Methods in representation theory and operator algebras

January 10, 2025

Abstract

This is the lecture note of the CIRM-IHP research school [CIRM2025].

1 Monica Nevins: Introduction to representation theory

1.1 Definitions

Definition 1.1.1. A *representation* of a group G is a pair (π, V) where V is a $\mathbb{C}$-vector space and π is a homomorphism $G \to \mathrm{GL}(V)$. If G is topological, the map $G \times V \to V$ has to be continuous. A morphism of G-representations between (π, V) and (σ, W) is a linear map $T : V \to W$ commuting with G-actions, and such T are called *intertwining operators*.

Example 1.1.2.
(1) Zero representation: $V = 0$.

(2) Trivial representation: $V = \mathbb{C}$ and $\pi : G \to \mathrm{GL}(V) = \mathbb{C}^\times$, $g \mapsto 1$, denoted by $\mathbb{1}$.

(3) $G = \mathrm{S}_3$. Permutation representation $\pi_P : \mathrm{S}_3 \to \mathrm{GL}(\mathbb{C}^3)$, sending g to its associated permutation matrix. Sign representation $\sigma : \mathrm{S}_3 \to \mathbb{C}^\times, g \mapsto \det(\pi_P(g))$.

A *subrepresentation* of (π, V) is a G-invariant subspace $W \subseteq V$. For example, $W = \mathbb{C}(1,1,1)$ is a subrepresentation of π_P. In other words, $T : (\mathbb{1}, \mathbb{C}) \to (\pi_P, \mathbb{C}^3)$, $1 \mapsto (1,1,1)$ lies in $\mathrm{Hom}_{\mathrm{S}_3}(\mathbb{C}, \mathbb{C}^3)$.

Example 1.1.3. Let $B = \{\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right)\} \subseteq \mathrm{GL}_2(\mathbb{C})$. The natural representation of B on $\mathbb{C}^2$ has a subrepresentation $W = \mathbb{C}(1,0)^{\mathrm{T}}$.

Definition 1.1.4. An *irreducible representation* of G is one with no non-trivial G-invariant closed subspace.

Example 1.1.5.
(1) Any 1-dimensional representation is irreducible.

(2) $(\pi_P, \mathbb{C}^3)$ is not irreducible.

Exercise 1.1.6. Any irreducible representation of a finite group is finite dimensional, and an irreducible representation of an abelian group is 1-dimensional.

Theorem 1.1.7 (Schur's lemma). *Suppose (π, V) and (σ, W) are irreducible representations of G, then $\dim \mathrm{Hom}_G(V, W) = 1$ if $\pi \simeq \sigma$, and 0 otherwise.*

The goals of representation theory:

- Classify all irreducible representations of G.
- Describe every representation of G in terms of its irreducible subrepresentations and irreducible subquotients.

1.2 Unitary representations

Definition 1.2.1. A representation (π, V) of G on a Hilbert space $(V, \langle\, ,\, \rangle)$ is *unitary* if π factors through $\mathrm{U}(V)$.

Theorem 1.2.2. *Any representation of a compact group G on a Hilbert space is unitarizable.*

Exercise 1.2.3. Let $W \subseteq V$ be a subrepresentation of a unitary representation, then $W^{\perp}$ is a subrepresentation of V and $V = W \oplus W^{\perp}$.

Exercise 1.2.4. Any finite dimensional (unitary) representation of a compact group is completely reducible.

1.3 Group algebras

Let G be a finite group. Define $\mathbb{C}[G] = \left\{ \sum_{g \in G} c_g g \,\middle|\, c_g \in \mathbb{C} \right\}$ to be the *group algebra*, equipped with the multiplication:

$$\sum_g c_g g \cdot \sum_h d_h h = \sum_k \left(\sum_g c_g d_{g^{-1}k} \right) k.$$

For a representation (π, V), we get a homomorphism of algebras:

$$\pi : \mathbb{C}[G] \to \mathrm{End}(V),\ \sum c_g g \mapsto \sum c_g \pi(g).$$

The group algebra $\mathbb{C}[G]$ is a left regular representation of G:

$$\lambda : G \to \mathrm{GL}(\mathbb{C}[G]),\ \lambda(g) \sum_h c_h h := \sum_h c_h gh = \sum_k c_{g^{-1}k} k.$$

It is also a C^*-algebra with the operator norm from λ and the involution sending $\sum c_g g$ to $\sum \overline{c_{g^{-1}}} g$.

Theorem 1.3.1. *(1) Every irreducible representation of G occurs as a subrepresentation of the group algebra $(\lambda, \mathbb{C}[G])$ with multiplicity equal to its degree.*

(2)

$$\mathbb{C}[G] = \bigoplus_{(\sigma, W)\ \mathit{irreducible}} W \otimes \mathrm{Hom}_G(W, \mathbb{C}[G]),\ w \otimes T \mapsto T(w).$$

(3) $\mathbb{C}[G] \simeq \bigoplus_{(\sigma, W)} \mathrm{End}(W)$ as an algebra.

Exercise 1.3.2. $\mathbb{C}[\mathrm{S}_3] = \mathbb{1} \oplus \mathbb{C}_{sign} \oplus \mathrm{M}_2(\mathbb{C}^2)$ and $\pi_P = \mathbb{1} \oplus \mathbb{C}^2$.

1.4 Beyond finite groups

We view $\sum c_g g \in \mathbb{C}[G]$ as a function in $\mathrm{C}_c(G)$, whose value at g is c_g, and the multiplication as the convolution.

Now we drop the finite group assumption. For a representation $\pi : G \to \mathrm{GL}(V)$, we have a homomorphism of algebras:

$$\pi : \mathrm{C}_c(G) \to \mathrm{End}(V),\ \pi(f)v = \int_G f(g)\pi(g)v dg.$$

The left regular representation

$$\lambda : G \to \mathrm{B}(\mathrm{L}^2(G)),\ (\lambda(g)f)\,(k) := f(g^{-1}k)$$

is an analogue of $G \to \mathrm{GL}(\mathbb{C}[G])$. We denote the closure of $\lambda(\mathrm{C}_c(G))$ by $\mathrm{C}^*_r(G)$.

When G is compact, we have the *Peter-Weyl theorem* and this case behaves like in the finite group setting. When G is not compact,

- not every irreducible representation is unitary, or finite dimensional;
- not every unitary representation occurs in $\mathrm{L}^2(G)$, for instance $\mathbb{1}$.

1.5 Induction and restriction

Let H be a subgroup of G and (σ, W) a representation of G. The restriction $\mathrm{Res}^G_H(\sigma) := (\sigma|_H, W)$ is a representation of H, but it is usually not irreducible even if σ is.

Definition 1.5.1 (Induction)**.** Let (σ, W) be a representation of H.

- If G is finite, $V = \mathrm{Ind}^G_H(\sigma) := \mathbb{C}[G] \otimes_{\mathbb{C}[H]} W \supseteq W$ is a representation of G. We have $\dim V = [G : H] \dim W$.
- For a general G, consider the vector bundle $G \times_H W$ over G/H. The induction is defined via sections of this vector bundle:

$$\begin{aligned} \mathrm{Ind}^G_H W :&= \left\{ f : G \to W \,\middle|\, f(gh) = \sigma(h^{-1})f(g) \right\} \\ (\pi(g)f)(k) &= f(g^{-1}k). \end{aligned}$$

Suppose that G is compact.

Proposition 1.5.2 (Frobenius reciprocity)**.**

$$\begin{aligned} \mathrm{Hom}_G(W, \mathrm{Ind}^G_H U) &= \mathrm{Hom}_H(\mathrm{Res}^G_H W, U) \\ T &\mapsto T'(w) = T(w)(1_G). \end{aligned}$$

2 Tyrone Crisp: Tempered representations from the point of view of C^*-algebras

The goal of this course: for a real or p-adic reductive group G, compute its reduced group C^*-algebra $\mathrm{C}^*_c(G)$.

Theorem 2.0.1 (Wassermann)**.** *Let G be a real reductive group. There is a Morita equivalence*

$$\mathrm{C}^*_r(G) \sim \bigoplus_{[P,\sigma]} \mathrm{C}_0(\mathfrak{a}^*_P / W'_\sigma) \rtimes R_\sigma.$$

2.1 Lecture 1

Definition 2.1.1. A *C^*-algebra* is an algebra A over $\mathbb{C}$, with

- a conjugate-linear involution $* : A \to A$ satisfying $(ab)^* = b^* a^*$;
- a norm $\| \ \|$ in which A is complete; $\|ab\| \leq \|a\|\|b\|$ and $\|a^* a\| = \|a\|^2$.

Example 2.1.2. Let X be a locally compact Hausdorff space, the space

$$\mathrm{C}_0(X) := \{f : X \to \mathbb{C} \text{ continuous} \mid f(x) \to 0 \text{ at } \infty\}$$

is a C^*-algebra.

For a Hilbert space, $\mathrm{B}(H)$ is a C^*-algebra.

Theorem 2.1.3. *Every C^*-algebra is isomorphic to a subalgebra of some* $\mathrm{B}(H)$*.*

Example 2.1.4. For the ideal of compact operators $\mathrm{K}(H) \subset \mathrm{B}(H)$, $C_0(X, \mathrm{K}(H))$ is also a C^*-algebra.

Example 2.1.5. For a C^*-algebra A equipped with an action of a finite group W, we have two new C^*-algebras:

- the *fixed-point algebra* A^W;
- the crossed product $A \rtimes W := \{\sum_{w \in W} a_w w \mid a_w \in A\}$.

Let X be a locally compact Hausdorff space, H a Hilbert space and W a finite group acting on X by homeomorphisms. Let $\{I_{w,x} \in \mathrm{U}(H) \mid w \in W, x \in X\}$ be unitary operators such that

- $I_{w_1, w_2 x} I_{w_2, x} = I_{w_1 w_2, x}$ (in particular, $I_{1,x} = \mathrm{id}_H$).
- For each $w \in W$, $x \mapsto I_{w,x}$ is continuous in the strong operator topology.

Let W act on $\mathrm{C}_0(X, \mathrm{K}(H))$ by

$$\beta_w(f)x := I_{w, w^{-1}x} f(w^{-1}x) I_{w^{-1}, x}.$$

The fixed-point algebra $\mathrm{C}_0(X, \mathrm{K}(H))^W$ will be the second-most important example of a C^*-algebra in these lectures.

Example 2.1.6. $W = \{1, w\}$ acts on $X = \mathbb{R}$ by $wx = -x$. $H = \mathbb{C}^2$ so $\mathrm{K}(H) = \mathrm{M}_2(\mathbb{C})$. $I_{w,x} = \left(\begin{smallmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{smallmatrix}\right)$. We have $\mathrm{C}_0(\mathbb{R}, \mathrm{M}_2)^W \simeq \mathrm{C}^0_C([0,\infty), \mathrm{M}_2)$.

Let G be a locally compact group with a left Haar measure dg.

Definition 2.1.7. The reduced group C^*-algebra is

$$\mathrm{C}^*_r(G) := \overline{\lambda(\mathrm{C}_c(G))}^{\|\ \|_{operator}},$$

where $\lambda : \mathrm{C}_c(G) \to \mathrm{B}(\mathrm{L}^2(G))$.

Definition 2.1.8. A representation of a C^*-algebra A is a homomorphism $\pi : A \to \mathrm{B}(H)$ for some Hilbert space H. The *spectrum* $\widehat{A}$ is the set of equivalence classes of irreducible representations of A. The *Jacobson topology* on $\widehat{A}$ has one open subset $\{\pi \mid \pi(J) \neq 0\}$ for an ideal J.

A *state* on A is a bounded linear $\varphi : A \to \mathbb{C}$ with $\varphi(a^*a) \geq 0$ and $\|\varphi\| = 1$. **Gelfand-Naimark-Segal construction**: given a state φ, define

- $J_\varphi = \{a \in A \mid \varphi(aa^*) = 0\}$,
- $H_\varphi = \overline{A/J_\varphi}$.
- $\pi_\varphi(a)(b + J_\varphi) = ab + J_\varphi$.

It is irreducible if φ is *pure*, *i.e.* not a convex combination of other states.

Theorem 2.1.9. *Let A be a C^*-algebra.*

- *Every irreducible representation is equivalent to a GNS representation.*
- *If $a \neq b \in A$, then $\pi(a) \neq \pi(b)$ for some π.*
- *We have a nice induction from a representation of a subalgebra of A.*

Theorem 2.1.10. *The followings are equivalent:*

- *π is irreducible;*
- *$\pi \simeq \pi_\varphi$ for φ pure;*
- *Schur's lemma;*
- *$\pi(A)$ is dense in $\mathrm{B}(H)$.*

A unitary representation $\pi : G \to \mathrm{U}(H)$ extends to $\mathrm{C}_c(G) \to \mathrm{B}(H)$. It extends to $\mathrm{C}^*_r(G)$ if and only if $\|\pi(f)\| \leq \|\lambda(f)\|_{operator}$. Denote by $\widehat{G}_r \subseteq \widehat{G}$ to be those that extend to $\mathrm{C}^*_r(G)$.

Theorem 2.1.11. $\widehat{G}_r \simeq \widehat{\mathrm{C}^*_r(G)}$.

Example 2.1.12. When G is abelian or compact, $\widehat{G}_r = \widehat{G}$. In general, $\pi \in \widehat{G}_r$ if and only if π is *tempered*, *i.e.* its K-finite matrix coefficients are $\mathrm{L}^{2+\varepsilon}$ modulo the center.

Strategy for computing $\mathrm{C}^*_r(G)$: match up tempered representations with representations of simpler C^*-algebras.

Theorem 2.1.13. $\widehat{\mathrm{C}_0(X)} \simeq X$, $\mathrm{ev}_x \leftrightarrow x$.

Theorem 2.1.14. *Every irreducible representation of* $\mathrm{K}(H)$ *is equivalent to the identity representation* $\mathrm{K}(H) \hookrightarrow \mathrm{B}(H)$.

Theorem 2.1.15. $\mathrm{C}_0(\widehat{X, \mathrm{K}(H)}) \simeq X$, $\mathrm{ev}_x \leftrightarrow x$.

Consider $X, H, W, I_{w,x}$ as before. Note that $w \mapsto I_{w,x}$ is a unitary representation of $W_x = \{w \in W \mid wx = x\}$, and $\mathrm{ev}_x(\mathrm{C}_0(X, \mathrm{K}(H))^W) = \mathrm{K}(H)^{W_x}$.

