A spectral gap for obstacle scattering in 2D

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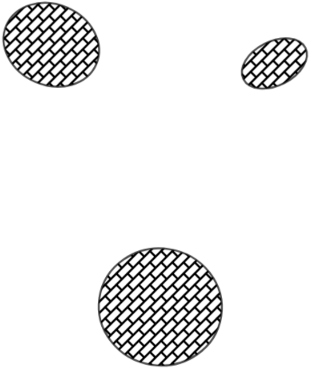
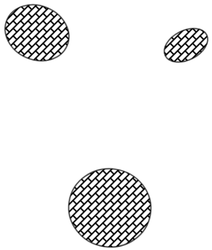
Obstacle scattering



Obstacle scattering

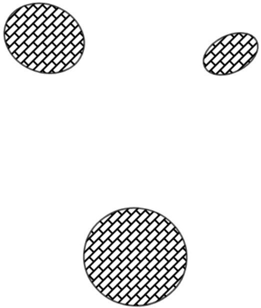


A set of N obstacles $\mathcal{O}_i$:

- smooth boundary
- strictly convex
- Non-eclipse condition :

$$\forall i \neq j \neq k, \overline{\mathcal{O}_i} \cap \text{conv}(\overline{\mathcal{O}_j} \cup \overline{\mathcal{O}_k}) = \emptyset$$

Obstacle scattering



$$\mathcal{O} = \bigcup_i \mathcal{O}_i$$

We study

$$-\Delta \text{ on } \mathbb{R}^d \setminus \mathcal{O}$$

with Dirichlet boundary condition.

Resonances

$$R(\lambda) \coloneqq (-\Delta - \lambda^2)^{-1} : H^2(\mathbb{R}^d \setminus \mathcal{O}) \to L^2(\mathbb{R}^d \setminus \mathcal{O})$$

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Theorem (see for instance the book of Dyatlov-Zworski)

$R(\lambda)$ extends meromorphically to a family of operators $H^2_{comp} \to L^2_{loc}$ with poles of finite rank to all $\mathbb{C}$ if the dimension is odd, to the log plane (and in particular to $\mathbb{C} \setminus (-i\mathbb{R})$) if d is even.

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Definition

The poles are the resonances.

Resonant states

A resonance $\lambda = k - i\gamma$ ($\gamma > 0$) comes with a **resonant states** u_λ : it is an outgoing solution (but not L^2) of the equation

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Interpretation : set $v_\lambda(t,x) = u_\lambda(x)e^{-it\lambda}$. It solves the wave equation :

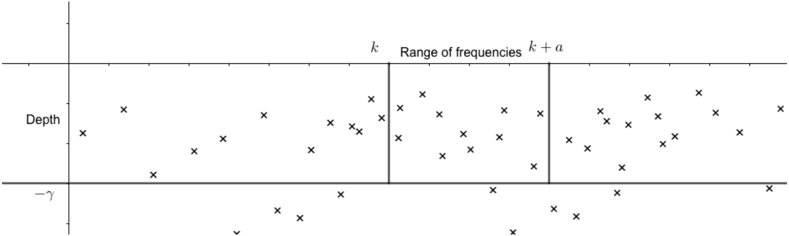
$$(\partial_t^2 - \Delta)v_\lambda = 0$$

$k = \operatorname{Re}\lambda \to$ frequency
$\gamma = -\operatorname{Im}\lambda \to$ decay rate

Distribution of resonances

Question

- $a > 0, \gamma > 0$ being fixed, number of resonances in boxes $[k, k+a] - i\gamma[0,1]$, as $k \to +\infty$.



