^{l} \tilde{c}_{l,m}(s, x)\partial_x^m a(x)$$

where $a_m(s, x) = e^{-sa_0}\partial_x^m(e^{sa_0})$ is a sum of terms of the form

$$s^i \prod_{j=1}^{i} a_0^{(k_j)} \text{ with } (k_1, \ldots, k_i) \in (\mathbb{N}^*)^i \text{ and } k_1 + \cdots + k_i = m$$

and $\tilde{c}_{l,m}(s, x) = \binom{m}{l} c_l(x)a_{l-m}(s, x)$. It is not hard to see that $|\partial_x^p a_m(s, x)| \leq C_{m,p}(1+|s|)^m h^{-\delta(m+p)}$ so that we have

$$|\partial_x^p \tilde{c}_{l,m}(x)| = \binom{m}{l} \left| \sum_{p_1+p_2=p} \binom{p}{p_1} c_l^{(p_1)}(x)a_{l-m}^{(p_2)}(s, x) \right|$$

$$\leq C_{p,l,m} \sup_{p_1+p_2=p} h^{\delta(l-p_1)\delta}h^{-(l-m+p_2)\delta} \leq C_{p,l,m}h^{(m-p)\delta}$$

which shows that the term in front of ∂_x^m has the correct behavior to be of the form (5.21) and we can set $\tilde{c}_m(s, x) = \sum_{l \leq 2k} \tilde{c}_{l,m}(s, x)$ so that

$$A_k(s)u(x) = \sum_{m \leq 2k} \tilde{c}_m(s, x)\partial_x^m \left(ue^{-i\beta_n x^2/2h}\right) e^{i\beta_n x^2/2h}$$

Let us now analyze the action of $A_k(s)$ on states of the form $c(x)\Lambda_{\alpha_n h}(\Psi_0)(x)e^{i\beta_n \frac{x^2}{2h}}$. We claim that we can write

(5.22) $$A_k(s)\left(c(x)\Lambda_{\alpha_n h}(\Psi_0)(x)e^{i\beta_n \frac{x^2}{2h}}\right) = d_k(s, x)\Lambda_{\alpha_n h}(\Psi_0)(x)e^{i\beta_n \frac{x^2}{2h}}$$

where

(5.23) $$|\partial_x^p d_k(s, x)| \leq C_{k,p}h^{-p\delta}(1+|s|)^k \sup_{y\in\mathbb{R}, m\leq 2k+p} |c^{(m)}(y)| \left(1 + \frac{x^2}{\alpha_n h}\right)^k.$$

To see that, let us write

$$\tilde{c}_m(s, x)\partial_x^m(c(x)\Lambda_{\alpha_n h}(\Psi_0)) = \tilde{c}_m(s, x) \sum_{l=0}^{m} \binom{m}{l} c^{(m-l)}(x)(\Lambda_{\alpha_n h}(\Psi_0))^{(l)}(x)$$

$$= \underbrace{\left(\sum_{l=0}^{m} \binom{m}{l} \tilde{c}_m(s, x)c^{(m-l)}(x)\frac{Q_l((\alpha_n h)^{-1/2}x)}{(\alpha_n h)^{l/2}}\right)}_{C_m(s, x)} \Lambda_{\alpha_n h}(\Psi_0)(x)$$

where $Q_l \in \mathbb{R}[X]$ are some polynomials of degree l. We hence have,

$$\left|\partial_x^{(p)} C_m(s, x)\right| = \left|\partial_x^p \left(\sum_{m_1+m_2=m} \binom{m}{m_1} \tilde{c}_m(s, x)c^{(m_1)}(x)\frac{Q_{m_2}((\alpha_n h)^{-1/2}x)}{(\alpha_n h)^{m_2/2}}\right)\right|$$

$$\leq C_{m,p} \sup_{\substack{m_1+m_2=m \\ p_1+p_2+p_3=p}} |\partial_x^{p_1}\tilde{c}_m(s, x)||c^{(m_1+p_2)}(x)|(\alpha_n h)^{-m_2/2-p_3/2}|Q_{m_2}^{(p_3)}((\alpha_n h)^{-1/2}x)|(1+|s|)^m$$

$$\leq C_{m,p} \sup_{\substack{m_1+m_2=m \\ p_1+p_2+p_3=p}} h^{(m-p_1)\delta}|c^{(m_1+p_2)}(x)|(\alpha_n h)^{-m_2/2-p_3/2}(1+|s|)^m \left(1 + \frac{x^2}{\alpha_n h}\right)^{(m_2-p_3)/2}$$

$$\leq C_{m,p} \sup_{y\in\mathbb{R}, l\leq m+p} |c^{(l)}(y)| \sup_{\substack{m_1+m_2=m \\ p_1+p_2+p_3=p}} h^{(l-p_1)\delta}h^{-(m_2+p_3)\delta}(1+|s|)^m \left(1 + \frac{x^2}{\alpha_n h}\right)^{m/2}$$

$$\leq C_{m,p} \sup_{y\in\mathbb{R}, l\leq m+p} |c^{(l)}(y)|h^{-\delta p}(1+|s|)^m \left(1 + \frac{x^2}{\alpha_n h}\right)^{m/2}$$

and the claim is proved, with $d_k(s,x) = \sum_{m \leq 2k} C_m(s,x)$.

We now analyze precisely the iteration formula (A.3) in Appendix A.2. We use the notations of this appendix f(in particular, for the remainders $\tilde{r}_{j,N}$ and R_N). Let $K_0 > 0$ be such that $|h_0| \leq K_0 \log 1/h$ so that $|e^{ta_0}| \leq h^{-tK_0}$. For $j \geq 0$, we have

$$q_j(e^{ta_0}u) \leq h^{-|t|K_0}c_{0,j}q_j(u)$$

This is obvious for $j = 0$. For $j \geq 1$, it comes from the fact that the derivatives of h_0 satisfy $|\partial^\alpha h_0| \leq C_\alpha h^{-|\alpha|\delta}$ for $\alpha \neq 0$ and the definition of q_j in (5.20).

Leading term. For our leading term in the expansion we want in Proposition 5.2, we simply have

$$u_0(t) = e^{ta_0}u$$

As a consequence,

$$q_j(\tilde{r}_{0,N}(t)) \leq C_{j,N}\tilde{h}^N q_{j+2N+M}(e^{ta_0}u) \leq C_{j,N}\tilde{h}^N c_{0,j+2N+M}h^{-|t|K_0}q_{j+2N+M}(u) \leq C_{d,j,N}\tilde{h}^N h^{-|t|K_0}N_\infty(P)$$

Iteration. By induction, using the formulas (A.3) and (5.22), we see that if the initial state is $u = \Lambda_{\alpha_n h}(P\Psi_0)e^{i\beta_n\frac{x^2}{2h}}$ then

$$v_k(t) = f_k(t,x)\Lambda_{\alpha_n h}(\Psi_0)(x)e^{i\beta_n\frac{x^2}{2h}}$$

where $|\partial_x^p f_k(s,x)| \leq C_{k,p}h^{-p\delta}(1+|s|)^k\left(1+\frac{x^2}{\alpha_n h}\right)^{k/2}N_\infty(P)$. When $p = 0$, it gives the required estimate for $|f_k(x)|$ in Proposition 5.2. It follows that

$$q_j(e^{tA_0}v_k(t)) \leq h^{-|t|M_0}c_{0,j}q_j(v_k(t)) \leq h^{-|t|M_0}c_{0,j}(1+|t|)^k h^{-|t|M_0}\tilde{C}_{d,j,k}(1+|t|)^k N_\infty(P)$$

Moreover, we can estimate

$$q_j(\tilde{r}_{k,N}(t)) \leq C_{j,N-k}\tilde{h}^{N-k}q_{j+2(N-k)+M}(e^{tA_0}v_k(t)) \leq C_{d,N,j,k}\tilde{h}^{N-k}h^{-|t|M_0}(1+|t|)^k N_\infty(P)$$

Conclusion. We find that for $j \in \mathbb{N}$,

$$q_j\left(\sum_{k=0}^{N-1}\tilde{h}^k \tilde{r}_{k,N}(t)\right) \leq \tilde{h}^N h^{-|t|M_0}\sum_{k=0}^{N-1}C_{d,N,j,k}(1+|t|)^k N_\infty(P)$$

$$\leq C_{d,j,N}\tilde{h}^N h^{-|t|M_0}(1+|t|)^{N-1}N_\infty(P)$$

Integrating (A.4), and recalling that $||\cdot|| \leq Cq_0$ in $\mathcal{C}$, we have

$$||R_{N-1}(t)|| \leq \int_0^{|t|}||A||\,||R_{N-1}(s)||ds + C_{d,0,N}\tilde{h}^N h^{-|t|M_0}(1+t^2)^{N-1}N_\infty(P)$$

By a version of Gronwall's lemma, we can find a constant $C_{N,d}$ such that

$$||R_{N-1}(t)|| \leq C_{N,d}\tilde{h}^N e^{|t|\max(K_0|\log h|,||A||)}(1+t^2)^N N_\infty(P)$$

Since, $||A|| = O(\log h)$, there exists $K_1 > 0$ such that $e^{|t|\max(K_0|\log h|,||A||)} \leq h^{-|t|K_1}$ and it concludes the proof of Proposition 5.2. $\qquad\square$

Combining Proposition 5.1, (5.18) and Proposition 5.2, we deduce the following expansion :

Corollary 5.2.

$$(5.24) \quad A_{q_n}e^{-tG}\mathfrak{M}^{n-1}M^{n_0}A_q B_q'\varphi_{\hat{\rho}} = A_{q_n}\sum_{2j+k+l+2m<2N}h^{j+k/2+l\varepsilon+m\varepsilon}u_n^{(j,k,l,m)}$$

$$+ O\left(h^{-K-t(K_0+K_1)}h^{2N\varepsilon}(\log h)^N\right)$$

where

$$T(\hat{\rho}_n)^* u_n^{(j,k,l,m)}(x) = e^{tg(\rho)-th_0(x,\beta_n x)}f_n^{(j,k,l,m)}(x)\Lambda_{\alpha_n h}(\Psi_0)(x)e^{\frac{i\beta_n x^2}{2h}}$$

where we have, for all $x \in \mathbb{R}$,

$$\left|f_n^{(j,k,l,m)}(x)\right| \leq C_{j,k,l,m}n^{2j+k}\Pi_{\alpha,n}(\rho)\left(J_{\mathbf{q}}^u\right)^{3k}h^{-k\varepsilon}\left(1+\frac{x^2}{\alpha_n h}\right)^{m/2}$$

Concerning the leading term, $f_n^{(0,0,0,0)}$ is constant equal to $\pi_{\alpha,n}(\rho)$.

Proof. In the expansion of Proposition 5.1, we transform the states $u_n^{(j,k,l)}$ using formula (5.18). Finally we use Proposition 5.2 on each such state. For $u_n^{(j,k,l)}$, we keep the $N_{j,k,l}$ first terms of the expansion, where $N_{j,k,l} = N - j - \lceil (k+l)/2 \rceil$. It gives a remainder term $r_n^{(j,k,l)}$ satisfying

$$
\begin{aligned}
||r_n^{(j,k,l)}||_{L^2} &\le C_{N,j,k,l} h^{j+k/2+l\varepsilon} h^{2\varepsilon N_{j,k,l}} N_\infty \left(\Phi_n \left(P_n^{(j,k,l)} \right) \right) h^{-t(K_0+K_1)} \\
&\le C_{N,j,k,l} h^{j+k/2+l\varepsilon} h^{2\varepsilon N_{j,k,l}} h^{-k\varepsilon} n^{2j+k} \Pi_{\alpha,n}(\rho) \left(J_{\mathbf{q}}^u \right)^{3k} h^{-t(K_0+K_1)} \\
&\le C_{N,j,k,l} h^{j+k/2+l\varepsilon} h^{2\varepsilon N_{j,k,l}} n^{2j+k} h^{-k\varepsilon} h^{-k\frac{1-4\varepsilon}{2(1+\varepsilon)}} h^{-K-t(K_0+K_1)} \\
&\le C_{N,j,k,l} \left(\log 1/h \right)^N h^{2\delta j} h^{2\varepsilon N} h^{k\left(1/2 - 2\varepsilon - \frac{1-4\varepsilon}{2(1+\varepsilon)}\right)} h^{-K-t(K_0+K_1)}
\end{aligned}
$$

