Exceptional theta correspondence $\mathbf{F}_4 \times \mathbf{PGL}_2$ for level one automorphic representations

Yi Shan

January 31, 2025

Abstract

Let $\mathbf{F}_4$ be the unique (up to isomorphism) connected semisimple algebraic group over $\mathbb{Q}$ of type F_4, with compact real points and split over $\mathbb{Q}_p$ for all primes p. A conjectural computation [Sha24, Proposition 6.3.6] predicts the existence of a family of level one automorphic representations of $\mathbf{F}_4$, which are expected to be functorial lifts of cuspidal representations of $\mathbf{PGL}_2$ associated with Hecke eigenforms. In this paper, we study the exceptional theta correspondence for $\mathbf{F}_4 \times \mathbf{PGL}_2$, and establish the existence of such a family of automorphic representations for $\mathbf{F}_4$. Motivated by [Pol23], our main tool is a notion of "exceptional theta series" on $\mathbf{PGL}_2$, arising from certain automorphic representations of $\mathbf{F}_4$. These theta series are holomorphic modular forms on $\mathbf{SL}_2(\mathbb{Z})$, with explicit Fourier expansions, and span the entire space of level one cusp forms.

Contents

1 Introduction

Since the last century, automorphic representations of general linear groups and classical groups have been widely studied. For those of *exceptional groups*, *i.e.* algebraic groups with Lie type $\mathrm{G}_2, \mathrm{F}_4, \mathrm{E}_6, \mathrm{E}_7$ or E_8, most of the known results are about the smallest exceptional group $\mathbf{G}_2$, either split or anisotropic. In this paper, we will study a family of automorphic representations for $\mathbf{F}_4$, the unique (up to isomorphism) connected semisimple algebraic group over $\mathbb{Q}$ of type F_4, with compact real points and split over $\mathbb{Q}_p$ for every prime p.

1.1 Motivation from [Sha24]

In [Sha24], we compute the number of *level one* automorphic representations for $\mathbf{F}_4$, *i.e.* unramified at every finite place, with any given arbitrary archimedean component. Furthermore, the *discrete global Arthur parameters* of these automorphic representations are classified *conjecturally*, admitting the existence of the (level one) Langlands group and Arthur's multiplicity formula [Art89]. In particular, we conjecture the existence of a specific family of automorphic representations for $\mathbf{F}_4$, which are related to classical modular forms for $\mathbf{SL}_2(\mathbb{Z})$. Before recalling this statement, we introduce some notations:

- Let ϖ_4 be the highest weight of the 26-dimensional irreducible representation of $\mathbf{F}_4(\mathbb{R})$.
- There is a morphism $\mathbf{Sp}_6(\mathbb{C})\times\mathbf{SL}_2(\mathbb{C}) \to \widehat{\mathbf{F}_4}(\mathbb{C}) = \mathbf{F}_4(\mathbb{C})$ whose kernel is a cyclic group of order 2, the image of this morphism is a maximal proper regular closed subgroup of $\mathbf{F}_4(\mathbb{C})$ (see [Sha24, §4.3.2]). Denote by ι the morphism:

$$\mathbf{SL}_2(\mathbb{C}) \times \mathbf{SL}_2(\mathbb{C}) \xhookrightarrow{\text{(principal embedding, id)}} \mathbf{Sp}_6(\mathbb{C}) \times \mathbf{SL}_2(\mathbb{C}) \to \mathbf{F}_4(\mathbb{C}).$$

- Denote by e_p the conjugacy class of $\begin{pmatrix} p^{1/2} & \\ & p^{-1/2} \end{pmatrix}$ in $\mathbf{SL}_2(\mathbb{C})$.

Conjecture A. *[Sha24, Proposition 6.3.6] Let π be the level one algebraic automorphic representation of $\mathbf{PGL}_2$ associated to a cuspidal Hecke eigenform of weight $2n+12$ for $\mathbf{SL}_2(\mathbb{Z})$, and c_p the Satake parameter of π_p, viewed as a semisimple conjugacy class in $\widehat{\mathbf{PGL}_2}(\mathbb{C}) = \mathbf{SL}_2(\mathbb{C})$. There exists a level one automorphic representation Π of $\mathbf{F}_4$ such that:*

- *$\Pi_\infty \simeq \mathrm{V}_{n\varpi_4}$, the irreducible representation of $\mathbf{F}_4(\mathbb{R})$ with highest weight $n\varpi_4$;*
- *for every prime p, the Satake parameter of Π_p is the conjugacy class of $\iota(e_p, c_p)$.*

Motivated by the Langlands functoriality principle, the automorphic representation Π in Conjecture A is expected to be a functorial lift of π with respect to the embedding

$$i : \widehat{\mathbf{PGL}_2} = \mathbf{SL}_2 \xhookrightarrow{(1,\mathrm{id})} \mathbf{Sp}_6 \times \mathbf{SL}_2 \hookrightarrow \widehat{\mathbf{F}_4}. \tag{1.1}$$

One useful tool for constructing functorial lifts is the *theta correspondence*, which studies the restriction of a *minimal representation* to reductive dual pairs. There exists a reductive dual pair $\mathbf{PGL}_2 \times \mathbf{F}_4$ inside certain algebraic group $\mathbf{E}_7$ of Lie type E_7 (see §2 for more details). For the theta correspondence associated with this dual pair over a characteristic 0 local field, one already has the following results (see also §3):

- Over $\mathbb{R}$, Gross and Savin describe the restriction of the minimal representation of $\mathbf{E}_7(\mathbb{R})$ to $\mathbf{PGL}_2(\mathbb{R}) \times \mathbf{F}_4(\mathbb{R})$ [GS98, Proposition 3.2], which shows that the theta lift $\Theta(\pi_\infty)$ of π_∞ is isomorphic to $\mathrm{V}_{n\varpi_4}$;

- Over a p-adic field, this theta correspondence is studied by Karasiewicz and Savin in [Sav94; KS23]. In particular, they demonstrate that the theta lift $\Theta(\pi_p)$ of the unramified tempered principal series representation π_p is irreducible and has the desired Satake parameter $\iota(e_p, c_p)$.

Based on these local results, it is natural to expect that the functorial lift Π is exactly the global theta lift $\Theta(\pi)$ of π to $\mathbf{F}_4$. The main result in this paper confirms this expectation:

Theorem B. *(Theorem 6.4.2) The global theta lift $\Theta(\pi)$ is a non-zero irreducible automorphic representation of $\mathbf{F}_4$, and satisfies the local-global compatibility of theta correspondence $\Theta(\pi) \simeq \otimes'_v \Theta(\pi_v)$. In particular, Conjecture A holds.*

1.2 Exceptional theta series

Our main tool is to develop a notion of "*exceptional theta series*", motivated by Pollack's construction of Siegel modular forms for $\mathbf{Sp}_6(\mathbb{Z})$. This is a variant of the classical weighted theta series developed by Jacobi and Hecke, and gives an explicit theta lift from certain automorphic forms of $\mathbf{F}_4$ to $\mathbf{PGL}_2$.

1.2.1 Classical theta series

We first recall the classical construction of theta series. Let L be an even unimodular lattice in the Euclidean space $\mathbb{R}^n$, *i.e.* a self-dual lattice for any element v of which the scalar product $v.v$ is even. A well-known result states that the series

$$\vartheta_L(z) = \sum_{v \in L} q^{\frac{v.v}{2}}, \text{ where } q = e^{2\pi i z},\ z \in \mathcal{H} = \{x + iy \mid y > 0\},$$

is a modular form of level $\mathbf{SL}_2(\mathbb{Z})$ and weight $n/2$. One can weight this theta series by a homogeneous harmonic polynomial P of degree d over $\mathbb{R}^n$ [Hec40]:

$$\vartheta_{L,P}(z) = \sum_{v \in L} P(v) q^{\frac{v.v}{2}}, \tag{1.2}$$

and the resulting weighted theta series is a modular form for $\mathbf{SL}_2(\mathbb{Z})$ of weight $\frac{n}{2} + d$. It is a cusp form when $d > 0$, and Waldspurger shows in [Wal79a] that for a fixed pair of integers (n, d), the space $\mathrm{S}_{\frac{n}{2}+d}(\mathbf{SL}_2(\mathbb{Z}))$ of weight $\frac{n}{2} + d$ cusp forms is spanned by:

$$\{\vartheta_{L,P} \mid L \subseteq \mathbb{R}^n \text{ is an even unimodular lattice, and } P \in \mathscr{H}_d(\mathbb{R}^n)\},$$

where $\mathscr{H}_d(\mathbb{R}^n)$ is the space of homogeneous harmonic polynomials of degree d over $\mathbb{R}^n$.

1.2.2 Corresponding structures in the exceptional case

We want to produce a family of modular forms analogous to (1.2), starting from automorphic representations for $\mathbf{F}_4$ with archimedean component $\mathrm{V}_{n\varpi_4}$. The table below highlights the corresponding structures in the classical and exceptional settings:

	classical case	exceptional case
underlying space	Euclidean space $\mathbb{R}^n$	Euclidean Albert $\mathbb{R}$-algebra $\mathrm{J}_{\mathbb{R}}$
group of automorphisms	$\mathbf{O}_n(\mathbb{R})$	$\mathbf{F}_4(\mathbb{R})$
integral structure	even unimodular lattice	Albert lattice
homogeneous polynomials	harmonic polynomials	a polynomial model of $\mathrm{V}_{n\varpi_4}$

Table 1: Comparison between classical and exceptional cases

We briefly explain the objects appearing in Table 1, and the details will be provided in §2.2 and §2.3:

- The 27-dimensional *Euclidean Albert* $\mathbb{R}$*-algbera (or Euclidean exceptional Jordan* $\mathbb{R}$*-algebra)* $\mathrm{J}_{\mathbb{R}} = \mathrm{Her}_3(\mathbb{O}_{\mathbb{R}})$ is the space of "Hermitian" 3-by-3 matrices over the *real octonion division algebra* $\mathbb{O}_{\mathbb{R}}$, equipped with the distinguished element $\mathrm{I} = \mathrm{diag}(1,1,1)$, the adjoint map $\# : \mathrm{J}_{\mathbb{R}} \to \mathrm{J}_{\mathbb{R}}$, and the determinant $\det : \mathrm{J}_{\mathbb{R}} \to \mathbb{R}$. Precisely, together with these structures, $\mathrm{J}_{\mathbb{R}}$ is a *cubic Jordan* $\mathbb{R}$*-algebra* and furthermore it is an *Albert* $\mathbb{R}$*-algebra*. We call it Euclidean because its underlying vector space admits a symmetric inner product $(A,B) = \frac{1}{2}\mathrm{Tr}(AB+BA)$ that is positive definite.
- The group of Albert $\mathbb{R}$-algebra automorphisms of $\mathrm{J}_{\mathbb{R}}$ is the real points $\mathbf{F}_4(\mathbb{R})$ of $\mathbf{F}_4$, *i.e.* $\mathbf{F}_4(\mathbb{R}) = \{g \in \mathrm{GL}(\mathrm{J}_{\mathbb{R}}) \,|\, g\mathrm{I} = \mathrm{I},\ \det(gA) = \det(A),\ \text{for any } A \in \mathrm{J}_{\mathbb{R}}\}$.
- By an *Albert lattice*, we mean a lattice $J \subseteq \mathrm{J}_{\mathbb{R}}$ satisfying that $\mathrm{I} \in J$, J is stable under $\#$, $\det(J) \subseteq \mathbb{Z}$, and $(J, \mathrm{I}, \#, \det)$ is an *Albebrt* $\mathbb{Z}$*-algebra*.
- In §4.1.2, we describe a polynomial model $\mathrm{V}_n(\mathrm{J}_{\mathbb{C}})$ of $\mathrm{V}_{n\varpi_4}$: the space spanned by degree n homogeneous polynomials over $\mathrm{J}_{\mathbb{R}}$ of the form:

$$X \mapsto (X,A)^n\,,\ \text{where } 0 \neq A \in \mathrm{J}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C},\ A^2 = 0,\ \mathrm{Tr}(A) = 0.$$

1.2.3 Weighting the theta series constructed by Elkies-Gross

The starting point of the exceptional theta series associated with $\mathrm{J}_{\mathbb{R}}$ is the work of Elkies and Gross [EG96].

Let $\mathcal{J}$ be the set of Albert lattices, and equip it with the natural $\mathbf{F}_4(\mathbb{R})$-action. This set is the disjoint union of two $\mathbf{F}_4(\mathbb{R})$-orbits [Gro96, Proposition 5.3]. We take a set of representatives $\{J_1, J_2\}$ for these two orbits, where $J_1 = \mathrm{J}_{\mathbb{Z}}$ (see Example 2.2.9) is taken as the base point of $\mathcal{J}$. For $J = J_1$ or J_2, Elkies and Gross construct the following theta series:

$$\vartheta_J(z) = 1 + 240 \sum_{\substack{J \ni T \geq 0,\\ \mathrm{rank}(T)=1}} \sigma_3(\mathrm{c}_J(T))q^{\mathrm{Tr}(T)},\ q = e^{2\pi i z},\ z \in \mathcal{H},$$

where $\mathrm{c}_J(T)$ is the largest integer c such that $T/c \in J$, and $\sigma_3(n) = \sum_{d|n} d^3$. This theta series is a modular form of weight 12 for $\mathbf{SL}_2(\mathbb{Z})$. Moreover,

$$\vartheta_{J_1} = E_{12} - \frac{65520}{691}\Delta,\ \vartheta_{J_2} = E_{12} + \frac{432000}{691}\Delta,$$

where E_{12} is the normalized Eisenstein series of weight 12, and Δ is the discriminant modular form.

Remark 1.2.1. The coefficient $240\sigma_3(\mathrm{c}(T))$ appearing in the Fourier expansion of ϑ_J comes from *Kim's modular form* F_{Kim}, an Eisenstein series on the exceptional tube domain $\mathcal{H}_{\mathrm{J}}$ (see §4.2.1), which is constructed in [Kim93] and serves as our source for producing theta series.

We extend the construction of Elkies-Gross to *weighted exceptional theta series* as follows:

Theorem C. *(Theorem 5.1.2, Corollary 5.1.5) For any Albert lattice $J \in \mathfrak{J}$ and a polynomial $P \in \mathrm{V}_n(\mathrm{J}_{\mathbb{C}})$, the theta series*

$$\vartheta_{J,P}(z) := \sum_{\substack{J \ni T \geq 0, \\ \mathrm{rank}(T)=1}} \sigma_3(\mathrm{c}_J(T))P(T)q^{\mathrm{Tr}(T)} \tag{1.3}$$

is a modular form of weight $2n+12$ for $\mathbf{SL}_2(\mathbb{Z})$. When $n = \deg(P) > 0$, $\vartheta_{J,P}$ is a cusp form.

Our proof of Theorem C follows Pollack's method for the proof of [Pol23, Theorem 1.1.1]. For the automorphic form (or precisely, *algebraic modular form*) of $\mathbf{F}_4$ associated with J and P, we construct its global theta lift to $\mathbf{PGL}_2$, taking certain (iterated) differential of Kim's modular form F_{Kim} as the kernel function. Then we show that this global theta lift arises from a holomorphic modular form, whose Fourier expansion is exactly (1.3).

Remark 1.2.2. Here we explain briefly how we describe the global theta lift from $\mathbf{F}_4$ to $\mathbf{PGL}_2$ in terms of exceptional theta series, and more details can be found in §4.1.1. The space $\mathcal{A}_{\mathrm{V}_{n\varpi_4}}(\mathbf{F}_4)$ of *level one "vector-valued" automorphic form of* $\mathbf{F}_4$ *with weight* $\mathrm{V}_{n\varpi_4}$ can be identified with the space of functions $f : \mathfrak{J} \to \mathrm{V}_n(\mathrm{J}_{\mathbb{C}})$ satisfying $f(gJ) = g.f(J)$ for any $g \in \mathbf{F}_4(\mathbb{R})$ and $J \in \mathfrak{J}$. The global theta lift of f to $\mathbf{PGL}_2$ is the modular form

$$\frac{1}{|\Gamma_1|}\vartheta_{J_1,f(J_1)} + \frac{1}{|\Gamma_2|}\vartheta_{J_2,f(J_2)} \in \mathrm{M}_{2n+12}(\mathbf{SL}_2(\mathbb{Z})),$$

where Γ_i is the automorphism group of the Albert $\mathbb{Z}$-algebra J_i, $i = 1, 2$.

1.3 Strategy towards Theorem B

Now we illustrate our strategy for the proof of Theorem B.

Let $\varphi \simeq \otimes\varphi_v \in \pi \simeq \otimes'_v\pi_v$ be the automorphic form of $\mathbf{PGL}_2$ associated to a Hecke eigenform $f \in \mathrm{S}_{2n+12}(\mathbf{SL}_2(\mathbb{Z}))$. We want to show that its global theta lift $\Theta_\phi(\varphi)$, with respect to some vector ϕ in the *minimal representation* of $\mathbf{E}_7(\mathbb{A})$, is non-zero. For this goal, we compute the $\mathbf{Spin}_9$*-period integral* of $\Theta_\phi(\varphi)$, where $\mathbf{Spin}_9$ is a maximal proper regular closed subgroup of $\mathbf{F}_4$. The $\mathbf{Spin}_9$-period of an automorphic form f on $[\mathbf{F}_4] = \mathbf{F}_4(\mathbb{Q})\backslash\mathbf{F}_4(\mathbb{A})$ is defined as follows, where dg is taken to be the Tamagawa measure:

$$\mathcal{P}_{\mathbf{Spin}_9}(f) := \int_{\mathbf{Spin}_9(\mathbb{Q})\backslash\mathbf{Spin}_9(\mathbb{A})} f(g)dg.$$

Remark 1.3.1. One motivation for considering this $\mathbf{Spin}_9$-period is the global conjecture of Sakellaridis-Venkatesh [SV17]. The homogeneous $\mathbf{F}_4$-space $\mathbf{X} = \mathbf{Spin}_9\backslash\mathbf{F}_4$ is a spherical variety whose *dual group* is $\mathbf{G}^\vee_{\mathbf{X}} = \mathbf{SL}_2$, equipped with the embedding $i : \mathbf{G}^\vee_{\mathbf{X}} \to \widehat{\mathbf{F}_4}$ as described in (1.1). Roughly speaking, the conjecture of Sakellaridis-Venkatesh predicts that the cuspidal automorphic representations of $\mathbf{F}_4$ with non-zero $\mathbf{Spin}_9$-periods arise from functorial lifts with respect to the embedding $i : \widehat{\mathbf{PGL_2}} \to \widehat{\mathbf{F}_4}$. Therefore, we expect the global theta lift $\Theta_\phi(\varphi)$ to have a non-zero $\mathbf{Spin}_9$-period (for some suitable choice of ϕ).

Using a see-saw duality argument, an *exceptional Siegel-Weil formula* that we prove in §6.1 and a standard calculation of Rankin-Selberg integral (§6.2), we rewrite the $\mathbf{Spin}_9$-period of $\Theta_\phi(\varphi)$ as an Eulerian integral over $\mathbf{PGL}_2(\mathbb{A})$. Moreover, we prove the following result, which verifies the prediction of Sakellaridis-Venkatesh [SV17, §17; Sak21, Table 1] for the global period associated with spherical variety $\mathbf{Spin}_9\backslash\mathbf{F}_4$:

Theorem D. *(Corollary 6.3.2) For any smooth, holomorphic and spherical vector $\phi \simeq \otimes_v \phi_v$ in the minimal representation $\Pi_{\min} \simeq \otimes'_v \Pi_{\min,v}$ of $\mathbf{E}_7(\mathbb{A})$, the $\mathbf{Spin}_9$-period integral of $\Theta_\phi(\varphi)$ is equal to:*

$$\mathcal{P}_{\mathbf{Spin}_9}(\Theta_\phi(\varphi)) = \frac{\mathrm{L}(\pi, \frac{5}{2})\mathrm{L}(\pi, \frac{11}{2})}{\zeta(4)\zeta(8)} \cdot I_\infty(\phi_\infty, \varphi_\infty),$$

where $\mathrm{L}(\pi, s) = \mathrm{L}(f, \frac{2n+11}{2} + s)$ is the standard automorphic L-function of π (as an Euler product over all the finite places), and $I_\infty(\phi_\infty, \varphi_\infty)$ is an integral over $\mathbf{PGL}_2(\mathbb{R})$.

The L-factor in Theorem D is non-zero by the theory of Rankin-Selberg, thus the non-vanishing of $\mathcal{P}_{\mathbf{Spin}_9}(\Theta_\phi(\varphi))$ is equivalent to that of $I_\infty(\phi_\infty, \varphi_\infty)$.

For any Hecke eigenform f in $\mathrm{S}_{2n+12}(\mathbf{SL}_2(\mathbb{Z}))$, the associated automorphic form $\varphi \simeq \otimes_v \varphi_v$ in $\pi \simeq \otimes'_v \pi_v$ satisfies that φ_∞ is the unique (up to some scalar) lowest weight holomorphic vector of the discrete series $\mathcal{D}(2n+12) \simeq \pi_\infty$. Therefore, fixing a vector $\phi \in \Pi_{\min}$ as in Theorem D, $\mathcal{P}_{\mathbf{Spin}_9}(\Theta_\phi(\varphi)) \neq 0$ for *any* such φ, if and only if it holds for *one* such φ. Hence to prove Theorem B it suffices to find a vector $\phi \in \Pi_{\min}$ satisfying the conditions in Theorem D and that $\mathcal{P}_{\mathbf{Spin}_9}(\Theta_\phi(\varphi)) \neq 0$, where φ is the automorphic form associated to certain Hecke eigenform $f \in \mathrm{S}_{2n+12}(\mathbf{SL}_2(\mathbb{Z}))$.

Our proof of the existence of $\phi \in \Pi_{\min}$ relies on an automorphic form of $\mathbf{F}_4$ that is invariant under $\mathbf{Spin}_9(\mathbb{R})$ and has a non-zero global theta lift to $\mathbf{PGL}_2$. As mentioned in §1.2, in this paper the global theta lifting from $\mathbf{F}_4$ to $\mathbf{PGL}_2$ is realized via exceptional theta series. If we take $J = J_1 = \mathrm{J}_\mathbb{Z}$ and P_n the unique non-zero $\mathbf{Spin}_9(\mathbb{R})$-invariant polynomial in $\mathrm{V}_n(\mathrm{J}_\mathbb{C})$, $n \geq 2$, then Theorem C gives us a weight $2n+12$ cusp form, which can be verified to be non-zero by analyzing the Fourier coefficient of q (Theorem 5.2.1). This implies that the automorphic form for $\mathbf{F}_4$ associated to $\mathrm{J}_\mathbb{Z}$ and P_n is the desired one!

As a corollary of Theorem B, we have the following analogue of Waldspurger's result for classical theta series:

Theorem E. *(Corollary 6.4.3) For any $n > 0$, the space $\mathrm{S}_{2n+12}(\mathbf{SL}_2(\mathbb{Z}))$ is spanned by the set of weighted exceptional theta series $\{\vartheta_{J,P} \mid J = J_1$ or $J_2, P \in \mathrm{V}_n(\mathrm{J}_\mathbb{C})\}$.*

We end the introduction with a short summary of the contents of this paper. We recall the necessary preliminaries on exceptional groups in §2, and the results on local theta correspondences in §3. We establish the global theta correspondence in §4, then study the Fourier expansions of exceptional theta series and prove Theorem C in §5. The last section §6 is for the proof of Theorem B, Theorem D and Theorem E.