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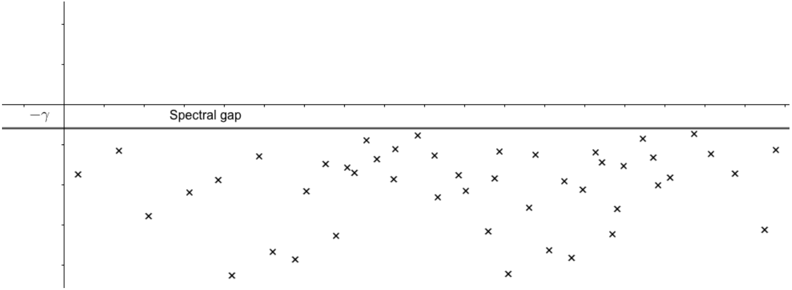
- Spectral gaps : does there exists $\gamma > 0$ such that there is **NO** resonances in $[1, +\infty[-i[0, \gamma]$?



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Applications : (exponential) decay of the local energy for the wave equation outside the obstacles , resonance expansion of scattered waves

Semiclassical setting

These are high-frequency problems : set

$$\operatorname{Re}\lambda = h^{-1} \quad , \quad h \text{ semiclassical parameter}$$

We now write $h\lambda = 1 + z$, $z \in D(0, Ch)$ i.e. we study

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Semiclassical *: the classical dynamics of the system influences the high-frequency behavior of the quantum problem*

Classical dynamic

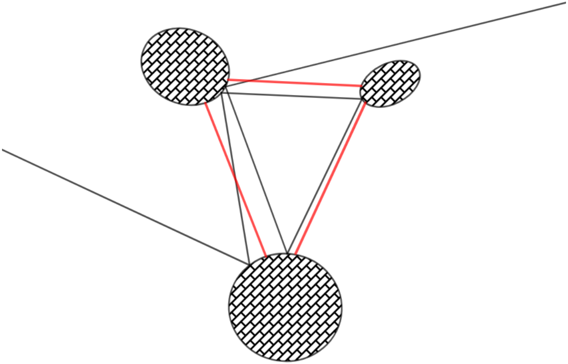

Figure: The billiard flow outside the obstacles

Billiard map

A reduced dynamics, at **discrete** time, living on the t**angent space of the boundary**. Each pair of obstacle $\mathcal{O}_i, \mathcal{O}_j$ gives a symplectic relation $\mathcal{B}_{ij}$
Let's define :

$$S^*\partial\mathcal{O}_j = \{(x,\xi) \in T^*\mathbb{R}^2, x \in \partial\mathcal{O}_j, |\xi| = 1\}$$
$$B^*\partial\mathcal{O}_j = \{(y,\eta) \in T^*\partial\mathcal{O}_j, |\eta| \leq 1\}$$

$\pi_j : S^*\partial\mathcal{O}_j \to B^*\partial\mathcal{O}_j$ the orthogonal projection on each fiber

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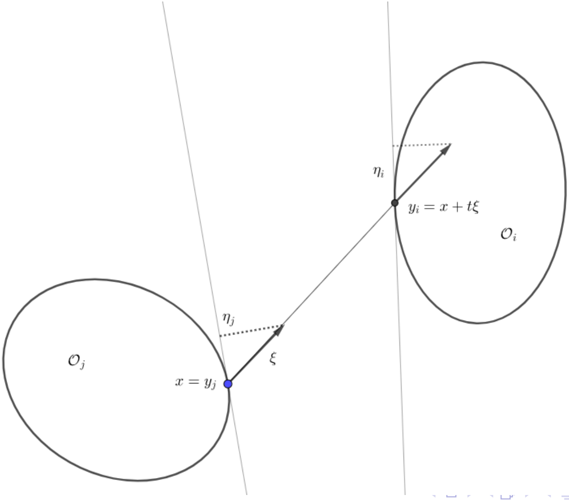

Trapped set

We consider the billiard flow ϕ^t on $S^*(\mathbb{R}^d \setminus \mathcal{O}) = \{(x,\xi), x \in \mathbb{R}^d \setminus \mathcal{O}, |\xi| = 1\}$ and we define the outgoing (Γ_+) and incoming (Γ_-) tails

$$\Gamma_\pm = \left\{(x,\xi), \phi^t(x,\xi) \text{ stays bounded as } t \to \mp\infty\right\}$$