But we have $\frac{1}{2} - 2\varepsilon - \frac{1-4\varepsilon}{2(1+\varepsilon)} \ge 0$ (assuming that $\varepsilon \le 1/4$, which is not a problem since we work with ε small). Hence, $||r_n^{(j,k,l)}|| \le C_{N,j,k,l} \left(\log 1/h \right)^N h^{-K-t(K_0+K_1)}$. As a consequence, gathering all the remainders $r_n^{(j,k,l)}$ together and adding them to $e^{-tG} B'_{q_n} R_n^{(N)}$, we obtain a remainder term controlled by $C_N h^{-K-t(K_0+K_1)} h^{2N\varepsilon} \left(\log 1/h \right)^N$ as expected. $\qquad\square$

5.5. **Crucial estimates for the terms of the expansion.** In the expansion of Corollary 5.2, the leading term is given by

$$
u_n^0 := T^*(\hat{\rho}_n) u_n^{(0,0,0,0)} = \exp \left(tg(\rho) - th_0(x, \beta_n x) \right) \Lambda_{\alpha_n h}(\Psi_0)(x) e^{\frac{i\beta_n x^2}{2h}} \pi_{\alpha,n}(\rho)
$$

As a consequence of the Corollary 5.2, the other terms have the form

$$
T^*(\hat{\rho}_n) = u_n^{(j,k,l,m)}(x) = f_n^{(j,k,l,m)}(x) \frac{u_n^0(x)}{\pi_{\alpha,n}(\rho)}
$$

with

$$
\left| f_n^{(j,k,l,m)}(x) \right| \le C_{j,k,l,m} n^{2j+k} \left(J_{q_0 \dots q_n}^u \right)^{3k} \Pi_{\alpha,n}(\rho) h^{-k\varepsilon} \left(1 + \frac{x^2}{\alpha_n h} \right)^{m/2}
$$

so that, denoting

$$
v_n = \frac{u_n^0}{\pi_{\alpha,n}} = \exp \left(tg(\rho) - th_0(x, \beta_n x) \right) \Lambda_{\alpha_n h}(\Psi_0)(x) e^{\frac{i\beta_n x^2}{2h}}
$$

we have

$$
(5.25) \qquad \frac{\left| \left| h^{j+k/2+l\varepsilon+m\varepsilon} u_n^{(j,k,l,m)} \right| \right|_{L^2}}{\Pi_{\alpha,n}(\rho) \times \left| \left| \left(1 + \frac{x^2}{\alpha_n h} \right)^{m/2} v_n \right| \right|_{L^2}} \le C_{j,k,l,m} h^{j+k/2+l\varepsilon+m\varepsilon} n^{2j+k} \left(J_{q_0 \dots q_n}^u \right)^{3k} h^{-k\varepsilon}
$$

Recalling that $\left(J_{q_0 \dots q_n}^u \right)^{3k} h^{-k\varepsilon} \le C_k e^{3kn\lambda_{\max}(1+\varepsilon)} h^{-k\varepsilon} \le C_k h^{-k\frac{1-4\varepsilon}{2(1+\varepsilon)}-\varepsilon} \ll C_k h^{-k/2}$, we see that the right hand side in the above inequality tends to 0 when $h \to 0$. As a consequence, it is enough to control

$$
\left| \left| \left(1 + \frac{x^2}{\alpha_n h} \right)^{m/2} v_n \right| \right|_{L^2}
$$

This is what we do in the rest of this subsection.

5.5.1. *Reduction to a compact interval.* Note that since $\mathrm{WF}_h(A_q)$ is compact, there exists $\chi \in C_c^\infty(\mathbb{R})$ such that

$$
A_q B'_q = A_q B'_q \chi(x) + O(h^\infty)
$$

and it is possible to choose a single χ for all the A_q. Indeed, recall that $\mathrm{WF}_h(A_q) = \mathrm{supp}(\chi_q) \Subset \mathcal{W}_q \subset B(\rho_q, 2\varepsilon_0)$ and that κ_q is well-defined in a neighborhood of ρ_q of fixed size ε_1 bigger than ε_0. There exists a $\Xi_q \in \Psi_0(\mathbb{R}^2)$ such that

$$
A_q B'_q \Xi_q = A_q B'_q + O(h^\infty) \quad ; \quad \mathrm{WF}_h(\Xi_q) \Subset \kappa_q(\mathcal{W}_q)
$$

In particular, $\mathrm{diam}\left(\mathrm{WF}_h(\Xi_q) \right) = O(\varepsilon_0)$. Hence, it is enough to fix $\chi \in C_c^\infty(\mathbb{R})$ such that $\chi = 1$ in a neighborhood of $\pi_x(\mathrm{WF}_h(\Xi_q))$ for all $q \in \mathcal{A}$ and such that $\mathrm{supp}\,\chi \subset [-C\varepsilon_0, C\varepsilon_0]$ for some large constant C independent of ε_0. As a consequence, we focus on χv_n.

We set

(5.26)
$$\boxed{\zeta_n(x) = \kappa_{q_n}^{-1}(\hat{\rho}_n + (x, \beta_n x)) \in \mathcal{W}_{q_n}}$$

It describes a curve, preimage by κ_{q_n} of the line $\hat{\rho}_n + (x, \beta_n x)$. To ensure that $\zeta_n(x)$ is well defined, $\hat{\rho}_n + (x, \beta_n x)$ has to be at distance at most ε_1 of $\kappa_{q_n}(\rho_{q_n}) = 0$. We claim that we may choose ε_0 small enough so that

(5.27)
$$x \in \text{supp}(\chi) \implies \zeta_n(x) \text{ is well defined}$$

Indeed, $x \in \text{supp}\,\chi \implies |x| \leq C\varepsilon_0$, so that $d(\hat{\rho}_n, \kappa_{q_n}(\rho_{q_n})) = O(\varepsilon_0)$ and we choose ε_0, ensuring the good definition of $\zeta_n(x)$.

5.5.2. *Control of the norm of v_n.* Our goal is to control the norm of χv_n, which allows to control the leading term. In fact, as already explained, to control the higher order terms, it is also necessary to control the norm of $\chi \left(1 + \frac{x^2}{\alpha_n h}\right)^{m/2} v_n$. Let us note $\widetilde{\Psi}_m(x) = \pi^{-1/4}(1+x^2)^{m/2}e^{-x^2/2}$.

$$\left\|\left(1 + \frac{x^2}{\alpha_n h}\right)^{m/2} \chi v_n\right\|_{L^2}^2 = \int_{\mathbb{R}} |\chi(x)|^2 e^{2tg(\rho) - 2tg(\zeta_n(x))} |\Lambda_{\alpha_n h}(\widetilde{\Psi}_m)(x)|^2 dx$$

$$= \int_{\mathbb{R}} \left|\chi((\alpha_n h)^{1/2}x)\right|^2 e^{2tg(\rho) - 2tg(\zeta_n((\alpha_n h)^{1/2}x))} (1+x^2)^m e^{-x^2} dx$$

We have

Lemma 5.11.

$$\mathbb{1}_{|x| \geq \alpha_n^{1/2} h^{\delta_0}} \left(1 + \frac{x^2}{\alpha_n h}\right)^{m/2} \chi(x) v_n(x) = O(h^\infty)_{L^2}$$

The constants in $O(h^\infty)$ depend on m and ε, but neither on ρ nor n as soon as $n \sim \vartheta_\varepsilon \log 1/h$.

Proof. Since $g = O(\log h)$, we have

$$e^{2tg(\rho) - 2tg(\zeta_n((\alpha_n h)^{1/2}x))} \leq \exp(O(\log(h))) = O(h^{-C})$$

for some C depending on t and g. We also have $|\chi| = O(1)$. Hence, after a change of variable $x = \alpha_n^{1/2} h^{1/2} y$, it suffices to estimate

$$\int_{\mathbb{R}} \mathbb{1}_{[-\alpha_n^{1/2} h^{\delta_0}, \alpha_n^{1/2} h^{\delta_0}]} (\alpha_n^{1/2} h^{1/2} y)(1+y^2)^m e^{-y^2} dy = \int_{|y| \geq h^{\delta_0 - 1/2}} (1+y^2)^m e^{-y^2} dy$$

Since $\delta_0 < 1/2$, we conclude by using the standard estimate

$$\int_{|y| > \lambda} (1+y^2)^m e^{-y^2} dy = O_m(\lambda^{-\infty})$$

$\square$

A very important consequence of this lemma is that we only need to focus on $\zeta_n(x)$ where $|x| \leq \alpha_n^{1/2} h^{\delta_0}$. In particular,

$$|x| \leq \alpha_n^{1/2} h^{\delta_0} \implies d(\zeta_n(x), \rho_n) \leq C\alpha_n^{1/2} h^{\delta_0}$$

Recall that $\alpha_n^{1/2} \leq Ce^{n\lambda_{\max}(1+\varepsilon)} \leq Ch^{-\frac{1-4\varepsilon}{6(1+\varepsilon)}}$ and recall that δ_0 is such that

(5.28)
$$\delta_0 - \frac{1-4\varepsilon}{6(1+\varepsilon)} \geq \frac{1}{3}$$

ensuring that $\alpha_n^{1/2} h^{\delta_0} \leq Ch^{1/3}$. It will turn out to be important.

5.5.3. *Control outside an h^δ-neighborhood of $\mathcal{T}$.* We first treat the case where ρ lies outside an h^δ-neighborhood of $\mathcal{T}$ (in fact, we will be slightly less precise in the unstable direction, see the Lemmata below). The following estimate strongly relies on the structure of the escape function g. The escape property of g has been used in [NSZ14] to damp the symbol of the Fourier integral operator $M(h)$ and they prove that the norm of $M_t(h)$ outside an h^δ-neighborhood of $\mathcal{T}$ can be smaller than any arbitrary ε as soon as t is well chosen. Here, we want to obtain strong polynomial decay of the form h^L for some $L = L(t)$ as large as we want if t is sufficiently large. This will be possible since we propagate on logarithmic times $n(h)$.

We are interested in controlling the term

$$d(x) := \exp\left(tg(\rho) - tg(\zeta_n(x))\right)$$

which controls the norm of χv_n. Indeed, since $||\widetilde{\Psi}_m||_\infty < +\infty$, we have

$$\left\|\left(1 + \frac{x^2}{\alpha_n h}\right)^{m/2} \chi v_n\right\|_{L^2}^2 \leq \int_{|x|\leq\alpha_n^{1/2}h^{\delta_0}} d(x)^2 |\Lambda_{\alpha_n h}\widetilde{\Psi}_m(x)|^2 dx + O(h^\infty)$$

$$\leq C_m(\alpha_n h)^{-1/2}\int_{|x|\leq\alpha_n^{1/2}h^{\delta_0}} d(x)^2 dx + O(h^\infty)$$

In virtue of the construction of g in (3.19), we have

$$(5.29)\qquad d(x) = (R_-(x)R_+(x))^t\,,\; R_-(x) := \frac{Mh^{2\delta} + \hat{\varphi}_-(\rho)}{Mh^{2\delta} + \hat{\varphi}_-(\zeta_n(x))}\,,\; R_+(x) := \frac{Mh^{2\delta} + \hat{\varphi}_+(\zeta_n(x))}{Mh^{2\delta} + \hat{\varphi}_+(\rho)}$$

(These terms depend on ρ and h, but we voluntarily omit this dependence to alleviate the notations). Recall also that $\rho_n = F^{nn_0}(\rho) = \kappa_{q_n}^{-1}(\hat{\rho}_n) = \zeta_n(0)$.

Proposition 5.3. Estimates for R_-. There exists a global constant $C > 0$ (i.e. depending only F and ε trough the choice of the partition of unity, but independent of ρ, h and $\mathbf{q}$) such that for all $x \in [-\alpha_n^{1/2}h^{\delta_0}, \alpha_n^{1/2}h^{\delta_0}]$, we have

- If $d(\rho, \mathcal{T}_-) \geq h^\delta$, $R_-(x) \leq C\left(J_\mathbf{q}^u\right)^{-2}$;
- If $d(\rho, \mathcal{T}_-) \leq h^\delta$, $R_-(x) \leq C$.

Proof. We pick $x \in [-\alpha_n^{1/2}h^{\delta_0}, \alpha_n^{1/2}h^{\delta_0}]$.