Theorem 2.1.16. *The maps* $\xi \otimes t \mapsto (\dim H_\rho)^{1/2} t(\xi)$ *give an isomorphism:*

$$\bigoplus_{\rho \in \widehat{W_x}} H_\rho \otimes \mathrm{HS}(\rho, I_x)^{W_x} \simeq H.$$

Theorem 2.1.17.

- *The irreducible representations of* $\mathrm{C}_0(X, \mathrm{K}(H))^W$ *are*

$$\pi_{x,\rho} : \mathrm{C}_0(X, \mathrm{K}(H))^W \xrightarrow{\mathrm{ev}_x} \mathrm{K}(H)^{W_x} \xrightarrow{k \mapsto k \otimes -} \mathrm{K}(\mathrm{HS}(\rho, I_x)^{W_x}).$$

- *Two representations* $\pi_{x_1,\rho_1}, \pi_{x_2,\rho_2}$ *are equivalent if and only if there exists some* $w \in W$ *such that* $x_2 = wx_1$ *and* $\rho_2 \simeq w\rho_1 : v \mapsto \rho_1(w^{-1}vw)$.

A C^*-algebra A is *CCR* if $\pi(A) \subseteq \mathrm{K}(H_\pi)$ for every $\pi \in \widehat{A}$. The examples that we have seen are all CCR.

Theorem 2.1.18 (Harish-Chandra,Bernstein)**.** *If G is a real and p-adic reductive group, then* $\mathrm{C}^*_r(G)$ *is CCR.*

A subalgebra $B \subseteq A$ is *separating* if the restrictions of irreducible representations remain irreducible, and the restrictions of inequivalent representations remain inequivalent. We say that A has the *Stone-Weierstrass property* if B separating implies $B = A$.

Theorem 2.1.19 (Kaplansky)**.** *Every CCR algebra has the SWP.*

Remark 2.1.20. This is a tool for computing the range of a Fourier transform.

2.2 Lecture 2

Plan: replace $\mathrm{C}^*_r(G)$ by a simpler C^*-algebra that is Morita equivalent to $\mathrm{C}^*_r(G)$. This is reasonable since Morita equivalent C^*-algebras have the same K-theory and representations.

For left (*resp.* right) Hilbert module of a C^*-algebra, we use the notation of inner product $[\,,\,]$ (*resp.* $\langle\,,\,\rangle$).

Example 2.2.1. Let H be a Hilbert space. It is a right $\mathbb{C}$-module, and a left Hilbert $\mathrm{B}(H)$-module ($[\xi,\eta] : \zeta \mapsto \xi\langle\eta,\zeta\rangle$). The left module structure is not *full*, *i.e.* $\overline{\operatorname{span}(\{[\xi|\eta]\})} \neq A$.

Example 2.2.2. $\mathrm{C}_0(X,H)$ is

- a full right Hilbert $\mathrm{C}_0(X)$-module;
- a full left Hilbert $\mathrm{C}_0(X,\mathrm{K}(H))$-module.

Example 2.2.3. Let E be a left Hilbert A-module. It is a left Hilbert A^W-module

$$^W[\xi|\eta] := \frac{1}{|W|}\sum_{w\in W}\beta_w([\xi|\eta]).$$

If E is full over A, then it is also full over A^W.

Example 2.2.4. $\pi : W \to \mathrm{U}(H)$ a representation, $\mathrm{C}^*_r(W) = \mathbb{C} \rtimes W$. H is a left Hilbert $\mathrm{K}(H)^W$-module:

$$^W[\xi|\eta] = \frac{1}{|W|}\sum_{w\in W}\pi(w)[\xi|\eta]\pi(w)^{-1},$$

and a right Hilbert $\mathrm{C}^*_r(W)$-module:

$$\xi.w = \pi(w^{-1})\xi,\ \langle\xi,\eta\rangle_W := \frac{1}{|W|}\sum_{w\in W}\langle\xi,\pi(w)\eta\rangle w.$$

Given a A-B bimodule E. If $\pi : B \to \mathrm{B}(V)$ is a Hilbert representation of B, then $E\otimes_B V$ is a Hilbert representation of A:

$$\langle\xi_E\otimes\xi_V,\eta_E\otimes\eta_V\rangle := \langle\xi_V,\pi(\langle\xi_E,\eta_E\rangle)\eta_V\rangle.$$

Example 2.2.5. Let $H \subseteq G$ be a closed unimodular subgroup, then $\mathrm{C}_c(G)$ is a $\mathrm{C}_c(G)$-$\mathrm{C}_c(H)$ bimodule:

$$\langle\xi,\eta\rangle(h) := \int_G \overline{\xi}(g)\eta(gh)dg.$$

Complete it to get a $\mathrm{C}^*(G)$-$\mathrm{C}^*(H)$ bimodule E. The *unitary induction* is given by $E\otimes_{\mathrm{C}_*} : \mathsf{URep}(H) \to \mathsf{URep}(G)$.

An A-B bimodule E is a *Morita equivalence* if

- E is a full left Hilbert A-module and a full right Hilbert B-module;
- $[\xi b|\eta] = [\xi|\eta b^*]$, $\langle a\xi,\eta\rangle = \langle\xi,a^*\eta\rangle$;
- $[\xi|\eta]\zeta = \xi\langle\eta,\zeta\rangle$.

We say A, B are (strongly) Morita equivalent, denoted by $A \sim_M B$, which is an equivalence relation.

- $A \sim_M B \Rightarrow \widehat{A} \simeq \widehat{B}$ and $K_*(A) \simeq K_*(B)$.

- $A \sim_M B \Leftrightarrow A \otimes \mathrm{K}(H) \simeq B \otimes \mathrm{K}(H)$ assuming countable approximate identities.
- $A \sim_M B \Leftrightarrow$ equivalent categories of operator modules. If they have 1, we can replace by the categories of (algebraic) modules.
- Equivalence of categories URep does not imply (strong) Morita equivalence.

Let $H \subseteq G$ be closed, and E the induction bimodule.

Theorem 2.2.6 (Rieffel). *THe induction bimodule E can be made into a Morita equivalence between $\mathrm{C}^*(H)$ and $\mathrm{C}_0(G/H) \rtimes G$.*

Corollary 2.2.7 (Mackey). *Unitary induction gives an equivalence between $\mathsf{URep}(H)$ and the category of unitary representations G admitting a compatible representation of $\mathrm{C}_0(G/H)$.*

Example 2.2.8. $A \rtimes K$, A abelian, K compact. Let $\pi : A \rtimes K \to \mathrm{U}(H)$ be irreducible. It is an irreducible representation of $\mathrm{C}_0(K_\varphi) \rtimes K \simeq \mathrm{C}_0(K/K_\varphi) \rtimes K$, and is induced from K_φ.

Example 2.2.9. $\mathrm{K}(H) \sim_M \mathbb{C}$.

In a Morita equivalence, we always have $[\xi|\eta] = |\xi\rangle\langle\eta|$.
Now given W, $\pi : W \to \mathrm{U}(H)$ and $\mathrm{K}(H)^W$.

Theorem 2.2.10. *H is a Morita equivalence between $\mathrm{K}(H)^W$ and the ideal*

$$J := \overline{\operatorname{span}}\,\{\langle\xi,\eta\rangle_W\} \subseteq \mathrm{C}^*_r(W),$$

and

$$J = \bigoplus_{\rho \in \widehat{W}, \overline{\rho} \subseteq \pi} \mathrm{K}(H_\rho).$$

Theorem 2.2.11. *Given $X, H, W, I_{w,x}$ as before, then $\mathrm{C}_0(X,H)$ is a Morita equivalence between $\mathrm{C}_0(X,\mathrm{K}(H))^W$ and certain ideal in $\mathrm{C}_0(X) \rtimes W$.*

Set $W_x = \{w \in W \mid wx = x\}$ and $W'_x = \{x \in W_x \mid I_{w,x} \in \mathbb{C}\mathrm{id}_H\}$.

- *Normalisation*:
- *Completeness*: for all x, the unitary representation $I_{-,x} : W_x \to \mathrm{U}(H)$ contains every $\rho \in \widehat{W_x/W'_x}$.

$C(X,W,I)$ things

Theorem 2.2.12. *$\mathrm{C}_0(X,H)$ can be made into a Morita equivalence between $\mathrm{C}_0(X,\mathrm{K}(H))^W$ with the ideal*

$$C(X,W,I) = \left\{ \sum_{w \in W} f_w w \in \mathrm{C}_0(X) \rtimes W \,\middle|\, f_{w'w}(x) = f_w(x), \forall x \in X, w \in W, w' \in W'_X \right\}.$$

Corollary 2.2.13. *Suppose that $W = W' \rtimes R$, where for each x we have $W'_x = W_x \cap W'$, then*

$$\mathrm{C}_0(X,\mathrm{K}(H))^W \sim_M \mathrm{C}_0(X/W') \rtimes R.$$

2.3 Lecture 3

The main reference is [CCH16]. The Langlands decomposition $G = M_G \times A_G$ where M_G has compact center and $\exp : \mathfrak{a}_G \to A_G$.

An irreducible unitary representation σ of M is *square-integrable* if for all $\xi, \eta \in H_\sigma$, the matrix coefficient $c_{\xi,\eta}(m) = \langle \sigma(m)\xi, \eta \rangle$ is in $\mathrm{L}^2(M)$.

Theorem 2.3.1. $\widehat{M}_{\mathrm{L}^2} \subseteq \widehat{M}_r$.

Theorem 2.3.2. *If $\sigma \in \widehat{M}_{\mathrm{L}^2}$, then $\sigma(\mathrm{C}^*_r(M)) = \mathrm{K}(H_\sigma)$.*

For each $\chi \in \mathfrak{a}^*$ and $\sigma \in \widehat{M}_{\mathrm{L}^2}$, we define an irreducible unitary representation $\sigma \otimes \chi : G \to \mathrm{U}(H_\sigma)$, $ma \mapsto \sigma(m)\chi(a)$.

Theorem 2.3.3. *For $f \in \mathrm{C}_c(G)$ and $\chi \in \mathfrak{a}^*$, let*

$$\pi_{G,\sigma}(f)(\chi) := (\sigma \otimes \chi)(f) = \int_M \int_A f(ma)\sigma(m)\chi(a) da dm.$$

This map $\pi_{G,\sigma}$ extends to a homomorphism of C^*-algebras:

$$\pi_{G,\sigma} : \mathrm{C}^*_r(G) \to \mathrm{C}_0(\mathfrak{a}^*, \mathrm{K}(H_\sigma)).$$

Proof. Study functions of the form $ma \mapsto f_M(m) f_A(a)$, which form a dense subset. □

Not every irreducible tempered representation is of this form, but it can be obtained by parabolic induction from some $\sigma \otimes \chi$ of a parabolic subgroup.

Now let $P = L_P N_P = M_P A_P N_P$ a parabolic subgroup of G.

Definition 2.3.4 (Parabolic induction)**.** For $\sigma \in (\widehat{M_P})_{\mathrm{L}^2}$ and $\chi \in \mathfrak{a}_P^*$, $\mathrm{Ind}_P^G(\sigma \otimes \chi)$ is the unitary representation of G induced from $\sigma \otimes \chi$.

Compact picture: $G = KP$ implies that $\mathrm{Ind}_P^G(\sigma \otimes \chi) \simeq \mathrm{Ind}_{K \cap P}^K(\sigma)$ over K. Fix σ, then all these parabolic inductions are isomorphic as K-representations.

Theorem 2.3.5. *We have a homomorphism of C^*-algebras:*

$$\pi_{P,\sigma} : \mathrm{C}^*_r(G) \to \mathrm{C}_0(\mathfrak{a}_P^*, \mathrm{K}(\mathrm{Ind}_P^G H_\sigma)).$$

Theorem 2.3.6 (Complete Fourier transform)**.** *We have an injective homomorphism of C^*-algebras:*

$$\bigoplus \pi_{P,\sigma} : \mathrm{C}^*_r(G) \to \bigoplus_{[P,\sigma]} \mathrm{C}_0(\mathfrak{a}_P^*, \mathrm{K}(\mathrm{Ind}_P^G H_\sigma)).$$

Question: What is the image of this Fourier transform? We need to understand the intertwining operators between $\mathrm{Ind}_P^G(\sigma \otimes \chi)$'s.

Theorem 2.3.7 (Bruhat)**.** *The intertwining operators between $\mathrm{Ind}_P^G(\sigma \otimes \chi)$'s are controlled by a certain finite group.*

Fix $P = MAN$, then

- W_P the Weyl group associated to A_P.
- For each $\sigma \in \widehat{M}_{\mathrm{L}^2}$, $W_\sigma := \{w \in W_P \mid w\sigma \simeq \sigma\}$.
- For each $\chi \in \mathfrak{a}_P^*$, $W_{\sigma,\chi} := \{w \in W_\sigma \mid w\chi = \chi\}$.

Theorem 2.3.8 (Knapp-Stein). *There are unitary operators:*

$$I_{w,\chi} \in \mathrm{U}(\mathrm{Ind}_P^G H_\sigma),\ w \in W_\sigma, \chi \in \mathfrak{a}_P^*,$$

satisfying

- $\chi \mapsto I_{w,\chi}$ *is continuous in the strong operator topology,*
- $I_{w_1,w_2\chi} I_{w_2,\chi} = I_{w_1 w_2,\chi}$,
- $I_{w,\chi}$ *is an intertwining operator* $\mathrm{Ind}_P^G(\sigma \otimes \chi) \to \mathrm{Ind}_P^G(\sigma \otimes w\chi)$.

Define an action of W_σ on $\mathrm{C}_0(\mathfrak{a}_P^*, \mathrm{K}(\mathrm{Ind}_P^G H_\sigma))$: $\beta_w(f)(\chi) = I_{w,w^{-1}\chi} f(w^{-1}\chi) I_{w^{-1}\chi}$, and we have

$$\pi_{P,\sigma}(\mathrm{C}_r^*(G)) \subseteq \mathrm{C}_0(\mathfrak{a}_P^*, \mathrm{K}(\mathrm{Ind}_P^G H_\sigma))^{W_\sigma}.$$

Theorem 2.3.9. *The Fourier transform*

$$\bigoplus \pi_{P,\sigma} : \mathrm{C}_r^*(G) \to \bigoplus_{[P,\sigma]} \mathrm{C}_0(\mathfrak{a}_P^*, \mathrm{K}(\mathrm{Ind}_P^G H_\sigma))^{W_\sigma}$$

is an isomorphism of C^*-algebras.