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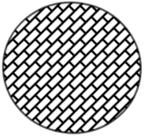
Let's note $\mathcal{K}_j := \Gamma \cap S^*\partial\mathcal{O}_j$, $K_j = \pi_j(\mathcal{K}_j)$ and $K = \bigcup \mathcal{K}_j$.
Claim :

- $K_j \subset \{(y,\eta) \in T^*\partial\mathcal{O}_j, |\eta| < 1\}$ i.e. **no trapped glancing rays**.
- The billiard map is a smooth canonical transformation in a neighborhood $V \subset B^*\partial\mathcal{O}$ of K ; $\mathcal{B} : V \to \mathcal{B}(V)$.

Spectral gaps and trapped set

- **1 obstacle** : no trapping. $\exists$ a spectral gap (see the works of Lax, Phillips, Morawetz, Raltson, Strauss, Melrose, etc., in the 70's)



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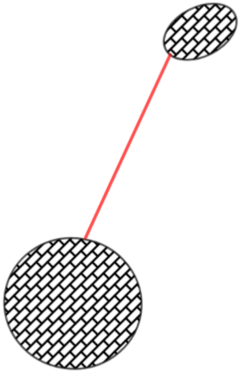
- **2 obstacles** : one single closed periodic orbit. $\exists$ a spectral gap. In fact, the asymptotic distribution is well understood : it is asymptotically close to a lattice (see Ikawa 80, Ch. Gerard 88)



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- **3 and more obstacles** : the trapped set is much more complicated.

Trapped set for $N \geq 3$ obstacles

Elements to understand the complexity of the trapped set :

- (Morita 91) Bijection between trapped orbits and sequences in

$$\left\{ (\alpha_n)_{n\in\mathbb{Z}} \in \{1,\dots,N\}^{\mathbb{Z}} ; \alpha_{n+1} \neq \alpha_n \right\}$$

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- Hyperbolicity of the billiard map on the trapped set : if $\rho \in K \subset V$, $\exists$ a continuous decomposition
$$T_\rho V = E_u(\rho) \oplus E_s(\rho)$$
where there exists $C > 0$ and $\lambda > 0$ such that
  - $d_\rho \mathcal{B}(E_u(\rho)) = E_u(\mathcal{B}(\rho))$; $d_\rho \mathcal{B}(E_s(\rho)) = E_s(\mathcal{B}(\rho))$;
  - for every $v \in E_u(\rho), n \in \mathbb{N}, ||d_\rho \mathcal{B}^{-n}(v)|| \leq Ce^{-\lambda n}$
  - for every $v \in E_s(\rho), n \in \mathbb{N}, ||d_\rho \mathcal{B}^{n}(v)|| \leq Ce^{-\lambda n}$;

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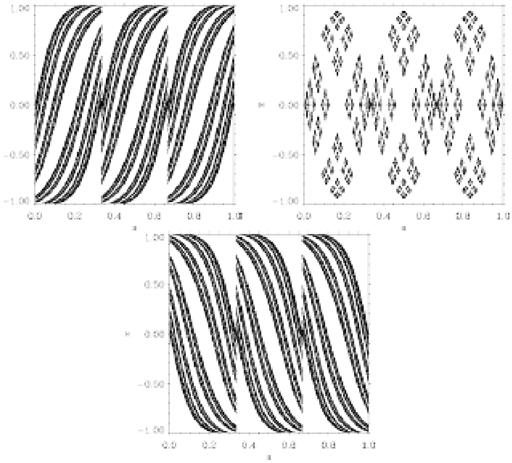
- Topological pressure $P(s)$ associated to $-s \log J_u$:
  - $s \mapsto P(s)$ strictly decreasing
  - $P(0) > 0$: *topological entropy*
  - $P(1) = -\gamma_{cl} < 0$ with γ_{cl} classical escape rate.
  - *Bowen's formula in dimension 2* : if $2s_0 = \dim_{Haus} K = \dim_{upper-box} K$, then s_0 is the unique root of $P(s) = 0$.