- We assume that $d(\rho, \mathcal{T}_-) \geq h^\delta$. By Lemma 4.2 and (5.17), $d(\rho_n, \mathcal{T}_-) \geq C^{-1}J_\mathbf{q}^u d(\rho, \mathcal{T}_-) \geq C^{-1}\alpha_n^{1/2}h^\delta$. Then, we have

$$d(\zeta_n(x), \mathcal{T}_-) \geq d(\rho_n, \mathcal{T}_-) - d(\zeta_n(x), \rho_n) \geq C^{-1}\alpha_n^{1/2}d(\rho, \mathcal{T}_-) - C\alpha_n^{1/2}h^{\delta_0} \geq \tilde{C}^{-1}\alpha_n^{1/2}d(\rho, \mathcal{T}_-)$$

   since $\delta < \delta_0$ and $h^\delta \leq d(\rho, \mathcal{T}_-)$. Recall that $\hat{\varphi}_-(\zeta) \sim h^{2\delta} + d(\zeta, \mathcal{T}_-)^2$. Hence,

$$R_-(x) \leq \frac{C_1 d(\rho, \mathcal{T}_-)^2}{C_2\alpha_n d(\rho, \mathcal{T}_-)^2} \leq C\alpha_n^{-1} \leq C\left(J_\mathbf{q}^u\right)^{-2}$$

- The second point is much easier (and in fact very crude at this stage) : if $d(\rho, \mathcal{T}_-) \leq h^\delta$, the numerator $Mh^{2\delta} + \hat{\varphi}_-(\rho)$ is smaller that $Ch^{2\delta}$. Concerning the denominator, we simply use the fact that $\hat{\varphi}_- \geq 0$ to bound it from below by $Mh^{2\delta}$, and we deduce that $R_-(x) \leq C$.
$\qquad\square$

We now come to the case of R_+. Before, we need to understand more precisely the Lagrangian space $\{(x, \beta_n x)\}$. We expect it to be a good first order approximation of an unstable manifold. This is the content of the following lemma :

Lemma 5.12. There exists a global constant $C > 0$ such that the following holds : there exists $\zeta_n^\star \in \mathcal{T}$ such that for all $x \in [-\alpha_n^{1/2}h^{\delta_0}, \alpha_n^{1/2}h^{\delta_0}]$, we have

$$d(\zeta_n(x), \mathcal{T}_+) \leq d(\zeta_n(x), W_u(\zeta_n^\star)) \leq C\left(h^{1/3} + \left(J_\mathbf{q}^u\right)^{-1}\right)d(\rho, \mathcal{T}_+) + Ch^{\delta_0}$$

Remark. The different terms which compose the error above appear at different places in the proof. One of this term is due to the fact that it is a first order approximation of an unstable manifold : we need to control the error term in this approximation. It turns out that as soon as $|x| \leq \alpha_n^{1/2}h^{\delta_0} = O(h^{1/3})$, this error is $O(|x|^2) = O(h^{2/3})$. Depending on ρ (and $\mathbf{q}$), the main term of the error can differ. As we will see, when $d(\rho, \mathcal{T}) \geq h^{\delta_1}$, the term h^{δ_0} is negligible.
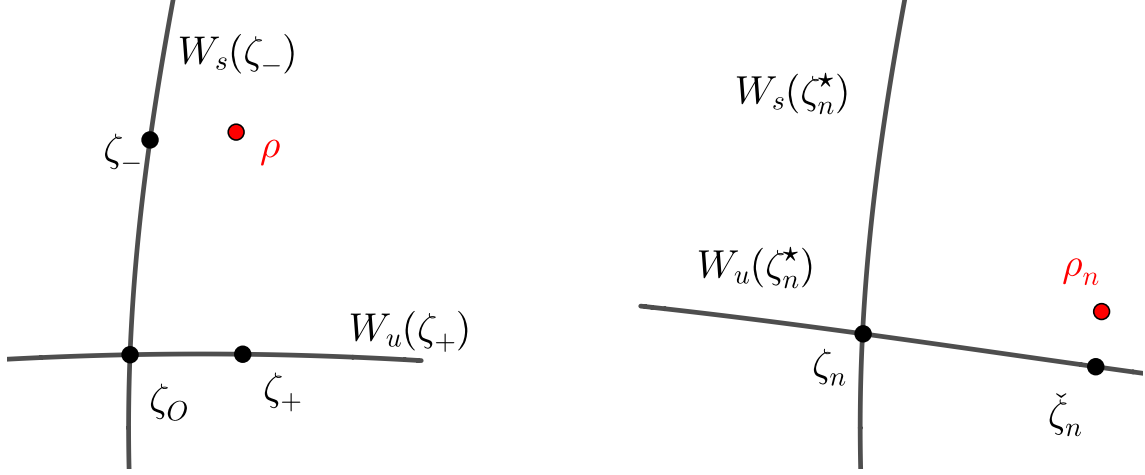
FIGURE 7. The points introduced in Step 1 of the proof of Lemma 5.12.

Proof. **Step 1 : ρ_n is close to a reference unstable manifold $W_u(\zeta_n^\star)$.** (See Figure 7)

In this first step, we want to show that ρ_n is close to an unstable manifold $W_u(\zeta_n^\star)$. As in the proof of Lemma 4.2, we consider a point $\zeta_+ \in \mathcal{T}$ such that $d(\rho, \mathcal{T}_+) = d(\rho, W_u(\zeta_+))$ and such that for all $0 \le i \le n$, $d(F^{in_0}(\zeta_+), F^{in_0}(\rho)) \le \varepsilon_2$ for some small ε_2 depending on ε_0. Note also that, by the third point of Lemma 4.2, $d(\rho, \mathcal{T}_-) \le C_2 \left(J_{\mathbf{q}}^u\right)^{-1} \varepsilon_0$. Let's fix a point ζ_- such that $d(\rho, \mathcal{T}_-) = d(\rho, W_s(\zeta_-))$ and let's consider ζ_O the unique point in $W_u(\zeta_+) \cap W_s(\zeta_-)$. Then, we still have $d(\rho, \mathcal{T}_+) = d(\rho, W_u(\zeta_O))$ and we also have

$$d(\rho, \zeta_O)^2 \sim d(\rho, \mathcal{T}_-)^2 + d(\rho, \mathcal{T}_+)^2 \le C \left(\left(J_{\mathbf{q}}^u\right)^{-1} \varepsilon_0 \right)^2 + d(\rho, \mathcal{T}_+)^2$$

For $0 \le i \le n$, set $\zeta_i^\star = F^{in_0}(\zeta_O)$. We have

$$d(\rho_n, W_u(\zeta_n^\star)) \le C \left(J_{\mathbf{q}}^u\right)^{-1} d(\rho, W_u(\zeta_O)) \le C \left(J_{\mathbf{q}}^u\right)^{-1} d(\rho, \mathcal{T}_+)$$

and

$$d(\rho_n, \zeta_n^\star)^2 \le C \left(J_{\mathbf{q}}^u\right)^2 d(\rho, \mathcal{T}_-)^2 + C \left(J_{\mathbf{q}}^u\right)^{-2} d(\rho, \mathcal{T}_+)^2 \le C\varepsilon_0$$

Let us fix $\check{\zeta}_n \in W_u(\zeta_n^\star)$ such that

$$d(\rho_n, W_u(\zeta_n^\star)) = d(\rho_n, \check{\zeta}_n)$$

Step 2 : The curve $\zeta_n(x)$ is close to the (unstable) tangent space $E_u(\check{\zeta}_n)$.

Step 2-a : First approximation. (See Figure 8).

We now want to show that the curve is a rather good approximation of the tangent space of $W_u(\zeta_n^\star)$ at $\check{\zeta}_n$. To do so, we make the following observation (recall the notations of (5.15) and the definition of β_n in (5.16)).

$$\mathbf{v}_n := \begin{pmatrix} 1 \\ \beta_n \end{pmatrix} = \alpha_n^{-1/2} d_{\hat{\rho}} F^{(n)}(\mathbf{v}_n') \ ; \ \mathbf{v}_n' = \alpha_n^{-1/2} \begin{pmatrix} a_n \\ b_n \end{pmatrix}$$

and note that $||\mathbf{v}_n'|| = 1$ (since $\alpha_n^2 = a_n^2 + b_n^2$). We compare this vector $\mathbf{v}_n$ to $\mathbf{w}_n := \alpha_n^{-1/2} d_{\hat{\zeta}} F^{(n)}(\mathbf{v}_n')$ where $\hat{\zeta} = \kappa_{q_0}(\check{\zeta}_0)$ with $\check{\zeta}_0 = F^{-nn_0}(\check{\zeta}_n)$. Arguing as in the proof of Lemma 5.8, we can show that

$$||d_{\hat{\rho}} F^{(n)} - d_{\hat{\zeta}} F^{(n)}|| \le C J_{\mathbf{q}}^u d(\rho, \check{\zeta}_0)$$

By the triangular inequality, $d(\rho, \check{\zeta}_0) \le d(\rho, \zeta_O) + d(\zeta_O, \check{\zeta}_0)$ where the first term is controlled by $C \left(\left(J_{\mathbf{q}}^u\right)^{-1} \varepsilon_0 + d(\rho, \mathcal{T}_+) \right)$. For the second term, we use the fact that $\check{\zeta}_0 \in W_u(\zeta_O)$ and $d(\check{\zeta}_n, F^{nn_0}(\zeta_O)) \le C\varepsilon_0$, this gives $d(\zeta_O, \check{\zeta}_0) \le C \left(J_{\mathbf{q}}^u\right)^{-1} \varepsilon_0$. As a consequence, we find that

$$||d_{\hat{\rho}} F^{(n)} - d_{\hat{\zeta}} F^{(n)}|| \le C J_{\mathbf{q}}^u d(\rho, \mathcal{T}_+) + C\varepsilon_0$$

Finally, recalling that $\alpha_n^{1/2} \sim J_{\mathbf{q}}^u$ - so that $\mathbf{v}_n$ and $\mathbf{w}_n$ are close from being normalized -, we get that $||\mathbf{v}_n - \mathbf{w}_n|| \le C d(\rho, \mathcal{T}_+) + C\varepsilon_0 \left(J_{\mathbf{q}}^u\right)^{-1}$.
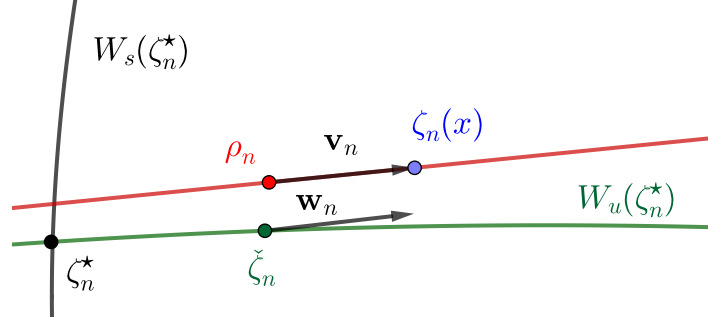
FIGURE 8. The curve $\zeta_n(x)$ (in red) passing through ρ_n is close to an unstable manifold $W_u(\zeta_n^\star)$ (in green). $W_u(\zeta_n^\star)$ is well approximated, near $\check{\zeta}_n$, by its tangent space at $\check{\zeta}_n$, spanned by a vector close to $\mathbf{w}_n$.

Let's now define $\widetilde{\zeta}_n(x)$ by

$$\kappa_{q_n}\left(\widetilde{\zeta}_n(x)\right) = \kappa_{q_n}\left(\check{\zeta}_n\right) + x\mathbf{w}_n$$

We have (recall that $\zeta_n(x) = \kappa_{q_n}^{-1}(\hat{\rho}_n + x\mathbf{v}_n)$)

$$\begin{aligned}
d(\widetilde{\zeta}_n(x), \zeta_n(x))) &\leq Cd(\kappa_{q_n}\left(\check{\zeta}_n\right) + x\mathbf{w}_n, \hat{\rho}_n + x\mathbf{v}_n) \\
&\leq Cd(\check{\zeta}_n, \rho_n) + |x| \|\mathbf{w}_n - \mathbf{v}_n\| \\
&\leq C\left(J_{\mathbf{q}}^u\right)^{-1} d(\rho, \mathcal{T}_+) + |x|\left(Cd(\rho, \mathcal{T}_+) + \varepsilon_0\left(J_{\mathbf{q}}^u\right)^{-1}\right) \\
&\leq C\left(J_{\mathbf{q}}^u\right)^{-1} d(\rho, \mathcal{T}_+) + CJ_{\mathbf{q}}^u h^{\delta_0}\left(d(\rho, \mathcal{T}_+) + C\varepsilon_0\left(J_{\mathbf{q}}^u\right)^{-1}\right) \text{ if } |x| \leq \alpha_n^{1/2} h^{\delta_0} \\
&\leq C\left(\left(J_{\mathbf{q}}^u\right)^{-1} + h^{1/3}\right) d(\rho, \mathcal{T}_+) + Ch^{\delta_0}
\end{aligned}$$

where we use the fact that $J_{\mathbf{q}}^u h^{\delta_0} \leq Ch^{1/3}$. We will now control the distance of $\widetilde{\zeta}_n(x)$ to $\mathcal{T}_+$.

Step 2-b : Comparison with the tangent space. (See Figure 8).