2.4 Lecture 4

Theorem 2.4.1 (Knapp-Stein). *Let $W'_\sigma := W'_{\sigma,0}$.*

(1) There is a subgroup $R_\sigma \subseteq W_\sigma$ such that $W_\sigma = W'_\sigma \rtimes R_\sigma$.

(2) $W'_{\sigma,\chi} = W_{\sigma,\chi} \cap W'_\sigma$.

(3) The $I_{w,\chi}$'s can be chosen so that they satisfy the normalisation and completeness conditions.

Corollary 2.4.2 (Wassermann). *For each real reductive group G, we have*

$$\mathrm{C}_r^*(G) \sim_M \bigoplus_{[P,\sigma]} \mathrm{C}_0(\mathfrak{a}_P^*/W'_\sigma) \rtimes R_\sigma.$$

Example 2.4.3. Let $G = \mathrm{SL}_2(\mathbb{R})$, then

$$\mathrm{C}_r^*(G) \sim_M \mathrm{C}_0(\mathbb{Z}\backslash\{0\}) \oplus \mathrm{C}_0([0,\infty)) \oplus \mathrm{C}_0(\mathbb{R}) \rtimes W,$$

where $W \simeq \mathbb{Z}/2\mathbb{Z}$.

Now let G be a p-adic reductive group.

Theorem 2.4.4. $\widehat{G}_{\mathrm{L}^2} \neq 0$.

Define $X_G = \{\chi : G \to \mathrm{U}(\mathbb{C}) \mid \chi(g) = 1 \text{ if contained in a compact subgroup}\}$, which is a compact torus. We get a Fourier transform:

$$\pi_{G,\sigma} : G \to \mathrm{C}(X_G, \mathrm{K}(H_\sigma)).$$

The complete Fourier transform is also injective due to Harish-Chandra and Bernstein.

- We define W_σ as a subgroup of $X_P \rtimes W_P$.
- $I_{w_1,w_2x} I_{w_2,x} = \gamma_{P,\sigma}(w_1,w_2) I_{w_1w_2,\chi}$ for some 2-cocycle. So we deal with projective representations and twisted crossed products.
- We need to keep track of a projective character $w \mapsto i_{w,\chi}$ of $W'_{\sigma,\chi}$ for each χ.

Theorem 2.4.5 (Plymen,Harish-Chandra)**.** *The Fourier transform*

$$\bigoplus \pi_{P,\sigma} : \mathrm{C}^*_r(G) \to \bigoplus_{[P,\sigma]} \mathrm{C}_0(X_P, \mathrm{K}(\mathrm{Ind}_P^G H_\sigma))^{W_\sigma}$$

is an isomorphism.

Theorem 2.4.6. *The bimodule $\mathrm{C}(X_P, \mathrm{Ind}_P^G H_\sigma)$ gives a Morita equivalence between $\mathrm{C}^*_r(G)_{(P,\sigma)}$ with the ideal $\mathrm{C}(X_P, W_\sigma, I)$.*

Some calculation of the K-theory for this reduced group C^*-algebra, with an example of $\mathrm{Ind}_{P_{\min}}^{\mathrm{Sp}(4,\mathbb{Q}_p)} \mathbb{1}$.

3 Omar Mohsen: Representation theory of nilpotent groups and Kirillov's orbit method

3.1 Lecture 1

Let V be a commutative monoid (abelian group without inverse). Define

$$K(V) := \{(a,b) \mid a,b \in V\} / \sim,$$

where $(a,b) \sim (c,d)$ if there exists $f \in V$ such that $a+d+f = b+c+f$.

Proposition 3.1.1. *$K(V)$ is an abelian group, and we have $V \to K(V)$, $a \mapsto [(a,0)]$.*

In this lecture, we will write $[(a,b)]$ as $a-b$.
Let A be a C^*-algebra with a unit.

Definition 3.1.2. If E is an A-module, we say that E is *finitely generated projective* if there exists an A-module F such that $E \oplus F = A^n$ for some n.

If E, F are finitely generated projective, then so is $E \oplus F$. When $A = \mathrm{C}(X)$ for a compact Hausdorff X.

Theorem 3.1.3. *Finitely generated projective A-modules are in bijection with vector bundles over X. If $L \to X$ is a vector bundle, then $E = \Gamma(L)$ is a finitely generated projective module.*

Definition 3.1.4. Define $V(A) := \{[E] \mid E \text{ is finitely generated projective}\}$, which is a commutative monoid with identity $[0]$, and the group law $[E] + [F] = [E \oplus F]$.

Definition 3.1.5.

$$K_0(A) := K(V(A)) = \{[E] - [F] \mid E, F \text{ are finitely generated projective}\} .$$

Example 3.1.6. $K_0(\mathrm{B}(H)) = 0$ for any infinite dimensional Hilbert space H.

Let $\varphi : A \to B$ be a $*$-homomorphism. We can define a map $\varphi_* : K_0(A) \to K_0(B)$ by sending $[E]$ to $[\varphi_*(E)]$, where $\varphi_*(E) = E \otimes_A B$ is a right B-module.

If $E \oplus F = A^n$, let $L : A^n \to A^n$ be the projection over E. In fact, any finitely generated projective module appears as $E = pA^n$ for some n and some projection $p : A^n \to A^n$.

- A (self-adjoint) projection $p \in \mathrm{M}_n(A)$ is an element such that $p^2 = p$ and $p^* = p$.
- Let p and q are two projections, then $pA^n \simeq qA^n$ if and only if there exist $x, y \in \mathrm{M}_n(A)$ such that $xy = p$ and $yx = q$ (*Von Neumann relation*). We write $p \sim_v q$ if they satisfy the Von Neumann relation.
- We have
$$V(A) = \bigcup_n \mathrm{proj}(\mathrm{M}_n(A)) / \sim_v .$$

Definition 3.1.7. For projections $p, q \in \mathrm{M}_n(A)$, define $p \sim_s q$ if there exists $z \in \mathrm{U}(\mathrm{M}_n(A))$ such that $zpz^{-1} = q$.

Remark 3.1.8. If $p \sim_v q$ with x invertible, then $p \sim_s q$. The relation $p \sim_s q$ implies $p \sim_v q$, but the converse fails.

Proposition 3.1.9. *If $p \sim_v q$, then $\left(\begin{smallmatrix} p & 0 \\ 0 & 0 \end{smallmatrix}\right) \sim_s \left(\begin{smallmatrix} q & 0 \\ 0 & 0 \end{smallmatrix}\right)$ as projections in $\mathrm{M}_{2n}(A)$.*

Proof. Take $E = pA^n$ and $E' = qA^n$, which are isomorphic since $p \sim_v q$. There exist F, F' such that $E \oplus F = A^n$ and $E' \oplus F' = A^n$. We have $E \oplus (F \oplus A^n) \simeq A^{2n}$, where $F \oplus A^n \simeq F \oplus E' \oplus F' \simeq F \oplus E \oplus F' \simeq F' \oplus A^n$, thus we get an isomorphism between the complements of E, E' in A^{2n}. □

Corollary 3.1.10.
$$V(A) = \bigcup_n \mathrm{proj}(\mathrm{M}_n(A)) / \sim_s .$$

Proposition 3.1.11. *If p, q are projections in $\mathrm{M}_n(A)$, such that $\|p - q\| < 1/4$, then $p \sim_s q$.*

Proof. Take $z = 2pq - p - q + 1$. This element satisfies $pz = pq$ and $zq = pq$. We need z to be unitary, which can be implied by $\|2pq - p - q\| < 1$. This inequality follows from $\|p - q\| < 1/4$. □

Let φ_0, φ_1 be two homomorphism from A to B. If there exists a homomorphism

$$\widetilde{\varphi} : A \to B[0,1] = \{f : [0,1] \to B \text{ continuous}\}$$

such that $\mathrm{ev}_0 \circ \widetilde{\varphi} = \varphi_0$ and $\mathrm{ev}_1 \circ \widetilde{\varphi} = \varphi_1$, then $\varphi_{0,*} = \varphi_{1,*}$.

3.2 Lecture 2

If we have C^*-algebras $A_1 \xrightarrow{\varphi_1} B \xleftarrow{\varphi_2} A_2$ such that φ_1 is surjective, then we have the fiber product $C = A_1 \times_B A_2$.

Proposition 3.2.1. *Let E_1, E_2 be finitely generated modules over A_1, A_2 respectively, and $L : \varphi_{1,*}E_1 \simeq \varphi_{2,*}E_2$, then*

$$\mathcal{M}(E_1, E_2, L) := \{(e_1, e_2) \in E_1 \times E_2 \,|\, L(e_1 \otimes 1) = e_2 \otimes 1\}$$

is finitely generated projective, and all finitely generated projective C-modules come this way.

Proof. One can assume that $E_1 = A_1^n$ and $E_2 = A_2^n$, using the complement trick yesterday.

Lemma 3.2.2. *If $L \in \mathrm{GL}_n(B)$ has invertible preimage under φ_1, then $\mathcal{M}(E_1, E_2, L) \simeq C^n$.*

Proof of Lemma 3.2.2. Take a basis $e_1, \ldots, e_n$ the corresponding basis of E_1, and $\widetilde{e}_1, \ldots, \widetilde{e}_n$ of E_2. Let K be the element in $\mathrm{GL}_n(A_1)$ such that $\varphi_1(K) = L^{-1}$. The elements $(\sum_j K_{i,j}e_i, \widetilde{e}_i)$ lie in $\mathcal{M}(E_1, E_2, L)$ and generate C^n. □

One has

$$\mathcal{M}(A_1^n, A_2^n, L) \oplus \mathcal{M}(A_1^n, A_2^n, L^{-1}) = \mathcal{M}(A_1^{2n}, A_2^{2n}, \mathrm{diag}(L, L^{-1})).$$

The projectivity of $\mathcal{M}(E_1, E_2, L)$ follows from the following lemma, where every matrix on the right hand side can be lifted to A_1:

Lemma 3.2.3.

$$\begin{pmatrix} L & 0 \\ 0 & L^{-1} \end{pmatrix} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -L^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

□

Now we get a sequence $K_0(C) \xrightarrow{(\pi_1,\pi_2)} K_0(A_1) \oplus K_0(A_2) \xrightarrow{(\varphi_1,\varphi_2)} K_0(B)$.

Exercise 3.2.4. This sequence is exact.

We will define a map $K_1(B) \to K_0(C)$ such that this makes a longer exact sequence, and K_1 should come from automorphisms.

Definition 3.2.5. Let A be a unital C^*-algebra. Define $K_1(A)$ to be the abelian group $\bigcup_n \pi_0(\mathrm{GL}_n(A))$, the product defined as

$$[M_1] \cdot [M_2] = [M_1 M_2],\ M_1, M_2 \in \mathrm{GL}_n(A).$$

Lemma 3.2.6 (Whitehead)**.** *If $x, y \in \mathrm{GL}_n(A)$, then*

$$\begin{pmatrix} xy & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} yx & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}, \begin{pmatrix} y & 0 \\ 0 & x \end{pmatrix}$$

are all in the same connected component of $\mathrm{GL}_{2n}(A)$.

Proof. Idea: using rotations in $\mathrm{GL}_{2n}(A)$. Set $R_\theta = \left(\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right)$ and $M_\theta = \mathrm{diag}(y, 1) \cdot R_{-\theta} \cdot \mathrm{diag}(1, x) \cdot R_\theta$, and $M_{k\pi/2}$, $k = 0, 1, 2, 3$ give the matrices in the lemma. □

Alternatively, one can define $K_1(A)$ by

$$\{[(E, L)] \mid E \text{ is a finitely generated projective module}, L \in \mathrm{Aut}(E)\} \;/\; \sim,$$

where $(E, L) \sim (E', L')$ if there exists (F, K) such that $(E, L) + (F, K)$ and $(E', L') + (F, K)$ are homotopic.

3.3 Lecture 3

The boundary map $K_1(B) \to K_0(C)$ is given by: for $L \in \mathrm{GL}_n(B)$,

$$\partial(L) = [\mathcal{M}(A_1^n, A_2^n, L)] - [C^n].$$

Example 3.3.1. $K_1(\mathrm{C}(S^1)) = \mathbb{Z}$.

The space $A \otimes \mathrm{C}(S^1)$ is the space of functions $f : S^1 \to A$, which is a C^*-algebra. It is the fiber product of two copies of $A \otimes \mathrm{C}([0, 1])$ over $A \otimes A$. From the exact sequence,

$$K_1(A) \simeq \ker\left(K_0(A \otimes \mathrm{C}(S^1)) \xrightarrow{\mathrm{ev}_1} K_0(A)\right).$$

Theorem 3.3.2 (Bott)**.**

$$\begin{aligned} \beta : K_0(A) &\simeq \ker\left(K_1(A \otimes \mathrm{C}(S^1)) \xrightarrow{\mathrm{ev}_1} K_1(A)\right) \\ [P] \in \mathrm{Proj}(\mathrm{M}_n(A)) &\mapsto [z \mapsto zP + (1 - P) \in \mathrm{GL}_n(A)]. \end{aligned}$$

Proof of the surjectivity. Suppose $f : S^1 \to \mathrm{GL}_n(A)$ such that $[f] \in \ker(K_1(A \otimes \mathrm{C}(S^1)) \to K_1(A))$. We may assume that $f(z) = z^{-m}(a_0 + za_1 + \cdots + z^{m'} a_{m'})$, then $[f] = [z^{-m}] + [a_0 + \cdots + z^{m'} a_{m'}]$.

Since $[z^{-m}] = -m[z] = -m\beta(1)$, now we assume $f(z) = a_0 + \cdots + a_m z^m \in \mathrm{GL}_n(A)$. Define

$$\mu(z) = \begin{pmatrix} a_0 & a_1 & \cdots & a_m \\ -z & 1 & & \\ & \ddots & & \\ & & -z & 1 \end{pmatrix}.$$

Using

$$\begin{pmatrix} 1 & & -a_m \\ & \ddots & \\ & & 1 \end{pmatrix} \mu(z) \begin{pmatrix} 1 & & & & \\ 0 & \ddots & & & \\ \vdots & & \ddots & & \\ \vdots & & & \ddots & \\ 0 & z & \cdots & z & 1 \end{pmatrix}$$

we can assume that $f(z) = a_0 + za_1$, and $f(1) = a_0 + a_1$ is invertible and homotopic to Id. So $(a_0 + a_1)^{-1} f(z) = (a_0 + a_1)^{-1} a_0 + z(a_0 + a_1)^{-1} a_1 = (1 - a) + za$.