A few pictures

Obstacles : Three disks at the vertices of an equilateral triangle.
Figures from *Wada Basin Boundaries in Chaotic Scattering*, available on the web page *Chaos at Maryland*.



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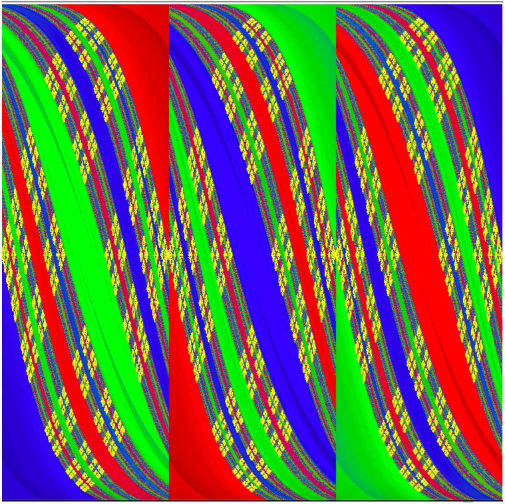
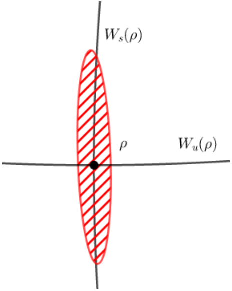


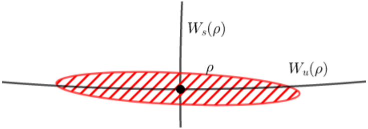
Unstable/stable manifolds

Theorem (see for instance Katok-Hasselblatt)

There exists two families of local manifolds $W_u(\rho)$ and $W_s(\rho)$ for $\rho \in K$ such that

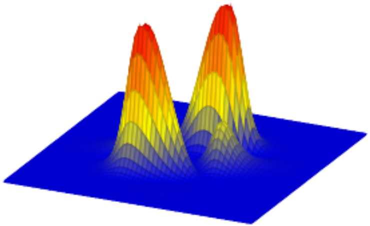
- $T_\rho W_*(\rho) = E_*(\rho)$;
- if $\rho' \in W_s(\rho), d(F^n(\rho), F^n(\rho')) \to 0$ when $n \to \infty$;
- if $\rho' \in W_u(\rho), d(F^{-n}(\rho), F^{-n}(\rho')) \to 0$ when $n \to \infty$;





Other open hyperbolic systems

- Convex co-compact hyperbolic surfaces : $-\Delta_g$ on (M, g)
- Scattering by a compactly supported potential : $-h^2\Delta + V$ with $V \in \mathcal{C}_c^\infty(\mathbb{R}^d, \mathbb{R})$



Conjectures and results

Conjecture (Zworski 17)

Suppose that the system has a compact hyperbolic trapped set. Then, there exists a spectral gap.

Results :

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- Holds under the pressure condition $P(1/2) < 0$ (in any dimension) (Ikawa 88), also in potential scattering (Nonnenmacher-Zworski 09)

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Results :

- Holds under the pressure condition $P(1/2) < 0$ (in any dimension) (Ikawa 88), also in potential scattering (Nonnenmacher-Zworski 09)
- For convex co-compact hyperbolic surfaces : recent results with a **fractal uncertainty principle** (Dyatlov-Zahl 16, Dyatlov-Bourgain 17 and 18)

A spectral gap for obstacle scattering in 2D

Theorem (V., WIP)

$d = 2$. Suppose that the obstacles have smooth strictly convex boundary and satisfy the non-eclipse condition. Then, there exists a spectral gap.

- This result also applies in potential scattering under certain dynamical assumptions.