In this step, we want to show that $\mathbf{w}_n$ is close to a vector spanning $d\kappa_{q_n}(T_{\check{\zeta}_n} W_u(\zeta_n^\star))$. To do so, we use Lemma 3.5. If ε_0 is small enough (depending on the parameter ε_1 appearing in Lemma 3.5), we can ensure that the vector $\mathbf{v}_n$ is suffienctly close to $\mathbb{R} \times \{0\}$ and hence, $\left(d_{\check{\zeta}_0}\kappa_0\right)^{-1}\mathbf{v}_n'$ is sufficiently close to $T_{\check{\zeta}_0} W_u(\zeta_O)$, so that we can apply this lemma with initial vector $\left(d_{\check{\zeta}_0}\kappa_0\right)^{-1}\mathbf{v}_n'$. To alleviate the notations, let's note $\mathcal{L} = \kappa_{q_n}(W_u(\zeta_n^\star))$, $m = \kappa_{q_n}(\check{\zeta}_n)$. By applying Lemma 3.5 and sending the result in the chart κ_{q_n}, we obtain that

$$d\left(\frac{\mathbf{w}_n}{\|\mathbf{w}_n\|}, T_m\mathcal{L}\right) \leq C\left(J_{\mathbf{q}}^u\right)^{-2}$$

since $\|\mathbf{w}_n\| \leq C$, the same is true for $\mathbf{w}_n$. Let's pick $\mathbf{w}_n' \in T_m\mathcal{L}$ such that $\|\mathbf{w}_n - \mathbf{w}_n'\| \leq C\left(J_{\mathbf{q}}^u\right)^{-2}$. We now define $Z_n(x)$ by the relation

$$\kappa_{q_n}\left(Z_n(x)\right) = \kappa_{q_n}\left(\widetilde{\zeta}_n(x)\right) + \mathbf{w}_n' x$$

If $|x| \leq \alpha_n^{1/2} h^{\delta_0}$, it is clear that

$$d(Z_n(x), \widetilde{\zeta}_n(x)) \leq |x| \|\mathbf{w}_n - \mathbf{w}_n'\| \leq C\left(J_{\mathbf{q}}^u\right)^{-2} \alpha_n^{1/2} h^{\delta_0} \leq C\alpha_n^{-1/2} h^{1/2} \ll h^{\delta_0}$$

Gathering the steps 2-a and 2-b, we see that

$$d(\zeta_n(x), \mathcal{T}_+) \leq d(Z_n(x), \mathcal{T}_+) + C\left(\left(J_{\mathbf{q}}^u\right)^{-1} + h^{1/3}\right) d(\rho, \mathcal{T}_+) + Ch^{\delta_0}$$

Step 3 : The tangent space is a good approximation. The only remaining point is to control $d(Z_n(x), \mathcal{T}_+)$. We observe that $\mathbf{w}_n' \in T_m\mathcal{L}$. Hence, by standard results of differential geometry, $d(m + x\mathbf{w}_n', \mathcal{L}) \leq Cx^2$ where C depends on $\|\mathbf{w}_n'\|$ and on the curvature of $\mathcal{L}$ - which can be controlled independently of the base point $\widetilde{\zeta}_n$ of this unstable manifold. As a consequence,

if $|x| \leq \alpha_n^{1/2} h^{\delta_0} \leq h^{1/3}$, $d(m + x\mathbf{w}_n', \mathcal{L}) \leq Ch^{2/3} \ll Ch^{\delta_0}$. This shows that $d(Z_n(x), \mathcal{T}_+) \leq Cd(m + x\mathbf{w}_n', \mathcal{L}) \leq Ch^{\delta_0}$ and concludes the proof of the lemma. $\qquad\square$

This Lagrangian being well understood, we can now come to the estimates for R_+ :

Proposition 5.4. Estimates for R_+. There exists a global constant $C > 0$ such that for all $x \in [-\alpha_n^{1/2} h^{\delta_0}, \alpha_n^{1/2} h^{\delta_0}]$, we have

- If $d(\rho, \mathcal{T}_+) \geq h^{\delta_1}$, $R_+(x) \leq Ch^{2\varepsilon}$;
- If $d(\rho, \mathcal{T}_+) \leq h^{\delta_1}$, $R_+(x) \leq C$

(for some constant $C > 0$).

Proof. Recall that $\delta_1 = \delta - \varepsilon$. We pick $x \in [-\alpha_n^{1/2} h^{\delta_0}, \alpha_n^{1/2} h^{\delta_0}]$. Here, we will use the inequality $d(\rho_n, \mathcal{T}_+) \leq C \left(J_{\mathbf{q}}^u \right)^{-1} d(\rho, \mathcal{T}_+)$ and the result of the previous lemma, namely,

$$d(\zeta_n(x), \mathcal{T}_+) \leq C \left(h^{1/3} + \left(J_{\mathbf{q}}^u \right)^{-1} \right) d(\rho, \mathcal{T}_+) + Ch^{\delta_0}$$

Recall that $J_{\mathbf{q}}^u \geq C_\varepsilon e^{n(h)\lambda_{\min}(1-\varepsilon)} \geq C_\varepsilon h^{-\vartheta_\varepsilon \lambda_{\min}(1-\varepsilon)}$.
We choose some $0 < \beta < \min(1/3, \vartheta_\varepsilon \lambda_{\min}(1-\varepsilon))$, which ensures that

$$d(\zeta_n(x), \mathcal{T}_+) \leq Ch^{\beta} d(\rho, \mathcal{T}_+) + Ch^{\delta_0}$$

Note that since we work with ε small, it is harmless to assume that $\varepsilon < \beta$. We treat the two points separately :

- For this first point, we distinguish two cases :

  First case : $h^{\delta_1} \leq d(\rho, \mathcal{T}_+) \leq h^{\delta-\beta}$. In this context, one has $d(\zeta_n(x), \mathcal{T}_+) \leq Ch^{\beta} h^{\delta-\beta} + Ch^{\delta_0} \leq Ch^{\delta}$. As a consequence, $\hat{\varphi}_+(\zeta_n(x)) \leq Ch^{2\delta}$. We also have $\hat{\varphi}_+(\rho) \geq C^{-1}(h^{2\delta} + h^{2\delta_1}) \geq C^{-1} h^{2\delta_1}$ which gives

  $$R_+(x) \leq \frac{(M + C)h^{2\delta}}{Mh^{2\delta} + C^{-1}h^{2\delta_1}} \leq Ch^{2(\delta-\delta_1)} = Ch^{2\varepsilon}$$

  Second case: $d(\rho, \mathcal{T}_+) \geq h^{\delta-\beta}$. In this context, we have $d(\rho, \mathcal{T}_+)^2 \gg h^{2\delta}$ so that we can bound the denominator $Mh^{2\delta} + \hat{\varphi}_+(\rho)$ from below by $C^{-1}d(\rho, \mathcal{T}_+)^2$. Concerning the numerator, we have

  $$d(\zeta_n(x), \mathcal{T}_+) \leq Ch^{\beta} d(\rho, \mathcal{T}_+) + Ch^{\delta_0} \leq Ch^{\beta} d(\rho, \mathcal{T}_+)$$

  since $h^{\beta} d(\rho, \mathcal{T}_+) \geq h^{\beta} h^{\delta-\beta} \geq h^{\delta} \gg h^{\delta_0}$. We deduce also that $\hat{\varphi}_+(\zeta_n(x)) \leq Ch^{2\beta} d(\rho, \mathcal{T}_+)^2$. As a consequence,

  $$R_+(x) \leq \frac{Ch^{2\beta} d(\rho, \mathcal{T}_+)^2}{C^{-1}d(\rho, \mathcal{T}_+)^2} \leq Ch^{2\beta} \ll h^{2\varepsilon}$$

- We now assume that $d(\rho, \mathcal{T}_+) \leq h^{\delta_1}$. As in the first case above, we can bound the numerator by $Ch^{2\delta}$. Concerning the denominator, we simply use the fact that $\hat{\varphi}_+ \geq 0$ to bound it from below by $Mh^{2\delta}$, and this gives, as expected

  $$R_+(x) \leq C$$

$\qquad\square$

Let's recap these two estimates and their implications concerning $d(x)$ (and recall that by definition, $\beta > \varepsilon$ and $J_{\mathbf{q}}^u \geq C^{-1} h^{-\beta}$)

$$d(\rho, \mathcal{T}_-) \geq h^{\delta} \implies d(x) \leq (Ch^{2\beta})^t \ll h^{2t\varepsilon}, \forall x \in [-\alpha_n^{1/2} h^{\delta_0}, \alpha_n^{1/2} h^{\delta_0}]$$

$$d(\rho, \mathcal{T}_-) \leq h^{\delta} \text{ and } d(\rho, \mathcal{T}_+) \geq h^{\delta_1} \implies d(x) \leq \left(Ch^{2\varepsilon} \right)^t, \forall x \in [-\alpha_n^{1/2} h^{\delta_0}, \alpha_n^{1/2} h^{\delta_0}]$$

As a consequence, the L^2 norm of χv_n is very small when ρ lies outside the neighborhood of $\mathcal{T}$ defined before Proposition 4.2 :

(5.30) $$\mathcal{T}_{\delta, \delta_1} = \left\{ \rho, d(\rho, \mathcal{T}_-) \leq h^{\delta}, d(\rho, \mathcal{T}_+) \leq h^{\delta_1} \right\}$$

Indeed, we obviously have

Proposition 5.5. For all $L > 0$, there exists $t = t(\varepsilon, L)$ such that the following holds. Assume that $\rho \notin \mathcal{T}_{\delta,\delta_1}$. Then,

$$\int_{|x| \leq \alpha_n^{1/2} h^{\delta_0}} d(x)^2 dx \leq C h^L$$

5.5.4. *Crucial estimates in $\mathcal{T}_{\delta,\delta_1}$.* We now turn to the crucial estimate which helps to control the L^2 norm of χv_n when $\rho \in \mathcal{T}_{\delta,\delta_1}$.

Proposition 5.6. Assume that $\rho \in \mathcal{T}_{\delta,\delta_1}$. Then,

$$\int_{|x| \leq \alpha_n^{1/2} h^{\delta_0}} d(x)^2 dx \leq C \left(J_{\mathbf{q}}^u\right)^{d_H + \varepsilon} h^{(\delta_0 - \delta)(d_H + \varepsilon)} h^\delta$$

Proof. **Step 0 : A simple estimates for $d(x)$.** First recall from Proposition 5.4, $\rho \in \mathcal{T}_{\delta,\delta_1} \implies d(x) \leq C R_-(x)^t$. Moreover, the numerator in $R_-(x)$ is bounded by $C h^{2\delta}$ and since $\hat{\varphi}_-(\zeta_n(x)) \geq C h^{2\delta} + C d(\zeta_n(x), \mathcal{T}_-)^2$, we find that

$$d(x) \leq C \left(1 + \left(\frac{d(\zeta_n(x), \mathcal{T}_-)}{h^\delta}\right)^2\right)^{-t}$$

Step 1 : The mass is supported in an h^δ-neighborhood of $\mathcal{T}$. We use Lemma 5.12 which asserts that there exists $\zeta_n^\star$ such that

(5.31) $d(\zeta_n(x), W_u(\zeta_n^\star)) \leq C h^\beta d(\rho, \mathcal{T}_+) + C h^{\delta_0} \ll h^\delta$

with β defined in the proof of Proposition 5.4. Recall that in the chart κ_{q_n}, $\kappa_{q_n}(\zeta_n(x)) = \hat{\rho}_n + (x, \beta_n x)$. Moreover, if ε_0 is small enough, we may assume that $\kappa_{q_n}(W_u(\zeta_n^\star))$ can be written as the graph of a function :

$$\kappa_{q_n}(W_u(\zeta_n^\star)) = \{(x, G_u(x)), x \in I_u\}$$

where I_u is a small interval of size $\sim \varepsilon_0$ and G_u a smooth function with bounded C^∞ norms (with bounds depending only on F and the charts). Since $d(\rho_n, W_u(\zeta_n^\star)) \ll h^\delta$, up to translating, we may assume that $\hat{\rho}_n = (0, \xi_n)$ and $|G_u(0) - \xi_n| \ll h^\delta$. In particular, if h is small enough, we may assume that $[-\alpha_n^{1/2} h^{\delta_0}, \alpha_n^{1/2} h^{\delta_0}] \subset I_u$. Finally, if ε_0 is small enough, we can also assume that $|G_u'(x)| \leq 1/4$ if $|x| \leq 2\alpha_n^{1/2} h^{\delta_0} \ll 1$ (recall that the chart κ_q is centered at a point ρ_q such that $\kappa_q(E_u(\rho_q)) = \mathbb{R} \times \{0\}$). We now set

$$X(\mathcal{T}) = \{x \in [-2\alpha_n^{1/2} h^{\delta_0}, 2\alpha_n^{1/2} h^{\delta_0}], \kappa_{q_n}^{-1}(x, G_u(x)) \in \mathcal{T}\}$$