Since $f(z)$ is invertible for any $z \in S^1$, if $\lambda \in \mathrm{Spec}(a)$, then $1 - \lambda + z\lambda \neq 0$ for any $z \in S^1$. This is equivalent to that $\mathrm{Re}(\lambda) \neq 1/2$. Now define $g(z) = 0$ if $\mathrm{Re}(z) \leq 1/2$ and 1 if $\mathrm{Re}(z) > 1/2$, and $P = g(a)$. It suffices to show that $\beta(P)$ is homotopic to $(1 - a) + za$. $\square$

If A is non-unital, we define $K_0(A) := \ker(K_0(A^+) \xrightarrow{a+\lambda 1 \mapsto \lambda} K_0(\mathbb{C}) = \mathbb{Z})$, and $K_1(A) := K_1(A^+)$.

Let G be a simply-connected nilpotent Lie group.

- $\exp : \mathfrak{g} \to G$ is a diffeomorphism.
- If $\mathfrak{h} \subseteq \mathfrak{g}$ is a Lie subalgebra, then $\exp(\mathfrak{h})$ is a closed subgroup of G, i.e. connected subgroups of G are closed.

Goal: classification of unitary irreducible representations of G.

For a closed subgroup H, $\chi : H \to S^1$ is unitary, and we take $\Pi = \mathrm{Ind}_H^G(\chi)$. If $H \subsetneq H'$, χ admits an extension to H'. The differential of χ takes values in $i\mathbb{R}$, and $\chi([h, h]) = 0$.

Take $\xi : \mathfrak{g} \to \mathbb{R}$, and look at $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$, $(v, w) \mapsto \xi([v, w])$, which is anti-symmetric. The kernel of B is $\{v \mid B(v, w) = 0, \forall w\}$. The codimension of $\ker(B)$ is even. We look for $\mathfrak{h} \subseteq \mathfrak{g}$ such that

(1) $\mathfrak{h}$ is a Lie subalgebra;

(2) $B(\mathfrak{h}, \mathfrak{h}) = 0$;

(3) $\dim \mathfrak{h} = \dim(\ker B) + \frac{1}{2}\mathrm{codim}(\ker B)$.

Theorem 3.3.3. *There exists such an $\mathfrak{h}$ (not unique).*

Theorem 3.3.4. *For such an H, the induction $\pi_\xi := \mathrm{Ind}_H^G(e^{i\xi})$ is irreducible and unitary, and its isomorphism class is independent of the choice of $\mathfrak{h}$.*

Theorem 3.3.5. *All irreducible unitary representations come from this construction.*

In conclusion, we have a surjection $\mathfrak{g}^* \to \widehat{G}$.

3.4 Lecture 4

Theorem 3.4.1 (Kirillov). *The map* $\mathrm{Ad}^*(G)\backslash\mathfrak{g}^* \to \widehat{G}$, $\xi \mapsto \pi_\xi$ *is bijective.*

Theorem 3.4.2 (Brown). *The map* $\mathrm{Ad}^*(G)\backslash\mathfrak{g}^* \to \widehat{G}$ *is a homeomorphism, where the topology on* $\widehat{G}$ *is the Fell topology.*

Recall a subquotient of A is I/J for ideals $I \subseteq A$, $J \subseteq I$.

Theorem 3.4.3. *There is a bijection between locally closed subsets of* $\widehat{A}$*, and isomorphism classes of subquotients of* A*.*

Remark 3.4.4. Given an irreducible unitary representation $\pi : I \to \mathrm{B}(H)$, one can extend it uniquely to $A \to \mathrm{B}(H)$.

Recall that if $\pi : A \to \mathrm{U}(H)$ is a unitary representation, then

$$\mathrm{supp}(\pi) = \left\{ [\pi'] \,\middle|\, \ker(\pi) \subseteq \ker(\pi') \right\}.$$

Theorem 3.4.5. *Let* $\mathfrak{h}$ *be a subalgebra of* $\mathfrak{g}$ *and* $\ell : \mathfrak{h} \to \mathbb{R}$ *a linear map such that* $\ell|_{[\mathfrak{h},\mathfrak{h}]} = 0$*, then*

$$\mathrm{supp}(\mathrm{Ind}_H^G e^{i\ell}) = \overline{\mathrm{Ad}^*(G)\left\{\xi \in \mathfrak{g}^* \mid \xi|_{\mathfrak{h}} = \ell\right\}}/\mathrm{Ad}^*(G).$$

Particularly, if $\ell = 0$*, then*

$$\mathrm{supp}(\mathrm{L}^2(G/H)) = \overline{\bigcup_{g\in G} (g\mathfrak{h}g^{-1})^{\perp}}/\mathrm{Ad}^*(G).$$

Theorem 3.4.6. *(1)* $\mathcal{S}(G)$ *is a* $*$*-subalgebra of* $\mathrm{C}^*(G)$*.*

(2) $\mathcal{S}(G)$ *is closed under smooth functional calculus.*

- *(i) If* $f \in \mathcal{S}(G)$ *and* $g : U \to \mathbb{C}$, $g(0) = 0$ *holomorphic on an open neighborhood of* $\mathrm{Spec}(f)$*, then* $g(f) \in \mathcal{S}(G)$*.*
- *(ii) If* $f \in \mathcal{S}(G)$ *is normal and* $g : W \to \mathbb{C}$ *smooth on an open neighborhood of* $\mathrm{Spec}(f)$ *and* $g(0) = 0$*, then* $g(f) \in \mathcal{S}(G)$*.*

Definition 3.4.7. A C^*-algebra A is called *liminal* (or CCR) if for any $[\pi] \in \widehat{A}$, $\pi(A) = \mathrm{K}(H_\pi)$, and is of *type I* if for any $[\pi]$ one has $\mathrm{K}(H) \subseteq \pi(A)$.

Theorem 3.4.8 (Dixmier). C^*G *is liminal. In fact if* $f \in \mathcal{S}(G)$ *and* $\pi \in \widehat{G}$*, then* $\pi(f)$ *is a trace class operator:*

$$\mathrm{Tr}\,\pi(f) = \int_{\mathcal{O}} \widehat{f \circ \exp}\, d\mu,$$

where $\mathcal{O} \subseteq \mathfrak{g}^*$ *is the corresponding co-adjoint orbit.*

If $\mathcal{O} \subset \mathfrak{g}^*$ is a co-adjoint orbit, then $\mathcal{O}$ is a symplectic smooth manifold: fix $\xi \in \mathcal{O}$, then $\mathcal{O} \simeq G/\mathrm{Stab}(\xi)$ and $\mathrm{T}_\xi\mathcal{O} = \mathfrak{g}/\ker B_\xi$. $\mathcal{O}$ is also a closed subset of $\mathfrak{g}^*$, which is equivalent to that $\widehat{G}$ is T_1 (consequence of C^*G being liminal). There exists a measure μ on $\widehat{G}$ such that for any $f \in \mathcal{S}(G)$,

$$f(1) = \int_{\widehat{G}} \mathrm{Tr}(\pi(f))\, d\mu(\pi).$$

If we replace f with $f^* \star f$, then

$$\int_G |f(g)|^2 dg = \int_{\widehat{G}} \|\pi(f)\|^2_{\mathrm{HS}} d\mu(\pi).$$

Theorem 3.4.9 (Beltita-Beltita-Ludwig)**.** *(Fourier transform of C^*-algebras of nilpotent Lie groups) There exist ideals $0 \subseteq I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n = \mathrm{C}^*G$ corresponding to $\widehat{G} = V_1 \cup \cdots \cup V_n$ with each subset locally closed is Hausdorff, such that*

$$I_{i+1}/I_i \simeq \mathrm{C}_0(V_i) \otimes \mathrm{K}(H).$$

All H's have infinite dimension, except one.

Conjecture 3.4.10. *If $J \subseteq I \subseteq \mathrm{C}^*G$ such that $\widehat{I/J}$ is Hausdorff, then I/J is Morita equivalent to $\mathrm{C}_0(\widehat{I/J})$.*

4 Hang Wang: Group C^*-algebras and their K-theory

The aim is Connes-Kasparov isomorphism as K-theoretic Mackey analogy.

4.1 Lecture 1

4.1.1 What is Mackey analogy?

Let G be a connected Lie group, and K the maximal compact subgroup of G. Let $\mathfrak{g}$ and $\mathfrak{k}$ be their real Lie algebras. The space $\mathfrak{g}/\mathfrak{k}$ is a metric space and an abelian group. For any $k \in K$, we have the adjoint action $\mathrm{Ad}(k)$ on $\mathfrak{g}/\mathfrak{k}$.

Definition 4.1.1. The *motion group* is defined to be

$$G_0 := K \ltimes (\mathfrak{g}/\mathfrak{k}),\ (k_1, v_1) \cdot (k_2, v_2) = (k_1 k_2, \mathrm{Ad}_{k_2^{-1}}(v_1) + v_2).$$

Example 4.1.2. For $G = \mathrm{SL}(2, \mathbb{R})$ and $K = \mathrm{SO}(2, \mathbb{R}) \subseteq G$, $\mathfrak{k} = \left\{ \left(\begin{smallmatrix} 0 & t \\ -t & 0 \end{smallmatrix} \right) \,\middle|\, t \in \mathbb{R} \right\} \subseteq \mathfrak{g}$, and $\mathfrak{g}/\mathfrak{k} = \left\{ \left(\begin{smallmatrix} a & b \\ b & a \end{smallmatrix} \right) \,\middle|\, a, b \in \mathbb{R} \right\} \simeq \mathbb{R}^2$. The action of $\left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix} \right)$ on $(a, b)^T \in \mathfrak{g}/\mathfrak{k}$ is the natural one. The motion group $G_0 = \mathrm{SO}(2) \ltimes \mathbb{R}^2$ is the group of rigid motions.

Mackey analogy is a prediction of a 1-1 correspondence between:

$$\widehat{G}_t = \{ \text{ tempered representations } \}/\simeq$$

and $\widehat{G_0}$ (unitary dual).

4.1.2 Structure of the unitary dual of G_0

Let K be a compact Lie group, X an abelian group, and $\alpha : K \to \mathrm{Aut}(X)$. One studies the unitary dual of $\widehat{K \ltimes X}$. The unitary dual $\widehat{X}$ is isomorphic to $X^* = \mathrm{Hom}(X, \mathrm{U}(1))$. For $\varphi \in \widehat{X}$, take $K_\varphi = \{k \in K \mid \varphi(k.x) = \varphi(x), \forall x \in X\}$ to be the isotopy subgroup.

Remark 4.1.3. When G is complex semisimple, K_φ is connected.

Given $\varphi \in \widehat{X}$ and $(\tau, W_\tau) \in \widehat{K_\varphi}$, we construct the following representation of $K_\varphi \ltimes X$:

$$\begin{aligned} \tau \otimes \varphi : K_\varphi \ltimes X &\to \mathrm{GL}(W_\tau) \\ (k, x) &\mapsto \varphi(x)\tau(k). \end{aligned}$$

Induce $\tau \otimes \varphi$ to G_0:

$$\pi_{\tau,\varphi} := \mathrm{Ind}_{K_\varphi \ltimes X}^{K \ltimes X} \tau \otimes \varphi.$$

Theorem 4.1.4 (Mackey). *Let G be complex semisimple and $X = \mathfrak{g}/\mathfrak{k}$. The representation $\pi_{\tau,\varphi}$ lies in $\widehat{G_0}$, and the map*

$$\bigsqcup_{\varphi \in \widehat{X}} \widehat{K_\varphi} \to \widehat{G_0},\ (\tau, \varphi) \mapsto \pi_{\tau,\varphi}$$

is surjective. Moreover, $\pi_{\tau_1,\varphi_1} \simeq \pi_{\tau_2,\varphi_2}$ if and only if there exists $k \in K$ such that $\varphi_2(x) = \varphi_1(k.x)$ and $\tau_2 = \tau_1 \circ \mathrm{Ad}(k)$.

In summary,

$$\widehat{K \ltimes X} \simeq \left(\bigsqcup_{\varphi \in \widehat{X}} \widehat{K_\varphi} \right) / K.$$

Now let $X = \mathfrak{g}/\mathfrak{k}$. In the following we assume that G is connected complex semisimple. We use the Iwasawa decomposition $G = KAN$, where A is the abelian component, and N is the unipotent subgroup. Let M be the centralizer of A in K, and $B = MAN$ the Borel subgroup.

Example 4.1.5. If $G = \mathrm{SL}(3, \mathbb{C})$, we have: $K = \mathrm{SU}(3)$,

$$A = \{\mathrm{diag}(x_1, x_2, x_3) \mid x_1 > 0, x_1x_2x_3 = 1\} \simeq \mathbb{R}^2,$$

$$M = \left\{ \mathrm{diag}(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}) \,\middle|\, \theta_i \in \mathbb{R}, \theta_1 + \theta_2 + \theta_3 = 0 \right\} \simeq (S^1)^2,$$

and $N = \left\{ \begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix} \right\}$.

We write X as $\mathfrak{a} \oplus \mathfrak{a}^\perp$. For $\varphi \in (\mathfrak{g}/\mathfrak{k})^*$, there exists a w in the Weyl group $W := \mathrm{N}_K(M)/M$ such that $\varphi' = \varphi \circ \mathrm{Ad}(w)$ has zero restriction to $\mathfrak{a}^\perp$, called the *balanced character*. The character φ' can be identified with $\varphi'|_{\mathfrak{a}} \in \widehat{\mathfrak{a}}$.

Remark 4.1.6. For a balanced character φ', then for any $k \in M \subseteq K$, $k.x = x$ for any $x \in \mathfrak{a}$, thus $k \in M \subseteq K_\varphi$. Hence M is a maxiaml torus of K_φ.

If K is a general connected compact Lie group with a maximal torus T, then $\widehat{K} \simeq \widehat{T}/W$, where W is the Weyl group. Applying this to K_φ, we have $\widehat{K_\varphi} = \widehat{M}/W_\varphi$, where $W_\varphi = W(K_\varphi, M)$.