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- Only in dimension 2.
- Based on a reduction to open quantum maps (Nonnenmacher-Sjöstrand-Zworski 11 and 14)
- Use of the Fractal Uncertainty Principle where the unstable/stable distribution is not smooth (but $C^{1+\varepsilon}$) (Dyatlov-Jin-Nonnenmacher 19)

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Open quantum maps

Definition

An open quantum map quantizing the billiard map is a matrix of operators $T = T(h) = (T_{ij})$ where each $T_{ij} : L^2(Y_j) \to L^2(Y_i)$ is a Fourier integral operator associated to the symplectic relation $\mathcal{B}_{ij}$ restricted to $Y_i \times Y_j$.

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Remarks

- Here, $Y_k \subset \partial \mathcal{O}_k$ is an open interval containing the reduced trapped set.
- *Being a FIO* means that if $a \in \mathcal{C}_c^\infty(Y_i)$,

$$T_{ij}^* \mathrm{Op_h}(a) T_{ij} = \mathrm{Op_h}(b) + O(h)$$

  where $b(\rho_j) = |\alpha_{ij}|^2(\rho_j) \times a \circ \mathcal{B}_{ij}(\rho_j)$.
- We say that T is *microlocally unitary near the trapped set* if $\alpha_{ij} \in \mathcal{C}_c^\infty(\partial \mathcal{O}_i)$ satisfies $|\alpha_{ij}| \equiv 1$ in a neighborhood of K_j

Reduction to an open quantum map

Theorem (Nonnenmacher-Sjöstrand-Zworski 14)

There exists a holomorphic family of operators $z \in D(0, Ch) \mapsto M(z; h)$ such that $M(z; h) = \Pi_h M_0(z; h) \Pi_h + O(h^L)$ (with L fixed but large) where

- (i) $M_0(0)$ is an open quantum map quantizing the billiard map microlocally unitary near the trapped set ;
- (ii) $M_0(z) = M_0(0) \, \mathrm{Op_h} \left(e^{i \frac{z}{h} \tau} \right) + O(h^{1-\varepsilon})$ for any $\varepsilon > 0$, where τ is a return time function
- (iii) Π_h is a finite rank projector with rank of order h^{-2}
- (iv) $\Pi_h M_0(z) \Pi_h = M_0(z) + O(h^K)$ for some $K > 0$.

and such that $h^{-1}(1 + z)$ is a resonance if and only if $\det(\mathrm{Id} - M(z; h)) = 0$.

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Remarks

Assume that τ is constant, there is a heuristic correspondence
$h^{-1}(1+z)$ resonance $\leftrightarrow e^{-i \frac{z}{h} \tau}$ eigenvalue of $M(0)$

Spectral gap of depth $\gamma \leftrightarrow$ the spectral radius of $M(0)$ is smaller than $e^{-\gamma \tau}$

Toy models

It motivates the study of open quantum maps for toy models. For instance, the baker's map on the torus.

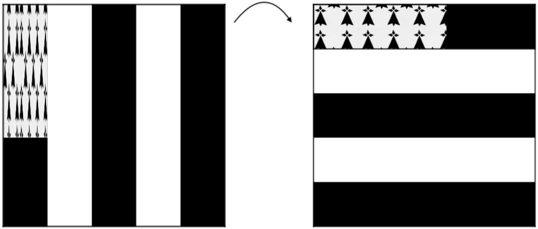

Figure: A classical baker's map

Here, $P(1/2) = \frac{\log 3}{\log 5} - \frac{1}{2} > 0$.

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Its quantum counterpart : in dimension $N = (2\pi h)^{-1} = 5^k$

$$B_N = \mathcal{F}_N^* \begin{pmatrix} \mathcal{F}_{N/5} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathcal{F}_{N/5} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathcal{F}_{N/5} \end{pmatrix}$$

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Theorem (Dyatlov-Jin 17)

There exists $\gamma > 0$ such that

$$\limsup_{k \to +\infty} \rho_{spec}(B_N) \leq e^{-\gamma} < 1$$

Ideas of proof for the theorem

Theorem (V., WIP)

$d = 2$. Suppose that the obstacles have smooth strictly convex boundary and satisfy the non-eclipse condition. Then, there exists a spectral gap.