Acknowledgements

The author thanks Gaëtan Chenevier for proposing this interesting project, and also thanks Edmund Karasiewicz, Nhat Hoang Le, Chuhao Huang, Aaron Pollack, Gordan Savin and Jialiang Zou for stimulating discussions. Special thanks go to Wee Teck Gan for pointing out the strategy towards Conjecture A to the author, the invitation for a visit to the National University of Singapore, and a lot of helpful discussions.

2 Preliminaries on exceptional groups

In this section we recall the definitions of two reductive algebraic groups $\mathbf{F}_4$ and $\mathbf{E}_7$ over $\mathbb{Q}$ and construct the following two reductive dual pairs [1] inside $\mathbf{E}_7$:

$$\mathbf{F}_4 \times \mathbf{PGL}_2 \text{ and } \mathbf{Spin}_9 \times \mathbf{SO}_{2,2}.$$

2.1 Octonions

We first recall the notion of octonions, which are crucial for defining exceptional groups.

Definition 2.1.1. An *octonion algebra* over a commutative ring k is a locally free k-module C of rank 8, equipped with a non-degenerate quadratic form $N : C \to k$ and a (possibly non-associative) k-algebra structure admitting a 2-sided identity element e, such that $N(xy) = N(x)N(y)$, $x, y \in C$. The quadratic form N is referred as the *norm* on C.

Now we recall some basic properties of octonion algebras, for which we refer to [SV00]. There is a unique anti-involution of algebra $x \mapsto \overline{x}$ called the *conjugation* on C, with the property that $N(x) = x\overline{x} = \overline{x}x$. The *trace* is defined as the linear map $\mathrm{Tr} : C \to k$, $x \mapsto x+\overline{x}$. The symmetric bilinear form associated with N is $\langle x, y\rangle := N(x+y)-N(x)-N(y) = \mathrm{Tr}(x\overline{y})$.

Although the multiplication law of C is not associative, it is still *trace-associative* in the sense that $\mathrm{Tr}((xy)z) = \mathrm{Tr}(x(yz))$ for all $x, y, z \in C$, and we can define a trilinear form: $\mathrm{Tr}(xyz) := \mathrm{Tr}((xy)z) = \mathrm{Tr}(x(yz))$.

When considering octonion algebras over $\mathbb{R}$, we have the following classification result:

Proposition 2.1.2. *[Ada96, Theorem 15.1] Up to $\mathbb{R}$-algebra isomorphism, there is a unique octonion algebra $\mathbb{O}_{\mathbb{R}}$ over $\mathbb{R}$ whose norm N is positive definite, which is named as the* real octonion division algebra.

We choose a basis $\{\mathrm{e}_0, \mathrm{e}_1, \ldots, \mathrm{e}_7\}$ as in [Gro96, §4], where e_0 is the 2-sided identity element. Identify the real numbers $\mathbb{R}$ with the subalgebra $\mathbb{R}\mathrm{e}_0$ of $\mathbb{O}_{\mathbb{R}}$, and denote the identity element e_0 by 1. On $\mathbb{O}_{\mathbb{R}}$, the conjugation is defined by $\overline{1} = 1$ and $\overline{\mathrm{e}_i} = -\mathrm{e}_i$ for each i. For any element $x = \sum\limits_{i=0}^{7} x_i \mathrm{e}_i \in \mathbb{O}_{\mathbb{R}}$, one has $\mathrm{N}(x) = \sum_{i=0}^{7} x_i^2$ and $\mathrm{Tr}(x) = 2x_0$.

Definition 2.1.3. *Cayley's definite octonion algebra* $\mathbb{O}_{\mathbb{Q}}$ is the sub-$\mathbb{Q}$-algebra of $\mathbb{O}_{\mathbb{R}}$, generated by $\{\mathrm{e}_1, \ldots, \mathrm{e}_7\}$, which is an octonion $\mathbb{Q}$-algebra with the norm $\mathrm{N}|_{\mathbb{O}_{\mathbb{Q}}}$.

[1] Actually we do not prove in this paper that $\mathbf{Spin}_9 \times \mathbf{SO}_{2,2}$ is indeed a reductive dual pair, instead we only give a homomorphism $\mathbf{Spin}_9 \times \mathbf{SO}_{2,2} \to \mathbf{E}_7$, whose kernel is a central cyclic group of order 2.

The following definition gives an integral structure of Cayley's definite octonion algebra:

Definition 2.1.4. *Coxeter's integral order* $\mathbb{O}_{\mathbb{Z}}$ in $\mathbb{O}_{\mathbb{Q}}$ is the lattice spanned by $\mathbb{Z} \oplus \mathbb{Z}\mathrm{e}_1 \oplus \cdots \oplus \mathbb{Z}\mathrm{e}_7$ and

$$\mathrm{h}_1 = (1 + \mathrm{e}_1 + \mathrm{e}_2 + \mathrm{e}_4)/2,\ \mathrm{h}_2 = (1 + \mathrm{e}_1 + \mathrm{e}_3 + \mathrm{e}_7)/2,$$
$$\mathrm{h}_3 = (1 + \mathrm{e}_1 + \mathrm{e}_5 + \mathrm{e}_6)/2,\ \mathrm{h}_4 = (\mathrm{e}_1 + \mathrm{e}_2 + \mathrm{e}_3 + \mathrm{e}_5)/2,$$

which is an octonion $\mathbb{Z}$-algebra with the norm $\mathrm{N}|_{\mathbb{O}_{\mathbb{Z}}}$.

2.2 Albert algebras

In this section, we will not generally define either an Albert algebra or a (cubic) Jordan algebra, where precise definitions and details can be found in [GPR23]. Instead, we recall some examples and properties of Albert algebras that are important for us.

2.2.1 Hermitian 3-by-3 matrices over octonion algebras

Given an octonion algebra C over a commutative ring k, we consider the space $\mathrm{Her}_3(C)$ consisting of "Hermitian matrices" in $\mathrm{M}_3(C)$, *i.e.* matrices of the form

$$[a, b, c\,; x, y, z] := \begin{pmatrix} a & z & \overline{y} \\ \overline{z} & b & x \\ y & \overline{x} & c \end{pmatrix},\ a, b, c \in k,\ x, y, z \in C,$$

equipped with the following structures, where the maps are all polynomial laws in the sense of [Rob63]:

- the identity matrix $\mathrm{I} = \mathrm{diag}(1, 1, 1)$,
- the adjoint map $\# : \mathrm{Her}_3(C) \to \mathrm{Her}_3(C)$, which is a quadratic map over k:

$$\begin{pmatrix} a & z & \overline{y} \\ \overline{z} & b & x \\ y & \overline{x} & c \end{pmatrix} \mapsto \begin{pmatrix} bc - N(x) & \overline{xy} - cz & zx - b\overline{y} \\ xy - c\overline{z} & ca - N(y) & \overline{yz} - ax \\ \overline{zx} - by & yz - a\overline{x} & ab - N(z) \end{pmatrix}, \tag{2.1}$$

- and the determinant, which is a cubic form over k:

$$\det([a, b, c\,; x, y, z]) := abc + \mathrm{Tr}(xyz) - aN(x) - bN(y) - cN(z). \tag{2.2}$$

One can construct more polynomial laws from these structures:

- There exists a symmetric bilinear form on $\mathrm{Her}_3(C)$:

$$(A, B) := (\nabla_A \det)(\mathrm{I}) \cdot (\nabla_B \det)(\mathrm{I}) - (\nabla_A \nabla_B \det)(\mathrm{I}).$$

  If $A = [a, b, c\,; x, y, z]$ and $B = [a', b', c'\,; x', y', z']$, then

$$(A, B) = aa' + bb' + cc' + \langle x, x' \rangle + \langle y, y' \rangle + \langle z, z' \rangle.$$

- The *trace* $\mathrm{Tr} : \mathrm{Her}_3(M) \to k$ is defined as $\mathrm{Tr}(A) = (A, \mathrm{I})$.

- The linearization of # gives a symmetric cross product $A \times B := (A+B)^{\#} - A^{\#} - B^{\#}$.

With these structures, we can define the rank of a matrix in $\mathrm{Her}_3(C)$:

Definition 2.2.1. The *rank* of $A \in \mathrm{Her}_3(C)$ is defined as follows:

- If $A = 0$, then $\mathrm{rank}(A) = 0$;
- If $A \neq 0$ and $A^{\#} = 0$, then $\mathrm{rank}(A) = 1$;
- If $A \neq 0, A^{\#} \neq 0$ and $\det(A) = 0$, then $\mathrm{rank}(A) = 2$;
- Otherwise, $\mathrm{rank}(A) = 3$.

2.2.2 Euclidean exceptional Jordan $\mathbb{R}$-algebra and its $\mathbb{Q}$-structure

One of the most important Albert algebras appearing in this article is the following:

Definition 2.2.2. The *Euclidean exceptional Jordan* $\mathbb{R}$*-algebra* (or *Euclidean Albert* $\mathbb{R}$*-algebra*) is defined to be $\mathrm{J}_{\mathbb{R}} := \mathrm{Her}_3(\mathbb{O}_{\mathbb{R}})$, where $\mathbb{O}_{\mathbb{R}}$ is the real octonion division algebra.

The space $\mathrm{J}_{\mathbb{R}}$ is a commutative but not associative $\mathbb{R}$-algebra with respect to the $\mathbb{R}$-bilinear multiplication law $A \circ B := \frac{1}{2}(AB + BA)$, where AB and BA denote the matrix multiplication, and I is its 2-sided identity element. One can easily check that the symmetric bilinear form $(\ ,\ )$ satisfies $(A, B) = \mathrm{Tr}(A \circ B)$ for any $A, B \in \mathrm{J}_{\mathbb{R}}$, and it is positive definite.

Definition 2.2.3. A matrix $A = [a, b, c\,; x, y, z] \in \mathrm{J}_{\mathbb{R}}$ is *positive semi-definite* if its seven minor determinants

$$a, b, c, bc - \mathrm{N}(x), ca - \mathrm{N}(y), ab - \mathrm{N}(z), \det(A)$$

are all non-negative, and we write $A \geq 0$. When these minor determinants are all positive, we call A *positive definite* and write $A > 0$.

Similarly to Definition 2.1.3, this algebra $\mathrm{J}_{\mathbb{R}}$ admits a rational structure:

Definition 2.2.4. The *Euclidean exceptional Jordan* $\mathbb{Q}$*-algebra* $\mathrm{J}_{\mathbb{Q}}$ is the sub-$\mathbb{Q}$-algebra of $\mathrm{J}_{\mathbb{R}}$ consisting of $[a, b, c\,; x, y, z], a, b, c \in \mathbb{Q}, x, y, z \in \mathbb{O}_{\mathbb{Q}}$ equipped with the multiplication $\circ$.

Notation 2.2.5. Here we fix some elements in $\mathrm{J}_{\mathbb{Q}}$ that will be used frequently in this paper:

$$\mathrm{E}_1 := [1, 0, 0\,; 0, 0, 0],\ \mathrm{E}_2 := [0, 1, 0\,; 0, 0, 0],\ \mathrm{E}_3 := [0, 0, 1\,; 0, 0, 0].$$

2.2.3 Albert algebras over $\mathbb{Z}$

Let k be a commutative ring. *Albert* k-algebras are defined in [GPR23, Definition 7.1] Roughly speaking, an Albert k-algebra is a projective k-module J together with a distinguished point 1_J, a quadratic map $\# : J \to J$ and a cubic form $d : J \to k$ (as polynomial laws in the sense of [Rob63]) satisfying certain equations, such that for some faithfully flat k-algebra K, $J \otimes_k K$ is isomorphic to $\mathrm{Her}_3(C_K)$ as *Jordan* K*-algebras*, where C_K is an octonion K-algebra. For any ring homomorphism $k \to k'$, the scalar extension $J \otimes_k k'$ of an Albert k-algebra J is an Albert k'-algebra.

Definition 2.2.6. [GPR23, Lemma 10.3] An isomorphism of Albert k-algebras $\phi : J \to J'$ is a k-module isomorphism such that $\phi(1_J) = 1_{J'}$ and $d_{J'} \circ \phi = d_J$ [2] as polynomial laws.

Example 2.2.7. The space of 3-by-3 Hermitian matrices $\mathrm{Her}_3(C)$ defined in §2.2.1 is an Albert k-algebra. In particular, $\mathrm{J}_{\mathbb{R}}$ and $\mathrm{J}_{\mathbb{Q}}$ defined in and §2.2.2 are Albert algebras over $\mathbb{R}$ and $\mathbb{Q}$ respectively.

Here are several classification results in [SV00, §5.8; GPR23, §11, §14] about Albert algebras that will be useful for us:

(1) There is a unqiue isomorphism class of Albert $\mathbb{R}$-algebras that are *Euclidean*, *i.e.* the associated symmetric bilinear form is positive definite, and this class is represented by $(\mathrm{J}_{\mathbb{R}}, \mathrm{I}, \#, \det)$ defined in §2.2.2.
(2) Euclidean Albert $\mathbb{Q}$-algebras are also unique up to isomorphism.
(3) Albert $\mathbb{Z}_p$-algebras are unique up to isomorphism.
(4) There are exactly two isomorphism classes of Euclidean Albert $\mathbb{Z}$-algebras.

In this article, we concentrate on the following family of Euclidean Albert $\mathbb{Z}$-algebras:

Definition 2.2.8. An *Albert lattice* of $\mathrm{J}_{\mathbb{R}}$ is a lattice $J \subseteq \mathrm{J}_{\mathbb{R}}$ satisfying:

- The identity matrix $\mathrm{I} = \mathrm{diag}(1, 1, 1) \in \mathrm{J}_{\mathbb{R}}$ is contained in J;
- It is stable under the adjoint map $\#$ defined in (2.1);
- The cubic form det defined in (2.2) takes integral values on J;
- Together with $\mathrm{I}, \#$ and det, J is an Albert $\mathbb{Z}$-algebra.

Denote the set of Albert lattices inside $\mathrm{J}_{\mathbb{R}}$ by $\mathscr{J}$.

Example 2.2.9. Let $\mathrm{J}_{\mathbb{Z}} := \mathrm{Her}_3(\mathbb{O}_{\mathbb{Z}})$, *i.e.* the rank 27 lattice

$$\{[a, b, c\,; x, y, z] \in \mathrm{J}_{\mathbb{Q}} \,|\, a, b, c \in \mathbb{Z}, x, y, z \in \mathbb{O}_{\mathbb{Z}}\}$$

inside $\mathrm{J}_{\mathbb{Q}}$. It satisfies the conditions in Definition 2.2.8, thus it is an Albert lattice.

Example 2.2.10. An Albert $\mathbb{Z}$-algebra not isomorphic to $(\mathrm{J}_{\mathbb{Z}}, \mathrm{I}, \#, \det)$ defined in Example 2.2.9 is constructed as follows, following [Gro96, §4; GPR23, Definition 14.1]. Take

$$\mathrm{E} = [2, 2, 2; \beta, \beta, \beta],\ \beta = \frac{1}{2}(-1 + \mathrm{e}_1 + \mathrm{e}_2 + \cdots + \mathrm{e}_7) \in \mathbb{O}_{\mathbb{Z}}.$$

This element $\mathrm{E} \in \mathrm{J}_{\mathbb{Z}}$ is positive definite and has determinant 1. Under the adjoint map $\#$ on $\mathrm{J}_{\mathbb{R}}$ defined as (2.1), we have $\mathrm{E}^{\#} = [2, 2, 2\,; \overline{\beta}, \overline{\beta}, \overline{\beta}] \in \mathrm{J}_{\mathbb{Z}}$. Using this element, we define another quadratic map $\#^{\mathrm{E}}$ on $\mathrm{J}_{\mathbb{Z}}$ by $X^{\#^{\mathrm{E}}} := (\mathrm{E}^{\#}, X^{\#})\mathrm{E}^{\#} - \mathrm{E} \times X^{\#}$. Set $\mathrm{J}_{\mathbb{Z}}^{(\mathrm{E})} := (\mathrm{J}_{\mathbb{Z}}, \mathrm{E}^{\#}, \#^{\mathrm{E}}, \det)$, where det is still the restriction of $\det : \mathrm{J}_{\mathbb{R}} \to \mathbb{R}$ to $\mathrm{J}_{\mathbb{Z}}$. This "*isotopy*" $\mathrm{J}_{\mathbb{Z}}^{(\mathrm{E})}$ is an Albert $\mathbb{Z}$-algebra [GPR23, Corollary 13.11], and it is not isomorphic to $(\mathrm{J}_{\mathbb{Z}}, \mathrm{I}, \#, \det)$ as Albert $\mathbb{Z}$-algebras [EG96, Proposition 5.5].

The associated symmetric bilinear form $(\ ,\ )$ on $\mathrm{J}_{\mathbb{Z}}^{(\mathrm{E})}$ is positive definite [EG96, Proposition 2.10], thus $\mathrm{J}_{\mathbb{Z}}^{(\mathrm{E})}$ is Euclidean. By the classification result about Euclidean Albert $\mathbb{R}$-algebras listed above, we have an isomorphism $\varphi : \mathrm{J}_{\mathbb{Z}}^{(\mathrm{E})} \otimes_{\mathbb{Z}} \mathbb{R} \to \mathrm{J}_{\mathbb{R}}$ of Albert $\mathbb{R}$-algebras. Its image $\varphi(\mathrm{J}_{\mathbb{Z}}^{(\mathrm{E})})$ is an Albert lattice of $\mathrm{J}_{\mathbb{R}}$ in the sense of Definition 2.2.8.

[2] Here $\circ$ means the composition, not the multiplication defined in §2.2.2.

Question. *Can we find a simpler description of Albert lattices of $\mathrm{J}_{\mathbb{R}}$? For example, is it true that a unimodular lattice $J \subset \mathrm{J}_{\mathbb{R}}$ such that J contains I as a characteristic vector and J is stable under # (or equivalently, under $A \mapsto A^2$) is an Albert lattice in $\mathrm{J}_{\mathbb{R}}$?*

2.3 $\mathbf{F}_4$

We start to define exceptional algebraic groups.

Definition 2.3.1. Define $\mathbf{F}_4$ to be the closed subgroup of the algebraic $\mathbb{Q}$-group $\mathbf{GL}_{\mathrm{J}_{\mathbb{Q}}}$, that (as a functor) sends a commutative $\mathbb{Q}$-algebra R to the group

$$\mathbf{F}_4(R) := \{g \in \mathrm{GL}(\mathrm{J}_{\mathbb{Q}} \otimes_{\mathbb{Q}} R) \,|\, g(A \circ B) = g(A) \circ g(B), \text{ for any } A, B \in \mathrm{J}_{\mathbb{Q}} \otimes_{\mathbb{Q}} R\}\,.$$

By [SV00, Theorem 7.2.1], $\mathbf{F}_4$ is a semisimple and simply-connected $\mathbb{Q}$-group of Lie type F_4. The real points $\mathrm{F}_4 := \mathbf{F}_4(\mathbb{R})$ of $\mathbf{F}_4$ is contained in the isometry group $\mathrm{O}(\mathrm{J}_{\mathbb{R}}, \mathrm{q})$ of the positive definite quadratic form q, thus it is compact. For every prime p, $\mathbf{F}_4$ is split over $\mathbb{Q}_p$. By [SV00, Proposition 5.9.4], the $\mathbb{Q}$-group $\mathbf{F}_4$ coincides with the algebraic group consisting of the Albert algebra automorphisms of $\mathrm{J}_{\mathbb{Q}}$, *i.e.* sending any commutative $\mathbb{Q}$-algebra R to

$$\{g \in \mathrm{GL}(\mathrm{J}_{\mathbb{Q}} \otimes_{\mathbb{Q}} R) \,|\, g(\mathrm{I}) = g, \det(gA) = \det(A), \text{ for any } A \in \mathrm{J}_{\mathbb{Q}} \otimes_{\mathbb{Q}} R\}\,.$$

With this coincidence, we construct *reductive $\mathbb{Z}$-models* of $\mathbf{F}_4$ in the sense of [Gro96] as group of Albert algebra automorphisms of elements $J \in \mathfrak{J}$.

Definition 2.3.2. Given an Albert lattice $J \in \mathfrak{J}$, define $\mathbf{Aut}_{J/\mathbb{Z}}$ to be the $\mathbb{Z}$-group scheme sending a commutative $\mathbb{Z}$-algebra R to the group

$$\mathbf{Aut}_{J/\mathbb{Z}}(R) := \{g \in \mathrm{GL}(J \otimes_{\mathbb{Z}} R) \,|\, g(\mathrm{I}) = \mathrm{I}, \det(gA) = \det(A), \text{ for any } A \in J \otimes_{\mathbb{Z}} R\}\,.$$

If we take J to be $\mathrm{J}_{\mathbb{Z}}$ defined in Example 2.2.9, we denote the group scheme $\mathbf{Aut}_{\mathrm{J}_{\mathbb{Z}}/\mathbb{Z}}$ by $\mathcal{F}_{4,\mathrm{I}}$.

The following result shows that $\mathbf{Aut}_{J/\mathbb{Z}}$ is a reductive $\mathbb{Z}$-model of $\mathbf{F}_4$:

Proposition 2.3.3. *[GPR23, Lemma 9.1] For any choice of Albert lattice $J \in \mathfrak{J}$, the group scheme $\mathbf{Aut}_{J/\mathbb{Z}}$ is smooth and its fiber $\mathbf{Aut}_{J/\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z}$ is semisimple for every prime p. Moreover, the generic fiber of $\mathbf{Aut}_{J/\mathbb{Z}}$ is $\mathbf{F}_4$.*

In [Gro96, Proposition 5.3], Gross proves that there are exactly two $\mathbf{F}_4(\mathbb{Q})$-orbits on the equivalence classes of reductive $\mathbb{Z}$-models of $\mathbf{F}_4$ in the *genus* of $\mathcal{F}_{4,\mathrm{I}}$. From now on we fix a reductive $\mathbb{Z}$-model $\mathcal{F}_{4,\mathrm{I}}$ of $\mathbf{F}_4$, and we have the following formulation of the double cosets space $\mathbf{F}_4(\mathbb{Q})\backslash\mathbf{F}_4(\mathbb{A})/\mathcal{F}_{4,\mathrm{I}}(\widehat{\mathbb{Z}})$.