Theorem 4.1.7.

$$\widehat{G_0} = K \ltimes \widehat{(\mathfrak{g}/\mathfrak{h})} = \bigsqcup_{\varphi \in \widehat{\mathfrak{a}}/W} \widehat{M}/W_\varphi = (\widehat{M} \times \widehat{\mathfrak{a}})/W.$$

4.1.3 Tempered dual of G

Assume that G is complex semisimple with Iwasawa decomposition $G = KAN$.

Definition 4.1.8. For $\sigma \in \widehat{M}$ and $\varphi \in \widehat{A}$, the *principal series* associated to (σ, φ) is

$$P_{\sigma,\varphi} := \mathrm{Ind}_{MAN}^{G} \sigma \otimes \varphi \otimes \mathbb{1}.$$

Under our assumption, $P_{\sigma,\varphi}$ is irreducible, and $P_{\sigma_1,\varphi_1} \simeq P_{\sigma_2,\varphi_2}$ if and only if there exists $w \in W = \mathrm{N}_G(MA)/MA$ such that $w(\sigma_1, \varphi_1) = (\sigma_2, \varphi_2)$. All tempered representations of G are such principal series.

Theorem 4.1.9.

$$\left(\widehat{M} \times \widehat{A}\right)/W \simeq \widehat{G}_t,\ (\sigma, \varphi) \mapsto P_{\sigma,\varphi}.$$

Via the exponential map $\mathfrak{a} \to A$, one has $\widehat{G_0} \simeq \widehat{G}_t$.

Example 4.1.10. For $G = \mathrm{SL}(3, \mathbb{C})$, $\widehat{M} \simeq \mathbb{Z}^2$ and $\widehat{A} \simeq \mathbb{R}^2$. The tempered dual $\widehat{G}_t$ is $(\widehat{M} \times \widehat{A})/W$. A fundamental domain of $\widehat{M}/W$ is $\left\{(m_1, m_2) \in \mathbb{Z}^2 \,\middle|\, m_2 \geq m_1 \geq 0\right\}$, and

$$\widehat{G}_t \simeq \left(\bigsqcup_{m_2 > m_1 > 0} \mathbb{R}^2\right) \sqcup \left(\bigsqcup_{m_2 = m_1 \text{ or } m_1 = 0} \mathbb{R}^2/\mathbb{Z}_2\right) \sqcup (\mathbb{R}^2/S_3)_{(0,0)}.$$

4.2 Lecture 2

4.2.1 Cartan decomposition

Let G be a non-compact semisimple Lie group with finite center. Let K be a maximal compact subgroup of G, then:

(1) There is a homomorphism $\Theta : G \to G$ with $\theta = d\Theta : \mathfrak{g} \to \mathfrak{g}$ such that $\Theta^2 = \mathrm{Id}$. This homomorphism is called a *Cartan decomposition*.

(2) Let $\mathfrak{k}$ and $\mathfrak{p}$ be the $1, -1$ eigenspaces of θ. Then we have $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, and we call $(\mathfrak{k}, \mathfrak{p})$ a *Cartan pair*.

(3) The morphism $K \times \mathfrak{p} \to G$, $(k, X) \mapsto k \exp X$ is a diffeomorphism. We write $G = KP$.

Example 4.2.1.
- If G is a complex semisimple Lie group, then $\mathfrak{p} = \mathfrak{k} + i\mathfrak{k}$ and $\theta(g) = -\overline{g}^T$. For instance, if $G = \mathrm{SL}(2, \mathbb{C})$, then $\mathfrak{k} = \mathfrak{su}(2)$.

- If $G = \mathrm{SL}(2,\mathbb{R})$, then $\mathfrak{sl}(2) = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{p}$ consists of symmetric matrices. In this case, $\theta(g) = -\overline{g}^T$.

The motion group $G_0 = K \ltimes (\mathfrak{g}/\mathfrak{k}) = K \ltimes \mathfrak{p}$, and we can view $\mathfrak{p}$ as the tangent space of G/K.

The Killing form $B(x,y) = \mathrm{Tr}(\mathrm{ad}(x) \circ \mathrm{ad}(y))$ is positive definite, and makes G/K a G-invariant Riemannian manifold.

4.2.2 Tangent groupoid (family of groups connecting G and G_0)

Define $\mathcal{G} = K \times \mathfrak{p} \times [0,1]$ to be a family of groups with parameter $t \in [0,1]$, with multiplication defined for each t:

- When $t = 0$, $(k_1, v_1) \circ_0 (k_2, v_2) := (k_1 k_2, \mathrm{Ad}(k_2^{-1})v_1 + v_2)$, thus $\mathcal{G}_0 = G_0$.
- When $t > 0$, we first define a diffeomorphism $\varphi_t : K \times \mathfrak{p} \to G$, $(k, X) \mapsto k\exp(tX)$. Set
$$(k_1, v_1) \circ_t (k_2, v_2) := \varphi_t^{-1} \left(\varphi_t(k_1, v_1) \circ \varphi_t(k_2, v_2).\right)$$

Exercise 4.2.2. $(k_1, v_1) \circ_t (k_2, v_2) \xrightarrow{t\to 0} (k_1, v_1) \circ_0 (k_2, v_2)$.

Remark 4.2.3. We can perform this construction whenever there is a submanifold inclusion $S \subseteq M$. For each $t > 0$, the fiber is N, and the fiber at $t = 0$ is the normal bundle $\mathrm{N}(S, M)$. If we take N to be $M \times M$ with $M = M^\Delta \subseteq N$, the fiber at $t = 0$ is $\mathrm{T}M$. A sequence $(x_n, y_n, t_n) \in M \times M \times (0,1]$ tends to $(X_x, 0) \in \mathrm{T}M$ if $\frac{y_n - x_n}{t_n} \to X_x$.

Remark 4.2.4. We have a diffeomorphism

$$\begin{aligned} K \times \mathfrak{a} \times \mathfrak{n} &\to G \\ (k, X, Y) &\mapsto k \exp X \exp Y. \end{aligned}$$

Using this we can perform a similar thing $K \times \mathfrak{a} \times \mathfrak{n} \times [0,1] \to \mathcal{G}$. When $t > 0$, it sends (k, X, Y, t) to $k\exp(tX)\exp(tY)$, and when $t = 0$, it sends that to $(k, X+Y, 0)$.

4.2.3 Continuous field of C^*-algebras

We write the fiber of $\mathcal{G}$ over t as G_t. Take the completion of $\mathrm{C}_c(G_t)$, and we get $\mathrm{C}_r^*(G_t)$. Consider continuous sections in this family, *i.e.* $f : [0,1] \to \{\mathrm{C}_r^*(G_t)\}_{t\in[0,1]}$ such that $t \mapsto \|f(t)\|_{\mathrm{C}_r^*(G_t)}$ is continuous. We define

$$\|f\| := \sup_{t\in[0,1]} \|f(t)\|_{\mathrm{C}_r^*(G_t)}.$$

The completion of continuous sections in this norm becomes a C^*-algebra, denoted by $\mathrm{C}_r^* G_{[0,1]}$.

The evaluation map ev_0 sending $f \in \mathrm{C}_r^* G_{[0,1]}$ to $f(0)$ is a C^*-algebra homomorphism, and we denote its kernel by $\mathrm{C}_r^* G_{(0,1]}$. We have a short exact sequence:

$$0 \to \mathrm{C}_r^* G_{(0,1]} \to \mathrm{C}_r^* G_{[0,1]} \to \mathrm{C}_r^*(G_0) \to 0.$$

For any f in the kernel, one can construct a homotopy $H(t,s) = f(st)$ between 0 and $f(t)$, thus the K-theory of $C_r^* G_{(0,1]}$ is 0. On the K-theory level, the algebras $C_r^* G_{[0,1]}$ and $C_r^*(G_0)$ are the same, thus $K_1(C_r^*(G_0)) \xleftarrow[\simeq]{\mathrm{ev}_0} K_1(C_r^* G_{[0,1]}) \xrightarrow{\mathrm{ev}_1} K_1(C_r^*(G))$. This also exists for $i = 2$. Connes-Kasparov morphism states that $K_i(C_r^*(G_0)) \simeq K_i(C_r^*(G))$. The point is that the K-theory of $C_r^*(G_0)$ is easy to compute. One has

$$\begin{aligned} K_i(C_r^*(G_0)) &\simeq K_i\left(C_0(\mathfrak{p}^*, \mathrm{K}(L^2(K))^K)\right) \\ &\simeq K_i(C_0(\mathfrak{p}^*) \rtimes K) \\ &\simeq K_i^K(C_0(\mathfrak{p}^*)) \\ &\simeq K_K^i(\mathfrak{p}^*). \end{aligned}$$

In certain nice situation, this is isomorphic to $K_K^{i+\dim \mathfrak{p}^*}(\mathrm{pt})$ [1], and isomorphic to the representation ring $\mathrm{R}(K)$ if $r + \dim \mathfrak{p}^*$ is even.

4.3 Lecture 3

4.3.1 Some calculation of $K(C_r^*(G))$

Let $G = KAN$ be a complex simisimple Lie group, then

$$C_r^*(G) \simeq C_0(\widehat{M} \times \widehat{A}, \mathrm{K}(L^2(K)))^W \simeq \bigoplus_{\sigma \in \widehat{M}/W} C_0(\widehat{A}/W_\sigma, \mathrm{K}(\mathrm{Ind}_M^G H_\sigma)),$$

where $W_\sigma = \{w \in W \mid w\sigma \simeq \sigma\}$. This C^*-algebra is Morita equivalent to

$$\bigoplus_{\sigma \in \widehat{M}/W} C_0(\widehat{A}/W_\sigma),$$

thus $K_i(C_r^*(G)) \simeq \bigoplus_{\sigma \in \widehat{M}/W} K^i(\widehat{A}/W_\sigma)$.

Set $\widehat{A} \simeq \mathbb{R}^n$. If $W_\sigma = 1$, then $K^i(\mathbb{R}^n) \xrightarrow[\simeq]{\text{Bott periodicity}} K^{i+n}(\mathrm{pt})$ equals $\mathbb{Z}$ if $i \equiv n \bmod 2$, and 0 otherwise.

Example 4.3.1. Let $G = \mathrm{SL}(3, \mathbb{C})$, then $A \simeq \mathbb{R}^2, M \simeq (S^1)^2, W \simeq S_3$. The fundamental domain $\widehat{M}/W = \left\{(m_1, m_2) \in \mathbb{Z}^2 \,\middle|\, m_2 \geq m_1 \geq 0\right\}$. We have

$$K_0(C_r^*(\mathrm{SL}(3,\mathbb{C}))) \simeq \bigoplus_{m_2 \geq m_1 \geq 0} K^0(\mathbb{R}^2/W_\sigma) \simeq \bigoplus_{m_2 > m_1 > 0} \mathbb{Z},$$

since $K^0(\text{half plane}) = 0$, and trivial K_1.

Example 4.3.2. Let $G = \mathrm{SL}_2(\mathbb{C})$, then $M \simeq \mathbb{Z}$, $A \simeq \mathbb{R}$, $W = \mathbb{Z}/2\mathbb{Z}$ and we have

$$K_1(C_r^*(\mathrm{SL}(2,\mathbb{C}))) \simeq \bigoplus_m K^1(\mathbb{R}/W_m) = K^1(\mathbb{R}/W) \oplus \bigoplus_{m>0} K^1(\mathbb{R}) = \bigoplus_{m>0} \mathbb{Z},$$

and trivial K_0.

[1] Here the subscript K means K-equivariant.

Now instead of a general real reductive group, we look at $\mathrm{SL}(2,\mathbb{R})$. In this case, we have two parabolic subgroups: the Borel P_1 and $P_2 = G$.

$$C_r^*(\mathrm{SL}(2,\mathbb{R})) \sim_M \left(\bigoplus_{\sigma\in(\widehat{M_2})_{ds}} \mathbb{C}\right) \oplus C_0(\mathbb{R}/\mathbb{Z}_2) \oplus C_0(\mathbb{R}) \rtimes \mathbb{Z}_2.$$

Considering the K-theory:

$$K_0(C_r^*(\mathrm{SL}(2,\mathbb{R}))) = \left(\bigoplus_{\sigma\in\widehat{G}_{ds}} \mathbb{Z}\right) \oplus \mathbb{Z}.$$

Here we use $K_0(C_0(\mathbb{R}) \rtimes \mathbb{Z}_2)$. The algebra $C_0(\mathbb{R}) \rtimes \mathbb{Z}_2$ has a model

$$A = \{f : [0,\infty) \to \mathrm{M}_2(\mathbb{C}) \mid f(\infty) = 0, f(0) = \left(\begin{smallmatrix} * & \\ & * \end{smallmatrix}\right)\}.$$

Take $A_0 = \{f \in A \mid f(0) = f(\infty) = 0\}$, which is Morita equivalent to $C_0(0,1)$. The evaluation map ev_0 gives a short exact sequence:

$$0 \to A_0 \to A \xrightarrow{\mathrm{ev}_0} \mathbb{C}^2 \to 0$$

is used to calculate $K_0(A)$.

4.3.2 $K(C_r^*(G_0))$

Recall that given $f \in C_c(G)$, one can define the generalized Fourier transform

$$\widehat{f} : \widehat{G} \to \{\mathrm{B}(H_\sigma)\}_{\pi\in\widehat{G}_t},$$
$$\widehat{f}(\pi) = \int_G f(g)\pi(g)dg.$$

We have $G_0 = K \ltimes \mathfrak{p}$. We define the Fourier transform

$$C_r^*(G_0) \xrightarrow{\simeq} C_0(\mathfrak{p}^*, \mathrm{K}(\mathrm{L}^2(K)))^K,$$
$$f \mapsto \widehat{f}(\varphi)(k_1,k_2) : \int_{\mathfrak{p}} f(k_1k_2^{-1}x)\varphi(k_2^{-1}(x))dx,$$

where we identify an operator with its integral kernel. For any $h \in K$, $\widehat{f}(h^{-1}\varphi)(k_1h, k_2h) = \widehat{f}(\varphi)(k_1,k_2)$. For the right hand side,

$$C_0(\mathfrak{p}^*, \mathrm{K}(\mathrm{L}^2(K)))^K \simeq C_0(\mathfrak{p}^*) \rtimes K.$$

If K acts on X trivially,

$$C(X, \mathrm{K}(\mathrm{L}^2(K)))^K = C(X) \otimes \mathrm{K}(\mathrm{L}^2(K))^K = C(X) \otimes C_r^*(K) \simeq C(X) \rtimes K,$$

In Lecture 2, we have $K_i(C_r^*(G_0)) \simeq \mathrm{K}$ under some nice conditions: where nice means K-action on $\mathfrak{p}^*$ preserves the orientation. In this situation, the K-action factors through $K \to \mathrm{SO}(\mathfrak{p}^*)$. Assume this lifts to $K \to \mathrm{Spin}(\mathfrak{p}^*)$, then

$$K_K^{\dim \mathfrak{p}^*}(\mathfrak{p}^*) \simeq \mathrm{R}(K).$$

Remark 4.3.3. Removing the conditions, we refer to a paper of Echterhoff-Pfante.