Aim : $\rho_{spec}(M(0;h)) \leq e^{-\gamma} < 1$.

- It is enough to show that if $n \sim \alpha |\log h|$: $||M^n|| = O(h^\beta)$ as $h \to 0$.

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- Basic idea : M FIO associated to $\mathcal{B}$: M^n FIO associated to $\mathcal{B}^n$.
  True for n **fixed** (Egorov's theorem)
- One would like to extend the property to $n \sim \alpha |\log h|$: hopeless for large α but if possible, we could write

  $$M^n = \mathrm{Op_h}(\chi_+) M^n + \mathrm{O}(h^\infty);\, M^n = M^n \, \mathrm{Op_h}(\chi_-) + \mathrm{O}(h^\infty)$$

  where $\chi_\pm \equiv 1$ on $\widetilde{\Gamma}_\pm$ and

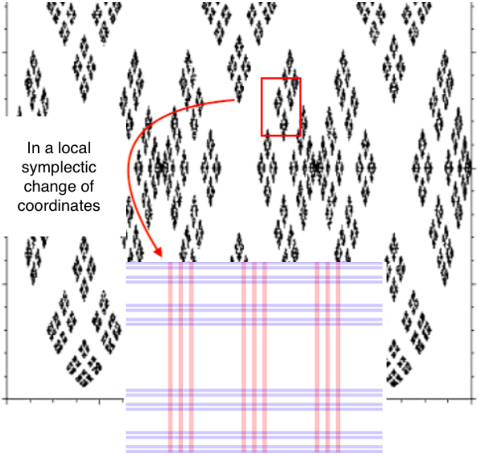
  $$\mathrm{supp}(\chi_\pm) \subset \widetilde{\Gamma}_\pm(Ch^{\delta\alpha})$$

  where δ related to the contraction/expansion rate of the classical flow and $\widetilde{\Gamma}_\pm = \bigcup_j \pi_j \left(\Gamma_\pm \cap S^* \partial \mathcal{O}_j\right)$.

Ideas of proof

We write the (informal) computation.

$$M^{2n} = M^n \underbrace{\mathrm{Op_h}(\chi_-)\,\mathrm{Op_h}(\chi_+)}_{\Downarrow} M^n + O(h^\infty)$$



More rigorous skecth of proof

Aim : $||M^N|| = O(h^\beta)$

- We break the symmetry $N = 2n = n + n$, by writing. $N = n_- + n_+$. $n_- \sim \rho_- |\log h|$ short (resp. $n_+ \sim \rho_+ |\log h|$ long) logarithmic time :

$$M^N = M^{n_-} M^{n_+}$$

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  $$M^{n_-} = M^{n_-} \mathrm{Op_h}(\chi_-) + \mathrm{O}(h^\infty)$$
- n_+ too long to use standard semiclassical tools : the study of M^{n_+} is much more technical.

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- We focus on sufficiently small pieces $U_1, \dots, U_J$ covering the trapped set.

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$$M^{n_+} = \sum_{\mathbf{q} \in \{1,\dots,J\}^{n_+}} M_{\mathbf{q}}$$

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- study the microlocolization of $M_{\mathbf{q}}$
- control the interaction between two terms of the sum.

Partition

Instead, we want to apply a Cotlar-Stein type estimate : we want to construct a partition of $\{1, \dots, J\}^{n_+}$, let's call it $V_1, \dots, V_{K(h)}$, to write

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We want to ensure that there exists K_0 independant of h such that if $k \in \{1, \dots, K(h)\}$, $M_k^* M_j = M_k M_j^* = O(h^\infty)$ holds for all but at most K_0 j.