Proposition 2.3.4. *There is a bijection $\mathfrak{J} \xrightarrow{\simeq} \mathbf{F}_4(\mathbb{Q})\backslash\mathbf{F}_4(\mathbb{A})/\mathcal{F}_{4,\mathrm{I}}(\widehat{\mathbb{Z}})$ sending the base point $\mathrm{J}_{\mathbb{Z}}$ to the double coset of the identity of $\mathbf{F}_4(\mathbb{A})$.*

Proof. For any $J \in \mathfrak{J}$, the Albert $\mathbb{Q}$-algebras $J \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\mathrm{J}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$ are isomorphic, so there exists an element $g_\infty \in \mathbf{F}_4(\mathbb{R})$ inducing $J \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\simeq} \mathrm{J}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$. Set $J' = g_\infty(J)$, which is an Albert $\mathbb{Z}$-algebra inside $\mathrm{J}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q} = \mathrm{J}_{\mathbb{Q}}$. Since $J' \otimes_{\mathbb{Z}} \mathbb{Z}_p$ and $\mathrm{J}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}_p$ are isomorphic as Albert

$\mathbb{Z}_p$-algebras, we can choose an element $g_p \in \mathbf{F}_4(\mathbb{Q}_p)$ that induces $\mathrm{J}_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{Z}_p \xrightarrow{\simeq} J' \otimes_\mathbb{Z} \mathbb{Z}_p$. For almost all prime numbers p, we have $J' \otimes_\mathbb{Z} \mathbb{Z}_p = \mathrm{J}_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{Z}_p$, so the element g_p lies in $\mathcal{F}_{4,\mathrm{I}}(\mathbb{Z}_p)$ for almost all p.

In this way, we associate with $J \in \mathcal{J}$ an element $(g_\infty, g_2, g_3, \dots) \in \mathbf{F}_4(\mathbb{A})$, and it can be easily verified that its image in $\mathbf{F}_4(\mathbb{Q})\backslash\mathbf{F}_4(\mathbb{A})/\mathcal{F}_{4,\mathrm{I}}(\widehat{\mathbb{Z}})$ does not depend on the choice of g_∞ and g_p. So we have a well-defined map $\mathcal{J} \to \mathbf{F}_4(\mathbb{Q})\backslash\mathbf{F}_4(\mathbb{A})/\mathcal{F}_{4,\mathrm{I}}(\widehat{\mathbb{Z}})$, and its inverse is:

$$(g_v) \mapsto g_\infty^{-1}\left(\bigcap_p \left(g_p\left(\mathrm{J}_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{Z}_p\right) \cap \mathrm{J}_\mathbb{Q}\right)\right) \in \mathcal{J}. \qquad \square$$

Notation 2.3.5. We choose a set of representatives $\{1, \gamma_\mathrm{E}\}$ of $\mathbf{F}_4(\mathbb{Q})\backslash\mathbf{F}_4(\mathbb{A}_f)/\mathcal{F}_{4,\mathrm{I}}(\widehat{\mathbb{Z}})$, and denote by $\mathrm{J}_\mathrm{E} \subseteq \mathrm{J}_\mathbb{Q}$ the Albert lattice corresponding to γ_E. Equipped with the natural $\mathbf{F}_4(\mathbb{R})$-action, $\mathcal{J}$ is the disjoint union of the $\mathbf{F}_4(\mathbb{R})$-orbits of $\mathrm{J}_\mathbb{Z}$ and J_E.

2.3.1 An algebraic group of type E_6

If we remove the condition of fixing the identity element I in the definition of $\mathcal{F}_{4,\mathrm{I}}$, we get the following group of type E_6:

Definition 2.3.6. Define $\mathbf{M}_\mathrm{J}$ to be the $\mathbb{Z}$-group scheme sending any commutative ring R to

$$\left\{(\lambda(g), g) \in R^\times \times \mathrm{GL}(\mathrm{J}_\mathbb{Z} \otimes_\mathbb{Z} R) \,\middle|\, \det(gA) = \lambda(g)\det(A), \text{ for any } A \in \mathrm{J}_\mathbb{Z} \otimes_\mathbb{Z} R\right\},$$

and $\mathbf{M}_\mathrm{J}^1$ to be $\ker\lambda$.

By [Con15, Proposition 6.5], $\mathbf{M}_\mathrm{J}^1$ is a simply-connected semisimple group scheme of type E_6, and its generic fiber has $\mathbb{Q}$-rank 2.

Remark 2.3.7. Notice that the bilinear form $(\ ,\ )$ on $\mathrm{J}_\mathbb{Z} \otimes_\mathbb{Z} R$ is not $\mathbf{M}_\mathrm{J}(R)$-invariant. For any $m \in \mathbf{M}_\mathrm{J}(R)$, we denote by m^* the unique element in $\mathbf{M}_\mathrm{J}(R)$ such that $(m(X), m^*(Y)) = (m^*(X), m(Y)) = (X, Y)$ for any $X, Y \in \mathrm{J}_\mathbb{Z} \otimes R$.

Observe that we have already seen two Albert $\mathbb{Z}$-algebras $\mathrm{J}_\mathbb{Z}^{(\mathrm{E})}$ and J_E that are both not isomorphic to $\mathrm{J}_\mathbb{Z}$ and their extensions to $\mathbb{Q}$ are isomorphic to $\mathrm{J}_\mathbb{Q}$, by the classification result listed in §2.2.3 they are isomorphic, although they have different distinguished points. This fact gives us an element that will be used in the proof of Theorem 5.1.2:

Lemma 2.3.8. *There exists an element $\delta \in \mathbf{M}_\mathrm{J}^1(\mathbb{Q})$ that induces an isomorphism of Albert $\mathbb{Z}$-algebras $\mathrm{J}_\mathbb{Z}^{(\mathrm{E})} \xrightarrow{\simeq} \mathrm{J}_\mathrm{E}$. Moreover, if we denote the image of δ under the diagonal embedding $\mathbf{M}_\mathrm{J}^1(\mathbb{Q}) \hookrightarrow \mathbf{M}_\mathrm{J}^1(\mathbb{A}) = \mathbf{M}_\mathrm{J}^1(\mathbb{R}) \times \mathbf{M}_\mathrm{J}^1(\mathbb{A}_f)$ by $(\delta_\infty, \delta_f)$, then $\delta_\infty(\mathrm{J}_\mathbb{Z}) = \mathrm{J}_\mathrm{E}$, $\delta_\infty(\mathrm{E}) = \mathrm{I}$ and $\delta_f^{-1}\gamma_\mathrm{E} \in \mathbf{M}_\mathrm{J}^1(\widehat{\mathbb{Z}})$.*

Proof. Since the Albert $\mathbb{Z}$-algebras $\mathrm{J}_\mathbb{Z}^{(\mathrm{E})}, \mathrm{J}_\mathrm{E}$ contained in $\mathrm{J}_\mathbb{Q}$ are isomorphic, there is a $\mathbb{Q}$-linear isomorphism δ of $\mathrm{J}_\mathbb{Q}$ such that $\delta(\mathrm{J}_\mathbb{Z}^{(\mathrm{E})}) = \mathrm{J}_\mathrm{E}$, $\delta(\mathrm{E}) = \mathrm{I}$ and $\det(\delta A) = \delta(A)$ for any $A \in \mathrm{J}_\mathbb{Q}$. In other words, δ is our desired element in $\mathbf{M}_\mathrm{J}^1(\mathbb{Q})$. The properties of δ_∞ follows immediately. Forgetting the Albert algebra structures, $\delta_f^{-1}\gamma_\mathrm{E} : \mathrm{J}_\mathbb{Z} \otimes_\mathbb{Z} \widehat{\mathbb{Z}} \to \mathrm{J}_\mathbb{Z}^{(\mathrm{E})} \otimes_\mathbb{Z} \widehat{\mathbb{Z}}$ is a linear automorphism of $\mathrm{J}_\mathbb{Z} \otimes_\mathbb{Z} \widehat{\mathbb{Z}}$ preserving the determinant, thus $\delta_f^{-1}\gamma_\mathrm{E} \in \mathbf{M}_\mathrm{J}^1(\widehat{\mathbb{Z}})$. $\square$

2.4 $\mathbf{E}_7$

Now we recall the definition of $\mathbf{E}_7$, a larger algebraic group over $\mathbb{Q}$ containing $\mathbf{F}_4$, and our main references are [Pol20, §2.2; KY16, §3] [3].

Consider the 56-dimensional vector space $\mathrm{W_J} = \mathrm{J}_\mathbb{Q} \oplus \mathbb{Q} \oplus \mathrm{J}_\mathbb{Q} \oplus \mathbb{Q}$ [4], equipped with the following structures:

- A symplectic form: for $w_i = (X_i, \xi_i, X_i', \xi_i') \in \mathrm{W_J}$, $i = 1, 2$,
$$\langle w_1, w_2 \rangle_\mathrm{J} := \xi_1\xi_2' - \xi_2\xi_1' + (X_1, X_2') - (X_2, X_1');$$
- A quartic form: for $w = (X, \xi, X', \xi') \in \mathrm{W_J}$,
$$\mathrm{Q}(w) = (\xi\xi' - (X, X'))^2 + 4\xi \det(X) + 4\xi' \det(X') - 4(X^\#, X'^\#).$$

Definition 2.4.1. Define $\mathbf{H}_\mathrm{J}$ to be the algebraic subgroup of $\mathbf{GL}_{\mathrm{W_J}}$ consisting of elements that preserve the forms $\langle\, ,\, \rangle_\mathrm{J}$ and Q up to some similitude $\nu : \mathbf{H}_\mathrm{J} \to \mathbf{G}_\mathrm{m}$, *i.e.*

$$\mathbf{H}_\mathrm{J} = \left\{ (\nu(g), g) \in \mathbf{G}_\mathrm{m} \times \mathbf{GL}_{\mathrm{W_J}} \,\middle|\, \langle gv, gw \rangle_\mathrm{J} = \nu(g)\langle v, w \rangle_\mathrm{J}, \mathrm{Q}(gv) = \nu(g)^2\mathrm{Q}(v), \forall v, w \in \mathrm{W_J} \right\}.$$

Define $\mathbf{H}_\mathrm{J}^1$ to be the kernel of ν, which is simply-connected and has $\mathbb{Q}$-rank 3 and Lie type E_7 [Fre54], and $\mathbf{E}_7$ to be the adjoint group of $\mathbf{H}_\mathrm{J}$.

Remark 2.4.2. The center of $\mathbf{H}_\mathrm{J}$ consists of scalars, and it contains a specific element $\iota^2 = -\mathrm{Id}_{\mathrm{W_J}}$, where $\iota \in \mathbf{H}_\mathrm{J}$ is defined as

$$\iota(X, \xi, X', \xi') = (-X', -\xi', X, \xi). \tag{2.3}$$

In [Gro96], we know that $\mathbf{E}_7$ has a unique (up to equivalence) reductive $\mathbb{Z}$-models, and we will also denote this $\mathbb{Z}$-group scheme by $\mathbf{E}_7$ when there is no confusion. Note that $\mathbf{E}_7(\mathbb{Z})$ is the stabilizer in $\mathbf{E}_7(\mathbb{R})$ of the lattice $\mathrm{J}_\mathbb{Z} \oplus \mathbb{Z} \oplus \mathrm{J}_\mathbb{Z} \oplus \mathbb{Z} \subseteq \mathrm{W_J}$.

2.4.1 Siegel parabolic subgroup of $\mathbf{E}_7$

Definition 2.4.3. The *Siegel parabolic subgroup* $\mathbf{P}_{\mathrm{J,sc}}$ of $\mathbf{H}_\mathrm{J}^1$ is defined as the stabilizer of the line $\mathbb{Q}(0, 1, 0, 0) \subseteq \mathrm{W_J}$. A Levi subgroup of $\mathbf{P}_{\mathrm{J,sc}}$ can be defined as the subgroup that also stabilizes $\mathbb{Q}(0, 0, 0, 1)$. Denote by $\mathbf{P}_\mathrm{J}$ the image of $\mathbf{P}_{\mathrm{J,sc}}$ in $\mathbf{E}_7$.

This Levi subgroup is isomorphic to $\mathbf{M}_\mathrm{J}$, and the action of $(\lambda(m), m) \in \mathbf{M}_\mathrm{J}$ on $\mathrm{W_J}$ is

$$m(X, \xi, X', \xi') = (m^*X, \lambda(m)\xi, mX', \lambda(m)^{-1}\xi').$$

The unipotent radical $\mathbf{N}_\mathrm{J}$ of $\mathbf{P}_{\mathrm{J,sc}}$ is abelian and satisfies $\mathbf{N}_\mathrm{J}(\mathbb{Q}) \simeq \mathrm{J}_\mathbb{Q}$, and any element of $\mathbf{N}_\mathrm{J}(\mathbb{Q})$ has the following form:

$$\mathrm{n}(A)(X, \xi, X', \xi') = \left(X + \xi' A, \xi + (A, X') + (A^\#, X) + \xi' \det(A), X' + A \times X + \xi' A^\#, \xi'\right), \; A \in \mathrm{J}_\mathbb{Q}.$$

We have the Levi decomposition $\mathbf{P}_{\mathrm{J,sc}} = \mathbf{M}_\mathrm{J}\mathbf{N}_\mathrm{J}$, and the action of $\mathbf{M}_\mathrm{J}$ on $\mathbf{N}_\mathrm{J}$ is given by the following lemma:

[3] Notice that there are some slight mistakes in [KY16, §3] and the correction is in [KY23, §2].

[4] In [Pol20], Pollack considers the space $\mathbb{Q} \oplus \mathrm{J}_\mathbb{Q} \oplus \mathrm{J}_\mathbb{Q}^\vee \oplus \mathbb{Q}$. An element $(X, \xi, X', \xi') \in \mathrm{W_J}$ corresponds to $(a, b, c, d) = (\xi', X, (-, X'), \xi)$ under the notations of Pollack.

Lemma 2.4.4. *For any $m \in \mathbf{M}_{\mathrm{J}}(\mathbb{Q}) \subseteq \mathbf{P}_{\mathrm{J,sc}}$ and $A \in \mathrm{J}_{\mathbb{Q}}$, we have the following identity:*

$$m\mathrm{n}(A)m^{-1} = \mathrm{n}\left(\lambda(m)m^{*}A\right).$$

Proof. This follows from a direct calculation using the property: for any $m \in \mathbf{M}_{\mathrm{J}}(\mathbb{Q})$ and $X, Y \in \mathrm{J}_{\mathbb{Q}}$, we have $m(X \times Y) = \lambda(m)(m^{*}X) \times (m^{*}Y)$. □

The Levi subgroup of $\mathbf{P}_{\mathrm{J}} \subseteq \mathbf{E}_7$ induced by $\mathbf{M}_{\mathrm{J}}$ is the quotient of $\mathbf{M}_{\mathrm{J}}$ by μ_2, where μ_2 is generated by the element $X \mapsto -X$ in $\mathbf{M}_{\mathrm{J}}$. We identify this Levi subgroup with $\mathbf{M}_{\mathrm{J}}$ via the short exact sequence:

$$1 \to \mu_2 \to \mathbf{M}_{\mathrm{J}} \xrightarrow{m \mapsto \lambda(m)m^{*}} \mathbf{M}_{\mathrm{J}} \to 1. \tag{2.4}$$

Hence we still have the Levi decomposition $\mathbf{P}_{\mathrm{J}} \simeq \mathbf{M}_{\mathrm{J}}\mathbf{N}_{\mathrm{J}}$, but with a different action:

$$m\mathrm{n}(A)m^{-1} = \mathrm{n}(mA), \text{ for any } m \in \mathbf{M}_{\mathrm{J}}(\mathbb{Q}), A \in \mathrm{J}_{\mathbb{Q}}.$$

Remark 2.4.5. For any $A \in \mathrm{J}_{\mathbb{Q}}$, we define $\mathrm{n}^{\vee}(A) = \iota\mathrm{n}(-A)\iota^{-1}$. Set $\overline{\mathbf{N}_{\mathrm{J}}} = \iota\mathbf{N}_{\mathrm{J}}\iota^{-1}$, then $\overline{\mathbf{P}_{\mathrm{J,sc}}} = \mathbf{M}_{\mathrm{J}}\overline{\mathbf{N}_{\mathrm{J}}}$ is the parabolic subgroup opposite to $\mathbf{P}_{\mathrm{J,sc}}$. The action of $\mathbf{M}_{\mathrm{J}}$ on $\overline{\mathbf{N}_{\mathrm{J}}}$ is:

$$m\mathrm{n}^{\vee}(A)m^{-1} = \mathrm{n}^{\vee}\left(\lambda(m)^{-1}mA\right), \text{ for any } m \in \mathbf{M}_{\mathrm{J}}(\mathbb{Q}), A \in \mathrm{J}_{\mathbb{Q}}.$$

2.4.2 The Lie algebra $\mathfrak{e}_7$

Denote the Lie algebra of $\mathbf{H}^1_{\mathrm{J}}(\mathbb{C})$ by $\mathfrak{e}_7$, which admits a decomposition

$$\mathfrak{e}_7 = \mathrm{n}^{\vee}_{\mathrm{L}}(\mathrm{J}_{\mathbb{C}}) \oplus \mathfrak{m}_{\mathrm{J}} \oplus \mathrm{n}_{\mathrm{L}}(\mathrm{J}_{\mathbb{C}}), \tag{2.5}$$

where

- $\mathfrak{m}_{\mathrm{J}} = \mathrm{Lie}(\mathbf{M}_{\mathrm{J}}(\mathbb{C}))$;
- for any $A \in \mathrm{J}_{\mathbb{C}}$, define $\mathrm{n}_{\mathrm{L}}(A)$ to be the element in $\mathrm{Lie}(\mathbf{N}_{\mathrm{J}}(\mathbb{C}))$ such that $\exp(\mathrm{n}_{\mathrm{L}}(A)) = \mathrm{n}(A)$, and denote $\mathrm{Lie}(\mathbf{N}_{\mathrm{J}}(\mathbb{C}))$ by $\mathrm{n}_{\mathrm{L}}(\mathrm{J}_{\mathbb{C}})$;
- for any $A \in \mathrm{J}_{\mathbb{C}}$, define $\mathrm{n}_{\mathrm{L}}(A)$ to be the element in $\mathrm{Lie}(\overline{\mathbf{N}_{\mathrm{J}}}(\mathbb{C}))$ such that $\exp(\mathrm{n}_{\mathrm{L}}(A)) = \mathrm{n}^{\vee}(A)$, and denote $\mathrm{Lie}(\overline{\mathbf{N}_{\mathrm{J}}}(\mathbb{C}))$ by $\mathrm{n}^{\vee}_{\mathrm{L}}(\mathrm{J}_{\mathbb{C}})$.

Besides this decomposition, we also have the Cartan decomposition of $\mathfrak{e}_7$. Let $\mathrm{K}_{\mathrm{E}_7}$ be the subgroup of $\mathbf{H}^1_{\mathrm{J}}(\mathbb{R})$ that fixes the line in $\mathrm{W}_{\mathrm{J}} \otimes \mathbb{C}$ spanned by $(i\mathrm{I}, -i, -\mathrm{I}, 1)$, which is a maximal compact subgroup of $\mathbf{H}^1_{\mathrm{J}}(\mathbb{R})$. Take $\mathfrak{k}_{\mathrm{E}_7}$ to be the complexified Lie algebra of $\mathrm{K}_{\mathrm{E}_7}$, then we have the following Cartan decomposition of $\mathfrak{e}_7$:

$$\mathfrak{e}_7 = \mathfrak{p}^{-}_{\mathrm{J}} \oplus \mathfrak{k}_{\mathrm{E}_7} \oplus \mathfrak{p}^{+}_{\mathrm{J}}, \tag{2.6}$$

where $\mathfrak{p}^{+}_{\mathrm{J}} \oplus \mathfrak{p}^{-}_{\mathrm{J}}$ is the natural decomposition of the -1 eigenspace for the Cartan involution. We have the following relation between these two decompositions (2.5) and (2.6) of $\mathfrak{e}_7$:

Proposition 2.4.6. *[Pol23, Proposition 6.1.1] There exists an element $\mathrm{C}_h \in \mathbf{H}^1_{\mathrm{J}}(\mathbb{C})$, called the* Cayley transform*, satisfying:*

(1) $\mathrm{C}_h^{-1}\mathrm{n}_{\mathrm{L}}(\mathrm{J}_{\mathbb{C}})\mathrm{C}_h = \mathfrak{p}^{+}_{\mathrm{J}}$*;*

(2) $\mathrm{C}_h^{-1}\mathrm{n}_{\mathrm{L}}^{\vee}(\mathrm{J}_{\mathbb{C}})\mathrm{C}_h = \mathfrak{p}_{\mathrm{J}}^-$;
(3) $\mathrm{C}_h^{-1}\mathfrak{m}_{\mathrm{J}}\mathrm{C}_h = \mathfrak{k}_{\mathrm{E}_7}$.

By Proposition 2.4.6, we make the following identifications:

- Identify the factor $\mathfrak{p}_{\mathrm{J}}^+$ as $\mathrm{J}_{\mathbb{C}}^{\vee}$, via the map
$$\mathfrak{p}_{\mathrm{J}}^+ \ni \mathrm{X}_A^+ := i\mathrm{C}_h^{-1}\mathrm{n}_{\mathrm{L}}(A)\mathrm{C}_h \mapsto (-, A) \in \mathrm{J}_{\mathbb{C}},$$
and equip it with the following $\mathbf{M}_{\mathrm{J}}(\mathbb{C})$-action:
$$(m.\ell)(X) = \ell\left(\lambda(m)m^{-1}(X)\right), \text{ for any } m \in \mathbf{M}_{\mathrm{J}}(\mathbb{C}), \ell \in \mathrm{J}_{\mathbb{C}}^{\vee}, X \in \mathrm{J}_{\mathbb{C}}.$$
- Identify $\mathfrak{p}_{\mathrm{J}}^-$ as $\mathrm{J}_{\mathbb{C}}$, via the map
$$\mathfrak{p}_{\mathrm{J}}^- \ni \mathrm{X}_A^- := i\mathrm{C}_h^{-1}\mathrm{n}_{\mathrm{L}}^{\vee}(A)\mathrm{C}_h \mapsto A \in \mathrm{J}_{\mathbb{C}},$$
and equip it with the following $\mathbf{M}_{\mathrm{J}}(\mathbb{C})$-action:
$$m.X = \lambda(m)^{-1}m(X) \text{ for any } m \in \mathbf{M}_{\mathrm{J}}(\mathbb{C}), X \in \mathrm{J}_{\mathbb{C}}.$$

The natural $\mathbf{M}_{\mathrm{J}}(\mathbb{C})$-invariant pairing $\{-,-\} : \mathrm{J}_{\mathbb{C}} \times \mathrm{J}_{\mathbb{C}}^{\vee} \to \mathbb{C}$ can be extended to
$$\{-,-\} : \mathrm{J}_{\mathbb{C}}^{\otimes n} \times \left(\mathrm{J}_{\mathbb{C}}^{\vee}\right)^{\otimes n} \to \mathbb{C}, (X_1 \otimes \cdots \otimes X_n, \ell_1 \otimes \cdots \otimes \ell_n) \mapsto \frac{\sum_{\sigma \in S_n} \prod_{i=1}^n \{X_i, \ell_{\sigma(i)}\}}{n!}, \tag{2.7}$$
which factors through $\mathrm{Sym}^n \mathrm{J}_{\mathbb{C}} \times \mathrm{Sym}^n(\mathrm{J}_{\mathbb{C}}^{\vee})$.

Example 2.4.7. Identifying $\mathrm{Sym}^n(\mathrm{J}_{\mathbb{C}}^{\vee})$ with the space $\mathrm{P}_n(\mathrm{J}_{\mathbb{C}})$ of degree n homogeneous polynomials over $\mathrm{J}_{\mathbb{C}}$, the $\mathbf{M}_{\mathrm{J}}(\mathbb{C})$-action on it is $(m.P)(X) = P(\lambda(m)m^{-1}(X))$ for any $m \in \mathbf{M}_{\mathrm{J}}(\mathbb{C})$, $P \in \mathrm{P}_n(\mathrm{J}_{\mathbb{C}})$ and $T \in \mathrm{J}_{\mathbb{C}}$, and the pairing $\{T^{\otimes n}, P\}$ is equal to $P(T)$.