4.4 Lecture 4

Let G be a complex semisimple Lie group and $G_0 = K \ltimes \mathfrak{p}$ the motion group. We have

$$\widehat{G}_t = \bigsqcup_{\tau \in \widehat{M}/W} \widehat{A}/W_\sigma,$$

with $W = \mathrm{N}_G(MA)/MA$, and

$$\widehat{G_0} = \bigsqcup_{\sigma \in \widehat{M}/W} \widehat{\mathfrak{a}}/W,$$

with $W = \mathrm{N}_K(M)/M$.

Definition 4.4.1. For $\tau \in \widehat{K}$, it is a *K-type* for $\pi \in \widehat{G}$ if it appears in the decomposition of $\pi|_K$. A K-type τ of π is called a *minimal K-type* if the highest weight of τ is minimal among all K-types.

For any $\sigma \in \widehat{M}$, we have

$$\widehat{G}_t = \bigsqcup_{\tau_\sigma \in \widehat{K}} C_\sigma,\ \widehat{G_0} = \bigsqcup_{\tau_\sigma \in \widehat{K}} \mathcal{C}_\sigma,$$

where C_σ (resp. $\mathcal{C}_\sigma$) is the set consisting of π with minimal K-type τ_σ.

4.4.1 Compact group representation

Let K be a compact Lie group. The Fourier transform gives a C^*-algebra homomorphism:

$$\mathrm{C}^*_r(K) \to \mathrm{C}_0(\widehat{K}, \{\mathrm{M}_{d_\pi}(\mathbb{C})\}_{\pi \in \widehat{K}}),\ \mathrm{C}(K) \ni f \mapsto \widehat{f}(\pi) = \int_K f(k)\pi(k)dk.$$

For $(\pi, H_\pi) \in \widehat{K}$, let $u, v \in H_\pi$, and we define $\phi_{u,v}(x) := \langle \pi(x)u, v\rangle$. If $e_1, \ldots, e_{d_\pi}$ is an orthogonal normal basis of H_π, we set $\pi_{i,j} = \phi_{e_i, e_j}$ and $\pi(x) = \big(\pi_{i,j}(x)\big)_{1 \le i,j \le d_\pi}$. We have the orthogonal relation:

$$\int_K \pi_{i,j}(x)\pi_{i',j'}(x)dx = \frac{1}{d_\pi}\delta_{i,j}\delta_{i',j'}.$$

Let e_1 be a unit vector in the highest weight space of π. The matrix coefficient $p_\pi := d_\pi \pi_{1,1} \in \mathrm{C}(K)$ is an idempotent in $\mathrm{C}^*(K)$, and

$$\int_K p_\pi(x)p_{\pi'}(x)dx = d_\pi \delta_{\pi,\pi'}.$$

For $\tau \in \widehat{K}$, then $\widehat{p_\tau}(\eta_{i,j}) = \int_K p_\tau(x)\eta_{i,j}(x)dx = \delta_{\tau,\eta}\delta_{i,1}\delta_{j,1}$. So $\widehat{p_\tau} = (0, \ldots, (E_{11})_{\pi=\tau}, \ldots, 0)$.

Under the Fourier transform, the subalgebra $\mathrm{C}^*(K)p_\tau\mathrm{C}^*(K)$ correspond to

$$\mathrm{M}_{d_\tau}(\mathbb{C})E_{1,1}\mathrm{M}_{d_\tau}(\mathbb{C}) = \mathrm{M}_{d_\tau}(\mathbb{C}).$$

The subalgebra $\mathrm{C}^*(K)p_\tau\mathrm{C}^*(K)$ is Morita equivalent to $p_\tau\mathrm{C}^*(K)p_\tau$. Observe that under the Fourier transform, $p_\tau\mathrm{C}^*(K)p_\tau$ becomes $E_{11}\mathrm{M}_{d_\tau}(\mathbb{C})E_{11} \simeq \mathbb{C}$.

4.4.2 Idea of proof

Let $M \subseteq K$ be the maximal torus, then $\widehat{M}/W$ is in bijection with $\widehat{K}$. For any $\sigma \in \widehat{M}/W$, we get a projection $p_{\tau_\sigma} \in C_r^*(K)$. Consider the subalgebra $C_r^*(G)p_{\tau_\sigma}C_r^*(G) \subseteq C_r^*(G)$. It corresponds to the component $\widehat{A}/W_\sigma$, *i.e.* equals $C_0(\widehat{A}/W_{\sigma_\tau}, \mathrm{K}(H_{\sigma_\tau}))$

Fact: For $\sigma \in \widehat{M}, \varphi \in \widehat{A}$, $p_{\sigma,\varphi} \in \widehat{G}_t$ has lowest K-type τ_σ. The representation $\pi_{\sigma,\varphi} = \mathrm{Ind}_{K_\varphi \ltimes \mathfrak{a}}^{K \ltimes \mathfrak{a}}(\sigma \otimes \varphi) \in \widehat{G_0}$ has lowest K-type τ_σ.

The subalgebra $C_0(\widehat{A}/W_\sigma, \mathrm{K}(H_\sigma))$ is Morita equivalent to $p_{\tau_\sigma}C_r^*(G)p_{\tau_\sigma}$, which becomes $C_0(\widehat{A}/W_\sigma)$ under the Fourier transform.

In order to show $K_0(C_r^*(G_0)) \simeq K_0(C_r^*(G))$, it is equivalent to show that

$$\mathrm{ev}_1 : \bigoplus_{\sigma \in \widehat{M}/W} K_0(\mathcal{C}_\sigma) = K_0(C_r^*(G_{[0,1]})) \xrightarrow{\simeq} K_0(C_r^*(G)) = \bigoplus_{\sigma \in \widehat{M}/W} K_0(C_\sigma).$$

Using the Morita equivalence,

$$K_0(\mathcal{C}_\sigma) \simeq K_0(\mathcal{C}_\sigma p_\sigma \mathcal{C}_\sigma) \xrightarrow[Morita]{\simeq} K_0(p_\sigma \mathcal{C}_\sigma p_\sigma) \simeq K_0\left(C_0(\widehat{\mathfrak{a}}/W_\sigma) \times [0,1]\right),$$

and on the other side similarly $K_0(C_\sigma) \simeq K_0(C_0(\widehat{A}/W_\sigma))$.

5 Erik Van Den Ban: Harmonic analysis on non-Riemannian symmetric spaces

There are a lot of details missing in this note! It is painful to take the notes, and the slides can be found in the website of trimester program.

5.1 Lecture 1

Settings:

- G real connected semisimple Lie group with finite center
- σ involution of G
- G^σ the fixed group of σ
- $H \subseteq G^\sigma$ open subgroup
- $X = G/H$ semisimple symmetric space
- $\sigma_* = d\sigma : \mathfrak{g} \to \mathfrak{g}$ infinitesimal involution

One has the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ as eigenspaces of σ. The tangent space $\mathrm{T}_e(X) \simeq \mathfrak{g}/\mathfrak{h} \simeq \mathfrak{q}$, and the Killing form of $\mathfrak{q}$ makes X a pseudo-Riemannian symmetric space.

Example 5.1.1.
- Riemannian case: σ a Cartan involution, $H = K$, $\mathfrak{q} = \mathfrak{p}$.

- Group case: $G = G' \times G'$, $H = (G')^{\Delta}$, and G acts on G' by $(x,y)g = xgy^{-1}$, $G' \simeq G/H$.
- Hyperbolic case.

Lemma 5.1.2. *There exists a Cartan involution θ of $\mathfrak{g}$ that commutes with σ. The composition $\theta\sigma$ is also an involution.*

Theorem 5.1.3. *The map $K \times (\mathfrak{p} \cap \mathfrak{q}) \times (\mathfrak{p} \cap \mathfrak{h}) \to G$, $(k, X, Y) \mapsto k \exp X \exp Y$ is a diffeomorphism.*

Corollary 5.1.4. *The map $K \times (\mathfrak{p} \cap \mathfrak{q}) \to G$ induces a diffeomorphism $G/H \simeq K \times_{K \cap H} (\mathfrak{p} \cap \mathfrak{q})$, which is a vector bundle over $K/K \cap H$ with fiber $\mathfrak{p} \cap \mathfrak{q}$.*

If $\sigma = \theta$, $G/K \simeq K \times_K \mathfrak{p} = \{*\} \times \mathfrak{p}$.

Theorem 5.1.5. *There are finitely many* $\mathrm{Ad}(H)$*-conjugacy classes of Cartan subspaces of $\mathfrak{q}$. They all have the same dimension, called the* rank *of G/H.*

Fix $\mathfrak{a}_{\mathfrak{q}} \subset \mathfrak{p} \cap \mathfrak{q}$ a maximal abelina subspace.

Lemma 5.1.6. $\Sigma = \left\{\alpha \in \mathfrak{a}_{\mathfrak{q}}^* \setminus \{0\} \,\middle|\, \mathfrak{g}_\alpha \neq 0\right\}$ *is a possibly non-reduced root system.*

Fix Σ^+ positive system and Δ simple roots, and take $W = W(\mathfrak{a}_{\mathfrak{q}})$ the Weyl group.

Definition 5.1.7. $W_{K\cap H} :=$ the image of $\mathrm{N}_{K\cap H}(\mathfrak{a}_{\mathfrak{q}})$ in W.

Put $\mathfrak{g}_{\alpha\pm} := \mathfrak{g}_\alpha \cap \mathfrak{g}_\pm$, eigenspaces of $\sigma\theta$, and $m_\alpha^\pm$ the dimension. $G^+ := G^{\sigma\theta}$ is reductive, and Σ_+ is a root system of $(\mathfrak{g}_+, \mathfrak{a}_{\mathfrak{q}})$. Define Σ_+^+ and W_+.

Remark 5.1.8. $W_+ \subseteq W_{K\cap H}$ is an equality if and only if H is essentially connected.

Definition 5.1.9. Define $\mathfrak{a}_{\mathfrak{q}}^{\mathrm{reg}}$ to be $W.\mathfrak{a}_{\mathfrak{q}}^+$, and also $\mathfrak{a}_{\mathfrak{q},+}^{\mathrm{reg}} := W_{K\cap H}\mathfrak{a}_{\mathfrak{q},+}^+$.

Lemma 5.1.10. $G = K\overline{A_{\mathfrak{q},+}^+}H$*, with unique $\overline{A_{\mathfrak{q},+}^+}$-part.*

Corollary 5.1.11. *The space $X_+ = KA_{\mathfrak{q}}^{\mathrm{reg}}H$ is an open dense subset of X.*

Suppose $\mathcal{W} \subset \mathrm{N}_K(\mathfrak{a}_{\mathfrak{q}})$ is finite, then

$$X_+ = \bigsqcup_{v \in \mathcal{W}} KA_{\mathfrak{q}}^+ vH \Leftrightarrow \mathcal{W} \xrightarrow{1-1} W/W_{K\cap H}.$$

Definition 5.1.12. Define $\mathfrak{g}^d \subseteq \mathfrak{g}_{\mathbb{C}}$ by $\mathfrak{g}_+ \oplus i\mathfrak{g}_-$.

Put $\mathfrak{k}^d := \mathfrak{h}_{\mathbb{C}} \cap \mathfrak{g}^d$, $\mathfrak{p}^d := \mathfrak{g}_{\mathbb{C}} \cap \mathfrak{q}^d$, then $\mathfrak{g}^d = \mathfrak{k}^d \oplus \mathfrak{p}^d$ is a Cartan decomposition, with $\theta^d = \sigma_{\mathbb{C}}|_{\mathfrak{g}^d}$. Put $\sigma^d := \theta_{\mathbb{C}}|_{\mathfrak{g}^d}$ and $\mathfrak{h}^d := \mathfrak{k}_{\mathbb{C}} \cap \mathfrak{g}^d$, $\mathfrak{q}^d := \mathfrak{p}_{\mathbb{C}} \cap \mathfrak{g}^d$.

We construct a duality $(\mathfrak{g}, \sigma, \theta) \leftrightarrow (\mathfrak{g}^d, \sigma^d, \theta^d)$.

Example 5.1.13. The dual space of $\mathrm{GL}(n, \mathbb{R})/\mathrm{O}(n)$ is $\mathrm{U}(n)/\mathrm{O}(n)$, and in the group case, the dual of a compact Lie group $G = (G \times G)/G^{\Delta}$ is $G^{\mathbb{C}}/G$, where $G_{\mathbb{C}}$ is the complexification of G.

Definition 5.1.14. Let $\mathcal{D}(G/H)$ be the space of linear partial differential operators on $C^\infty(G/H)$, and $\mathbb{D}(G/H)$ its H-invariant subspace.

We have a map $R : \mathcal{U}(\mathfrak{g}) \to \mathrm{End}_{\mathbb{C}}(C^\infty(G))$, and it induces $r : \mathcal{U}(\mathfrak{g})^H \twoheadrightarrow \mathbb{D}(G/H)$, Suppose that $G \subseteq G_{\mathbb{C}}$, and let G^d, K^d be the analytic subgroups of $G_{\mathbb{C}}$ with Lie algebras $\mathfrak{g}^d, \mathfrak{k}^d$. One has $\mathcal{U}(\mathfrak{g})^H = \mathcal{U}(\mathfrak{g}^d)^{K^d}$. There exists a unique homomorphism of algebras $\mathbb{D}(G/H) \to \mathbb{D}(G^d/K^d)$, $D \mapsto {}^dD$ commuting with the identity of $\mathcal{U}$, and it is an isomorphism.