Let's cover $X(\mathcal{T})$ by N intervals of size $2h^\delta$, centered at points $x_1, \ldots, x_N \in X(\mathcal{T})$. In virtue of Lemma 3.6, we can choose N such that

$$N \leq C \left(J_{\mathbf{q}}^u h^{\delta_0 - \delta}\right)^{d_H + \varepsilon}$$

Each interval around x_i of size $O(h^\delta)$ supports a mass of order $O(h^\delta)$. Our aim in the following lines is to show that the weight of the integral supported at distance larger than h^δ of the x_i is also $O(h^\delta)$, so that we will be able to estimate the whole integral by $N h^\delta$, which would conclude the proof. Let us consider $x \in [-\alpha_n^{1/2} h^{\delta_0}, \alpha_n^{1/2} h^{\delta_0}]$ and assume that for all $1 \leq i \leq N$, $|x - x_i| \geq 2h^\delta$. Let us choose i such that $|x - x_i| = \min_{1 \leq k \leq N} |x - x_k|$. We claim that there exists $\nu > 0$, uniform with respect to ρ, h and $x \in [-\alpha_n^{1/2} h^{\delta_0}, \alpha_n^{1/2} h^{\delta_0}]$ such that

(5.32) $d(\zeta_n(x), \mathcal{T}_-) \geq \nu |x - x_i|$

Let's admit it for a while. For $i \in \{1, \ldots, N\}$, let's note

$$J_i = \{x \in [-\alpha_n^{1/2} h^{\delta_0}, \alpha_n^{1/2} h^{\delta_0}], |x - x_i| = \min_{1 \leq k \leq N} |x - x_k|\}$$
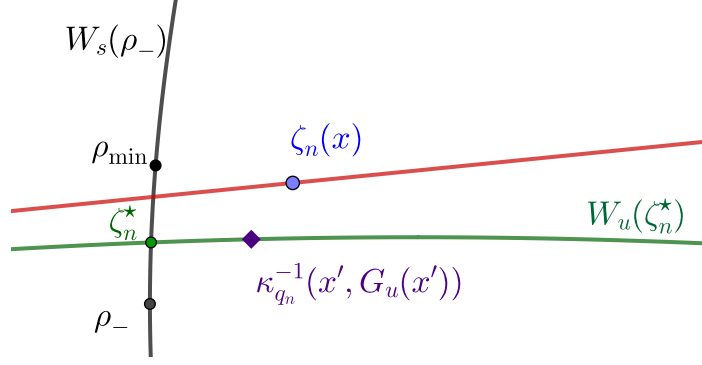
FIGURE 9. The points appearing in the proof of the claim 5.32. The curve $\zeta_n(x)$ is in red. The image of the important point $(x', G_u(x'))$ is a purple diamond.

These intervals form a partition of $[-\alpha_n^{1/2}h^{\delta_0}, \alpha_n^{1/2}h^{\delta_0}]$.

$$
\int_{|x|\leq\alpha_n^{1/2}h^{\delta_0}} d(x)^2 dx \leq \int_{|x|\leq\alpha_n^{1/2}h^{\delta_0}} C\left(1+\left(\frac{d(\zeta_n(x),\mathcal{T}_-)}{h^\delta}\right)^2\right)^{-2t} dx
$$

$$
\leq C\sum_{i=1}^N \int_{J_i} \left(1+\left(\frac{d(\zeta_n(x),\mathcal{T}_-)}{h^\delta}\right)^2\right)^{-2t} dx
$$

$$
\leq C\sum_{i=1}^N \left(\int_{|x_i-x|\leq 2h^\delta} 1 dx + \int_{|x_i-x|>2h^\delta} \left(1+\left(\frac{\nu|x-x_i|}{h^\delta}\right)^2\right)^{-2t} dx\right)
$$

$$
\leq C\sum_{i=1}^N \left(4h^\delta + \int_{|y|>2}\left(\frac{1}{1+(\nu y)^2}\right)^{2t} h^\delta dy\right)
$$

$$
\leq CNh^\delta
$$

Here, t is large enough and in particular, we may ensure that $t \geq 1$ so that the integral converges.

Step 2 : Proof of the claim (5.32). We argue by contradiction and assume that $d(\zeta_n(x),\mathcal{T}_-) \leq \nu|x-x_i|$ for some sufficiently small ν (with conditions specified below). Since $\mathcal{T}_-$ is made of local stable leaves near $\mathcal{T}$ (and $\zeta_n(x)$ lies in a small neighborhood of $\mathcal{T}$), we may chose $\rho_- \in \mathcal{T}$ such that $d(\zeta_n(x),\mathcal{T}_-) = d(\zeta_n(x), W_s(\rho_-))$. Let's still note $\rho_- \in \mathcal{T}$ the unique point of $W_u(\zeta_n^\star) \cap W_s(\rho_-)$ and let's write $\kappa_{q_n}(\rho_-) = (x_-, G_u(x_-))$. Again if ε_0 is small enough, all the stable leaves in κ_{q_n} can be written as graphs in the vertical variable : let us write

$$
\kappa_{q_n}(W_s(\rho_-)) = \{(H_s(\xi),\xi), \xi \in I_s\}
$$

where I_s is a small interval of size $O(\varepsilon_0)$ and H_s a smooth function with C^∞ norms bounded by constants only depending on the dynamics and the chart. Up to translating, we may assume that $(H_s(0),0) = \kappa_{q_n}(\rho_-) = (x_-,0)$. As for G_u, if ε_0 is small enough, we can assume that $|H_s'(\xi)| \leq 1$ for all $\xi \in I_s$. Finally, let us note $\rho_{\min} = \kappa_{q_n}^{-1}(H_s(\xi_{\min}),\xi_{\min})$ a point in $W_s(\rho_-)$ such that $d(\zeta_n(x),\rho_{\min}) = d(\zeta_n(x), W_s(\rho_-))$ (see Figure 9).

Since by (5.31), $d(\zeta_n(x), W_u(\zeta_n^\star)) \ll h^\delta$, we can find $x' \in I_u$ such that $||(x,\beta_n x)-(x',G_u(x'))|| \ll h^\delta$. This inequalitiy implies

$$
|x-x'| \ll h^\delta,
$$

$$
|\beta_n x - G_u(x')| \ll h^\delta,
$$

$$
|G_u(x) - G_u(x')| \ll h^\delta,
$$

$$
|G_u(x) - \beta_n x| \ll h^\delta.
$$

Since by assumption, $|x-x_i| \geq 2h^\delta$, when h is small enough, the inequality $|G_u(x)-\beta_n x| \leq \nu|x-x_i|$ holds. We have

$$
||(H_s(\xi_{\min}),\xi_{\min}) - (x,\beta_n x)|| \leq Cd(\zeta_n(x),\rho_{\min}) \leq Cd(\zeta_n(x),\mathcal{T}_-) \leq C\nu|x-x_i|
$$

From this we deduce that

$$|\xi_{\min}| \leq |\xi_{\min} - G_u(x)| + |G_u(x)| = |\xi_{\min} - G_u(x)| + |G_u(x) - G_u(x_-)|$$

$$\leq |\xi_{\min} - \beta_n x| + |\beta_n x - G_u(x)| + \frac{1}{4}|x - x_-|$$

$$\leq ||(H(\xi_{\min}), \xi_{\min}) - (x, \beta_n x)|| + \nu|x - x_i| + \frac{1}{4}|x - x_-|$$

$$\leq C\nu|x - x_i| + \frac{1}{4}|x - x_-|$$

Finally, we find that,

$$|x_- - x| \leq |x_- - H_s(\xi_{\min})| + |H_s(\xi_{\min}) - x|$$

$$\leq |H(0) - H(\xi_{\min})| + ||(H(\xi_{\min}), \xi_{\min}) - (x, \beta_n x)||$$

$$\leq |\xi_{\min}| + C\nu|x - x_i| \text{ (recall that } |H'| \leq 1)$$

$$\leq \frac{1}{4}|x - x_-| + C\nu|x - x_i|$$

From this, we deduce that

$$(5.33) \qquad\qquad |x - x_-| \leq \frac{4}{3}C\nu|x - x_i|$$

A first consequence of this inequality is that if ν is small enough so that $\frac{4\nu C}{3} \leq \frac{1}{4}$, we have

$$|x_-| \leq |x| + \frac{1}{4}|x - x_i| \leq \frac{5}{4}|x| + |x_i| \leq \frac{5}{4}\alpha_n^{1/2}h^{\delta_0} + \frac{2}{4}\alpha_n^{1/2}h^{\delta_0} \leq 2\alpha_n^{1/2}h^{\delta_0}$$

Since $\kappa_{q_n}^{-1}(x_-, G_u(x_-)) = \rho_- \in \mathcal{T}$, we deduce that $x_- \in X(\mathcal{T})$. In particular, there exists $j \in \{1, \ldots, N\}$ such that $|x_- - x_j| \leq h^{\delta}$. But then, we would have

$$|x_i - x| \leq |x_j - x| \leq |x_- - x_j| + |x_- - x| \leq h^{\delta} + \frac{1}{4}|x - x_i| \leq \frac{1}{2}|x - x_i| + \frac{1}{4}|x - x_i| < |x - x_i|$$

(recall that $|x - x_i| \geq 2h^{\delta}$) . This gives the required contradiction and concludes the proof of the claim (5.32).

$\square$

5.6. **End of the proof.** We can use Lemma 5.11, Proposition 5.5 and Proposition 5.6 to conclude the proof of Proposition 4.2. Indeed, since $||\widetilde{\Psi}_m||_\infty < +\infty$, we have

$$\left|\left|\left(1 + \frac{x^2}{\alpha_n h}\right)^{m/2} \chi v_n\right|\right|_{L^2}^2 \leq \int_{|x| \leq \alpha_n^{1/2}h^{\delta_0}} d(x)^2 |\Lambda_{\alpha_n h}\widetilde{\Psi}_m(x)|^2 dx + O(h^\infty)$$

$$\leq C_m(\alpha_n h)^{-1/2} \int_{|x| \leq \alpha_n^{1/2}h^{\delta_0}} d(x)^2 dx + O(h^\infty)$$

It gives a bound $C_m h^L$ when $\rho \notin \mathcal{T}_{\delta,\delta_1}$ (with L as large as necessary by choosing t large enough) and when $\rho \in \mathcal{T}_{\delta,\delta_1}$, we find that

$$\left|\left|\left(1 + \frac{x^2}{\alpha_n h}\right)^{m/2} \chi v_n\right|\right|_{L^2}^2 \leq C_m \left(J_{\mathbf{q}}^u\right)^{d_H - 1 + \varepsilon} h^{(\delta_0 - \delta)(d_H + \varepsilon) + \delta - 1/2}$$

When $m = 0$, it gives a control of the leading term, since we have

$$||u_n^0||_{L^2}^2 \leq \Pi_{\alpha,n}(\rho)^2 ||\chi v_n||_{L^2}^2$$

and since $\Pi_{\alpha,n}(\rho)^2 = O(h^{-L_2})$ for some $L_2 > 0$, so that for $\rho \notin \mathcal{T}_{\delta,\delta_1}$ we can have $||u_n^0||_{L^2}^2 = O(h^L)$ for any L by choosing t large enough.

It controls the first term of the expansion given by Corollary 5.2. We recall that the number of terms in this expansion is controlled by a integer $N \in \mathbb{N}$. For the other terms in the expansion given by Corollary 5.2, as already explained with (5.25), they all have their L^2 norms controlled by some

$$\varepsilon(h)\Pi_{\alpha,n}(\rho) \left|\left|\left(1 + \frac{x^2}{\alpha_n h}\right)^{m/2} \chi v_n\right|\right|_{L^2}$$

(A) The notations used to define the billiard map and the shadow map.