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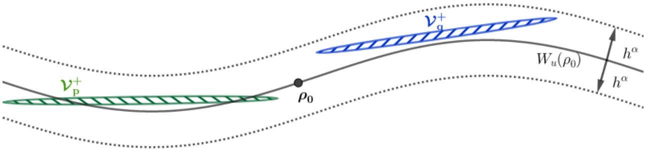
$$||M^N|| \leq C \sup_k ||M_k||$$

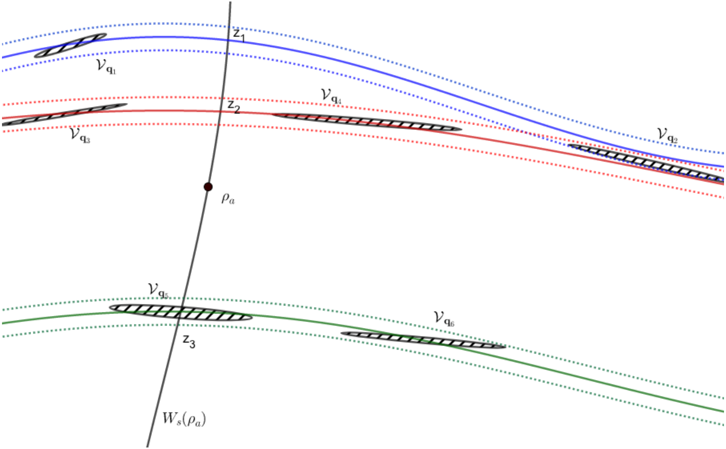
How to build this partition?

- Define in which situation two $M_{\mathbf{q}}$ can "interact "



How to build this partition?

- Define in which situation two $M_{\mathbf{q}}$ can "interact "
- Gather the pieces into "clouds" where a cloud interact with no more than K_0 other clouds.

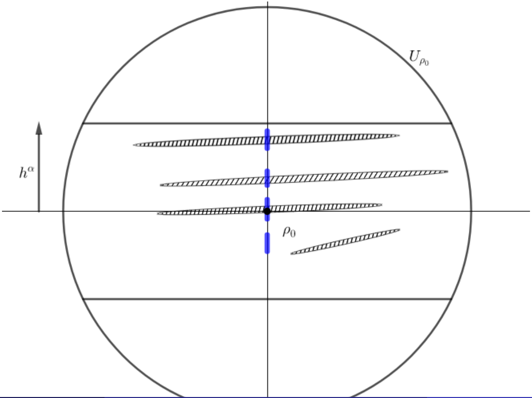


Reduction to a Fractal Uncertainty Principle

The bound $||M^N|| = O(h^\beta)$ reduces to a bound of the form :

$$\left|\left| \mathrm{Op_h}(\chi_-) \sum_{\mathbf{q} \in V_k} M_{\mathbf{q}} \right|\right| \leq Ch^\beta$$

Question : Can we find a coordinate chart in which the pieces of V_k are arranged like this ?



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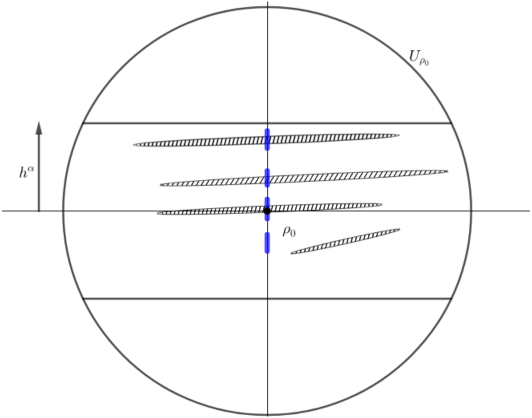

Claim : Yes. It relies on the $C^{1+\varepsilon}$ regularity of the unstable distribution.

Reduction to a Fractal Uncertainty Prinicple

Using this special change of coordinates, the bound $||M^N|| = O(h^\beta)$ reduces to a bound of the form :

$$\left|\left|\mathbb{1}_{X_-(h)}(hD_y)\mathbb{1}_{X_+(h)}(y)\right|\right| \leq Ch^\beta$$

where $X_-(h)$ (resp. $X_+(h)$) is an h^{α_-} (resp. h^{α_+}) neighborhood of fractal sets, namely, with an upper-box dimension strictly smaller than one.