We have the *Harish-Chandra isomorphism*: $\gamma^d : \mathbb{D}(G^d/K^d) \xrightarrow{\simeq} \mathbb{P}(\mathfrak{a}^{d,*})^{W(\mathfrak{g}^d,\mathfrak{a}^d)}$.

5.2 Lecture 2

When H is reductive, $X = G/H$ has a left invariant measure dx and $L^2(G/H, dx)$ carries the left regular representation $L_g\varphi(x) = \varphi(g^{-1}x)$. A goal of the harmonic analysis is to study the Plancherel decomposition of $L^2(G/H)$ in terms of irreducible unitary representations.

5.2.1 Basic representation theory

Setting: V is Fréchet (or complete locally convex space).

For a continuous representation (π, V), the space V^∞ of smooth vectors is a representation of $\mathcal{U}(\mathfrak{g})$. The subspace of K-finite smooth vectors $V^\infty \cap V_K$ is dense in V.

Definition 5.2.1. For $\delta \in \widehat{K}$, define $V[\delta]$ to be the image of $V_\delta \otimes \mathrm{Hom}_K(V_\delta, V)$. Then $V_K = \oplus_{\delta\in\widehat{K}} V[\delta]$. A representation V is *admissible* if $\dim V[\delta] < \infty$ for any $\delta \in \widehat{K}$.

Lemma 5.2.2. *If V is admissible, then $V_K \subseteq V^\infty$ (and this is a $(\mathfrak{g}, K)$-*module*).*

Lemma 5.2.3. *If (π, V) is admissible, then V_K is an admissible $(\mathfrak{g}, K)$-module. Furthermore,*

(1) the map $W \mapsto W \cap V_K$ defines a bijection between the closed invariant subspaces of V and V_K. The inverse is given by taking the closure.

(2) (π, V) is irreducible if and only if V_K is irreducible.

Definition 5.2.4. A *Harish-Chandra module* is a finitely generated admissible $(\mathfrak{g}, K)$-module.

A motivating result of Harish-Chandra: suppose that $(\pi, \mathcal{H})$ is irreducible unitary, then π is admissible. Two irreducible unitary representations are equivalent if their associated $(\mathfrak{g}, K)$-modules are equivalent.

Define $\mathfrak{Z} = \mathfrak{Z}(\mathfrak{g})$ to be the center of $\mathcal{U}(\mathfrak{g})$.

Theorem 5.2.5 (Harish-Chandra)**.** *Let $(\pi, \mathcal{H})$ be irreducible unitary. Then π is* quasi-simple*, i.e. $\mathfrak{Z}$ acts by scalars on V^∞ (through an* infinitesimal character $\chi \in \widehat{\mathfrak{Z}}$*).*

5.2.2 Back to $\mathbb{D}(G/H)$

For $D \in \mathbb{D}(G/H)$, we can define its *formal adjoint* $D' \in \mathbb{D}(G/H)$.

Theorem 5.2.6. *If $D = D'$, then D is essentially self-adjoint with operator core $\mathrm{L}^2(X)^\infty$.*

Definition 5.2.7. A *discrete series* of G/H is an irreducible unitary G-representation $(\pi, \mathcal{H})$ that admits G-equivariant $\mathcal{H} \to \mathrm{L}^2(G/H)$. For $\xi \in (G/H)^\wedge_{ds}$, we denote the isotopic space by $\mathrm{L}^2(G/H)_\xi$.

Lemma 5.2.8. *R induces an injective homomorphism $\mathfrak{Z} \hookrightarrow \mathbb{D}(G/H)$. Accordingly, $\mathbb{D}(G/H)$ is a finite $\mathfrak{Z}$-module.*

For each ξ, one can decompose the finite $\mathbb{D}(G/H)$-module $\mathrm{L}^2(G/H)^\infty_{\xi,K}$ into a direct sum of $(\mathfrak{g}, K)$-submodules on which $\mathbb{D}(G/H)$ acts by scalars.

For $\chi \in \mathbb{D}(G/H)^\wedge$, put $\xi_\chi(G/H)$ the space of smooth eigenfunctions of χ. Our goal is for each χ to describe the irreducible $(\mathfrak{g}, K)$-submodules of $\xi_\chi(G/H)_K \cap \mathrm{L}^2(G/H)^\infty$. The idea of Flensted-Jensen is to use the duality $G/H \leftrightarrow G^d/K^d$.

For simplicity, assume $G \subseteq G_{\mathbb{C}}$ and define G^d, K^d, H^d as Lie subgroups of $G_{\mathbb{C}}$ with corresponding Lie algebras.

Recall that $G_+ = \exp(\mathfrak{p} \cap \mathfrak{q})(K \cap H)$ is contained in $G \cap G^d$. For $f \in \mathrm{C}^\infty(G/H)_K$ and $x \in G_+$, the function $k \mapsto f(kx)$ has a unique analytic extension to $f_x : K_{\mathbb{C}} \to \mathbb{C}$.

Theorem 5.2.9 (F-J)**.** *There exists a unique map $\mathrm{C}^\infty(G/H)_K \to \mathrm{C}^\infty(G^d/K^d)_{H^d}$, $f \mapsto {}^d f$ such that*

- *${}^d f = f$ on G_+,*
- *for all $x \in G_+, h \in H^d$, ${}^d f(hx) = f_x(h)$.*

For all $D \in \mathbb{D}(G/H)$, ${}^d(Df) = {}^d D^d f$.

Corollary 5.2.10. *The duality $f \mapsto {}^d f$ fives $\xi_\chi(G/H)_K \hookrightarrow \xi_{{}^d\chi}(G^d/K^d)_{H^d}$, where ${}^d\chi$ is defined by ${}^d\chi({}^d D) = \chi(D)$.*

5.2.3 Poisson transform on G/K

Setting: $G = KAN$ and the minimal parabolic $P = MAN$. For $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, define $\chi_\lambda(D) = \gamma(D, \lambda) = (\gamma(D))(\lambda)$, which is a character in $\mathbb{D}(G/H)^\wedge$. Denote $\xi_{\chi_\lambda}(G/K)$ by $\xi_\lambda(G/K)$. For $\xi \in \mathfrak{a}^*_{\mathbb{C}}$, set $a^\xi := e^{\xi(\log a)}$, $a \in A$.

For $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, we define $\pi_\lambda = \mathrm{Ind}_P^G(\mathbb{1} \otimes (-\lambda) \otimes \mathbb{1})$ to be

$$\mathrm{C}^0(G/P; -\lambda) = \left\{ f \in \mathrm{C}^0(G) \,\middle|\, f(gman) = a^{\lambda - \rho_P} f(g) \right\}$$

with the action $\pi_\lambda(g) f(x) = f(g^{-1}x)$.

Definition 5.2.11. The *Poisson transform* $\mathcal{P}_\lambda : \mathrm{C}^0(G/P; -\lambda) \to \mathrm{C}^\infty(G/K)$ is defined by

$$\mathcal{P}_\lambda \varphi(x) = \int_K \varphi(xk) dk, \ x \in G.$$

Lemma 5.2.12. *$\mathcal{P}_\lambda$ maps $C^0(G/P;-\lambda)$ into $\xi_\lambda(G/K)$, and intertwines π_λ with L.*

The Poisson transform $\mathcal{P}_\lambda$ factors through $\mathrm{res} : C^0(G/P;-\lambda) \xrightarrow{\simeq} C(K/M)$, and we still denote that by $\mathcal{P}_\lambda$.

Definition 5.2.13. $\mathcal{B}'(K/M) := [C^\omega(K/M)dk]'$ (*hyperfunctions* in K/M) [2].

Theorem 5.2.14 (Helgason's conjecture; proved by Kashiwara-Kowata-Minewasa-Oshima-Okamoto-Tanaka)**.** *$\mathcal{P}_\lambda$ admits a unique extension to a continuous linear map $\mathcal{B}'(K/M) \to \xi_\lambda(G/K)$, which intertwines π_λ and L. For $\rho(\lambda) \neq 0$, this extension is a topological linear isomorphism.*

Theorem 5.2.15. $\operatorname{rank} G/H = \operatorname{rank} K/K \cap H$ *implies that* $(G/H)^\wedge_{ds} \neq \varnothing$.

Theorem 5.2.16 (Oshima-Matsuki,1982)**.** $(G/H)^\wedge_{rs} \neq \varnothing \Leftrightarrow \operatorname{rank} G/H = \operatorname{rank} K/K \cap H$.

5.3 Lecture 3

5.3.1 Parabolic induction

Definition 5.3.1. A *parabolic subgroup* of G is a subgroup P such that $P = N_G(\mathrm{Lie}(P))$.

Given such a P, take a maximal abelian $\mathfrak{a} \subseteq \mathfrak{g}$, $\Sigma^+(\mathfrak{g},\mathfrak{a})$ the positive system, $M = Z_K(\mathfrak{a})$ and we have the Iwasawa decomposition $G = KAN$. The group $P_0 = MAN$ is a minimal parabolic subgroup. One has $K \cap P_0 = M$, $G = KP_0 \simeq K \times_M P_0$ and $\mathfrak{k} \subseteq \mathfrak{g}$ induces a diffeomorphism $K/M \simeq G/P_0$. Every parabolic subgroup of G is K-conjugate to a *standard* parabolic subgroup.

For a parabolic $Q \subseteq G$, set $M_{1,Q} = Q \cap \theta(Q)$. The parabolic Q decomposes as $Q = M_{1,Q}N_Q$. Set $\mathfrak{a}_Q = Z(\mathfrak{m}_{1,Q}) \cap \mathfrak{p}$ and $A_Q = \exp \mathfrak{a}_Q$. We have $M_{1,Q} = M_Q A_Q = Z_G(\mathfrak{a}_Q)$. THe *Langlands decomposition* of Q is $Q = M_Q A_Q N_Q$.

Let $\mathcal{P}(A)$ to be set of parabolic subgroups containing A, and

Definition 5.3.2. Given $Q \in \mathcal{P}(A)$, define

$$\mathfrak{a}_Q^+ = \left\{ X \in \mathfrak{a}_Q \,\middle|\, \alpha(X) > 0, \forall \alpha \in \Sigma(\mathfrak{n}_Q, \mathfrak{a}_Q) \right\}.$$

For $X \in \mathfrak{a}$, define $\Sigma(X) = \{\alpha \in \Sigma \,|\, \alpha(X) > 0\}$.

Remark 5.3.3. Set $X \sim Y$ if $\Sigma(X) = \Sigma(Y)$, and this defines an equivalence relation on $\mathfrak{a}$.

Lemma 5.3.4. *$Q \mapsto \mathfrak{a}_Q^+$ gives a bijection from $\mathcal{P}(A)$ to $\mathfrak{a}/\sim$. The inverse is given by $\Phi \mapsto P_\Phi = M_{1,\Phi}N_\Phi$, where $M_{1,\Phi} = Z_G(\Phi)$, $\mathfrak{n}_\Phi = \sum_{\alpha \in \Sigma, \alpha|_\Phi > 0} \mathfrak{g}_\alpha$.*

The classes $\mathfrak{a}/\sim$ are facets, where G has the smallest dimension, and minimal parabolics have the maximal dimension.

Remark 5.3.5.

- $P \subseteq Q \Leftrightarrow \overline{\mathfrak{a}_P^+} \supseteq \mathfrak{a}_Q^+$;

[2]Here $C^\infty(K/M)dk$ stands for the space of real analytic densities.

- $\mathfrak{a}^+_{wPw^{-1}} = w(\mathfrak{a}^+_P)$, $w \in W(\Sigma)$.

Given:

- $\xi \in \widehat{M_P}, \lambda \in i\mathfrak{a}^*_P \hookrightarrow \widehat{A_P}$;
- $\xi \otimes \lambda$ is a unitary representation of $M_{1,P}$.
- The *unitary induction*: $\mathrm{Ind}_P^G(\xi \otimes \lambda)$.

The space of the unitary induction is

$$\mathrm{L}^2(P;\xi;\lambda) := \left\{ f \in \mathrm{L}^2(G, \mathcal{H}_P)_{loc} \,\middle|\, f(manx) = a^{\lambda+\rho_P}\xi(m)^{-1}f(x) \right\}.$$

This representation is unitary for $\lambda \in i\mathfrak{a}_P$, and

$$\begin{aligned} \mathrm{L}^2(P;\xi;\lambda) \times \mathrm{L}^2(P,\xi,-\overline{\lambda}) &\to \mathbb{C} \\ (f,g) &\mapsto \int_K \langle f(k), g(k)\rangle dk \end{aligned}$$

is G-equivariant.

The restriction gives a topological linear isomorphism

$$\mathrm{L}^2(P;\xi;\lambda) \simeq \mathrm{L}^2(K;\xi|_{K\cap M_P}) = \mathrm{Ind}_{K\cap M_P}^K(\xi|_{K\cap M_P}).$$

Theorem 5.3.6. $\mathrm{L}^2(P;\xi;\lambda)^\infty = \mathrm{C}^\infty(P;\xi,\lambda)$.

Define the dual $\mathrm{C}^{-\infty}(P;\xi;\lambda) = \overline{\mathrm{C}^\infty(P;\xi;-\overline{\lambda})'} \hookleftarrow \mathrm{C}^\infty(P;\xi;\lambda)$.
The idea is to construct $j \in \mathrm{C}^{-\infty}(P;\xi;\lambda)^H$, then have G-matrix coefficient

$$m_j : \mathrm{C}^\infty(P;\xi;-\overline{\lambda}) \hookrightarrow \mathrm{C}^\infty(G/H).$$

On open orbit $PvH \subseteq G$, one must have $j|_{PvJ} \in \mathrm{C}^\infty(PvH, \mathcal{H}_\xi^{-\infty})^H$ and
Consider parabolic subgroups stable under $\sigma\theta$.
I am lost here.

5.4 Lecture 4

Definition 5.4.1. The *unnormalized Fourier transform* ${}^u\widehat{f}$ of $f \in \mathrm{C}^\infty(G/H)$ is defined by

$${}^u\widehat{f}(P,\xi,\lambda) := \int_{G/H} f(x)\pi_{P,\xi,\lambda}(x)j(P,\xi,\lambda)dx \in V_P(\xi)^* \otimes \mathrm{C}^\infty(K;\xi|_{K_P}).$$

Example 5.4.2. If $H = K$, $\widehat{f}(P_\varnothing, 1, \lambda) = \pi_{P_\varnothing,1,\lambda}(f)\mathbb{1}_{P_\varnothing,\lambda}$.