(B) The billiard map. $B_{ij}^+(y_j, \eta_j) = (y_i, \eta_i)$.
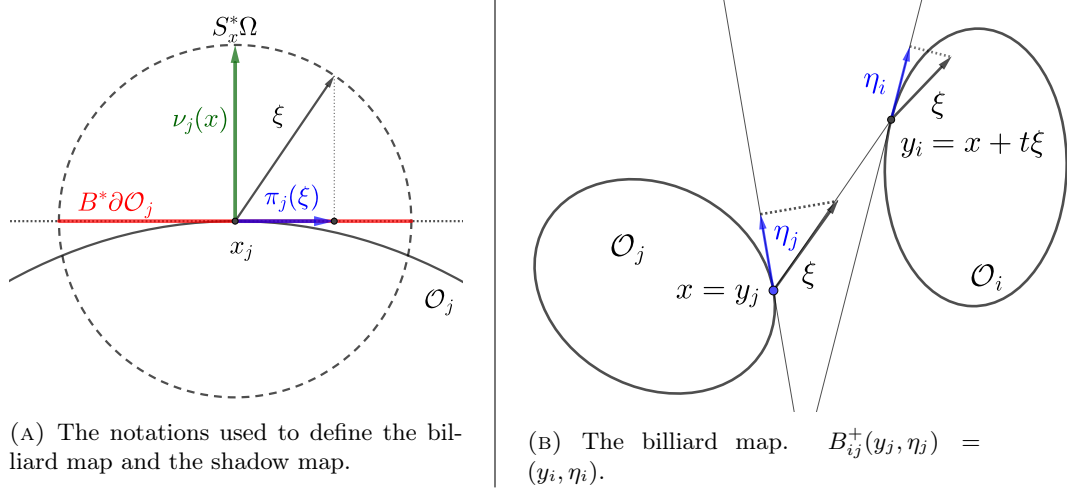
FIGURE 10. Definition of the billiard map

with $m \leq N$ and $\varepsilon(h) \to 0$ when $h \to 0$. Finally, we can choose $N = N(\varepsilon)$ such that the remainder has an L^2 norm $O(h^{2L})$. This concludes the proof of Proposition 4.2, and eventually of Theorem 3.

6. Proof of the fractal Weyl upper bounds in obstacle scattering and scattering by a potential

6.1. **Proof of Theorem 2.** Let us show how Theorem 3 implies Theorem 2. Suppose that the obstacles $\mathcal{O}_j$ are strictly convex, have smooth boundary and satisfy Ikawa condition of no-eclipse. We will use the results of [NSZ14] to apply Theorem 3 to the case where F is the billiard map. To be precise, let us introduce the following notations.

For $j \in \{1, \ldots, J\}$, let $B^* \partial \mathcal{O}_j$ be the co-ball bundle of $\partial \mathcal{O}_j$, $S^*_{\partial \mathcal{O}_j}$ be the restriction of $S^* \Omega$ to $\partial \mathcal{O}_j$, $\pi_j : S^*_{\partial \mathcal{O}_j} \to B^* \partial \mathcal{O}_j$ the natural projection and $\nu_j(x)$ be the outward normal vector at $x \in \partial \mathcal{O}_j$ (see Figure 10).

$\mathcal{B}$ is then the union of the maps $\mathcal{B}_{ij}$ corresponding to the reflection on two obstacles : for $(\rho_i, \rho_j) \in B^* \partial \mathcal{O}_i \times B^* \partial \mathcal{O}_j$ (with $\rho_i = (y_i, \eta_i), \rho_j = (y_j, \eta_j)$).

$$\rho_i = \mathcal{B}_{ij}(\rho_j) \iff \exists t > 0, \xi \in \mathbb{S}^1, x \in \partial \mathcal{O}_j$$
$$\pi_j(x, \xi) = \rho_j, \pi_i(x + t\xi, \xi) = \rho_i, \nu_j(x) \cdot \xi > 0, \nu_i(x + t\xi) \cdot \xi < 0.$$

It is a standard fact in the study of chaotic billiards (see for instance [CM00]) that the billiard map is hyperbolic due to the strict convexity assumption. Ikawa's condition ensures that the restriction of the dynamical system to the trapped set has a symbolic representation ([Mor91]). It is possible to restrict the study to a neighborhood of the trapped set. Since $\pi_y(\mathcal{T} \cap B^* \partial \mathcal{O}_j) \neq \partial \mathcal{O}_j$, it is possible to work with an interval $Y_j \subset \partial \mathcal{O}_j$ instead of the whole boundary. Moreover, it is known that $\mathcal{T}$ is compact and totally disconnected, so that the relation $\mathcal{B}$ satisfies the assumption of Theorem 3.

In [NSZ14], the author have shown that there exists a family $M_h(z)$ of open hyperbolic quantum maps associated with $\mathcal{B}$, depending holomorphically on $z \in \Omega(h) =] - R, R[+i] - C \log 1/h, R[$ for some fixed $R > 1$ and $C > 1$, and such that for h small enough and for $z \in \Omega(h)$, $\frac{1}{h} + z$ is a resonance if and only if $\det(1 - M_h(z)) = 0$, and the multiplicity of the resonance coincides with the multiplicity of z as a zero of $\det(1 - M_h(z))$. The construction of this operator relies on the study of the operators $M_0(z) : C^\infty(\partial \mathcal{O}) \to C^\infty(\partial \mathcal{O})$ defined as follows : for $1 \leq j \leq J$, let $H_j(z) : C^\infty(\partial \mathcal{O}_j) \to C^\infty(\mathbb{R}^2 \setminus \mathcal{O}_j)$ be the resolvent of the problem

$$\begin{cases} (-h^2 \Delta - (1 + hz)^2)(H_j(z)v) = 0 \\ H_j(z)v \text{ is outgoing} \\ H_j(z)v = v \text{ on } \partial \mathcal{O}_j \end{cases}.$$

Let γ_j be the restriction of a smooth function $u \in C^\infty(\mathbb{R}^2)$ to $C^\infty(\partial \mathcal{O}_j)$ and define $M_0(z)$ by :

$$M_0(z) = \begin{cases} 0 \text{ if } i = j \\ -\gamma_i H_j(z) \text{ otherwise.} \end{cases}$$

Using the analysis of Gérard ([Gé88], Appendix II) and restricting the study near the trapped set by the use of escape functions, the author transform M_0 into a Fourier integral operator associated with the billiard map (see Section 6 in [NSZ14]). Moreover, by analyzing the formula given in [Gé88], Appendix II) we see that the amplitude of $M_h(z)$ is related, via the solutions of the eikonal equation, to the distance between two collisions. In particular, near the trapped set, it is given by

(6.1) $\alpha_h(z)(\rho) = \exp\left(-t_{ret}(\rho)\operatorname{Im} z\right) + O\left(h^{1-}S_{0+}\right).$

For $\rho \in \mathcal{T}$, $t_{ret}(\rho)$ is described as follows : assume that $\rho = (x, \xi)$ and $(y, \eta) = \mathcal{B}(x, \xi)$, then $t_{ret}(\rho) = |x - y|$. t continues smoothly in a neighborhood of $\mathcal{T}$ and is called a *return time function*.

We can apply Theorem 3 to this family of open quantum maps and we find that, for any fixed $\varepsilon > 0$ and for $r \gg 1$ (with $h = r^{-1}$, recalling that the resonances are given by $1/h + z$ where z is a pole of $\det(1 - M_h(z))$), the number $N(r, \gamma)$ of resonances, counted with multiplicity, in $[r, r+1] - i[0, \gamma]$ satisfies

$$N(r, \gamma) \leq m_M\{|\operatorname{Re} z| < 2, \operatorname{Im} z \geq -\gamma\} \leq C_{\varepsilon, \gamma} r^{d_H - p(\gamma + \varepsilon)_+ + \varepsilon}$$

Here, $p(\beta)$ is given by

$$p(\beta) = -\frac{1}{6\lambda_{\max}} P(2\beta t_{ret} - \varphi_u).$$

Using the continuity of the pressure, we can choose $\varepsilon' > 0$ to ensure that

$$P(2(\gamma + \varepsilon')t - \varphi_u) \geq P(2\gamma t_{ret} - \varphi_u) + \varepsilon/2$$

and we may assume that $\varepsilon' \leq \varepsilon/2$. Applying the above formula with ε' , we find that

$$N(r, \gamma) \leq C_{\varepsilon, \gamma} r^{d_H - \sigma(\gamma) + \varepsilon}$$

with

(6.2) $\sigma(\gamma) = \max\left(0, -\frac{1}{6\lambda_{\max}} P(-\varphi_u + 2\gamma t_{ret})\right).$

To check that σ satisfies the properties listed in Theorem 3, we invoke the theory of Axiom A flows ([BR75]) : the map $s \mapsto P(-\varphi_u + st)$ is strictly increasing and has a unique root given by γ_{cl}. In particular, we deduce that $\sigma(\gamma) > 0$ for $\gamma < \gamma_{cl}/2$ and $\sigma(\gamma) = 0$ for $\gamma \geq \gamma_{cl}/2$, as expected. Finally, since the bound $N(r, \gamma) = O(r^{d_H})$ holds for any γ, we can change $\sigma(\gamma) - \varepsilon$ into $(\sigma(\gamma) - \varepsilon)_+ = \max(\sigma(\gamma) - \varepsilon, 0)$. This concludes the proof of Theorem 2.

6.2. **Proof of Theorem 4.** Let us show how Theorem 3 implies Theorem 4. The ideas are the same as for the case of obstacle scattering and rely on the reduction performed in [NSZ11].

We consider $V \in C_c^\infty(\mathbb{R}^2)$, $E_0 > 0$ and the semiclassical pseudodifferential operator $P_h = -h^2\Delta + V - E_0$. We note $p(x, \xi) = \xi^2 + V - E_0$ and we assume that

$$dp \neq 0 \text{ on } p^{-1}(0).$$

Let's note H_p the Hamiltonian vector field associated with p and $\Phi_t = \exp(tH_p)$ the corresponding Hamiltonian flow. Let's note K_0 the trapped set of Φ_t at energy 0 and we assume that Φ_t is hyperbolic on K_0 and K_0 is topologically one dimensional. More generally, we could work with more general Schrödinger operators in manifolds with Euclidean ends. We refer the reader to [NSZ11] (Section 2.1) for more general assumptions.

To apply Theorem 3, we use the results of [NSZ11] (Theorem 1 and 2). Under the assumptions above, there exists a smooth Poincaré hypersurface Σ for the flow Φ_t on the energy shell $p^{-1}(0)$ near K_0. Σ is made of several disjoint pieces Σ_j, $1 \leq j \leq J$. The reduced trapped set is now $\mathcal{T} := K_0 \cap \Sigma$, and if we write $2d_H + 1$ for the dimension of K_0, $\mathcal{T}$ has dimension

$$\dim \mathcal{T} = \dim K_0 - 1 = 2d_H.$$

The assumption that Σ is a smooth Poincaré hypersurface ensures that there exists $\varepsilon_{\min} > 0$ such that the map

$$(\rho, t) \in \Sigma \times] - \varepsilon_{\min}, \varepsilon_{\min}[\mapsto \Phi_t(\rho)$$

is a smooth diffeomorphism onto its image. We note t_{ret} the return time function on Σ : for $\rho \in \Sigma$,

$$t_{ret}(\rho) = \inf\{t > \varepsilon_{\min}, \Phi_t(\rho) \in \Sigma\} \in [\varepsilon_{\min}, +\infty]$$

$t_{ret} < +\infty$ in a neighborhood $U \subset \Sigma$ of $\mathcal{T}$. We then define the Poincaré return map F, which is an open hyperbolic map defined on an open subset of Σ :

$$F : \rho \in \Sigma \mapsto \Phi_{t_{ret}}(\rho) \in \Sigma$$

In [NSZ11], the authors construct a family of finite-dimensional matrices $(\mathbf{M}(z; h))$ for $z \in \Omega(h) =]-R, R[+i] - C \log 1/h, R[$ (with R fixed but large) such that for h small enough and for all $z \in \Omega(h)$,

$$\det(I - \mathbf{M}(z; h)) = 0 \iff hz \text{ is a resonace of } P_h$$

The matrices $\mathbf{M}(z; h)$ satisfy uniformly for $z \in \Omega(h)$ and for h small enough,

(6.3) $$\mathbf{M}(z; h) = \Pi_h M(z; h)\Pi_h + O(h^L)$$

where $L > 0$ can be chosen as large as necessary , Π_h is a finite rank projector and $M(z; h)$ is a family of open hyperbolic quantum maps associated with F (in the sense of Definition 2.4). The amplitude of $M(z; h)$ satisfies

$$\alpha_h(z)(\rho) = \exp(-t_{ret}(\rho) \operatorname{Im} z) + O\left(h^{1-} S_{0+}\right).$$

By their construction, $M(z; h)$ and Π_h satisfy, for some $L > 0$ as large as necessary, uniformly for $z \in \Omega(h)$ and for h small enough,

(6.4) $$\Pi_h M(z; h)\Pi_h = M(z; h) + O(h^L)$$

We can apply Theorem 3 to the family $M(z; h)$ of open quantum maps and we find that, for any fixed $\varepsilon > 0$ and $K > 0$ (with $K < R$) and for $h \ll 1$, the number $N_M(R, \gamma; h)$ of zeros of $\det(\operatorname{Id} - M(z; h))$ in $\{|\operatorname{Re} z| < R, \operatorname{Im} z \in [-\gamma, 0]\}$ satisfies

$$N_M(R, \gamma; h) \leq C_{R,\varepsilon,\gamma} h^{-d_H + p(\gamma+\varepsilon)_+ - \varepsilon}$$

Here, $p(\beta)$ is given by

$$p(\beta) = -\frac{1}{6\lambda_{\max}} P(2\beta t_{ret} - \varphi_u)$$

where φ_u is the unstable Jacobian associated with F. Here, it can also be obtained by differentiating the flow Φ_t. In fact, by inspecting the proof of Theorem 3 and by using (6.3) and (6.4), we see that the same conclusion holds for $\mathbf{M}$ instead of M. Indeed, in the formula (4.2) in Proposition 4.1, one can replace $M(z; h)$ by $\mathbf{M}(z; h)$ since $M(z; h)^{N(h)} = \mathbf{M}(z; h)^{N(h)} + O(|\log h| h^L)$ as soon as $N(h) = O(\log h)$. We now conclude as for the case of obstacle scattering in 6.1 and find that

$$N(R, \gamma; h) \leq C_{R,\gamma,\varepsilon} h^{-d_H + \sigma(\gamma) - \varepsilon}$$

where

(6.5) $$\sigma(\gamma) = \max\left(0, -\frac{1}{6\lambda_{\max}} P(-\varphi_u + 2\gamma t_{ret})\right)$$

APPENDIX A.