2.5 Dual pairs

Now we explain the two reductive dual pairs $\mathbf{F}_4 \times \mathbf{PGL}_2$ and $\mathbf{Spin}_9 \times \mathbf{SO}_{2,2}$ in $\mathbf{E}_7$.

2.5.1 $\mathbf{F}_4 \times \mathbf{PGL}_2$

We study first the centralizer of $\mathbf{F}_4$ in $\mathbf{M}_{\mathrm{J}}$. For any element g in the centralizer $\mathbf{C}_{\mathbf{M}_{\mathrm{J}}}(\mathbf{F}_4)$, it stabilizes the subspace $\mathrm{J}_{\mathbb{Q}}^{\mathbf{F}_4(\mathbb{Q})}$, which is a line spanned by I, thus $g(\mathrm{I})$ is a non-zero multiple of I. So we obtain a morphism $\mathbf{C}_{\mathbf{M}_{\mathrm{J}}}(\mathbf{F}_4) \to \mathbf{G}_{\mathrm{m}}$ by restricting to the line spanned by I.

- This morphism is injective, since the center of $\mathbf{F}_4$ is trivial;
- For any scalar $\lambda \in \mathbb{Q}^{\times}$, the map $X \mapsto \lambda X$ is an element of $\mathbf{C}_{\mathbf{M}_{\mathrm{J}}}(\mathbf{F}_4)(\mathbb{Q})$, thus morphism is also surjective.

Hence the centralizer of $\mathbf{F}_4$ in the Levi subgroup $\mathbf{M}_{\mathrm{J}}$ of $\mathbf{H}_{\mathrm{J}}^1$ is a rank 1 torus.

The centralizer of $\mathbf{F}_4$ in $\mathbf{P}_{\mathrm{J,sc}} = \mathbf{M}_{\mathrm{J}}\mathbf{N}_{\mathrm{J}}$ is generated by $\mathbf{C}_{\mathbf{M}_{\mathrm{J}}}(\mathbf{F}_4)$ and the subgroup $\{\mathrm{n}(x\mathrm{I}), x \in \mathbf{G}_{\mathrm{a}}\}$ of $\mathbf{N}_{\mathrm{J}}$, and it is isomorphic to the standard Borel subgroup of $\mathbf{SL}_2$ via:
$$(X \mapsto uX) \mapsto \begin{pmatrix} u & \\ & u^{-1} \end{pmatrix}, \mathrm{n}(x\mathrm{I}) \mapsto \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}.$$

Similarly, the centralizer of $\mathbf{F}_4$ in $\overline{\mathbf{P}_{\mathrm{J,sc}}}$ is isomorphic to the opposite Borel subgroup of $\mathbf{SL}_2$. As a consequence, we get a subgroup $\mathbf{F}_4 \times \mathbf{SL}_2$ inside $\mathbf{H}_{\mathrm{J}}^1$, which is a maximal proper subgroup of $\mathbf{H}_{\mathrm{J}}^1$ [KS23, Lemma 2.4], so it gives a reductive dual pair in $\mathbf{H}_{\mathrm{J}}^1$, and induces a dual pair $\mathbf{F}_4 \times \mathbf{GL}_2$ (*resp.* $\mathbf{F}_4 \times \mathbf{PGL}_2$) inside $\mathbf{H}_{\mathrm{J}}$ (*resp.* $\mathbf{E}_7$).

2.5.2 $\mathbf{Spin}_9 \times \mathbf{SO}_{2,2}$

By [Yok09, Theorem 2.7.4], the stabilizer of $\mathrm{E}_1 = [1,0,0\,;0,0,0]$ in $\mathbf{F}_4$ is isomorphic to $\mathbf{Spin}_9$, the spin group of a positive definite 9-dimensional quadratic space. In the sequel we refer to this group as $\mathbf{Spin}_9$. The 9-dimensional quadratic space can be found inside $\mathrm{J}_{\mathbb{Q}}$:

Lemma 2.5.1. *The group $\mathbf{Spin}_9$ preserves respectively the following subspaces of $\mathrm{J}_{\mathbb{Q}}$:*

$$\mathrm{J}_1 := \{[0,\xi,-\xi\,;x,0,0] \,|\, \xi \in \mathbb{Q}, x \in \mathbb{O}_{\mathbb{Q}}\}$$

and

$$\mathrm{J}_2 := \{[0,0,0\,;0,y,z] \,|\, y,z \in \mathbb{O}_{\mathbb{Q}}\}\,.$$

Proof. Since

$$\mathrm{J}_1 = \{X \in \mathrm{J}_{\mathbb{Q}} \,|\, \mathrm{E}_1 \circ X = 0, \mathrm{Tr}(X) = 0\}$$

and

$$\mathrm{J}_2 = \{X \in \mathrm{J}_{\mathbb{Q}} \,|\, 2\mathrm{E}_1 \circ X = X\}\,,$$

the lemma follows from the definition that $\mathbf{Spin}_9$ is the stabilizer of E_1 in $\mathbf{F}_4$. □

Notation 2.5.2. In this article, $\mathbf{SO}_{2,2}$ is defined to be the special orthogonal group of a split 4-dimensional quadratic space over $\mathbb{Q}$, and we define $\mathbf{Spin}_{2,2}, \mathbf{GSpin}_{2,2}$ similarly. Notice that $\mathbf{GSpin}_{2,2} \simeq \{(g_1,g_2) \in \mathbf{GL}_2 \times \mathbf{GL}_2, \det(g_1) = \det(g_2)\}$, $\mathbf{Spin}_{2,2} \simeq \mathbf{SL}_2 \times \mathbf{SL}_2$, and $\mathbf{SO}_{2,2} \simeq \mathbf{GSpin}_{2,2}/\mathbf{G}_{\mathrm{m}}^{\Delta} \simeq \mathbf{Spin}_{2,2}/\mu_2^{\Delta}$.

We study first the centralizer of $\mathbf{Spin}_9$ in the Levi subgroup $\mathbf{M}_{\mathrm{J}} \subseteq \mathbf{H}_{\mathrm{J}}^1$:

Lemma 2.5.3. *The centralizer $\mathbf{C}_{\mathbf{M}_{\mathrm{J}}}(\mathbf{Spin}_9)$ is an extension of $\mathbf{G}_{\mathrm{m}} \times \mathbf{G}_{\mathrm{m}}$ by μ_2.*

Proof. For any element $g \in \mathbf{C}_{\mathbf{M}_{\mathrm{J}}}(\mathbf{Spin}_9)$, it stabilizes the subspace $\mathrm{J}_{\mathbb{Q}}^{\mathbf{Spin}_9(\mathbb{Q})}$, which is spanned by E_1 and $\mathrm{I} - \mathrm{E}_1 = \mathrm{E}_2 + \mathrm{E}_3$. The rank 1 elements in this subspace are non-zero multiples of E_1, and the rank 2 elements are non-zero multiples of $\mathrm{E}_2 + \mathrm{E}_3$. As elements of $\mathbf{M}_{\mathrm{J}}$ preserve the rank, g acts on E_1 (*resp.* $\mathrm{E}_2 + \mathrm{E}_3$) by a scalar. So we obtain a morphism from $\mathbf{C}_{\mathbf{M}_{\mathrm{J}}}(\mathbf{Spin}_9)$ to $\mathbf{G}_{\mathrm{m}} \times \mathbf{G}_{\mathrm{m}}$, whose kernel is the center of $\mathbf{Spin}_9$, a cyclic group generated by the involution $[a,b,c\,;x,y,z] \mapsto [a,b,c\,;x,-y,-z]$ [Sha24, §4.3.1]. This morphism of algebraic groups is also surjective, since for any non-zero scalars λ,μ, we have the following element in $\mathbf{C}_{\mathbf{M}_{\mathrm{J}}}(\mathbf{Spin}_9)$:

$$m_{\lambda,\mu} : [a,b,c\,;x,y,z] \mapsto [\lambda^{-1}\mu^2 a, \lambda b, \lambda c\,; \lambda x, \mu y, \mu z].$$ □

Let $\mathbf{C}'$ be the subgroup of $\mathbf{C}_{\mathbf{M}_{\mathrm{J}}}(\mathbf{Spin}_9)$ consisting of $m_{\lambda,\mu}$, then we have the following commutative diagram:

$$\begin{tikzcd}
1 \arrow[r] & \mu_2 \arrow[r] \arrow[d, equal] & \mathbf{C}_{\mathbf{M}_{\mathrm{J}}}(\mathbf{Spin}_9) \arrow[r] & \mathbf{G}_{\mathrm{m}} \times \mathbf{G}_{\mathrm{m}} \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \mu_2 \arrow[r, "\mu \mapsto m_{1,\mu}"] & \mathbf{C}' \arrow[r, "m_{\lambda,\mu} \mapsto (\lambda^{-1}\mu^2,\lambda)"] \arrow[u, hook] & \mathbf{G}_{\mathrm{m}} \times \mathbf{G}_{\mathrm{m}} \arrow[r] & 1
\end{tikzcd},$$

which shows that $\mathbf{C}_{\mathbf{M}_{\mathrm{J}}}(\mathbf{Spin}_9) = \mathbf{C}'$ is a split torus of rank 2. The centralizer of $\mathbf{Spin}_9$ in $\mathbf{P}_{\mathrm{J,sc}}$ is generated by $\mathbf{C}_{\mathbf{M}_{\mathrm{J}}}(\mathbf{Spin}_9)$ and $\{\mathrm{n}(x\mathrm{E}_1 + y(\mathrm{E}_2 + \mathrm{E}_3)), x, y \in \mathbb{Q}\} \subseteq \mathbf{N}_{\mathrm{J}}$, and it is isomorphic to the standard Borel subgroup of $\mathbf{Spin}_{2,2} = \mathbf{SL}_2 \times \mathbf{SL}_2$ via:

$$m_{\lambda,\mu} \mapsto \left(\begin{pmatrix} \lambda & \\ & \lambda^{-1} \end{pmatrix}, \begin{pmatrix} \mu & \\ & \mu^{-1} \end{pmatrix}\right), \ \mathrm{n}(x\mathrm{E}_1 + y(\mathrm{E}_2 + \mathrm{E}_3)) \mapsto \left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & y \\ & 1 \end{pmatrix}\right). \quad (2.8)$$

Similarly, the centralizer of $\mathbf{Spin}_9$ in $\overline{\mathbf{P}_{\mathrm{J,sc}}}$ is isomorphic to the opposite Borel subgroup of $\mathbf{Spin}_{2,2}$, thus we get a morphism $\mathbf{Spin}_9 \times \mathbf{Spin}_{2,2} \to \mathbf{H}_{\mathrm{J}}^1$. The kernel of this morphism is $\{(\mathrm{id}, \mathrm{id}), (m_{1,-1}, m_{1,-1})\}$, and we denote by $\mathbf{Spin}_9 \times_{\mu_2} \mathbf{Spin}_{2,2}$ the quotient of $\mathbf{Spin}_9 \times \mathbf{Spin}_{2,2}$ by this kernel. The morphism $\mathbf{Spin}_9 \times_{\mu_2} \mathbf{Spin}_{2,2} \hookrightarrow \mathbf{H}_{\mathrm{J}}^1$ induces an embedding of $\mathbf{Spin}_9 \times_{\mu_2} \mathbf{GSpin}_{2,2}$ (*resp.* $\mathbf{Spin}_9 \times_{\mu_2} \mathbf{SO}_{2,2}$) into $\mathbf{H}_{\mathrm{J}}$ (*resp.* $\mathbf{E}_7$).

The centralizer $\mathbf{C}_{\mathbf{E}_7}(\mathbf{F}_4) \simeq \mathbf{PGL}_2$ is embedded into $\mathbf{SO}_{2,2} \subseteq \mathbf{C}_{\mathbf{E}_7}(\mathbf{Spin}_9)$ via the map induced from the diagonal embedding $\mathbf{GL}_2 \to \mathbf{GSpin}_{2,2}$.

3 Local theta correspondence

In this section we recall some results on the minimal representation of $\mathbf{E}_7$ and the local theta correspondences for the exceptional dual pairs constructed in §2.5.

3.1 Minimal representation of $\mathbf{E}_7$

The theory of theta correspondences studies the restrictions of minimal representations to reductive dual pairs, so we first recall the definition of the minimal representation of $\mathbf{E}_7(F)$ for $F = \mathbb{Q}_p$ or $\mathbb{R}$, and also some properties that will be used.

Definition 3.1.1. (i) The *minimal representation* $\Pi_{\mathrm{min},p}$ *of* $\mathbf{E}_7(\mathbb{Q}_p)$ is the unramified representation whose Satake parameter is the $\widehat{\mathbf{E}_7}(\mathbb{C})$-conjugacy class of $\varphi\begin{pmatrix} p^{1/2} & \\ & p^{-1/2} \end{pmatrix}$. Here the morphism $\varphi : \mathbf{SL}_2(\mathbb{C}) \to \widehat{\mathbf{E}_7}(\mathbb{C})$ corresponds to the subregular unipotent orbit of $\widehat{\mathbf{E}_7}(\mathbb{C}) = \mathbf{H}_{\mathrm{J}}^1(\mathbb{C})$.

(ii) Let Π^+ be the holomorphic representation of $\mathbf{H}_{\mathrm{J}}^1(\mathbb{R})$ with the smallest Gelfand-Kirillov dimension among non-trivial representations, and Π^- be the anti-holomorphic representation contragradient to Π^+. The *minimal representation* $\Pi_{\mathrm{min},\infty}$ *of* $\mathbf{E}_7(\mathbb{R})$ is the unique representation whose restriction to $\mathbf{H}_{\mathrm{J}}^1(\mathbb{R})$ is $\Pi^+ \oplus \Pi^-$.

The first property that we need is the following relation between the minimal representation and a principal series:

Proposition 3.1.2. *[Sav94, Proposition 6.1][Sah93] For $v = p$ or ∞, the minimal representation $\Pi_{\mathrm{min},v}$ of $\mathbf{E}_7(\mathbb{Q}_v)$ is the unique irreducible submodule of the normalized degenerate principal series*

$$\mathrm{Ind}_{\mathbf{P}_{\mathrm{J}}(\mathbb{Q}_v)}^{\mathbf{E}_7(\mathbb{Q}_v)} \delta_{\mathbf{P}_{\mathrm{J}}}^{-1/2} |\lambda|^2,$$

where $\delta_{\mathbf{P}_{\mathrm{J}}}$ is the modulus character of $\mathbf{P}_{\mathrm{J}}(\mathbb{Q}_v)$, and $\lambda : \mathbf{M}_{\mathrm{J}}(\mathbb{Q}_v) \to \mathbb{Q}_v^\times$ is the similitude character of $\mathbf{M}_{\mathrm{J}}(\mathbb{Q}_v)$.

The sections of $\mathrm{Ind}_{\mathbf{P}_{\mathrm{J}}(\mathbb{Q}_p)}^{\mathbf{E}_7(\mathbb{Q}_v)}\delta_{\mathbf{P}_{\mathrm{J}}}^{-1/2}|\lambda|^2$ are smooth functions $f : \mathbf{P}_{\mathrm{J}}(\mathbb{Q}_p) \to \mathbb{C}$ such that

$$f(pg) = |\lambda(p)|_p^2 f(g), \text{ for all } p \in \mathbf{P}_{\mathrm{J}}(\mathbb{Q}_p), g \in \mathbf{E}_7(\mathbb{Q}_p). \tag{3.1}$$

From now on, we identify $\Pi_{\mathrm{min},v}$ as the unique irreducible submodule of $\mathrm{Ind}_{\mathbf{P}_{\mathrm{J}}(\mathbb{Q}_v)}^{\mathbf{E}_7(\mathbb{Q}_v)}\delta_{\mathbf{P}_{\mathrm{J}}}^{-1/2}|\lambda|^2$, and normalize the spherical vector Φ_p in $\Pi_{\mathrm{min},v}$ by the condition that $\Phi_p(1) = 1$.

The second property is the $\mathrm{K}_{\mathrm{E}_7}$-types of the holomorphic part Π^+ of Π_{min}. The maximal compact subgroup $\mathrm{K}_{\mathrm{E}_7}$ of $\mathbf{H}_{\mathrm{J}}^1(\mathbb{R})$ is isomorphic to $\mathrm{E}_6 \times \mathrm{U}(1)$, where E_6 is the simply-connected compact Lie group of type E_6.

Definition 3.1.3. (1) Define $\mathrm{E}(n)$ to be the irreducible representation of the compact Lie group E_6 with highest weight $n\lambda$, where λ is the highest weight of $\mathfrak{p}_{\mathrm{J}}^+$ as a E_6-representation.
(2) For $n, k \in \mathbb{N}$, define $\mathrm{E}(n,k)$ to be the irreducible representation of $\mathrm{K}_{\mathrm{E}_7}$ such that its restriction to E_6 is isomorphic to $\mathrm{E}(n)$ and its restriction to $\mathrm{U}(1)$ is the character $z \mapsto z^k$.

The restriction of Π^+ to $\mathrm{K}_{\mathrm{E}_7}$ is given in [Wal79b]:

$$\Pi^+|_{\mathrm{K}_{\mathrm{E}_7}} \simeq \bigoplus_{n=0}^{\infty} \mathrm{E}(n, 2n+12). \tag{3.2}$$

3.2 p-adic correspondence for $\mathbf{F}_4 \times \mathbf{PGL}_2$

Over $\mathbb{Q}_p$, the exceptional theta correspondence for $\mathbf{F}_4 \times \mathbf{PGL}_2$ has been studied in [Sav94; KS23]. Now we recall some results that we need.

Definition 3.2.1. Let π be a smooth irreducible representation of $\mathbf{PGL}_2(\mathbb{Q}_p)$, then the maximal π-isotypic quotient of $\Pi_{\mathrm{min},p}$ admits an action of $\mathbf{F}_4(\mathbb{Q}_p)$ and factors as $\pi \boxtimes \Theta(\pi)$. We call $\Theta(\pi)$ the *big theta lift* of π, and its maximal semisimple quotient $\theta(\pi)$ the *small theta lift* of π.

Let $\mathbf{B}_0 = \mathbf{T}_0\mathbf{N}_0$ be the Borel subgroup of $\mathbf{PGL}_2$ consisting of upper triangular matrices, and $\overline{\mathbf{B}_0}$ be the opposite Borel subgroup. Let χ be a character of $\mathbf{T}_0(\mathbb{Q}_p) = \{\left(\begin{smallmatrix} t & \\ & 1\end{smallmatrix}\right), t \in \mathbb{Q}_p^\times\}$ satisfying $\chi = |-|^s \cdot \chi_0$, where $s \geq 0$ and χ_0 is a unitary character of $\mathbf{T}_0(\mathbb{Q}_p)$. When $s \neq \frac{1}{2}$ or $\chi_0^2 \neq 1$, the principal series $\mathrm{Ind}_{\overline{\mathbf{B}_0}(\mathbb{Q}_p)}^{\mathbf{PGL}_2(\mathbb{Q}_p)}(\chi)$ is irreducible. It turns out the theta lift of this principal series to $\mathbf{F}_4(\mathbb{Q}_p)$ is also a principal series. Before stating the result of Karasiewicz-Savin, we introduce a maximal parabolic subgroup of $\mathbf{F}_4$.

Definition 3.2.2. Using Bourbaki's labeling for simple roots of F_4, we define $\mathbf{Q}$ to be the maximal parabolic subgroup of $\mathbf{F}_4$ obtained by removing α_4 from the Dynkin diagram.

The Levi subgroup of $\mathbf{Q}$ is isomorphic to $\mathbf{GSpin}_7$, whose similitude map $\mathbf{GSpin}_7 \to \mathbf{GL}_1$ is given by the fundamental weight ϖ_4. Notice that $\widehat{\mathbf{Q}} \simeq \mathbf{GSp}_6 \simeq \mathbf{Sp}_6 \times \mathbf{G}_m$.

Proposition 3.2.3. *[KS23, Proposition 6.4] Let $\chi = |-|^s \cdot \chi_0$ be a character of $\mathbf{T}_0(\mathbb{Q}_p)$ such that χ_0 is unitary and $0 \leq s < 1/2$, then the big theta lift of $\mathrm{Ind}_{\overline{\mathbf{B}_0}(\mathbb{Q}_p)}^{\mathbf{PGL}_2(\mathbb{Q}_p)}(\chi)$ to $\mathbf{F}_4(\mathbb{Q}_p)$ is irreducible, and*

$$\Theta(\mathrm{Ind}_{\overline{\mathbf{B}_0}(\mathbb{Q}_p)}^{\mathbf{PGL}_2(\mathbb{Q}_p)}(\chi)) = \theta(\mathrm{Ind}_{\overline{\mathbf{B}_0}(\mathbb{Q}_p)}^{\mathbf{PGL}_2(\mathbb{Q}_p)}(\chi)) \simeq \mathrm{Ind}_{\mathbf{Q}(\mathbb{Q}_p)}^{\mathbf{F}_4(\mathbb{Q}_p)}(\chi \circ \varpi_4).$$

Remark 3.2.4. If χ is unramified, then Proposition 3.2.3 tells that the Satake parameter of $\theta(\mathrm{Ind}_{\mathbf{B}_0(\mathbb{Q}_p)}^{\mathbf{PGL}_2(\mathbb{Q}_p)}(\chi))$ is the $\widehat{\mathbf{F}_4}(\mathbb{C})$-conjugacy class of the image of (e_p, c_p) under the embedding $\mathbf{SL}_2 \times \mathbf{SL}_2 \to \mathbf{Sp}_6 \times \mathbf{SL}_2 \to \widehat{\mathbf{F}_4}$, where $c_p = \mathrm{diag}(\chi(p), \chi(p)^{-1})$ and $e_p = \mathrm{diag}(p^{1/2}, p^{-1/2})$.

3.3 Archimedean theta correspondence

For the dual pair $\mathbf{F}_4(\mathbb{R}) \times \mathbf{PGL}_2(\mathbb{R})$ inside $\mathbf{E}_7(\mathbb{R})$, we have the following result:

Proposition 3.3.1. *[GS98, Proposition 3.2] The restriction of $\Pi_{\min,\infty}$ to $\mathbf{F}_4(\mathbb{R}) \times \mathbf{PGL}_2(\mathbb{R})$ is isomorphic to*

$$\bigoplus_{n \geq 0} \mathrm{V}_{n\varpi_4} \boxtimes \mathcal{D}(2n+12),$$

where $\mathrm{V}_{n\varpi_4}$ is the irreducible representation of $\mathbf{F}_4(\mathbb{R})$ with highest weight $n\varpi_4$, and $\mathcal{D}(m)$ is the unitary completion of $d_{hol}(m) \oplus d_{anti\text{-}holo}(m)$, $d_{hol}(m)$ being the holomorphic discrete series representation of $\mathbf{SL}_2(\mathbb{R})$ with minimal $\mathbf{SO}_2(\mathbb{R})$ type m and $d_{anti\text{-}holo}(m)$ being its contragradient.