The Fourier transform intertwines L with $\mathbb{1} \otimes \pi_{P,\xi,\lambda}$.

Theorem 5.4.3 (Plancherel identity). *For $f \in \mathrm{C}_c^\infty(G/K)$,*

$$\|f\|^2_{\mathrm{L}^2(X)} = \sum_{P\in\mathbb{P}_\sigma}[W : W_P^*] \sum_{\xi\in X^\wedge_{P,*,ds}} \int_{i\mathfrak{a}^*_{P,\mathfrak{q}}} \|{}^u\widehat{f}(P,\xi,\lambda)\|^2_{\mathrm{HS}} d\mu_{P,\xi}(\lambda).$$

Remark 5.4.4. $V_P(\xi)$ plays the role of multiplicity space.

Suppose that $P \in \mathcal{P}_\sigma(A_\mathfrak{q})$, $\xi \in \widehat{M_P}$ and ξ has real infinitesimal character.

Theorem 5.4.5 (Knapp-Stein, Vogan-Wallach)**.** *There is a unique meromorphic family:*

$$\mathfrak{a}^*_{P,\mathbb{C}} \ni \lambda \mapsto A(\overline{P}, P, \xi, \lambda)$$

of intertwining operators $\pi_{P,\xi,\lambda} \to \pi_{\overline{P},\xi,\lambda}$ such that for $\langle \mathrm{Re}(\lambda), \alpha \rangle >> 0$ for each α, then for $f \in \mathrm{C}^\infty(P;\xi;\lambda)$,

$$A(Q, P, \xi, \lambda) f(x) = \int_{\overline{N_P} \cap N_Q} f(nx) dn.$$

Remark 5.4.6. $A(\overline{P}, P, \xi, \lambda) \circ A(P, \overline{P}, \xi, \lambda) = \eta(P, \overline{P}, \xi, \lambda) \cdot \mathrm{Id}$, with $\eta(P, \overline{P}, \xi, \cdot)$ a meromorphic function.

Lemma 5.4.7. *$\eta(P, \overline{P}, \xi, \lambda) \geq 0$ for $\lambda \in i\mathfrak{a}^*_P$.*

The *Plancherel measure* $d\mu_{P,\xi}(\lambda)$ is $\eta(\overline{P}, P, \xi, \lambda)^{-1} \cdot d\mu_P(\lambda)$, where $d\mu_P$ is the Lebesgue measure on $i\mathfrak{a}^*_P$.

Definition 5.4.8. We normalize j by

$$j^\circ(P, \xi, \lambda) := A(\overline{P}, P, \xi, \lambda)^{-1} j(\overline{P}, \xi, \lambda),$$

and define $\widehat{f}$ as ${}^u\widehat{f}$ but with j° in place of j.

Corollary 5.4.9. *For $f \in \mathrm{C}^\infty_c(G)$, $\widehat{f}(P, \xi, \lambda) = A(\overline{P}, P, \xi, \lambda)^{-1} {}^u\widehat{f}(\overline{P}, \xi, \lambda)$.*

Theorem 5.4.10 (Normalized Plancherel indentity)**.** *For $f \in \mathrm{C}^\infty_c(G/H)$,*

$$\|f\|^2_{\mathrm{L}^2(X)} = \sum_{P \in \mathbb{P}_\sigma} [W : W^*_P] \sum_{\xi \in X^\wedge_{P,*,ds}} \int_{i\mathfrak{a}^*_{P,\mathfrak{q}}} \|\widehat{f}(P, \xi, \lambda)\|^2_{\mathrm{HS}} d\mu_P(\lambda).$$

5.5 Lecture 5

I give up...

6 Toshiyuki Kobayashi: Basic questions in group-theoretic analysis on manifolds

(1) Is representation theory useful to the global analysis on the G-manifold X? Does the group sufficiently control the space of functions?

(2) What can we say about the "spectrum" on $\mathrm{L}^2(X)$?

Given a G-manifold X, G acts on $C^\infty(X)$ and $L^2(X, \nu_X)$, where ν_X is a G-invariant Radon measure. More generally, $L^2(X)$ is defined by using the half-density bundle or a multiplier representation built on the cocycle $c(g,x)$ where $g_*\nu_X = c(g,x)\nu_X$.

For any unitary representation Π, one has

$$\Pi \simeq \int_{\widehat{G}}^{\oplus} m_\pi \pi d_\mu(\pi),$$

where $m : \widehat{G} \to \mathbb{N} \cup \{\infty\}$.

- Smallest units of representations: irreducible ones;
- Smallest units of Lie groups: 1-dim abelian groups and simple Lie groups;
- Reductive Lie groups "are" products of abelian groups and simple Lie groups.

6.1 Spectral analysis

Let X be a (pseudo-)Riemannian manifold with a G-action.

- Spectral analysis of Δ_X: $L^2(X) \simeq \int \mathcal{H}_\lambda d\tau(\lambda)$;
- Representation theory: Plancherel decomposition. This induces the spectral decomposition if $m_\pi \leq 1$.

Example 6.1.1. $O(n+1)$ acts on S^n, $O(n,1)$ acts on $\mathbb{H}^n$ (hyperbolic), and $O(p,q)$ acts of the space of forms (pseudo-Riemannian).

Hint for rigorous formulation. In group representations:

- strong point: can distinguish inequivalent irreducible representations even they are infinite-dimensional.
- weak point: multiplicity.

For $\pi \in \mathrm{Irr}(G)$, consider the multiplicity $\dim \mathrm{Hom}_G(\pi, C^\infty(X))$ (infinite, finite, bounded, multiplicity free).

Let $G_{\mathbb{C}}$ be a complex reductive Lie group and B a Borel subgroup of $G_{\mathbb{C}}$. Suppose that $G_{\mathbb{C}}$ acts on a connected complex manifold $X_{\mathbb{C}}$.

Definition 6.1.2. $X_{\mathbb{C}}$ is *spherical* if B has an open orbit in $X_{\mathbb{C}}$.

Example 6.1.3. Grassmannian varieties, flag varieties and symmetric spaces.

For reductive $G \supseteq H$, consider $X = G/H$.

Theorem 6.1.4. *The followings are equivalent:*

(1) (Global analysis and representation theory) There exists $C > 0$ such that

$$\dim \mathrm{Hom}_G(\pi, C^\infty(X)) \leq C, \textit{ for any } \pi \in \mathrm{Irr}(G).$$

(2) (Complex geometry) $X_{\mathbb{C}}$ is spherical.

(3) (Algebra) The ring $\mathbb{D}_{G_{\mathbb{C}}}(X_{\mathbb{C}})$ is commutative.

(4) (Algebra) The ring $\mathbb{D}_{G_{\mathbb{C}}}(X_{\mathbb{C}})$ is a polynomial ring.

6.2 Branching problems

Example 6.2.1 (Induction). $\mathrm{Ind}_H^G \mathbb{1} \simeq \mathrm{C}^\infty(G/H), \mathrm{L}^2(G/H), \ldots$.

Example 6.2.2 (Restriction). $\pi' \otimes \pi''|_{G_1^\Delta} \simeq ?$.

Given $G' \subseteq G$, the *branching problem* is to understand how the restriction behaves, *i.e.*

$$[\Pi|_{G'} : \pi] := \dim \mathrm{Hom}_{G'}(\Pi_{G'}, \pi),\ \pi \in \mathrm{Irr}(G'), \Pi \in \mathrm{Irr}(G).$$

Theorem 6.2.3 (Uniformly bounded multiplicity criterion)**.** *For a pair $G \supseteq G'$ of real reductive groups, the followings are equivalent:*

(1) $\sup_\Pi \sup_\pi [\Pi|_{G'} : \pi] < \infty$.

(2) $(G_{\mathbb{C}} \times G'_{\mathbb{C}})/\mathrm{diag}(G'_{\mathbb{C}})$ *is spherical.*

(3) The ring $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})^{G'_{\mathbb{C}}}$ is commutative.

(4) The ring $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})^{G'_{\mathbb{C}}}$ is a polynomial ring.

Remark 6.2.4. We also have $(G \times G')\,/\mathrm{diag}(G')$ is spherical (replacing Borel subgroup by minimal parabolic subgroup) if and only if $[\Pi|_{G'}; \pi] < \infty$ for any π and Π.

6.3 Tempered homogeneous spaces

Let G be a locally compact group.

Definition 6.3.1. A unitary irreducible representation π of G is *tempered* if π is weakly contained in $\mathrm{L}^2(G)$.

A basic question: when is $\mathrm{L}^2(X)$ tempered? In other words, for which G-space X, $\mathrm{L}^2(X) \prec \mathrm{L}^2(G)$?

Suppose that G is a real reductive Lie group, one has

$$\mathrm{Irr}(G) \supseteq \widehat{G} \supseteq \widehat{G}_{temp}.$$

$\mathrm{Irr}(G)$ is classified by Langlands, $\widehat{G}_{temp}$ is classified by Knapp-Zuckerman, but $\widehat{G}$ is still mysterious over 70 years.

Even when G/H is a reductive symmetric space, the question involves a hard problem regarding vanishing conditions of cohomological parabolic inductions with singular parameters. How about more general space $X = G/H$?

Example 6.3.2. Let $G = \mathrm{GL}(p+q+r, \mathbb{R})$, and the subgroup $H = \mathrm{GL}(p) \times \mathrm{GL}(q) \times \mathrm{GL}(r)$, then $\mathrm{L}^2(G/H)$ is tempered if and only if

$$p \leq q+r+1,\ q \leq p+r+1,\ r \leq p+q+1.$$

Definition 6.3.3. A continuous G-action on X is *proper* if $G_S = \{g \in G \,|\, gS \cap S \neq \varnothing\}$ is compact for any compact subset $S \subseteq X$.

Theorem 6.3.4. *Let H be a connected subgroup of a real reductive Lie group G, then the followings are equivalent:*

(1) $\mathrm{L}^2(G/H)$ *is tempered.*

(2) $2\rho_{\mathfrak{h}} \leq \rho_{\mathfrak{g}}(Y)$ *for any* $Y \in \mathfrak{h}$.

Remark 6.3.5. For the example $G = \mathrm{GL}(p+q+r)$, this combinatorial condition is equivalent to $2\max(p,q,r) \leq p+q+r+1$.

Theorem 6.3.6. *Let $\mathfrak{g}$ be a complex reductive Lie algebra, then the followings are equivalent:*

(1) $\mathrm{L}^2(G/H)$ *is tempered.*

(2) $2\rho_{\mathfrak{h}} \leq \rho_{\mathfrak{g}}$.

(3) $\mathfrak{h}$ *has a solvable limit in* $\mathfrak{g}$.

(4) $\mathfrak{h}^{\perp} \cap \mathfrak{g}^*_{\mathrm{reg}} \neq 0$ *in* $\mathfrak{g}^*$.

7 Nigel Higson: C^*-algebras and tempered representation theory: a look backward and a look forward

7.1 Some (selective) history

- 1943, Gelfand-Naimark C^*-algebras
- 1946, Gelfand-Naimark unitary representations of $\mathrm{SL}(n,\mathbb{C})$ and Plancherel formula
- 1947, Segal $\mathrm{C}^*_r(G)$
- 1955-1975, Harish Chandra
- 1959, Bott periodicity, Atiyah-Hirzebruch K-theory
- 1965, Seeley C^*-algebra extension from pseudo-differential operators
- 1973, Brown-Donglas-Fillmore theory of

$$0 \to \chi(H) \to E \to \mathrm{C}(X) \to 0$$

- 1980, Pimsner-Voiculescu: $K_*(A_\theta)$
- 1984, Connes-Kasparov conjecture
- 1983, Kasparov (ICM) on K-theory and non-commutative geometry and representations:"At present this is a non-existent math region..."

7.2 The present

V.Lafforgue (1998-2000) Proof of Connes-Kasparov conjecture over p-adic fields using index theory, and a new proof for Harish-Chandra's classification of discrete series using Connes-Kasparov.

Remark 7.2.1. V.Lafforgue uses Kasparov's "dual Dirac" method (a left-inverse to Connes-Kasparov, following Lusztig and Atiyah). Note that for a discrete series π, H_π is projective over $C_r^*(G)$. He uses Weyl's $\sum(n_k)^2 = 1$ trick (in K-theory, not L^2).

7.3 Bradd-Higson-Yuncken paper [BHY24]

A tidied up picture of $\widehat{SL(2,\mathbb{R})}_{tempered}$, indexed by $\widehat{SO(2)}$ and $\widehat{\mathbb{Z}/2\mathbb{Z}}$, also for $SL(3,\mathbb{R})$.

Let $\mathfrak{a}$ be the one in the Iwasawa decomposition, and $\mathfrak{a}^*_{dom}$ the dominant chamber. Define $\mathfrak{a}^*_{I,+}$ to be facets of $\mathfrak{a}^*_{dom}$, and M_I to be the M-part of the centralizer of $\mathfrak{a}_I$ in G.

Definition 7.3.1. We have $\mathrm{Im}(\mathrm{InfChar}(\pi)) \in \mathfrak{a}^*_{dom}$. An unitary representation is *tempiric* if $\mathrm{Im}(\mathrm{InfChar}(\pi)) = 0$.

Theorem 7.3.2 (Bruhat 1954, Harish-Chandra 1960s, Vogan 2000)**.**

$$\widehat{G}_{tempered} = \bigsqcup_I \left(\widehat{M_I}\right)_{tempiric} \times \mathfrak{a}^*_{I,+}.$$

Theorem 7.3.3 (Vogan 1981)**.** *There is a natural bijection*

$$\left(\widehat{M_I}\right)_{tempiric} \simeq \widehat{K_I},$$

where K_I is the maximal compact subgroup of M_I.

Theorem 7.3.4 (Bradd-Higson-Yuncken)**.** *The followings are equivalent:*

(1) Connes-Kasparov isomorphism for every real reductive group G.

(2) The group morphism

$$\mathbb{Z}[\widehat{K}] \to \mathbb{Z}[\widehat{G}_{tempiric}],\ \tau \mapsto \sum_\pi \mathrm{mult}(\tau,\pi)\cdot\pi$$

is an isomorphism for every G.

8 Monica Nevins: Wonders of p-adic representation theory

Je suis fatiguée.

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