A.1. **Proofs of the missing Lemmas involving stationary phase expansions.** In this appendix, we give the missing proofs of Lemmas 2.5, 2.6 and 5.10. It relies on different uses of stationary phase theorems.

A.1.1. *Proof of of Lemma 2.5.* To alleviate the notations, let's note $q(x, \eta) = \langle D^2 \psi(x_1, \xi_0)(x, \eta), (x, \eta)\rangle$ and write it $q(x, \eta) = ux^2 + 2vx\eta + w\eta^2$. The metaplectic operator $\mathcal{M}(d_{\rho_0} F)$ admits the kernel

$$k(x, y) := \frac{|v|^{1/2}}{2\pi h} \int_{\mathbb{R}} e^{\frac{i}{h}\left(\frac{1}{2} q(x, \eta) - y\eta\right)} d\eta$$

and $\overline{k}(y, x)$ is the kernel of $\mathcal{M}(d_{\rho_0} F)^*$. We also note

$$\mathcal{M}_b u(x) = \frac{1}{2\pi h} \int_{\mathbb{R}^2} e^{\frac{i}{h}\left(\frac{1}{2} q(x, \eta) - y\eta\right)} \tilde{b}(x, \eta) u(y) dy d\eta$$

We have

$$
\begin{aligned}
\left(\mathcal{M}(d_{\rho_0}F)^*\mathcal{M}_b u\right)(x) &= \frac{|v|}{(2\pi h)^2}\int_{\mathbb{R}^4} e^{\frac{i}{h}\left(-\frac{1}{2}q(y,\eta)+x\eta+\frac{1}{2}q(y,\xi)-z\xi\right)}\tilde{b}(y,\xi)u(z)dyd\eta dzd\xi \\
&= \frac{1}{2\pi h}\int_{\mathbb{R}^2} u(z)e^{\frac{i}{h}(x-z)\xi}\underbrace{\left(\frac{|v|}{2\pi h}\int_{\mathbb{R}^2} e^{\frac{i}{h}\left(\frac{1}{2}q(y,\xi)-\frac{1}{2}q(y,\eta)+x(\eta-\xi)\right)}\tilde{b}(y,\xi)dyd\eta\right)}_{\check{b}(x,\xi)}dzd\xi \\
&= \mathrm{Op}_h^R(\check{b})u(x) \\
&= \mathrm{Op}_h(b)u(x)
\end{aligned}
$$

where Op_h^R denotes the right quantization, and by [Zwo12] (Theorem 4.13), $b(x,\xi) = e^{-\frac{ih}{2}\langle D_x, D_\xi\rangle}\check{b}(x,\xi)$. Let's analyze $\check{b}$:

$$
\begin{aligned}
\check{b}(x,\xi) &= \frac{|v|}{2\pi h}\int_{\mathbb{R}^2} e^{\frac{i}{h}\left(\frac{1}{2}q(y,\xi)-\frac{1}{2}q(y,\eta+\xi)+x\eta\right)}\tilde{b}(y,\xi)dyd\eta \\
&= \frac{|v|}{2\pi h}\int_{\mathbb{R}^2} e^{\frac{i}{h}\left(\frac{1}{2}w\eta^2-w\eta\xi-vy\eta+x\eta\right)}\tilde{b}(y,\xi)dyd\eta \\
&= \frac{1}{2\pi h}\int_{\mathbb{R}^2} e^{\frac{i}{2h}w\eta^2}e^{\frac{i}{h}(x-y)\eta}\tilde{b}(v^{-1}(y-w\xi),\xi)dyd\eta \quad \text{(change of variable } vy+w\xi \mapsto y) \\
&= e^{\frac{ih}{2}wD_x^2}\tilde{b}(v^{-1}(x-w\xi),\xi)
\end{aligned}
$$

In particular, if $w=0$, we directly find that $\check{b}(x,\xi) = \tilde{b}(v^{-1}x,\xi)$. Otherwise, it is represented by the formula ([Zwo12], Theorem 4.8):

$$
\check{b}(x,\xi) = \frac{e^{i\frac{\pi}{4}\frac{w}{|w|}}}{2\pi h|w|}\int_{\mathbb{R}} e^{-\frac{ih}{2w}y^2}\tilde{b}(v^{-1}(y+x-w\xi),\xi)dy
$$

As a consequence, we see that b is obtained from $\tilde{b}$ by composing 3 actions : the one of $e^{-\frac{ih}{2}\langle D_x, D_\xi\rangle}$, the change of variable $(x,\xi) \mapsto (v^{-1}(x-w\xi),\xi)$ and $e^{\frac{ih}{2}wD_x^2}$. The second one is obviously continuous from $S(\langle\rho\rangle^{3N})$ to $S(\langle\rho\rangle^{3N})$. We can now use [Zwo12] Theorem 4.17 (or more precisely, the estimates given in the proof) : both the action of $e^{\frac{ih}{2}wD_x^2}$ and $e^{-\frac{ih}{2}\langle D_x, D_\xi\rangle}$ are continuous from $S(\langle\rho\rangle^{3N})$ to $S(\langle\rho\rangle^{3N})$, and more precisely, there exists a universal integer M and universal constants C_α such that, for every $\alpha \in \mathbb{N}^2$, $(x,\xi) \in T^*\mathbb{R}$,

$$
|\partial^\alpha(L\tilde{b})(x,\xi)| \le C_\alpha \sup_{|\beta|\le|\alpha|+M} ||\langle\rho\rangle^{-3N}\partial^\beta\tilde{b}||\langle\rho\rangle^{3N}
$$

with L being either $e^{\frac{ih}{2}wD_x^2}$ or $e^{-\frac{ih}{2}\langle D_x, D_\xi\rangle}$. The same holds for the change of variable. This gives the required estimates for the symbol b and concludes the proof of the Lemma.

$\square$

A.1.2. *Proof of Lemma 2.6.* Fix $s \in [0,1]$ and recall that, with the notation q introduced above

$$
\tilde{R}_s u(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{i\left(\frac{1}{2}q(x,\eta)+sh^{1/2}r_3^\psi(x,\eta;h)-y\eta\right)}b_N(x,\eta)u(y)dyd\eta
$$

Let's introduce

$$
R_s = \Lambda_h\tilde{R}_s\Lambda_h^*
$$

and observe that the Schwartz kernel of R_s is given by

$$
k_s(x,y) = \frac{1}{2\pi h}\int_{\mathbb{R}} e^{\frac{i}{h}\left(\frac{1}{2}q(x,\eta)+s\rho_3^\psi(x,\eta;h)-y\eta\right)}\tilde{b}_N(x,\eta)u(y)d\eta
$$

where

$$
\rho_3^\psi(x,\eta) = h^{3/2}r_3^\psi(h^{-1/2}x, h^{-1/2}\eta) = \psi(x_1+x, \xi_0+\eta)-\psi(x_1,\xi_0)-x\partial_x\psi(x_1,\xi_0)-\eta\partial_\eta\psi(x_1,\xi_0)-\frac{1}{2}q(x,\eta)
$$

and $\tilde{b}_N(x,\eta) = b_N(h^{-1/2}x, h^{-1/2}\eta)$ which lies in $S_{0+}(\langle\rho\rangle^{3N})$. Let's note $\psi_s(x,\eta) = \frac{1}{2}q(x,\eta)+s\rho_3^\psi$ and remark that

$$
\partial_{x\eta}^2\psi_s = (1-s)\partial_{x\eta}^2\psi(x_1,\xi_0)+s\partial_{x\eta}^2\psi(x_1+x, \xi_0+\eta)
$$

Since $\partial_{x\eta}\psi$ does not vanish on $\Omega_x \times \Omega_\eta$, it has constant sign and hence, $\partial^2_{x\eta}\psi_s(x,\eta) \neq 0$ on $\Omega_x \times \Omega_\eta$. We now analyze the kernel K_s of $R^*_s R_s$ and find that this kernel is

$$
\begin{aligned}
K_s(x,y) &= \int_{\mathbb{R}} \overline{k}_s(z,x) k_s(z,y) dz \\
&= \int_{\mathbb{R}^3} \exp\left(\frac{i}{h}(\psi_s(z,\eta) - \psi_s(z,\xi) - y\eta + x\xi)\right) \overline{\tilde{b}_N(z,\xi)} \tilde{b}_N(z,\eta) d\eta d\xi dz \\
&= \int_{\mathbb{R}} d\xi e^{\frac{i}{h}(x-y)\xi} \underbrace{\int_{\mathbb{R}^2} \exp\left(\frac{i}{h}(\psi_s(z,\eta) - \psi_s(z,\xi) - y(\eta - \xi))\right) \overline{\tilde{b}_N(z,\xi)} \tilde{b}_N(z,\eta) d\eta dz}_{B_s(y,\xi)}
\end{aligned}
$$

which is the kernel of $\mathrm{Op}^R_h(B_s)$. To analyze B_s, we want to apply a stationary phase theorem and we need to know the stationary points in the variable (z,η), of the phase

$$
\Phi_s(z,\eta,y,\xi) = \psi_s(z,\eta) - \psi_s(z,\xi) - y(\eta - \xi)
$$

We have

$$
\partial_z \Phi_s(z,\eta,y,\xi) = \partial_x\psi_s(z,\eta) - \partial_x\psi_s(z,\eta) = \partial^2_{x\eta}\psi_s(z,\eta(z,\xi,\eta))(\eta - \xi)
$$

for some $\eta(z,\xi,\eta) \in [\eta,\xi]$. Hence, since $\partial^2_{x\eta}\psi$ does not vanish,

$$
\partial_z \Phi_s(z,\eta,y,\xi) = 0 \iff \xi = \eta
$$

We also have

$$
\partial_\eta \Phi_s(z,\eta,y,\xi) = \partial_\eta \psi_s(z,\eta) - y
$$

so that the equation $\partial_\eta \Phi_s(z,\xi,y,\xi) = 0$ has at most one solution, using again the fact that $\partial^2_{x\eta}\psi_s$ does not vanish. When there is no stationary point, a non stationary phase argument gives that $|B_s(y,\xi)| \leq O(h^\infty)\langle\rho\rangle^{6N}$. If there is a stationary point, it is given by a smooth function $z_s(y,\xi)$ locally around (y,ξ) and a stationary phase argument shows that $|B_s(y,\xi)| \leq C_M \langle\rho\rangle^{6N}$ where C_M depends on the first M semi-norms (for some universal integer M) of $\tilde{b}_N$. We can treat the derivatives of B_s by differentiating under the integral and integration by part to obtain the same estimates for $\partial^\alpha B_s$, involving derivatives of $\tilde{b}_N$ up to order $|\alpha|+M$. This shows that $B_s \in S(\langle\rho\rangle^{3N})$. We conclude the proof by passing from Op^R_h to Op_h as in the proof of Lemma 2.5 and we come back to $h = 1$ by standard scaling arguments. $\qquad\square$

A.1.3. *Proof of Lemma 5.10.* Let's write $u(x) = a(x)e^{i\beta_n \frac{x^2}{2h}}$ with a satisfying (5.19).