Before stating the result for $\mathbf{Spin}_9 \times \mathbf{SO}_{2,2}$, we define some notations for $\mathbf{Spin}_9(\mathbb{R})$.

Notation 3.3.2. Let λ_1 be the highest weight of the standard 9-dimensional representation of $\mathbf{Spin}_9(\mathbb{R})$, and λ_2 that of the 16-dimensional spinor representation. Denote by $\mathrm{U}_{m,n}$ the irreducible representation of $\mathbf{Spin}_9(\mathbb{R})$ with highest weight $m\lambda_1 + n\lambda_2$.

Proposition 3.3.3. *The restriction of $\Pi_{\min,\infty}$ to $\mathbf{Spin}_9(\mathbb{R}) \times \mathbf{SO}_{2,2}(\mathbb{R})$ is isomorphic to*

$$\bigoplus_{m,n \geq 0} \mathrm{U}_{m,n} \boxtimes \mathcal{D}(n+4) \boxtimes \mathcal{D}(2m+n+8),$$

where we view $\mathcal{D}(n+4) \boxtimes \mathcal{D}(2m+n+8)$ as a representation of $\mathbf{SO}_{2,2}(\mathbb{R})$.

Proof. The proof is parallel to the argument in [GS98, §3] for $\mathbf{G}_2 \times \mathbf{PGSp}_6$, using the branching laws in [Lep70]. □

4 Global theta correspondence

In this section, we recall an automorphic realization of the minimal representation of $\mathbf{E}_7(\mathbb{A})$, and then use it to define global theta lifts.

4.1 Automorphic forms

Let $\mathbf{G}$ be a connected reductive group over $\mathbb{Q}$ which admits a (reductive) $\mathbb{Z}$-model $\mathscr{G}$, in the sense of [Gro96]. Let $\widehat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$, $\mathbb{A}_f = \widehat{\mathbb{Z}} \otimes \mathbb{Q}$, and $\mathbb{A} = \mathbb{R} \times \mathbb{A}_f$. We fix a maximal compact subgroup K_∞ of $\mathbf{G}(\mathbb{R})$ and let $\mathfrak{g} = \mathbb{C} \otimes_{\mathbb{R}} \mathrm{Lie}(\mathbf{G}(\mathbb{R}))$.

For the simplicity we assume that the center of $\mathbf{G}$ is anisotropic, and denote the quotient space $\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})$ by $[\mathbf{G}]$. This topological space $[\mathbf{G}]$ admits a right invariant finite Haar measure μ, with respect to which we can define the space $\mathrm{L}^2([\mathbf{G}])$ of square-integrable functions on $[\mathbf{G}]$. The topological group $\mathbf{G}(\mathbb{A})$ acts on $\mathrm{L}^2([\mathbf{G}])$ by right translation, and the Petersson inner product makes it a unitary $\mathbf{G}(\mathbb{A})$-representation.

Definition 4.1.1. (1) An irreducible unitary representation π of $\mathbf{G}(\mathbb{A})$ is *(square-integrable) discrete automorphic* in the sense of [BJ79, §4.6], if π is isomorphic to a $\mathbf{G}(\mathbb{A})$-invariant closed subspace of $\mathrm{L}^2([\mathbf{G}])$. We denote by $\Pi_{\mathrm{disc}}(\mathbf{G})$ the set of equivalence classes of discrete automorphic representations of $\mathbf{G}$, and by $\mathrm{L}^2_{\mathrm{disc}}([\mathbf{G}])$ the discrete part of $\mathrm{L}^2([\mathbf{G}])$.
(2) An irreducible unitary representation π of $\mathbf{G}(\mathbb{A})$ has *level one* if π can be decomposed as $\pi = \pi_\infty \otimes \pi_f$, where π_∞ is an irreducible unitary representation of $\mathbf{G}(\mathbb{R})$ and π_f is a smooth irreducible representation of $\mathbf{G}(\mathbb{A}_f)$ such that $\pi_f^{\mathscr{G}(\widehat{\mathbb{Z}})} \neq 0$. We denote the subset of $\Pi_{\mathrm{disc}}(\mathbf{G})$ consisting of those with level one by $\Pi^{\mathrm{unr}}_{\mathrm{disc}}(\mathbf{G})$.
(3) The space of *(square-integrable) automorphic forms* $\mathcal{A}(\mathbf{G})$ is defined to be the space of $K_\infty \times \mathscr{G}(\widehat{\mathbb{Z}})$-finite and $\mathrm{Z}(\mathrm{U}(\mathfrak{g}))$-finite functions in the discrete spectrum $\mathrm{L}^2_{\mathrm{disc}}([\mathbf{G}])$.

Definition 4.1.2. (1) A square-integrable Borel function $f : [\mathbf{G}] \to \mathbb{C}$ is *cuspidal* if for the unipotent radical $\mathbf{U}$ of every proper parabolic subgroup of $\mathbf{G}$, we have

$$\int_{[\mathbf{U}]} f(ug)du = 0$$

for almost all $g \in \mathbf{G}(\mathbb{A})$. We denote the subspace of $\mathrm{L}^2([\mathbf{G}])$ consisting of the classes of cuspidal functions by $\mathrm{L}^2_{\mathrm{cusp}}([\mathbf{G}])$, and the subspace of $\mathcal{A}(\mathbf{G})$ consisting of cuspidal automorphic forms by $\mathcal{A}_{\mathrm{cusp}}(\mathbf{G})$.
(2) A discrete automorphic representation of $\mathbf{G}$ is *cuspidal* if it is a subrepresentation of $\mathrm{L}^2_{\mathrm{cusp}}([\mathbf{G}])$. Denote by $\Pi_{\mathrm{cusp}}(\mathbf{G})$ (*resp.* $\Pi^{\mathrm{unr}}_{\mathrm{cusp}}(\mathbf{G})$) the subset of $\Pi_{\mathrm{disc}}(\mathbf{G})$ (*resp.* $\Pi^{\mathrm{unr}}_{\mathrm{disc}}(\mathbf{G})$) consisting of cuspidal representations.

4.1.1 Automorphic forms of $\mathbf{F}_4$

Now we concentrate on the level one automorphic forms of $\mathbf{F}_4$, and describe them in a manner similar to the case for orthogonal groups [CL19, §4.4]. The adelic quotient $[\mathbf{F}_4]$ us compact, so $\mathrm{L}^2([\mathbf{F}_4]) = \mathrm{L}^2_{\mathrm{disc}}([\mathbf{F}_4]) = \mathrm{L}^2_{\mathrm{cusp}}([\mathbf{F}_4])$, and every automorphic representation of $\mathbf{F}_4$ is discrete and cuspidal.

A level one automorphic representation of $\mathbf{F}_4$ is generated by some automorphic form $\varphi \in \mathcal{A}(\mathbf{F}_4)^{\mathcal{F}_{4,\mathrm{I}}(\widehat{\mathbb{Z}})} \subseteq \mathrm{L}^2([\mathbf{F}_4])^{\mathcal{F}_{4,\mathrm{I}}(\widehat{\mathbb{Z}})}$. The latter space can be viewed as the space of square-integrable functions on $\mathbf{F}_4(\mathbb{Q})\backslash\mathbf{F}_4(\mathbb{A})/\mathcal{F}_{4,\mathrm{I}}(\widehat{\mathbb{Z}})$, endowed with the Radon measure that is the image of μ by the canonical map $[\mathbf{F}_4] \to \mathbf{F}_4(\mathbb{Q})\backslash\mathbf{F}_4(\mathbb{A})/\mathcal{F}_{4,\mathrm{I}}(\widehat{\mathbb{Z}})$. By the Peter-Weyl theorem, $\mathrm{L}^2([\mathbf{F}_4])^{\mathcal{F}_{4,\mathrm{I}}(\widehat{\mathbb{Z}})}$ can be decomposed into a direct sum of irreducible representations:

Lemma 4.1.3. *Denote by* $\mathrm{Irr}(\mathbf{F}_4(\mathbb{R}))$ *the set of equivalence classes of irreducible representations of* $\mathbf{F}_4(\mathbb{R})$*, then we have:*

$$\mathrm{L}^2([\mathbf{F}_4])^{\mathcal{F}_{4,\mathrm{I}}(\widehat{\mathbb{Z}})} \simeq \overline{\bigoplus_{V \in \mathrm{Irr}(\mathbf{F}_4(\mathbb{R}))}} V \otimes \mathcal{A}_V(\mathbf{F}_4),$$

where $\mathcal{A}_V(\mathbf{F}_4)$ *is defined as*

$$\left\{ f : \mathbf{F}_4(\mathbb{Q})\backslash\mathbf{F}_4(\mathbb{A})/\mathcal{F}_{4,\mathrm{I}}(\widehat{\mathbb{Z}}) \to V \,\middle|\, f(gh) = h^{-1}.f(g), \text{ for any } g \in \mathbf{F}_4(\mathbb{A}), h \in \mathbf{F}_4(\mathbb{R}) \right\}. \quad (4.1)$$

Under this isomorphism, an automorphic form $\varphi \in \mathcal{A}(\mathbf{F}_4)^{\mathcal{F}_{4,\mathrm{I}}(\widehat{\mathbb{Z}})}$ is identified with an element of $\bigoplus_{V \in \mathrm{Irr}(\mathbf{F}_4(\mathbb{R}))} V \otimes \mathcal{A}_V(\mathbf{F}_4)$. The number of $\pi \in \Pi_{\mathrm{disc}}^{\mathrm{unr}}(\mathbf{F}_4)$ such that $\pi_\infty \simeq V$, counted with multiplicities, is exactly $\dim \mathcal{A}_V(\mathbf{F}_4)$, which is computed explicitly in [Sha24].

Using Proposition 2.3.4, we identify $\mathbf{F}_4(\mathbb{Q}) \backslash \mathbf{F}_4(\mathbb{A}) / \mathcal{F}_{4,\mathrm{I}}(\widehat{\mathbb{Z}})$ with the set $\mathcal{J}$ of Albert lattices, and equip $\mathcal{J}$ with the corresponding right $\mathbf{F}_4(\mathbb{R})$-invariant Radon measure. We can thus identify $\mathrm{L}^2([\mathbf{F}_4])^{\mathcal{F}_{4,\mathrm{I}}(\widehat{\mathbb{Z}})}$ with $\mathrm{L}^2(\mathcal{J})$, equipped with the induced $\mathbf{F}_4(\mathbb{R})$ action:

$$(g.f)(J) = f(g^{-1}J), \text{ for any } g \in \mathbf{F}_4(\mathbb{R}), J \in \mathcal{J},$$

and identify $\mathcal{A}_V(\mathbf{F}_4)$ with the space

$$\{f : \mathcal{J} \to V \mid f(gJ) = g.f(J), \text{ for any } g \in \mathbf{F}_4(\mathbb{R}), J \in \mathcal{J}\}.$$

We will use either of these two formulations of $\mathcal{A}_V(\mathbf{F}_4)$, depending on convenience.

A function $f \in \mathcal{A}_V(\mathbf{F}_4)$ is determined by its values on the set of representatives $\{1, \gamma_{\mathrm{E}}\}$ for $\mathbf{F}_4(\mathbb{Q}) \backslash \mathbf{F}_4(\mathbb{A}_f) / \mathcal{F}_{4,\mathrm{I}}(\widehat{\mathbb{Z}})$ chosen in Notation 2.3.5. Furthermore, we have:

Lemma 4.1.4. *The evaluation map $f \mapsto (f(1), f(\gamma_{\mathrm{E}}))$ (or equivalently $f \mapsto (f(\mathrm{J}_{\mathbb{Z}}), f(\mathrm{J}_{\mathrm{E}}))$) induces an isomorphism of vector spaces:*

$$\mathrm{M}_V(\mathbf{F}_4) \simeq V^{\Gamma_{\mathrm{I}}} \oplus V^{\Gamma_{\mathrm{E}}},$$

where $\Gamma_{\mathrm{I}} = \mathcal{F}_{4,\mathrm{I}}(\mathbb{Z})$ is the automorphism group of the Albert algebra $\mathrm{J}_{\mathbb{Z}}$, and Γ_{E} is that of J_{E}.

4.1.2 A polynomial model of $\mathrm{V}_{n\varpi_4}$

In this paper, we focus on automorphic representations of $\mathbf{F}_4$ with archimedean component $V = \mathrm{V}_{n\varpi_4}$. Now we give a polynomial model of this family of irreducible representations.

When $n = 1$, a natural model for the 26-dimensional representation V_{ϖ_4} is the trace 0 part of $\mathrm{J}_{\mathbb{C}} \simeq \mathfrak{p}_{\mathrm{J}}^-$. We choose the realization dual to this one, *i.e.* the subspace of $\mathrm{P}_1(\mathrm{J}_{\mathbb{C}}) \simeq \mathfrak{p}_{\mathrm{J}}^+$ consisting of linear functions ℓ on $\mathrm{J}_{\mathbb{C}}$ such that $\ell(\mathrm{I}) = 0$.

For $n \geq 1$, $\mathrm{V}_{n\varpi_4}$ is a subrepresentation of $\mathrm{Sym}^n \mathrm{V}_{\varpi_4} \subseteq \mathrm{Sym}^n \mathfrak{p}_{\mathrm{J}}^+ = \mathrm{P}_n(\mathrm{J}_{\mathbb{C}})$, where the action of $\mathbf{F}_4(\mathbb{R})$ on $\mathrm{P}_n(\mathrm{J}_{\mathbb{C}})$ is given as:

$$(g.P)(X) = P(g^{-1}x), \text{ for any } g \in \mathbf{F}_4(\mathbb{R}), P \in \mathrm{P}_n(\mathrm{J}_{\mathbb{C}}) \text{ and } X \in \mathrm{J}_{\mathbb{C}}.$$

Definition 4.1.5. Define $\mathbb{X}$ to be the following $\mathbf{F}_4(\mathbb{C})$-orbit in $\mathrm{J}_{\mathbb{C}}$:

$$\mathbb{X} := \{A \in \mathrm{J}_{\mathbb{C}} \mid \mathrm{Tr}(A) = 0, \mathrm{rank}(A) = 1\} = \{A \in \mathrm{J}_{\mathbb{C}} \mid A \neq 0, \mathrm{Tr}(A) = 0, \mathrm{rank}(A) = 1\}.$$

For any $n \geq 1$, we define $\mathrm{V}_n(\mathrm{J}_{\mathbb{C}})$ to be the subspace of $\mathrm{P}_n(\mathrm{J}_{\mathbb{C}})$ spanned by polynomials of the form $X \mapsto (\mathrm{Tr}\,(X \circ A))^n$, $A \in \mathbb{X}$.

Lemma 4.1.6. *For any $n \geq 1$, $\mathrm{V}_n(\mathrm{J}_{\mathbb{C}})$ is an irreducible representation of $\mathbf{F}_4(\mathbb{R})$, and its highest weight is $n\varpi_4$.*

Proof. This lemma follows from the fact that $\mathbb{X}$ is the set of highest vectors in the irreducible $\mathbf{F}_4(\mathbb{R})$-representation $\{A \in \mathrm{J}_{\mathbb{C}}, \mathrm{Tr}(A) = 0\} \simeq \mathrm{V}_{\varpi_4}$, and $\mathbf{F}_4(\mathbb{R})$ acts on it transitively. ∎

4.2 Automorphic realization of minimal representation

Let $\Pi_{\min} = \otimes'_v \Pi_{\min,v}$ be the (adelic) minimal representation of $\mathbf{E}_7(\mathbb{A})$. To establish the global theta correspondence for dual pairs inside $\mathbf{E}_7$, we need to choose an automorphic realization of $\Pi_{\min}$, *i.e.* an $\mathbf{E}_7(\mathbb{A})$-equivariant embedding $\theta : \Pi_{\min} \hookrightarrow \mathrm{L}^2([\mathbf{E}_7])$. In this section, we follow [KY16, §6] to give θ via an explicit modular form constructed by Kim in [Kim93].

4.2.1 Exceptional modular forms

Definition 4.2.1. The *exceptional tube domain* $\mathcal{H}_\mathrm{J}$ of complex dimension 27 is the open subset of $\mathrm{J}_\mathbb{C} = \mathrm{J}_\mathbb{R} + i\mathrm{J}_\mathbb{R}$ consisting of $Z = X + iY$ with Y positive definite.

For any element $Z \in \mathrm{J}_\mathbb{C}$, set $\mathrm{r}_1(Z) := \big(Z, \det(Z), Z^\#, 1\big) \in \mathrm{W}_\mathrm{J} \otimes \mathbb{C}$. By [Pol20, Proposition 2.3.1], for any $g \in \mathbf{H}^1_\mathrm{J}(\mathbb{R})$ and $Z \in \mathcal{H}_\mathrm{J}$, there exist a unique scalar $J(g, Z) \in \mathbb{C}^\times$, which is called the *automorphy factor* for $\mathbf{H}^1_\mathrm{J}(\mathbb{R})$, and a unique $Z' \in \mathcal{H}_\mathrm{J}$ such that

$$g.\mathrm{r}_1(Z) = J(g, Z)\mathrm{r}_1(Z').$$

Definition 4.2.2. The action of $\mathbf{H}^1_\mathrm{J}(\mathbb{R})$-action on $\mathcal{H}_\mathrm{J}$ is defined as follows: for $g \in \mathbf{H}^1_\mathrm{J}(\mathbb{R})$ and $Z \in \mathcal{H}_\mathrm{J}$, $g.Z$ is the unique $Z' \in \mathcal{H}_\mathrm{J}$ satisfying $g.\mathrm{r}_1(Z) \in \mathbb{C}^\times \mathrm{r}_1(Z')$.

Example 4.2.3. We list the actions of some elements in $\mathbf{H}^1_\mathrm{J}(\mathbb{R})$:

- For $\mathrm{n}(A) \in \mathbf{N}_\mathrm{J}(\mathbb{R})$, $\mathrm{n}(A).Z = Z + A$ and $J(\mathrm{n}(A), Z) = 1$;
- For $m \in \mathbf{M}_\mathrm{J}(\mathbb{R})$, $m.(X + iY) = \lambda(m)(\lambda(X) + i\lambda(Y))$ and $J(m, Z) = \lambda(m)^{-1}$;
- For ι defined by (2.3), $\iota.Z = -Z^{-1}$ and $J(\iota, Z) = \det(Z)$.

The center $\pm 1 \simeq \langle \iota^2 \rangle$ of $\mathbf{H}^1_\mathrm{J}(\mathbb{R})$ acts trivially on $\mathcal{H}_\mathrm{J}$, and the group of holomorphic transformations of $\mathcal{H}_\mathrm{J}$ is $\mathbf{H}^1_\mathrm{J}(\mathbb{R})/\pm 1$, the connected component of $\mathbf{E}_7(\mathbb{R})$.

Definition 4.2.4. A holomorphic function $F : \mathcal{H}_\mathrm{J} \to \mathbb{C}$ is a *modular form of level* 1 *and weight* k if for any $Z \in \mathcal{H}_\mathrm{J}$ and $\gamma \in \mathbf{H}^1_\mathrm{J}(\mathbb{Z})$ we have

$$F(\gamma.Z) = J(\gamma, Z)^k \cdot F(Z).$$

Kim's modular form F_{Kim} is defined by the following Fourier expansion:

$$\mathrm{F}_{Kim}(Z) := 1 + 240 \sum_{\substack{\mathrm{J}_\mathbb{Z} \ni T \geq 0, \\ \mathrm{rank}(T) = 1}} \sigma_3\left(\mathrm{c}_{\mathrm{J}_\mathbb{Z}}(T)\right) e^{2\pi i (T, Z)}, \text{ for any } Z \in \mathcal{H}_\mathrm{J}, \tag{4.2}$$

where $\mathrm{c}_{\mathrm{J}_\mathbb{Z}}(T)$ is the content of T, *i.e.* the largest integer c such that $T/c \in \mathrm{J}_\mathbb{Z}$, and $\sigma_3(n) = \sum_{d|n} d^3$. The function F_{Kim} defined by (4.2) is a modular form of level 1 and weight 4.

4.2.2 Kim's automorphic form

Kim's modular form F_{Kim} gives rise to a level one automorphic form of $\mathbf{E}_7$. Using the strong approximation property of $\mathbf{E}_7$, we have the following natural homemorphisms:

$$\mathbf{E}_7(\mathbb{Q})\backslash\mathbf{E}_7(\mathbb{A})/\mathbf{E}_7(\widehat{\mathbb{Z}}) \simeq \mathbf{E}_7(\mathbb{Z})\backslash\mathbf{E}_7(\mathbb{R}) \simeq \mathbf{H}^1_\mathrm{J}(\mathbb{Z})\backslash\mathbf{H}^1_\mathrm{J}(\mathbb{R}),$$

thus we write any element $g \in \mathbf{E}_7(\mathbb{A})$ as $g = g_{\mathbb{Q}} g_\infty g_{\widehat{\mathbb{Z}}}$, where $g_{\mathbb{Q}} \in \mathbf{E}_7(\mathbb{Q})$, $g_{\widehat{\mathbb{Z}}} \in \mathbf{E}_7(\widehat{\mathbb{Z}})$ and $g_\infty \in \mathbf{E}_7(\mathbb{R})$ is the image of an element in $\mathbf{H}^1_{\mathrm{J}}(\mathbb{R})$ under the projection $\mathbf{H}_{\mathrm{J}}(\mathbb{R}) \to \mathbf{E}_7(\mathbb{R})$. In other words, g_∞ is an element of $\mathbf{H}^1_{\mathrm{J}}(\mathbb{R})/\pm 1$, the group of holomorphic automorphisms of $\mathcal{H}_{\mathrm{J}}$. Now for $g = g_{\mathbb{Q}} g_\infty g_{\widehat{\mathbb{Z}}} \in \mathbf{E}_7(\mathbb{A})$, we define

$$\Theta_{Kim}(g) := J(g_\infty, i\mathrm{I})^{-4} \cdot \mathrm{F}_{Kim}(g_\infty.i\mathrm{I}),$$

which is a well-defined [5] automorphic form of $\mathbf{E}_7$. Using the explicit action on $\mathcal{H}_{\mathrm{J}}$ given in Example 4.2.3, one gets the following:

Lemma 4.2.5. *The automorphic form $\Theta_{Kim} \in \mathcal{A}(\mathbf{E}_7)$ is invariant under $\mathbf{F}_4(\mathbb{R}) \times \mathbf{E}_7(\widehat{\mathbb{Z}})$.*

Now we use Θ_{Kim} to embed $\Pi_{\min}$ into $\mathrm{L}^2([\mathbf{E}_7])$:

Definition 4.2.6. Let $\Phi_p \in \Pi_{\min,p}$ be the normalized spherical vector, $\Phi_\infty \in \Pi^+ \subseteq \Pi_{\min,\infty}$ the unique (up to scalar) holomorphic vector with the minimal $\mathrm{K}_{\mathrm{E}_7}$-type, and $\Phi_0 := \Phi_\infty \otimes \Phi_f = \otimes_v \Phi_v \in \Pi_{\min}$. The automorphic realization $\theta : \Pi_{\min} \hookrightarrow \mathrm{L}^2([\mathbf{E}_7])$ is defined to be the unique $\mathbf{E}_7(\mathbb{A})$-equivariant map sending Φ_0 to Θ_{Kim}.