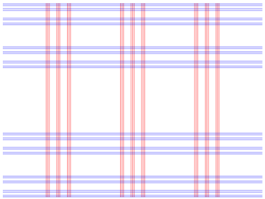

Figure: Ideal (but hopeless) case. In reality, we dont'h have perfect straight lines but a controlled deformation of such a picture.

Table of Contents

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Uncertainty principle

The h-Fourier transform $\mathcal{F}_h : L^2(\mathbb{R}) \to L^2(\mathbb{R})$:

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Definition

Let $X(h)$ and $Y(h)$ be two families of h-dependent sets. We say that X and Y satisfy an uncertainty principle with exponent β if there exists $C > 0$:

$$\left|\left|\mathbb{1}_{X(h)} \mathcal{F}_h \mathbb{1}_{Y(h)}\right|\right|_{L^2 \to L^2} \leq C h^{\beta}$$

Example

Uncertainty principle with $X(h) = Y(h) = [0, h]$

$$\left|\left|\mathbb{1}_{X(h)} \mathcal{F}_h \mathbb{1}_{Y(h)}\right|\right|_{L^2 \to L^2} \leq \frac{1}{2\pi} h^{1/2}$$

Fractal sets and porosity

Definition

Let $0 \leq \alpha_0 \leq \alpha_1 \leq +\infty$ and $\nu > 0$. We say that a closed set $X \subset \mathbb{R}$ is ν-porous on scale α_0 to α_1 if for every interval $I \subset \mathbb{R}$ with $|I| \in [\alpha_0, \alpha_1]$, there exists a subinterval $J \subset I$ such that

$$J \cap X = \emptyset \quad ; \quad |J| = \nu |I|$$

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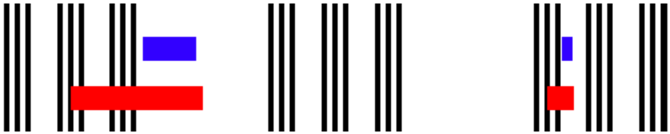
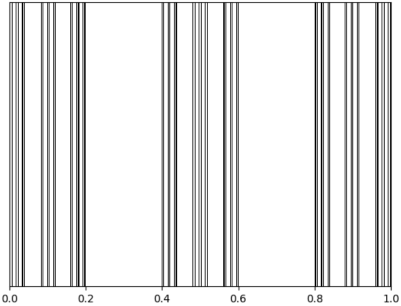


Examples



Upper-box dimension of a pre-compact set $X \subset \mathbb{R}$:

$$\delta_X = \limsup_{\varepsilon \to 0} -\frac{\log N_X(\varepsilon)}{\log \varepsilon}$$

where $N_X(\varepsilon)$ is the minimal number of balls of radius ε needed to cover X.

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The sets $X_\pm(h)$ are $\nu_\pm$ porous on scales $Ch^{\alpha_\pm}$ to 1.

Fractal uncertainty principle

Theorem (Dyatlov-Bourgain 18)

Let $\nu > 0$. Then there exists $\beta = \beta(\nu)$ such that, for all h-dependent families of sets $X = X(h)$, $Y = Y(h)$ which are ν-porous on scale h to 1, there exists $C > 0$ such that

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Proposition (Dyatlov-Jin-Nonnenmacher 19)

Let $\nu > 0$, $0 < \alpha_1, \alpha_2 \leq 1$. And assume than $\gamma := \alpha_1 + \alpha_2 - 1 > 0$. Then there exists $\beta = \beta(\nu)$ such that, for all h-dependent families of sets $X = X(h)$,(resp. $Y = Y(h)$) which are ν-porous on scale h^{α_1} (resp. h^{α_2}) to 1, there exists $C > 0$ such that

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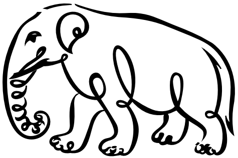

Thank you for your attention