$$
\begin{aligned}
\mathrm{Op}_h(m)u(x) &= \frac{1}{2\pi h} \int_{\mathbb{R}^2} m\left(\frac{x+y}{2}, \xi\right) e^{\frac{i}{h}(x-y)\xi} a(y) e^{i\beta_n \frac{y^2}{2h}} dy d\xi \\
&= \frac{1}{2\pi h} \int_{\mathbb{R}^2} m\left(x + \frac{y}{2}, \beta_n x + \xi\right) a(x+y) e^{-\frac{i}{h}y(\xi+\beta_n x)} e^{i\beta_n \frac{(x+y)^2}{2h}} dy d\xi \\
&= e^{i\beta_n \frac{x^2}{2h}} \underbrace{\frac{1}{2\pi h} \int_{\mathbb{R}^2} m\left(x + \frac{y}{2}, \beta_n x + \xi\right) a(x+y) e^{\frac{i}{2h}(\beta_n y^2 - 2y\xi)} dy d\xi}_{B(x)}
\end{aligned}
$$

To analyze $B(x)$, we invoke the stationary phase theorem in the quadratic case (see [Zwo12], Theorem 3.13) with the non singular symmetric matrix $Q_n = \begin{pmatrix} \beta_n & -1 \\ -1 & 0 \end{pmatrix}$ and we follow the proof of [Zwo12], Theorem 4.17. We fix a cut-off function $\chi \in C^\infty_c(\mathbb{R}^2)$ with $\mathrm{supp}\,\chi \subset B(0,1)$ and $\chi = 1$ in a neighborhood of 0. We write (with $\chi_1 = \chi, \chi_2 = 1 - \chi$)

$$
B(x) = B_1(x) + B_2(x) \quad ; \quad B_i(x) = \frac{1}{2\pi h} \int_{\mathbb{R}^2} \chi_i(y,\xi) m\left(x + \frac{y}{2}, \beta_n x + \xi\right) a(x+y) e^{\frac{i}{2h}(\beta_n y^2 - 2y\xi)} dy d\xi
$$

We also set $v_i(x) = B_i(x)e^{i\beta_n \frac{x^2}{2h}}$. By the stationary phase expansion, we can expand B_1 : for every $N \in \mathbb{N}$,

$$
B_1(x) = \sum_{k=0}^{N-1} \frac{h^k}{k!} \left(\frac{(Q_n^{-1}D, D)}{2i}\right)^k c(x,0,0) + R_N(x)
$$

$$c(x,y,\xi) = \chi(y,\xi)m\left(x + \frac{y}{2}, \beta_n x + \xi\right)a(x+y) \quad ; \quad D = \begin{pmatrix} D_y \\ D_\xi \end{pmatrix}$$

$$R_N(x) = O\left(h^N \sup_{y,\xi} \sup_{k+l \le 2N+2} |\partial_y^k \partial_\xi^l c(x,y,\xi)|\right)$$

We observe that :

- The first term of the expansion of B_1 is given by $m(x, \beta_n x)a(x)$;
- B_1 is smooth since we can derive under the integral and obtain the same kind of expansion;
- The k-th term, that is $c_k(x) = \frac{1}{k!}\left(\frac{(Q_n^{-1}D,D)}{2i}\right)^k c(x,0,0)$ is a sum of terms of the form $c_\alpha \partial^\alpha m(x, \beta_n x)a^{(l)}(x)$ with $\alpha \in \mathbb{N}^2$, $l \in \mathbb{N}$, $|\alpha| + l \le 2k$ and $c_\alpha \in \mathbb{R}$. The coefficients c_α of this sum depend on Q_n. Since $\beta_n = O(\varepsilon_0)$, these coefficients are bounded uniformly in n. As a consequence, there exists $c_{k,p} = c_{k,p}(m)$ such that for $p \in \mathbb{N}$ with $k + p > 0$,

$$q_p(c_k) \le c_{k,p} h^{-2k\delta} q_{2k+p}(u)$$

Hence we set $A_k u(x) = h^{2k\delta} c_k(x) i^{\beta_n \frac{x^2}{2h}}$, which has the required form in virtue of the expression of $c_k(x)$.

Concerning the remainder term, we have

$$R_N(x) \le C_N(m) h^N h^{-(2N+2)\delta} \sup_{|y|\le 1}\left(1 + \frac{(x+y)^2}{\alpha_n h}\right)^{-2}$$

It is not hard to see that

$$\sup_{|y|\le 1}\left(1 + \frac{(x+y)^2}{\alpha_n h}\right)^{-2} \le C\left(1 + \frac{x^2}{\alpha_n h}\right)^{-2}$$

We choose $M > 0$ such that $M(1-2\delta) - 2\delta > 0$, so that $R_{N+M}(x) \le C_N(m)h^{N(1-2\delta)}\sup_{|y|\le 1}\left(1 + \frac{(x+y)^2}{\alpha_n h}\right)^{-2}$. By writing, $B_1(x) = \sum_{k=0}^{N-1} c_k(x) + \sum_{k=N}^{N+M-1} c_k(x) + R_{N+M}(x)$, we see that

$$q_0\left(v_1 - \sum_{k=0}^{N-1} h^k A_k u\right) \le C_N h^{N(1-2\delta)} q_{2N+M}(u)$$

By differentiating under the integral, we can show similarly that

$$q_j\left(v_1 - \sum_{k=0}^{N-1} A_k u\right) \le C_N h^{N(1-2\delta)} q_{j+2N+M}(u)$$

It remains to analyze B_2. Since there is no stationary point in the integral defining B_2, we do repeated integration by part using the differential operator $L(y,\xi) = \frac{(Q_n(y,\xi),D)}{|Q_n(y,\xi)|^2}$ which satisfies $L\left(e^{\frac{i}{h}(Q_n(y,\xi),(y,\xi))}\right) = e^{\frac{i}{h}(Q_n(y,\xi),(y,\xi))}$. Set $c_2(x,y,\xi) = (1-\chi(y,\xi))m(x+y/2, \beta_n + y/2)a(x+y)$. Since $|Q(y,\xi)| \ge c(y^2+\xi^2)^{1/2}$ on $\operatorname{supp}(1-\chi)$, we observe that for $M \in \mathbb{N}$.

$$(L^*)^{2M} c_2(x,y,\xi) \le C_M(1+y^2+\xi^2)^{-M/2} h^{2(1-\delta)M} q_{2M}(u)(\alpha_n h)^{1/4}\left(1+(x+y)^2/\alpha_n h\right)^{-2}$$
$$\le C_M(1+y^2+\xi^2)^{-M/2} h^{2(1-\delta)M} q_{2M}(u)(\alpha_n h)^{1/4}\left(1+x^2/\alpha_n h\right)^{-2}\left(1+y^2/\alpha_n h\right)^{-2}$$
$$\le C_M(1+y^2+\xi^2)^{-M/2} h^{2(1-\delta)M} q_{2M}(u)(\alpha_n h)^{1/4}\left(1+x^2/\alpha_n h\right)^{-2}$$

Integrating over $\mathbb{R}^2$, we find that $|B_2(x)| \le C_M h^{2(1-\delta)M} q_{2M}(u)(\alpha_n h)^{1/4}\left(1+x^2/\alpha_n h\right)^{-2}$. In particular, with $M = N$, $q_0(v_2) \le C_N h^{N(1-2\delta)} q_{2N}(u)$. Similarly, we can show that $q_j(v_2) \le C_{j,N} h^{N(1-2\delta)} q_{j+2N}(u)$. Since $\operatorname{Op}_h(m)u = v_1 + v_2$, this concludes the proof of the Lemma 5.10.

$\square$

A.2. **Formulas for approximation of exponential.** We consider

- a Hilbert space H ($H = L^2(\mathbb{R})$ for applications in this article) ;
- a bounded operator $A : H \to H$;
- a parameter h;
- a "class" $\mathcal{C}$ of elements of H, that is a subspace of H.

We assume that for each $j \in \mathbb{N}$, there exists $A_j : \mathcal{C} \to \mathcal{C}$ such that, in some sense to be specified in applications, $Au \sim \sum_{j=0}^{\infty} h^j A_j u$. More precisely, we assume that for all $N \in \mathbb{N}$ and all $u \in \mathcal{C}$, we can write

$$Au = \sum_{j=0}^{N-1} h^j A_j u + h^N R_N(u)$$

We are interested in understanding the action of the operator e^{tA} on elements of $\mathcal{C}$. Recall that if $u_0 \in H$, $t \mapsto e^{tA} u_0$ is the solution of the Cauchy problem

$$\begin{cases} \frac{d}{dt} u(t) = Au(t) \\ u(0) = u_0 \end{cases}$$

Moreover, we assume that A_0 extends to a bounded operator on H, so that e^{tA_0} is a well-defined operator and we assume also that $e^{tA_0}(\mathcal{C}) \subset \mathcal{C}$ for all $t \in \mathbb{R}$. We introduce in this appendix formulas and notations to give an approximation of $e^{tA} u$. Of course, the interesting mathematical work lies in controlling the following terms and the accuracy of the expansion, which is done in applications. Let us fix an integer $N \in \mathbb{N}$ and an initial state $u \in \mathcal{C}$.

Leading term. For our leading term, we simply state $u_0(t) = e^{tA_0} u$. Then, we set $R_0(t) = e^{tA} u - e^{tA_0} u$. We have $\dot{R}_0(t) = Ae^{tA} u - A_0 e^{tA_0} u$. Hence, we have

$$(A.1) \qquad \dot{R}_0(t) = AR_0(t) + \sum_{j=1}^{N-1} h^j A_j e^{tA_0} u + \tilde{r}_{0,N}(t) \quad ; \quad \tilde{r}_{0,N}(t) = h^N R_N(u_0(t))$$

First correction. When $N = 1$, we stop. Otherwise, we can correct this first approximation by a term of order h. Of course, it is possible to write down directly a general formula for every j, but it seems to the author that the case $j = 1$ helps to understand the general case. Let's try the Ansatz $u_1(t) = e^{tA_0} v_1(t)$ and set

$$R_1(t) = e^{tA} u - e^{tA_0} (u + h v_1(t))$$

Then we have,

$$\dot{R}_1(t) = \dot{R}_0(t) - h e^{tA_0} (A_0 v_1(t) + v_1'(t))$$

$$= AR_0(t) + \sum_{j=1}^{N-1} h^j A_j e^{tA_0} u + \tilde{r}_{0,N}(t) - h A e^{tA_0} v_1(t) + h(A - A_0) e^{tA_0} v_1(t) - h e^{tA_0} v_1'(t)$$

$$= AR_1(t) + \sum_{j=1}^{N-1} h^j A_j e^{tA_0} u + \tilde{r}_{0,N}(t) + h(A - A_0) e^{tA_0} v_1(t) - h e^{tA_0} v_1'(t)$$

To cancel the term of order h in the sum, we set

$$(A.2) \qquad v_1(t) = \int_0^t e^{-sA_0} A_1 e^{sA_0} u \, ds$$

To proceed with our expansion, we need to assume that $v_1(t) \in \mathcal{C}$ for all $t \in \mathbb{R}$. This will be the case in the applications, with precise control on $v_1(t)$.

Higher order terms. For convenience, let's note $A_j(s) = e^{-sA_0} A_j e^{sA_0}$. We can construct by induction a family of functions $v_k(t)$ by setting $v_0(t) = u$ and for $1 \le k \le N - 1$,

$$(A.3) \qquad v_k(t) = \sum_{l=0}^{k-1} \int_0^t A_{k-l}(s) v_l(s) \, ds$$

For these formulas to hold, we assume this construction ensure that $v_k(t) \in \mathcal{C}$ for all $t \in \mathbb{R}$. It will be easily satisfied in applications. We also set

$$\tilde{r}_{k,N}(t) = (A - A_0)e^{tA_0}v_k(t) - \sum_{j=1}^{N-k-1} h^j A_j e^{tA_0} v_k(t)$$

and

$$R_k(t) = e^{tA}u - e^{tA_0} \sum_{l=0}^{k} h^l v_l(t)$$

$\tilde{r}_{k,N}(t)$ has to be seen as a term of order h^{N-k}. These formulas ensure that

$$\dot{R}_k(t) = AR_k(t) + \sum_{j=k+1}^{N-1} h^j \left(\sum_{l=0}^{k} A_{k-l}e^{tA_0}v_l(t) \right) + \sum_{j=0}^{k} h^j \tilde{r}_{j,N}(t)$$

In particular, when $k = N - 1$,

(A.4)
$$\dot{R}_{N-1}(t) = AR_{N-1}(t) + \sum_{j=0}^{N-1} h^j \tilde{r}_{j,N}$$

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