4.2.3 Constructing automorphic forms with non-minimal K_{E_7}-types

The holomorphic vector Φ_∞ lies in the minimal $\mathrm{K}_{\mathrm{E}_7}$-type of $\Pi^+ \subseteq \Pi_{\min,\infty}$, and we follow the method in [Pol20] to produce (holomorphic) automorphic forms with higher $\mathrm{K}_{\mathrm{E}_7}$-types.

For the two summands $\mathfrak{p}^\pm_{\mathrm{J}}$ in the Cartan decomposition (2.6) of $\mathfrak{e}_7$, choose a basis $\{\mathrm{X}_\alpha\}_\alpha$ of $\mathfrak{p}^+_{\mathrm{J}}$ and its dual basis $\{\mathrm{X}^\vee_\alpha\}_\alpha$ of $\mathfrak{p}^-_{\mathrm{J}}$ with respect to $\mathfrak{p}^+_{\mathrm{J}} \times \mathfrak{p}^-_{\mathrm{J}} \simeq \mathrm{J}^\vee_{\mathbb{C}} \times \mathrm{J}_{\mathbb{C}} \xrightarrow{\{-,-\}} \mathbb{C}$.

Definition 4.2.7. We define a linear differential operator $\mathrm{D} : \mathcal{A}(\mathbf{E}_7) \to \mathcal{A}(\mathbf{E}_7) \otimes \mathfrak{p}^-_{\mathrm{J}}$ by

$$\mathrm{D}\varphi(g) := \sum_\alpha (\mathrm{X}_\alpha \varphi)(g) \otimes \mathrm{X}^\vee_\alpha, \text{ for every } \varphi \in \mathcal{A}(\mathbf{E}_7),$$

which is independent of the choice of $\{\mathrm{X}_\alpha\}_\alpha$. For any integer $n \geq 0$, set D^n to be the n-times composition of D.

Applying the differential operator D^n defined in Definition 4.2.7 to Θ_{Kim}, we obtain

$$\Theta_n := \mathrm{D}^n \Theta_{Kim} \in \mathcal{A}(\mathbf{E}_7) \otimes (\mathfrak{p}^-_{\mathrm{J}})^{\otimes n},$$

whose coordinates belong to the $\mathrm{K}_{\mathrm{E}_7}$-type $\mathrm{E}(n, 2n+12)$ in (3.2).

Notation 4.2.8. (1) For any Albert lattice $J \in \mathcal{J}$, denote by J^+ the set of rank 1 and positive semi-definite elements in J, and set $\mathrm{a}_J(T) := \sigma_3(\mathrm{c}_J(T))$ for any $T \in J$, where $\mathrm{c}_J(T)$ is the content of T in J.

(2) For any element $T \in \mathrm{J}_{\mathbb{R}}$, denote by h_T the function:

$$g = g_{\mathbb{Q}} g_\infty g_{\widehat{\mathbb{Z}}} \in \mathbf{E}_7(\mathbb{A}) \mapsto J(g_\infty, i\mathrm{I})^{-4} \cdot e^{2\pi i (T, g_\infty . i\mathrm{I})},$$

where $g_{\mathbb{Q}} \in \mathbf{E}_7(\mathbb{Q}), g_{\widehat{\mathbb{Z}}} \in \mathbf{E}_7(\widehat{\mathbb{Z}})$ and g_∞ lies in the image of $\mathbf{H}^1_{\mathrm{J}}(\mathbb{R})$.

[5] Here we use the fact that $J(\gamma, Z) = \pm 1$ for any $\gamma \in \mathbf{H}^1_{\mathrm{J}}(\mathbb{Z})$ and $Z \in \mathcal{H}_{\mathrm{J}}$.

With these notations, for any $n \geq 1$, we rewrite Θ_n as:

$$\Theta_n(g) = 240 \sum_{T \in \mathrm{J}_{\mathbb{Z}}^+} \mathrm{a}_{\mathrm{J}_{\mathbb{Z}}}(T) \cdot \mathrm{D}^n h_T(g) = 240 \sum_{T \in \mathrm{J}_{\mathbb{Z}}^+} \mathrm{a}_{\mathrm{J}_{\mathbb{Z}}}(T) \cdot \mathrm{D}^n h_T(g_\infty). \tag{4.3}$$

We end this section by the following property of Θ_n:

Lemma 4.2.9. *For any $g_\infty \in \mathbf{H}_{\mathrm{J}}^1(\mathbb{R})$ and $h_\infty \in \mathbf{F}_4(\mathbb{R})$, we have $\Theta_n(g_\infty h_\infty) = h_\infty^{-1}.\Theta_n(g_\infty)$, where the action of h_∞^{-1} is applied on $(\mathfrak{p}_{\mathrm{J}}^-)^{\otimes n}$.*

Proof. By the definition of $\Theta_n = \mathrm{D}^n \Theta_{Kim}$, we have:

$$\begin{aligned}
\Theta_n(g_\infty h_\infty) &= \sum_{\alpha_1,\dots,\alpha_n} (\mathrm{X}_{\alpha_n} \cdots \mathrm{X}_{\alpha_1} \Theta_{Kim})(g_\infty h_\infty) \otimes \mathrm{X}_{\alpha_1}^\vee \otimes \cdots \otimes \mathrm{X}_{\alpha_n}^\vee \\
&= \sum_{\alpha_1,\dots,\alpha_n} \frac{d}{dt_n}\Big|_{t_n=0} \cdots \frac{d}{dt_1}\Big|_{t_1=0} \Theta_{Kim}(g_\infty h_\infty e^{t_n \mathrm{X}_{\alpha_n}} \cdots e^{t_1 \mathrm{X}_{\alpha_1}}) \otimes \mathrm{X}_{\alpha_1}^\vee \otimes \cdots \otimes \mathrm{X}_{\alpha_n}^\vee \\
&= \sum_{\alpha_1,\dots,\alpha_n} \frac{d}{dt_n}\Big|_{t_n=0} \cdots \frac{d}{dt_1}\Big|_{t_1=0} \Theta_{Kim}(g_\infty e^{t_n \mathrm{Ad}(h_\infty)\mathrm{X}_{\alpha_n}} \cdots e^{t_1 \mathrm{Ad}(h_\infty)\mathrm{X}_{\alpha_1}} h_\infty) \otimes \mathrm{X}_{\alpha_1}^\vee \otimes \cdots \otimes \mathrm{X}_{\alpha_n}^\vee \\
&= \sum_{\alpha_1,\dots,\alpha_n} \frac{d}{dt_n}\Big|_{t_n=0} \cdots \frac{d}{dt_1}\Big|_{t_1=0} \Theta_{Kim}(g_\infty e^{t_n h_\infty.\mathrm{X}_{\alpha_n}} \cdots e^{t_1 h_\infty.\mathrm{X}_{\alpha_1}}) \otimes \mathrm{X}_{\alpha_1}^\vee \otimes \cdots \otimes \mathrm{X}_{\alpha_n}^\vee,
\end{aligned}$$

where $h_\infty.\mathrm{X}_\alpha = \mathrm{Ad}(h_\infty)\mathrm{X}_\alpha$ and the last equality follows from Lemma 4.2.5. Since $\mathbf{F}_4(\mathbb{R})$ is a subgroup of the maximal compact subgroup $\mathrm{K}_{\mathrm{E}_7}$ of $\mathbf{E}_7(\mathbb{R})$, $\{h_\infty.\mathrm{X}_\alpha\}_\alpha$ also gives a basis of $\mathfrak{p}_{\mathrm{J}}^+$, and its dual basis of $\mathfrak{p}_{\mathrm{J}}^-$ is $\{h_\infty.\mathrm{X}_\alpha^\vee\}_\alpha$. As the differential operator D is independent of the choice of $\{\mathrm{X}_\alpha\}_\alpha$, we have:

$$\Theta_n(g_\infty h_\infty) = \sum_{\alpha_1,\dots,\alpha_n} (\mathrm{X}_{\alpha_n} \cdots \mathrm{X}_{\alpha_1} \Theta_{Kim})(g_\infty) \otimes h_\infty^{-1}.\mathrm{X}_{\alpha_1}^\vee \otimes \cdots \otimes h_\infty^{-1}.\mathrm{X}_{\alpha_n}^\vee = h_\infty^{-1}.\Theta_n(g_\infty). \qquad \square$$

4.3 Global theta lifts

Let $\mathbf{G} \times \mathbf{H}$ be one of the two reductive dual pairs given in §2.5, *i.e.* $\mathbf{G} \times \mathbf{H} = \mathbf{F}_4 \times \mathbf{PGL}_2$ or $\mathbf{Spin}_9 \times \mathbf{SO}_{2,2}$.

Definition 4.3.1. For $\varphi \in \mathcal{A}(\mathbf{H})$ and $\phi \in \Pi_{\mathrm{min}}$, the *global theta lift of φ with respect to ϕ* is the automorphic form of $\mathbf{G}$ defined by the following absolutely convergent integral:

$$\Theta_\phi(\varphi)(g) := \int_{[\mathbf{H}]} \theta(\phi)(gh)\overline{\varphi(h)}dh, \text{ for any } g \in \mathbf{G}(\mathbb{A}).$$

For a cuspidal automorphic representation $\pi \in \Pi_{\mathrm{cusp}}(\mathbf{H})$, its *global theta lift* $\Theta(\pi)$ is the $\mathbf{G}(\mathbb{A})$-subspace of $\mathrm{L}^2([\mathbf{G}])$ generated by $\{\Theta_\phi(\varphi) \,|\, \varphi \in \pi,\, \phi \in \Pi_{\mathrm{min}}\}$.

Remark 4.3.2. In this paper, we are always in the situation that either $[\mathbf{H}]$ is compact or $\varphi \in \mathcal{A}(\mathbf{H})$ is cuspidal. For the second case, the absolute convergence comes from the rapid decay of φ.

We also define the global theta lift of a "vector-valued automorphic form" $\alpha \in \mathcal{A}_{\mathrm{V}_{n\varpi}}(\mathbf{F}_4)$ defined as (4.1), which is compatible with Definition 4.3.1:

Definition 4.3.3. For a function $\alpha : \mathbf{F}_4(\mathbb{Q})\backslash\mathbf{F}_4(\mathbb{A})/\mathcal{F}_{4,\mathrm{I}}(\widehat{\mathbb{Z}}) \to \mathrm{V}_{n\varpi_4}$ in $\mathcal{A}_{\mathrm{V}_{n\varpi_4}}(\mathbf{F}_4)$, its *global theta lift* $\Theta(\alpha)$ is defined as:

$$\Theta(\alpha)(g) = \int_{[\mathbf{F}_4]} \{\Theta_n(gh), \alpha(h)\}dh, \text{ for any } g \in \mathbf{PGL}_2(\mathbb{A}), \tag{4.4}$$

where $\{-,-\} : \mathrm{J}_{\mathbb{C}}^{\otimes n} \times (\mathrm{J}_{\mathbb{C}}^{\vee})^{\otimes n} \to \mathbb{C}$ is the pairing defined in (2.7), and we view $\alpha(h) \in \mathrm{V}_{n\varpi_4}$ as a homogeneous polynomial over $\mathrm{J}_{\mathbb{C}}$.

5 Exceptional theta series

In this section, we compute the Fourier expansion of the theta lift $\Theta(\alpha)$ of $\alpha \in \mathcal{A}_{\mathrm{V}_{n\varpi_4}}(\mathbf{F}_4)$, and prove Theorem C in the introduction. From now on, we will identify α with its values $\alpha_{\mathrm{I}} \in \mathrm{V}_n(\mathrm{J}_{\mathbb{C}})^{\Gamma_{\mathrm{I}}}, \alpha_{\mathrm{E}} \in \mathrm{V}_n(\mathrm{J}_{\mathbb{C}})^{\Gamma_{\mathrm{E}}}$ at $1, \gamma_{\mathrm{E}}$ as in Lemma 4.1.4.

5.1 Fourier expansions of global theta lifts

Normalize the Haar measure dh of $\mathbf{F}_4(\mathbb{A})$ in (4.4) so that $\mathbf{F}_4(\mathbb{R})\mathcal{F}_{4,\mathrm{I}}(\widehat{\mathbb{Z}})$ has measure 1. Write $g \in \mathbf{PGL}_2(\mathbb{A})$ as $g = g_{\mathbb{Q}}g_{\infty}g_{\widehat{\mathbb{Z}}}$, where $g_{\mathbb{Q}} \in \mathbf{PGL}_2(\mathbb{Q}), g_{\widehat{\mathbb{Z}}} \in \mathbf{PGL}_2(\widehat{\mathbb{Z}})$ and g_{∞} is the image of an element in $\mathbf{SL}_2(\mathbb{R})$, then using Lemma 2.3.8, Lemma 4.2.9 and the $\mathbf{F}_4(\mathbb{R})$-invariance of $\{-,-\}$, we obtain:

$$\begin{aligned}\Theta(\alpha)(g) =& \frac{1}{|\Gamma_{\mathrm{I}}|}\int_{\mathbf{F}_4(\mathbb{R})\mathcal{F}_{4,\mathrm{I}}(\widehat{\mathbb{Z}})} \{\Theta_n(gh_{\infty}h_{\widehat{\mathbb{Z}}}), \alpha(h_{\infty}h_{\widehat{\mathbb{Z}}})\}dh + \frac{1}{|\Gamma_{\mathrm{E}}|}\int_{\mathbf{F}_4(\mathbb{R})\mathcal{F}_{4,\mathrm{I}}(\widehat{\mathbb{Z}})} \{\Theta_n(gh_{\infty}\gamma_{\mathrm{E}}h_{\widehat{\mathbb{Z}}}), \alpha(h_{\infty}\gamma_{\mathrm{E}}h_{\widehat{\mathbb{Z}}})dh\}\\ =& \frac{1}{|\Gamma_{\mathrm{I}}|}\int_{\mathbf{F}_4(\mathbb{R})\mathcal{F}_{4,\mathrm{I}}(\widehat{\mathbb{Z}})} \{h_{\infty}^{-1}.\Theta_n(g_{\infty}), h_{\infty}^{-1}.\alpha_{\mathrm{I}}\}dh + \frac{1}{|\Gamma_{\mathrm{E}}|}\int_{\mathbf{F}_4(\mathbb{R})\mathcal{F}_{4,\mathrm{I}}(\widehat{\mathbb{Z}})} \{h_{\infty}^{-1}.\Theta_n(\delta_{\infty}^{-1}g_{\infty}), h_{\infty}^{-1}.\alpha_{\mathrm{E}}\}\\ =& \frac{1}{|\Gamma_{\mathrm{I}}|}\{\Theta_n(g_{\infty}), \alpha_{\mathrm{I}}\} + \frac{1}{|\Gamma_{\mathrm{E}}|}\{\Theta_n(\delta_{\infty}^{-1}g_{\infty}), \alpha_{\mathrm{E}}\}.\end{aligned} \tag{5.1}$$

If the global theta lift $\Theta(\alpha) \in \mathcal{A}(\mathbf{PGL}_2)$ is non-zero, then the following result shows that it arises from a weight $2n+12$ classical holomorphic modular form on $\mathbf{SL}_2(\mathbb{Z})$:

Proposition 5.1.1. *Let $\mathcal{H} \subseteq \mathbb{C}$ be the Poincaré half plane, and $j : \mathbf{SL}_2(\mathbb{R}) \times \mathcal{H} \to \mathbb{C}^{\times}$ the automorphy factor given by $j\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), z\right) = cz + d$. For any $\alpha \in \mathcal{A}_{\mathrm{V}_{n\varpi_4}}(\mathbf{F}_4)$, the function*

$$f_{\Theta(\alpha)}(z) := j(g,i)^{2n+12}\Theta(\alpha)(g),\ z = g.i \in \mathcal{H},\ g \in \mathbf{SL}_2(\mathbb{R}),$$

is well-defined and is a level one holomorphic modular form of weight $2n+12$. Furthermore, it is a cusp form when $n > 0$.

We postpone the proof of Proposition 5.1.1 to §5.3, and prove the following main theorem on the Fourier expansion of $f_{\Theta(\alpha)}$:

Theorem 5.1.2. *(Theorem C in §1) Let $\alpha \in \mathcal{A}_{\mathrm{V}_{n\varpi_4}}(\mathbf{F}_4)$, $n > 0$ and $f_{\Theta(\alpha)}$ the cusp form associated to its global theta lift $\Theta(\alpha)$. Up to a non-zero constant, $f_{\Theta(\alpha)}$ has the following Fourier expansion:*

$$f_{\Theta(\alpha)}(z) = \frac{1}{|\Gamma_{\mathrm{I}}|}\sum_{T\in \mathrm{J}_{\mathbb{Z}}^{+}} \mathrm{a}_{\mathrm{J}_{\mathbb{Z}}}(T)\alpha_{\mathrm{I}}(T)q^{\mathrm{Tr}(T)} + \frac{1}{|\Gamma_{\mathrm{E}}|}\sum_{T\in \mathrm{J}_{\mathrm{E}}^{+}} \mathrm{a}_{\mathrm{J}_{\mathrm{E}}}(T)\alpha_{\mathrm{E}}(T)q^{\mathrm{Tr}(T)},\ q = e^{2\pi i z}.$$

Remark 5.1.3. The case when $n = 0$ is studied by Elkies and Gross in [EG96]. In this case $\alpha \in \mathcal{A}_{\mathbf{1}}(\mathbf{F}_4)$ can be identified as a pair of complex numbers. For α corresponding to $(|\Gamma_{\mathrm{I}}|, 0)$, $f_{\Theta(\alpha)} = E_{12} + \frac{432000}{691}\Delta$; for α corresponding to $(0, |\Gamma_{\mathrm{E}}|)$, $f_{\Theta(\alpha)} = E_{12} - \frac{65520}{691}\Delta$, where $E_{12}(z) = 1 + \frac{2}{\zeta(-11)}\sum_{n\geq 1}\sigma_{11}(n)q^n$ is the normalized weight 12 Eisenstein series, and $\Delta(z) = q\prod_{n\geq 1}(1-q^n)^{24}$ is the discriminant modular form.

Before proving Theorem 5.1.2, we state a result that will be used in the proof, whose proof is also postponed to §5.3.

Theorem 5.1.4. *Let $P \in \mathrm{V}_n(\mathrm{J}_{\mathbb{C}}) \simeq \mathrm{V}_{n\varpi_4}$ for any $n > 0$, T an element of $\mathrm{J}_{\mathbb{R}}$, and $h_T(g) = J(g_\infty, i\mathrm{I})^{-4}\cdot e^{2\pi i (T, g_\infty . i\mathrm{I})}$ the function given in Notation 4.2.8, then we have:*

$$\{(\mathrm{D}^n h_T)(g), P\} = (-4\pi)^n \cdot j(g,i)^{-2n-12} P\left(T\right) e^{2\pi i (T,\, g.i\mathrm{I})},\ \ \textit{for any } g \in \mathbf{SL}_2(\mathbb{R}).$$

Proof of Theorem 5.1.2. By (5.1), we have

$$f_{\Theta(\alpha)}(z) = j(g,i)^{2n+12}\left(\frac{1}{|\Gamma_{\mathrm{I}}|}\{\Theta_n(g), \alpha_{\mathrm{I}}\} + \frac{1}{|\Gamma_{\mathrm{E}}|}\{\Theta_n(\delta_\infty^{-1}g), \alpha_{\mathrm{E}}\}\right),\ z = g.i \in \mathcal{H}. \tag{5.2}$$

Using the Fourier expansion (4.3) of Θ_n and Theorem 5.1.4, the first term in (5.2) equals

$$\begin{aligned}\frac{1}{|\Gamma_{\mathrm{I}}|} j(g,i)^{2n+12}\{\Theta_n(g), \alpha_{\mathrm{I}}\} &= \frac{240}{|\Gamma_{\mathrm{I}}|} j(g,i)^{2n+12}\sum_{T\in \mathrm{J}_{\mathbb{Z}}^{+}} \mathrm{a}_{\mathrm{J}_{\mathbb{Z}}}(T)\{\mathrm{D}^n h_T(g), \alpha_{\mathrm{I}}\}\\ &= \frac{240(-4\pi)^n}{|\Gamma_{\mathrm{I}}|}\sum_{T\in \mathrm{J}_{\mathbb{Z}}^{+}} \mathrm{a}_{\mathrm{J}_{\mathbb{Z}}}(T)\alpha_{\mathrm{I}}(T)q^{(T,\mathrm{I})_{\mathrm{I}}},\end{aligned}$$

and the second term in (5.2) equals

$$\begin{aligned}\frac{1}{|\Gamma_{\mathrm{E}}|} j(g,i)^{2n+12}\{\Theta_n(\delta_\infty^{-1}g), \alpha_{\mathrm{E}}\} &= \frac{240}{|\Gamma_{\mathrm{E}}|} j(g,i)^{2n+12}\sum_{T\in \mathrm{J}_{\mathbb{Z}}^{+}} \mathrm{a}_{\mathrm{J}_{\mathbb{Z}}}(T)\{\mathrm{D}^n h_T(\delta_\infty^{-1}g), \alpha_{\mathrm{E}}\}\\ &= \frac{240}{|\Gamma_{\mathrm{E}}|} j(g,i)^{2n+12}\sum_{T\in \mathrm{J}_{\mathbb{Z}}^{+}} \mathrm{a}_{\mathrm{J}_{\mathbb{Z}}}(T)\left\{\mathrm{D}^n h_{\delta_\infty^* T}(g), \alpha_{\mathrm{E}}\right\}\\ &= \frac{240(-4\pi)^n}{|\Gamma_{\mathrm{E}}|}\sum_{T\in \mathrm{J}_{\mathbb{Z}}^{+}} \mathrm{a}_{\mathrm{J}_{\mathbb{Z}}}(T)\alpha_{\mathrm{E}}(\delta_\infty^* T)e^{2\pi i(\delta_\infty^* T, g.i\mathrm{I})}.\end{aligned}$$

Since $\mathbf{M}_{\mathrm{J}}^{1}(\mathbb{R})$ preserves the rank and stabilizes the set of positive semi-definite elements [EG96, Proposition 2.4], we have $\mathrm{J}_{\mathrm{E}}^{+} = \delta_\infty(\mathrm{J}_{\mathbb{Z}}^{+})$, thus

$$\sum_{T\in \mathrm{J}_{\mathbb{Z}}^{+}} \mathrm{a}_{\mathrm{J}_{\mathbb{Z}}}(T)\alpha_{\mathrm{E}}(\delta_\infty^* T)e^{2\pi i(\delta_\infty^* T, g.i\mathrm{I})} = \sum_{T\in \mathrm{J}_{\mathrm{E}}^{+}} \mathrm{a}_{\mathrm{J}_{\mathrm{E}}}(T)\alpha_{\mathrm{E}}(\delta_\infty^*\delta_\infty^{-1} T)e^{2\pi i(\delta_\infty^*\delta_\infty^{-1} T, g.i\mathrm{I})}.$$

The element $\delta_\infty^*\delta_\infty^{-1}$ is the archimedean part of $\delta^*\delta^{-1} \in \mathbf{M}_{\mathrm{J}}^1(\mathbb{Q})$. By Lemma 2.3.8, $\delta_f^{-1}\gamma_{\mathrm{E}} \in \mathbf{M}_{\mathrm{J}}^1(\widehat{\mathbb{Z}})$, so $\delta_f^*\delta_f^{-1} \in \gamma_{\mathrm{E}}^*\mathbf{M}_{\mathrm{J}}^1(\widehat{\mathbb{Z}})\gamma_{\mathrm{E}}^{-1} = \gamma_{\mathrm{E}}\mathbf{M}_{\mathrm{J}}^1(\widehat{\mathbb{Z}})\gamma_{\mathrm{E}}^{-1} = \mathrm{Aut}(\mathrm{J}_{\mathrm{E}} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}, \det)$. As a direct consequence, $\delta^*\delta^{-1}$ induces an automorphism of the lattice J_{E}, thus we have:

$$\sum_{T\in \mathrm{J}_{\mathrm{E}}^+} \mathrm{a}_{\mathrm{J}_{\mathrm{E}}}(T)\alpha_{\mathrm{E}}(\delta_\infty^*\delta_\infty^{-1}T)e^{2\pi i(\delta_\infty^*\delta_\infty^{-1}T, g.i\mathrm{I})} = \sum_{T\in \mathrm{J}_{\mathrm{E}}^+} \mathrm{a}_{\mathrm{J}_{\mathrm{E}}}(T)\alpha_{\mathrm{E}}(T)q^{\mathrm{Tr}(T)}. \qquad \square$$

A direct corollary of Theorem 5.1.2 is the following:

Corollary 5.1.5. *For any Albert lattice $J \in \mathcal{J}$ and any polynomial $P \in \mathrm{V}_n(\mathrm{J}_{\mathbb{C}})$, the (weighted) theta series*

$$\vartheta_{J,P}(z) := \sum_{T\in J^+} \mathrm{a}_J(T)P(T)q^{\mathrm{Tr}(T)},\ z \in \mathcal{H}, q = e^{2\pi i}, \tag{5.3}$$

is a modular form on $\mathbf{SL}_2(\mathbb{Z})$ of weight $2n+12$, and it is cuspidal if P is not constant.

Proof. Since the theta series (5.3) is invariant under the $\mathbf{F}_4(\mathbb{R})$-action on the pair (J, P) in the sense that $\vartheta_{gJ,gP} = \vartheta_{J,P}$, it suffices to prove the modularity for $J \in \{\mathrm{J}_{\mathbb{Z}}, \mathrm{J}_{\mathrm{E}}\}$. Here we give the proof for $J = \mathrm{J}_{\mathbb{Z}}$, and that for J_{E} is almost the same.

Let $\alpha : \mathcal{J} \to \mathrm{V}_n(\mathrm{J}_{\mathbb{C}})$ be the element in $\mathcal{A}_{\mathrm{V}_n(\mathrm{J}_{\mathbb{C}})}(\mathbf{F}_4)$ that is supported on the $\mathbf{F}_4(\mathbb{R})$-orbit of $\mathrm{J}_{\mathbb{Z}}$ and takes the value $\sum_{\gamma\in\Gamma_{\mathrm{I}}} \gamma.P$ at $\mathrm{J}_{\mathbb{Z}} \in \mathcal{J}$. By Theorem 5.1.2 and Remark 5.1.3, $f_{\Theta(\alpha)}$ is a modular form on $\mathbf{SL}_2(\mathbb{Z})$ of weight $2n + 12$. On the other hand, $\mathrm{J}_{\mathbb{Z}}^+$ is stable under the action of Γ_{I}, thus one has:

$$\begin{aligned} f_{\Theta(\alpha)}(z) &= \frac{1}{|\Gamma_{\mathrm{I}}|}\sum_{T\in\mathrm{J}_{\mathbb{Z}}^+} \mathrm{a}_{\mathrm{J}_{\mathbb{Z}}}(T)\left(\sum_{\gamma\in\Gamma_{\mathrm{I}}} P(\gamma^{-1}T)\right)q^{\mathrm{Tr}(T)} \\ &= \frac{1}{|\Gamma_{\mathrm{I}}|}\sum_{\gamma\in\Gamma_{\mathrm{I}}}\left(\sum_{T\in\mathrm{J}_{\mathbb{Z}}} \mathrm{a}_{\mathrm{J}_{\mathbb{Z}}}(\gamma T)P(T)q^{\mathrm{Tr}(\gamma T)}\right) \\ &= \vartheta_{\mathrm{J}_{\mathbb{Z}},P}(z) \end{aligned} \qquad \square$$

If we view $\alpha \in \mathcal{A}_{\mathrm{V}_{n\varpi_4}}(\mathbf{F}_4)$ as a function $\alpha : \mathcal{J} \to \mathrm{V}_n(\mathrm{J}_{\mathbb{C}})$, the modular form $f_{\Theta(\alpha)}$ can be written in the following forms:

$$f_{\Theta(\alpha)} = \frac{1}{|\Gamma_{\mathrm{I}}|}\vartheta_{\mathrm{J}_{\mathbb{Z}},\alpha(\mathrm{J}_{\mathbb{Z}})} + \frac{1}{|\Gamma_{\mathrm{E}}|}\vartheta_{\mathrm{J}_{\mathrm{E}},\alpha(\mathrm{J}_{\mathrm{E}})}.$$

5.2 Theta series attached to $\mathbf{Spin}_9(\mathbb{R})$-invariant polynomials

As an application of Theorem 5.1.2, we are going to show that for every weight k with $\mathrm{S}_k(\mathbf{SL}_2(\mathbb{Z})) \neq 0$, there exists a polynomial $P \in \mathrm{V}_{\frac{k-12}{2}}(\mathrm{J}_{\mathbb{C}})$ such that the weighted theta series $\vartheta_{\mathrm{J}_{\mathbb{Z}},P}$ defined as (5.3) is non-zero. This result will be used later in §6.4.

The $\mathrm{F}_4 \downarrow \mathrm{B}_4$ branching law [Lep70, §2, Theorem 7] says that $\dim \mathrm{V}_{n\varpi_4}^{\mathbf{Spin}_9(\mathbb{R})} = 1$ for any $n > 0$, where $\mathbf{Spin}_9$ is defined as the stabilizer of $\mathrm{E}_1 = [1, 0, 0\,; 0, 0, 0]$ in $\mathbf{F}_4$, thus the $\mathbf{Spin}_9(\mathbb{R})$-invariant polynomial in $\mathrm{V}_n(\mathrm{J}_{\mathbb{C}})$ is unique up to a non-zero scalar.

Theorem 5.2.1. *For $n \geq 2$ and any non-zero polynomial $P \in \mathrm{V}_n(\mathrm{J}_\mathbb{C})^{\mathbf{Spin}_9(\mathbb{R})}$, the weighted theta series $\vartheta_{\mathrm{J}_\mathbb{Z},P}$ is non-zero.*

Proof. We first construct an explicit polynomial $P_n \in \mathrm{V}_n(\mathrm{J}_\mathbb{C})^{\mathbf{Spin}_9(\mathbb{R})}$. In the real definite octonion algebra $\mathbb{O}_\mathbb{R}$, we pick three purely imaginary elements x_0, y_0, z_0 such that $\mathbb{R} \oplus \mathbb{R}x_0 \oplus \mathbb{R}y_0 \oplus \mathbb{R}z_0$ is isomorphic to Hamilton's quaternion algebra, *i.e.*

$$x_0^2 = y_0^2 = z_0^2 = -1 \text{ and } x_0 y_0 = -y_0 x_0 = z_0.$$

Take $x_1 = x_0$, $y_1 = \sqrt{-2}y_0$ and $z_1 = \sqrt{-2}z_0$, and choose $B = [2, -1, -1\,; x_1, y_1, z_1] \in \mathrm{J}_\mathbb{C}$. It can be easily verified that $B \in \mathbb{X}$, thus the polynomial $Q_n(X) := (\mathrm{Tr}(X \circ B))^n = (X, B)^n$ lies in $\mathrm{V}_n(\mathrm{J}_\mathbb{C})$, and take $P_n(X) := \int_{\mathbf{Spin}_9(\mathbb{R})} k.Q_n(X)dk = \int_{\mathbf{Spin}_9(\mathbb{R})} (X, kB)^n dk$ to be the average of Q_n over $\mathbf{Spin}_9(\mathbb{R})$. Now it suffices to show that the associated theta series $\vartheta_{\mathrm{J}_\mathbb{Z},P_n} \neq 0$.

Consider the first Fourier coefficient a_1 of $\vartheta_{\mathrm{J}_\mathbb{Z},P_n}$. The elements in $\mathrm{J}_\mathbb{Z}^+$ having contributions to the coefficient of q are E_1, E_2 and E_3, thus:

$$a_1 = \sum_{i=1}^{3} P_n(\mathrm{E}_i) = \int_{\mathbf{Spin}_9(\mathbb{R})} \left(\sum_{i=1}^{3} (\mathrm{E}_i, kB)^n \right) dk. \tag{5.4}$$

By Lemma 2.5.1, $\mathbf{Spin}_9(\mathbb{R})$ preserves the subspaces $\mathrm{J}_1 = \{[0, \xi, -\xi\,; x, 0, 0] \,|\, \xi \in \mathbb{R}, x \in \mathbb{O}_\mathbb{R}\}$ and $\mathrm{J}_2 = \{[0, 0, 0\,; 0, y, z] \,|\, y, z \in \mathbb{O}_\mathbb{R}\}$ respectively. So for any $k \in \mathbf{Spin}_9(\mathbb{R})$ we set:

$$\begin{aligned} k[0,0,0\,;x_1,0,0] &= [0, \xi(k), -\xi(k)\,; x(k), 0, 0] \in \mathrm{J}_1, \\ k[0,0,0\,;0,y_1,z_1] &= [0,0,0\,;0,y(k),z(k)] \in \mathrm{J}_2 \otimes \mathbb{C}. \end{aligned}$$

We have the equality $2\xi(k)^2 + \langle x(k), x(k) \rangle = \langle x_1, x_1 \rangle = 2$, as k preserves the inner product on $\mathrm{J}_\mathbb{R}$, which implies that $|\xi(k)| \leq 1$. The three diagonal entries of kB are $2, -1 + \xi(k)$ and $-1 - \xi(k)$, thus $\sum_{i=1}^{3} (\mathrm{E}_i, kB)^n = 2^n + (-1 + \xi(k))^n + (-1 - \xi(k))^n \in \mathbb{R}_{\geq 0}$. When we take $k = 1$, $\sum_{i=1}^{3} (\mathrm{E}_i, B)^n = 2^n + (-1)^n + (-1)^n$ is positive for any $n \geq 2$. Hence the integral in (5.4) is strictly positive, and as a consequence the weighted theta series $\vartheta_{\mathrm{J}_\mathbb{Z},P_n}$ is non-zero. $\square$

5.3 Proof of Theorem 5.1.4

In this section, we will prove Proposition 5.1.1 and Theorem 5.1.4, following a similar strategy to that of Pollack in [Pol23, §6].

We first define a basis $\{\mathrm{X}_\alpha\}_\alpha$ of $\mathfrak{p}_\mathrm{J}^+$ as follows: for any $A \in \mathrm{J}_\mathbb{C}$, write $\mathrm{X}_A := \mathrm{X}_A^+ = i\mathrm{C}_h^{-1}\mathrm{n}_\mathrm{L}(A)\mathrm{C}_h$ as in §2.4.2, which is an element of $\mathfrak{p}_\mathrm{J}^+$ by Proposition 2.4.6. Choose a $\mathbb{C}$-basis $\{e_1, \ldots, e_{27}\}$ of $\mathrm{J}_\mathbb{C}$, then we have a basis $\{\mathrm{X}_{e_i}\}_{1 \leq i \leq 27}$ of $\mathfrak{p}_\mathrm{J}^+$, and we denote its dual basis by $\{\mathrm{X}_{e_i}^\vee\}_{1 \leq i \leq 27}$. In [Pol23, §6.2], Pollack calculates the action of $\mathrm{X}_{A_n} \cdots \mathrm{X}_{A_1}$ on $h_T|_{\mathbf{M}_\mathrm{J}(\mathbb{R})}$. Before recalling his result, we explain some notations that will appear in the statement.

Let $\mathrm{T}(\mathrm{J}_\mathbb{C}) = \bigoplus_{k=0}^{\infty} \mathrm{J}_\mathbb{C}^{\otimes k}$ be the tensor algebra of $\mathrm{J}_\mathbb{C}$. Define a family of $\mathbf{F}_4(\mathbb{R})$-equivariant maps $\mathscr{P}_k : \mathrm{J}_\mathbb{C}^{\otimes k} \to \mathrm{T}(\mathrm{J}_\mathbb{C})$ inductively:

- let $\mathscr{P}_0 = 1$ be the constant map;

- for $k \geq 0$, define [6]

$$\mathscr{P}_{k+1}(A_1 \otimes \cdots \otimes A_k \otimes A_{k+1}) = \mathscr{P}_k(A_1 \otimes \cdots \otimes A_k) \otimes A_{k+1} + 4\,\mathrm{Tr}(A_{k+1})\mathscr{P}_k(A_1 \otimes \cdots \otimes A_k) \\ + A_{k+1} \circ \mathscr{P}_k(A_1 \otimes \cdots \otimes A_k) + \mathscr{P}_k(A_{k+1} \circ (A_1 \otimes \cdots \otimes A_k)),$$

  where $A \circ (A_1 \otimes \cdots \otimes A_r) := \sum_{j=1}^r A_1 \otimes \cdots \otimes (A \circ A_j) \otimes \cdots \otimes A_r$.

For any $T \in \mathrm{J}_{\mathbb{R}}$ and $m \in \mathbf{M}_{\mathrm{J}}(\mathbb{R})$, we define a linear form $w_{T,m}$ on $\mathrm{T}(\mathrm{J}_{\mathbb{C}})$ by:

$$w_{T,m}(A_1 \otimes \cdots \otimes A_r) = (-4\pi)^r \prod_{j=1}^r (T, m(A_j)), \text{ for any } r \geq 0.$$

Proposition 5.3.1. *[Pol23, Proposition 6.2.2] Let the notations be as above, then for any $m \in \mathbf{M}_{\mathrm{J}}(\mathbb{R})$ and $A_1, \ldots, A_n \in \mathrm{J}_{\mathbb{C}}$, we have*

$$\mathrm{X}_{A_n} \cdots \mathrm{X}_{A_1} h_T(m) = w_{T,\lambda(m)m^*}(\mathscr{P}_n(A_1 \otimes \cdots \otimes A_n))h_T(m).$$

Remark 5.3.2. There is a slight mistake in [Pol23, Proposition 6.2.2], whose correct formula should be

$$\mathrm{X}_{A_n} \cdots \mathrm{X}_{A_1} h_T(M(\delta, m)) = w_{T,m}(\mathscr{P}_n(A_1 \otimes \cdots \otimes A_n))h_T(M(\delta, m)),$$

where $M(\delta, m)$ is the element of $\mathbf{M}_{\mathrm{J}}(\mathbb{R})$ such that $M(\delta, m)\mathrm{n}(A)M(\delta, m)^{-1} = \mathrm{n}(m(A))$.

Observe that $\mathscr{P}_n(A_1 \otimes \cdots \otimes A_n)$ is the sum of $A_1 \otimes \cdots \otimes A_n$ with tensors of smaller degrees. The following lemma enables us to consider only the leading term of $\mathscr{P}_n$.

Lemma 5.3.3. *Let P be an element in $\mathrm{V}_n(\mathrm{J}_{\mathbb{C}}) \simeq \mathrm{V}_{n\varpi_4}$, then:*

$$\sum_{i_1,\ldots,i_n} \mathscr{P}_n(e_{i_1} \otimes \cdots \otimes e_{i_n})\{\mathrm{X}^\vee_{e_{i_1}} \otimes \cdots \otimes \mathrm{X}^\vee_{e_{i_n}}, P\} = \sum_{i_1,\ldots,i_n} e_{i_1} \otimes \cdots \otimes e_{i_n}\{\mathrm{X}^\vee_{e_{i_1}} \otimes \cdots \otimes \mathrm{X}^\vee_{e_{i_n}}, P\}. \tag{5.5}$$

Proof. Since the pairing $\{-,-\}$ is $\mathbf{F}_4(\mathbb{R})$-invariant and $\mathscr{P}_n$ is $\mathbf{F}_4(\mathbb{R})$-equivariant, for any $g \in \mathbf{F}_4(\mathbb{R})$, we have:

$$\begin{aligned} &\sum_{i_1,\ldots,i_n} \mathscr{P}_n(e_{i_1} \otimes \cdots \otimes e_{i_n})\{\mathrm{X}^\vee_{e_{i_1}} \otimes \cdots \otimes \mathrm{X}^\vee_{e_{i_n}}, g.P\} \\ =& \sum_{i_1,\ldots,i_n} \mathscr{P}_n(e_{i_1} \otimes \cdots \otimes e_{i_n})\{\mathrm{X}^\vee_{g^{-1}.e_{i_1}} \otimes \cdots \otimes \mathrm{X}^\vee_{g^{-1}.e_{i_n}}, P\} \\ =& \sum_{i_1,\ldots,i_n} \mathscr{P}_n(g.e_{i_1} \otimes \cdots \otimes g.e_{i_n})\{\mathrm{X}^\vee_{e_{i_1}} \otimes \cdots \otimes \mathrm{X}^\vee_{e_{i_n}}, P\} \\ =& \sum_{i_1,\ldots,i_n} g.\mathscr{P}_n(e_{i_1} \otimes \cdots \otimes e_{i_n})\{\mathrm{X}^\vee_{e_{i_1}} \otimes \cdots \otimes \mathrm{X}^\vee_{e_{i_n}}, P\}. \end{aligned}$$

Comparing this with the right-hand side of (5.5), it suffices to prove (5.5) for one non-zero vector in $\mathrm{V}_{n\varpi_4}$, so we take P to be $(\mathrm{Tr}(X \circ A))^n \in \mathrm{V}_n(\mathrm{J}_{\mathbb{C}})$ for an arbitrary $A \in \mathbb{X}$, as explained in §4.1.2.

[6] In [Pol23, §6.2], the Jordan product $A \circ B$ is denoted by $\frac{1}{2}\{A, B\}$, where $\{A, B\} = AB + BA$ is defined in [Pol20, §3.3.1].

Both sides of (5.5) are independent of the choice of the basis $\{e_i\}_{1\leq i\leq 27}$ of $\mathrm{J}_{\mathbb{C}}$, thus we choose a specific basis $\{e_i\}_{1\leq i\leq 27}$ such that $e_1 = A$. With this choice, it suffices to prove $\mathscr{P}_n(e_1^{\otimes n}) = e_1^{\otimes n}$, which follows from the inductive definition of $\mathscr{P}_n$ and the fact that $\mathrm{Tr}(e_1) = 0$, $e_1 \circ e_1 = 0$. $\square$

Proposition 5.3.4. *For $m \in \mathbf{M}_{\mathrm{J}}(\mathbb{R})$ and $P \in \mathrm{V}_n(\mathrm{J}_{\mathbb{C}}) \simeq \mathrm{V}_{n\varpi_4}$, we have*

$$\{\mathrm{D}^n h_T(m), P\} = (-4\pi)^n P\left(\lambda(m)m^{-1}T\right) h_T(m).$$

Proof. Combining Proposition 5.3.1 and Lemma 5.3.3 together, we have:

$$\begin{aligned}
\{\mathrm{D}^n h_T(m), P\} &= \sum_{i_1,\dots,i_n} \mathrm{X}_{e_{i_n}} \cdots \mathrm{X}_{e_{i_1}} h_T(m) \left\{\mathrm{X}_{e_{i_1}}^{\vee} \otimes \cdots \otimes \mathrm{X}_{e_{i_n}}^{\vee}, P\right\} \\
&= \sum_{i_1,\dots,i_n} w_{T,\lambda(m)m^*}(\mathscr{P}_n(e_{i_1} \otimes \cdots \otimes e_{i_n}))h_T(m) \left\{\mathrm{X}_{e_{i_1}}^{\vee} \otimes \cdots \otimes \mathrm{X}_{e_{i_n}}^{\vee}, P\right\} \\
&= h_T(m) \sum_{i_1,\dots,i_n} w_{T,\lambda(m)m^*}(e_{i_1} \otimes \cdots \otimes e_{i_n}) \left\{\mathrm{X}_{e_{i_1}}^{\vee} \otimes \cdots \otimes \mathrm{X}_{e_{i_n}}^{\vee}, P\right\} \\
&= (-4\pi)^n h_T(m) \sum_{i_1,\dots,i_n} \left(\prod_{j=1}^{n} (T, \lambda(m)m^*(e_{i_j}))\right) \left\{\mathrm{X}_{e_{i_1}}^{\vee} \otimes \cdots \otimes \mathrm{X}_{e_{i_n}}^{\vee}, P\right\} \\
&= (-4\pi)^n h_T(m) \sum_{i_1,\dots,i_n} \left(\prod_{j=1}^{n} \left(\lambda(m)m^{-1}T, e_{i_j}\right)\right) \left\{\mathrm{X}_{e_{i_1}}^{\vee} \otimes \cdots \otimes \mathrm{X}_{e_{i_n}}^{\vee}, P\right\} \\
&= (-4\pi)^n h_T(m) \left\{\left(\lambda(m)m^{-1}T\right)^{\otimes n}, P\right\} \\
&= (-4\pi)^n P\left(\lambda(m)m^{-1}T\right) h_T(m).
\end{aligned}$$ $\square$

To prove Theorem 5.1.4, we use the Iwasawa decomposition to write $g \in \mathbf{SL}_2(\mathbb{R})$ as:

$$g = tnk, \text{ where } t = \left(\begin{smallmatrix} u & \\ & u^{-1} \end{smallmatrix}\right), \; n = \left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right), \; k = \left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right).$$

By a direct calculation, we have the following:

Lemma 5.3.5. *For $A_1, \dots, A_n \in \mathrm{J}_{\mathbb{C}}$, we have the following identities:*

(1) $\mathrm{X}_{A_n} \cdots \mathrm{X}_{A_1} h_T(m\mathrm{n}(A)) = e^{2\pi i (T, \lambda(m)m^* A)} \mathrm{X}_{A_n} \cdots \mathrm{X}_{A_1} h_T(m)$, $\forall A \in \mathrm{J}_{\mathbb{C}}, m \in \mathbf{M}_{\mathrm{J}}(\mathbb{R})$;
(2) $\mathrm{X}_{A_n} \cdots \mathrm{X}_{A_1} h_T(gk) = J(k, i\mathrm{I})^{-4} (k.\mathrm{X}_{A_n}) \cdots (k.\mathrm{X}_{A_1}) h_T(g)$, $\forall k \in \mathrm{K}_{\mathrm{E}_7}, g \in \mathbf{H}_{\mathrm{J}}^1(\mathbb{R})$.

Proof of Theorem 5.1.4. Let the notations be as above. By Lemma 5.3.5, we have:

$$\begin{aligned}
\mathrm{D}^n h_T(g) &= \mathrm{D}^n h_T(tnk) \\
&= J(k, i\mathrm{I})^{-4} \sum_{i_1,\dots,i_n} (k.\mathrm{X}_{e_{i_1}} \cdots k.\mathrm{X}_{e_{i_n}}) h_T(tn) \otimes \mathrm{X}_{e_{i_1}}^{\vee} \otimes \cdots \otimes \mathrm{X}_{e_{i_n}}^{\vee} \\
&= J(k, i\mathrm{I})^{-4} e^{2\pi i (T, u^2 x \mathrm{I})} \sum_{i_1,\dots,i_n} (k.\mathrm{X}_{e_{i_1}} \cdots k.\mathrm{X}_{e_{i_n}}) h_T(t) \otimes \mathrm{X}_{e_{i_1}}^{\vee} \otimes \cdots \otimes \mathrm{X}_{e_{i_n}}^{\vee}. \\
&= j(k, i)^{-2n-12} e^{2\pi i (T, u^2 x \mathrm{I})} \cdot \mathrm{D}^n h_T(t),
\end{aligned}$$

where the last equality follows from $k.\mathrm{X}_A = (\cos\theta + i\sin\theta)^2\mathrm{X}_A = j(k,i)^{-2}\mathrm{X}_A$. Now we take the pairing of $\mathrm{D}^n h_T(g)$ with P, and use Proposition 5.3.4 to obtain the desired identity:

$$\begin{aligned}\{\mathrm{D}^n h_T(g), P\} &= j(k,i)^{-2n-12}e^{2\pi i(T,u^2x\mathrm{I})}(-4\pi)^n P\left(u^2T\right) J(t,i\mathrm{I})^{-4}e^{2\pi i(T,t.i\mathrm{I})}\\ &= (-4\pi)^n j(k,i)^{-2n-12}j(t,i)^{-12}u^{2n}P(T)e^{2\pi i(T,t.(i\mathrm{I}+x\mathrm{I}))}\\ &= (-4\pi)^n j(g,i)^{-2n-12}P\left(T\right)e^{2\pi i(T,g.i\mathrm{I})}.\end{aligned}$$

□

Proof of Proposition 5.1.1. To show that $f_{\Theta(\alpha)}(z) := j(g,i)^{2n+12}\Theta(\alpha)(g)$ is well-defined, it suffices to verify that for k in the maximal compact subgroup of $\mathbf{SL}_2(\mathbb{R})$, we have:

$$\Theta(\alpha)(gk) = j(k,i)^{-2n-12}\Theta(\alpha)(g), \text{ for any } g\in\mathbf{SL}_2(\mathbb{R}).$$

This follows from Lemma 5.3.5 and the identity $k.\mathrm{X}_A = j(k,i)^{-2}\cdot\mathrm{X}_A$. By the definition of $\Theta(\alpha)$ and Proposition 3.3.1, $f_{\Theta(\alpha)}$ is a level one holomorphic modular form with weight $2n+12$, and when $n>0$ it is a cusp form. □

6 Global theta lifts from $\mathbf{PGL}_2$ to $\mathbf{F}_4$

We look at the other direction of the global theta correspondence, *i.e.* from $\mathbf{PGL}_2$ to $\mathbf{F}_4$. Let $\pi\simeq\otimes'_v\pi_v$ be a level one algebraic cuspidal automorphic representation of $\mathbf{PGL}_2$ associated to a Hecke eigenform of $\mathbf{SL}_2(\mathbb{Z})$ with weight $2n+12$, $n>0$. We take an automorphic form $\varphi\in\pi$ corresponding to $\otimes'\varphi_v$ under the isomorphism $\pi\simeq\otimes'\pi_v$, such that:

- φ_∞ is the unique lowest weight holomorphic vector in the discrete series representation $\mathcal{D}(2n+12)$ of $\mathbf{PGL}_2(\mathbb{R})$;
- for each prime p, φ_p is chosen to be the normalized spherical vector in the principal series representation π_p of $\mathbf{PGL}_2(\mathbb{Q}_p)$.

Our goal is to prove $\Theta(\pi)\neq 0$. In other words, we need to find a vector $\phi\in\Pi_{\min}$ such that $\Theta_\phi(\varphi)\neq 0$. The strategy is to calculate the $\mathbf{Spin}_9$*-period* of the global theta lift $\Theta_\phi(\varphi)$:

$$\mathcal{P}_{\mathbf{Spin}_9}\left(\Theta_\phi(\varphi)\right) := \int_{[\mathbf{Spin}_9]}\Theta_\phi(\varphi)(g)dg.$$

As stated in Remark 1.3.1, one motivation for considering this period integral is the conjecture of Sakellaridis-Venkatesh.

Plugging the definition of the global theta lift $\Theta_\phi(\varphi)$ in this period integral and changing the order of integration, we obtain:

$$\begin{aligned}\mathcal{P}_{\mathbf{Spin}_9}(\Theta_\phi(\varphi)) &= \int_{[\mathbf{Spin}_9]}\int_{[\mathbf{PGL}_2]}\theta(\phi)\left(gh\right)\overline{\varphi(h)}dhdg\\ &= \int_{[\mathbf{PGL}_2]}\overline{\varphi(h)}\left(\int_{[\mathbf{Spin}_9]}\theta(\phi)\left(gh\right)dg\right)dh.\end{aligned}\tag{6.1}$$

6.1 Exceptional Siegel-Weil formula

The integral $\int_{[\mathbf{Spin}_9]} \theta(\phi)(gh)dg$ appearing in (6.1), as a function of $h \in \mathbf{SO}_{2,2}(\mathbb{A})$, is the global theta lift of the constant function on $[\mathbf{Spin}_9]$ to $\mathbf{SO}_{2,2}$. In this section, we will prove an *exceptional Siegel-Weil formula* for $\mathbf{Spin}_9 \times \mathbf{SO}_{2,2}$, which represents this theta lift as an Eisenstein series on $\mathbf{SO}_{2,2}$.

Definition 6.1.1. Let $\mathbf{B} = \mathbf{TN}$ be the Borel subgroup of

$$\mathbf{SO}_{2,2} = \mathbf{GSpin}_{2,2}/\mathbf{G}_{\mathrm{m}} = \{(g_1, g_2) \in \mathbf{GL}_2 \times \mathbf{GL}_2 \,|\, \det g_1 = \det g_2\} / \mathbf{G}_{\mathrm{m}}^{\Delta}$$

consisting of the equivalence classes of (g_1, g_2), where $g_1$and g_2 are upper triangular matrices. For $s_1, s_2 \in \mathbb{C}$, we define a character χ_{s_1,s_2} on $\mathbf{T}(\mathbb{A})$ by:

$$\chi_{s_1,s_2}\left(\left(\begin{smallmatrix} a_1 & \\ & b_1 \end{smallmatrix}\right), \left(\begin{smallmatrix} a_2 & \\ & b_2 \end{smallmatrix}\right)\right) := |a_1/b_1|^{\frac{s_1}{2}} \cdot |a_2/b_2|^{\frac{s_2}{2}},$$

and define $\mathrm{I}(s_1, s_2)$ to be the (normalized) degenerate principal series $\mathrm{Ind}_{\mathbf{B}(\mathbb{A})}^{\mathbf{SO}_{2,2}(\mathbb{A})} \chi_{s_1,s_2}$.

By Proposition 3.1.2, we identify the (adelic) minimal representation $\Pi_{\min}$ of $\mathbf{E}_7(\mathbb{A})$ as a subrepresentation of $\mathrm{Ind}_{\mathbf{P}_\mathrm{J}(\mathbb{A})}^{\mathbf{E}_7(\mathbb{A})} \delta_{\mathbf{P}_\mathrm{J}}^{-1/2} |\lambda|^2$.

Lemma 6.1.2. *The restriction of sections gives a morphism* $\mathrm{Ind}_{\mathbf{P}_\mathrm{J}(\mathbb{A})}^{\mathbf{E}_7(\mathbb{A})} \delta_{\mathbf{P}_\mathrm{J}}^{-1/2} |\lambda|^2 \to \mathrm{I}(3, 7)$.

Proof. A section $f \in \mathrm{Ind}_{\mathbf{P}_\mathrm{J}(\mathbb{A})}^{\mathbf{E}_7(\mathbb{A})} \delta_{\mathbf{P}_\mathrm{J}}^{-1/2} |\lambda|^2$ satisfies the functional equation (3.1). Combining the explicit morphisms (2.4) and (2.8), the image of $\left(\left(\begin{smallmatrix} a_1 & \\ & b_1 \end{smallmatrix}\right), \left(\begin{smallmatrix} a_2 & \\ & b_2 \end{smallmatrix}\right)\right) \in \mathbf{T}(\mathbb{A})$ in $\mathbf{M}_\mathrm{J} \subseteq \mathbf{E}_7$ has similitude $(a_1/b_1) \cdot (a_2/b_2)^2$, thus the restriction of f to $\mathbf{SO}_{2,2}(\mathbb{A})$ satisfies:

$$f(tng) = \chi_{4,8}(t) f(g), \text{ for any } t \in \mathbf{T}(\mathbb{A}), n \in \mathbf{N}(\mathbb{A}), g \in \mathbf{SO}_{2,2}(\mathbb{A}).$$

This shows that $f|_{\mathbf{SO}_{2,2}(\mathbb{A})}$ is a section of $\mathrm{Ind}_{\mathbf{B}(\mathbb{A})}^{\mathbf{SO}_{2,2}(\mathbb{A})} \delta_{\mathbf{B}}^{-1/2} \chi_{4,8} = \mathrm{I}(3, 7)$. $\square$

Lemma 6.1.2 gives us a $\mathbf{SO}_{2,2}(\mathbb{A})$-equivariant map:

$$\mathrm{Res} : \Pi_{\min} \hookrightarrow \mathrm{Ind}_{\mathbf{P}_\mathrm{J}(\mathbb{A})}^{\mathbf{E}_7(\mathbb{A})} \delta_{\mathbf{P}_\mathrm{J}}^{-1/2} |\lambda|^2 \to \mathrm{I}(3, 7).$$

Given a smooth vector $\phi \in \Pi_{\min}$, we have the following two automorphic forms on $\mathbf{SO}_{2,2}$:

- The theta integral:

$$\Theta_\phi(1)(g) = \int_{[\mathbf{Spin}_9]} \theta(\phi)(gh)dh, \text{ for any } g \in \mathbf{SO}_{2,2}(\mathbb{A}),$$

- The Eisenstein series associated to $\widetilde{\phi} := \mathrm{Res}(\phi) \in \mathrm{I}(3, 7)$:

$$E(\widetilde{\phi})(g) := \sum_{\gamma \in \mathbf{B}(\mathbb{Q}) \backslash \mathbf{SO}_{2,2}(\mathbb{Q})} \widetilde{\phi}(\gamma g), \text{ for any } g \in \mathbf{SO}_{2,2}(\mathbb{A}).$$

Theorem 6.1.3. *Let* $\Phi_f := \otimes_p \Phi_p$ *be the normalized spherical vector in* $\Pi_{\min,f}$ *chosen in §4.2, then for any smooth holomorphic vector* $\phi_\infty \in \Pi_{\min,\infty}$*, up to some scalar we have:*

$$E(\mathrm{Res}(\phi_\infty \otimes \Phi_f)) = \Theta_{\phi_\infty \otimes \Phi_f}(1).$$

Before proving this formula for any smooth vector $\phi_\infty \in \Pi_{\min,\infty}$, we verify it for the specific vector Φ_∞ chosen in §4.2.

Proposition 6.1.4. *For the vector $\Phi_0 = \Phi_\infty \otimes \Phi_f \in \Pi_{\min}$, up to some scalar we have:*

$$E(\mathrm{Res}(\Phi_0)) = \Theta_{\Phi_0}(1).$$

Proof. By the choice of Φ_0, $\mathrm{Res}(\Phi_0)_p$ is the normalized spherical vector of $\mathrm{I}(3,7)_p$ for each prime p, and $\mathrm{Res}(\Phi_0)_\infty$ is the unique holomorphic vector in $\mathrm{I}(3,7)_\infty$ with minimal $\mathrm{K}_{\mathrm{E}_7} \cap \mathbf{Spin}_{2,2}(\mathbb{R})$-type. As a result, the Eisenstein series $E(\mathrm{Res}(\Phi_0))$ is a non-zero multiple of the automorphic form associated to $E_4 \boxtimes E_8$, where E_k is the normalized holomorphic Eisenstein series in $\mathrm{M}_k(\mathbf{SL}_2(\mathbb{Z}))$.

On the other side, the global theta lift is a non-zero multiple of

$$(g_1, g_2) \in \mathbf{SO}_{2,2}(\mathbb{A}) \mapsto j(g_{1,\infty})^{-4} j(g_{2,\infty})^{-8} \mathrm{F}_{Kim}\left(\mathrm{diag}(g_{1,\infty}.i, g_{2,\infty}.i, g_{2,\infty}.i)\right),$$

where $(g_{1,\infty}, g_{2,\infty}) \in \mathbf{Spin}_{2,2}(\mathbb{R})$ is the archimedean component of (g_1, g_2) (up to some left translation by $\mathbf{SO}_{2,2}(\mathbb{Q})$). It suffices to show that $\mathrm{F}_{Kim}(\mathrm{diag}(z_1, z_2, z_2))$, as a function on $\mathcal{H} \times \mathcal{H}$, is a non-zero multiple of $E_4(z_1)E_8(z_2)$.

Since the space of modular forms $\mathrm{M}_k(\mathbf{SL}_2(\mathbb{Z}))$, $k = 4$ or 8, is 1-dimensional and spanned by E_k, it suffices to show that as a function for the variable z_1 (*resp.* z_2), $\mathrm{F}_{Kim}(\mathrm{diag}(z_1, z_2, z_2))$ is a modular form of weight 4 (*resp.* 8). The only hard part in the proof of the modularity is to show that

$$z_1^{-4}\mathrm{F}_{Kim}(\mathrm{diag}(-1/z_1, z_2, z_2)) = \mathrm{F}_{Kim}(\mathrm{diag}(z_1, z_2, z_2)) = z_2^{-8}\mathrm{F}_{Kim}(\mathrm{diag}(z_1, -1/z_2, -1/z_2)).$$

We only give the proof for the first equality here, and the second one can be proved similarly. From the explicit actions on $\mathcal{H}_\mathrm{J}$ given in Example 4.2.3, we have

$$\mathrm{diag}(-1/z_1, z_2, z_2) = \left(\mathrm{n}(\mathrm{E}_1) \cdot \iota \cdot \mathrm{n}(\mathrm{E}_1) \cdot \iota \cdot \mathrm{n}(\mathrm{E}_1)\right).\mathrm{diag}(z_1, z_2, z_2),$$

then the desired functional equation is implied by the modularity of F_{Kim}:

$$\begin{aligned}
&\mathrm{F}_{Kim}(\mathrm{diag}(-1/z_1, z_2, z_2))\\
=&J(\iota, \mathrm{diag}(z_1/(z_1+1), -1/z_2, -1/z_2))J(\iota^{-1}, \mathrm{diag}(z_1+1, z_2, z_2))\mathrm{F}_{Kim}(\mathrm{diag}(z_1, z_2, z_2))\\
=&\left(\frac{z_1}{(z_1+1)z_2^2}\right)^4 \cdot \left(-\left(z_1+1\right)z_2^2\right)^4 \cdot \mathrm{F}_{Kim}(\mathrm{diag}(z_1, z_2, z_2))\\
=&z_1^4\mathrm{F}_{Kim}(\mathrm{diag}(z_1, z_2, z_3)).
\end{aligned}$$

□

Proof of Theorem 6.1.3. For a smooth vector $\phi_\infty \in \Pi^+ \subseteq \Pi_{\min,\infty}$ whose restriction $\mathrm{Res}(\phi_\infty \otimes \Phi_f)$ to $\mathbf{SO}_{2,2}(\mathbb{A})$ vanishes, we know from Proposition 3.3.3 that it is orthogonal to the space $(\Pi^+)^{\mathbf{Spin}_9(\mathbb{R})}$, thus the theta lift $\Theta_{\phi_\infty\otimes\Phi_f}(1) = 0$.

Now we can assume that the smooth vector $\phi_\infty \in (\Pi^+)^{\mathbf{Spin}_9(\mathbb{R})}$ lies in the $\mathbf{Spin}_{2,2}(\mathbb{R})$-orbit of Φ_∞, then the theorem follows from Proposition 6.1.4 and the fact that the maps $E(\mathrm{Res}(-))$ and $\Theta_-(1)$ are both $\mathbf{SO}_{2,2}(\mathbb{A})$-equivariant. □

6.2 Unfolding the period integral

Take the smooth vector $\phi \in \Pi_{\min}$ to be $\phi_\infty \otimes \Phi_f$, where Φ_f is the normalized spherical vector and ϕ_∞ is a vector in $\Pi^+ \subseteq \Pi_{\min,\infty}$ such that $\widetilde{\phi} := \mathrm{Res}(\phi) \in \mathrm{I}(3,7)$ is non-zero. Using the Siegel-Weil formula Theorem 6.1.3 for $\mathbf{Spin}_9 \times \mathbf{SO}_{2,2}$, we write the period integral (6.1) as a Rankin-Selberg type integral and unfold it:

$$\begin{aligned}
\mathcal{P}_{\mathbf{Spin}_9}(\Theta_\phi(\varphi)) &= \int_{[\mathbf{PGL}_2]} \overline{\varphi(h)} E(\mathrm{Res}(\phi))(h^\Delta) dh \\
&= \int_{[\mathbf{PGL}_2]} \overline{\varphi(h)} \sum_{\mathbf{B}(\mathbb{Q})\backslash \mathbf{SO}_{2,2}(\mathbb{Q})} \widetilde{\phi}\left(\gamma h^\Delta\right) dh \\
&= \sum_{\gamma \in \mathbf{B}(\mathbb{Q})\backslash \mathbf{SO}_{2,2}(\mathbb{Q})/\mathbf{PGL}_2^\Delta(\mathbb{Q})} \int_{{}^\gamma\mathbf{G}(\mathbb{Q})\backslash \mathbf{PGL}_2(\mathbb{A})} \widetilde{\phi}(\gamma h^\Delta)\overline{\varphi(h)} dh,
\end{aligned} \tag{6.2}$$

where h^Δ denotes the image of $h \in \mathbf{PGL}_2(\mathbb{A})$ under $\mathbf{PGL}_2(\mathbb{A}) \to \mathbf{SO}_{2,2}(\mathbb{A})$, and the reductive subgroup ${}^\gamma\mathbf{G}$ of $\mathbf{PGL}_2$ is defined to be $\mathbf{PGL}_2^\Delta \cap \gamma^{-1}\mathbf{B}\gamma$.

By an easy calculation of orbits, the double coset in the summation of (6.2) has two orbits, represented by $1 = \left(\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right), \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\right)$ and $\gamma_0 = (w_0, 1) := \left(\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right), \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\right)$ respectively. For the first orbit, ${}^1\mathbf{G} = \mathbf{B}_0 = \mathbf{T}_0\mathbf{N}_0$ is the standard Borel subgroup of $\mathbf{PGL}_2$, and its contribution to the Rankin-Selberg integral (6.2) is zero since φ is cuspidal. For the second orbit, ${}^{\gamma_0}\mathbf{G} = \mathbf{T}_0$ is the maximal torus consisting of diagonal matrices, thus we have:

$$\mathcal{P}_{\mathbf{Spin}_9}(\Theta_\phi(\varphi)) = \int_{\mathbf{T}_0(\mathbb{Q})\backslash \mathbf{PGL}_2(\mathbb{A})} \widetilde{\phi}(\gamma_0 g^\Delta)\overline{\varphi(g)} dg. \tag{6.3}$$

Before calculating this integral, we make some normalization on the measure dg of $\mathbf{PGL}_2(\mathbb{A})$:

Notation 6.2.1. Fix a Haar measure dx on $\mathbb{Q}_p$ such that $dx(\mathbb{Z}_p) = 1$, and let $d^\times t$ be the Haar measure $(1-p^{-1})^{-1} \cdot \frac{dt}{|t|}$ on $\mathbb{Q}_p^\times$ so that $d^\times t(\mathbb{Z}_p^\times) = 1$. We choose the following left-invariant Haar measure db on $\mathbf{B}_0(\mathbb{Q}_p)$:

$$db := d^\times t dx = \frac{dtdx}{|t|}, \text{ for } b = \begin{pmatrix} t & \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \in \mathbf{B}_0(\mathbb{Q}_p).$$

On the hyperspecial subgroup $\mathbf{PGL}_2(\mathbb{Z}_p)$, we choose the invariant Haar measure dk such that the volume of $\mathbf{PGL}_2(\mathbb{Z}_p)$ is 1. Via the Iwasawa decomposition, we give $\mathbf{PGL}_2(\mathbb{Q}_p)$ the product measure $dg_p = dbdk$, which makes $\mathbf{PGL}_2(\mathbb{Z}_p)$ have measure 1. Take a non-trivial invariant Haar measure dg_∞ on $\mathbf{PGL}_2(\mathbb{R})$ and set $dg = \otimes_v' dg_v$.

The first step to calculate (6.3) is to rewrite it as an Euler product, for which we need the following:

Definition 6.2.2. Fix a non-trivial continuous unitary character $\psi = \psi_\infty \otimes \psi_f = \otimes_v \psi_v$ of $\mathbb{Q}\backslash\mathbb{A}$ such that the conductor of ψ_p is $\mathbb{Z}_p$ for each p and $\psi_\infty(x) = e^{2\pi i x}$ for all $x \in \mathbb{R}$. The *ψ-Whittaker coefficient* of $\varphi \in \mathcal{A}_{\mathrm{cusp}}(\mathbf{PGL}_2)$ is defined to be:

$$W_{\varphi,\psi}(g) := \int_{[\mathbf{N}_0]} \varphi(ng)\psi^{-1}(n) dn.$$

The global Whittaker function $W_{\varphi,\psi}$ satisfies $W_{\varphi,\psi}(ng) = \psi(n)W_{\varphi,\psi}(g)$ for any $g \in \mathbf{PGL}_2(\mathbb{A})$ and $n \in \mathbf{N}_0(\mathbb{A})$, and it factors as $W_{\varphi,\psi}(g) = \prod_v W_{\varphi_v,\psi_v}(g_v)$ [Cog04, Corollary 4.1.3], where W_{φ_p,ψ_p} is a spherical Whittaker function on $\mathbf{PGL}_2(\mathbb{Q}_p)$. We normalize the spherical vector $\varphi_p \in \pi_p$ so that $W_{\varphi_p,\psi_p}|_{\mathbf{PGL}_2(\mathbb{Z}_p)} = 1$.

Expanding the automorphic form φ along $\mathbf{N}_0$, the right-hand side of (6.3) becomes:

$$\int_{\mathbf{T}_0(\mathbb{Q})\backslash\mathbf{PGL}_2(\mathbb{A})} \widetilde{\phi}(\gamma_0 g^\Delta)\overline{\sum_{a\in\mathbb{Q}^\times} W_{\varphi,\psi}\left(\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix} g\right)}dg = \int_{\mathbf{PGL}_2(\mathbb{A})} \widetilde{\phi}(\gamma_0 g^\Delta)\overline{W_{\varphi,\psi}(g)}dg.$$

So far we have proved the following:

Proposition 6.2.3. *Let $\phi = \phi_\infty \otimes \Phi_f \in \Pi_{\min}$ be a smooth vector such that $\widetilde{\phi} = \mathrm{Res}(\phi) \neq 0$, then we have*

$$\mathcal{P}_{\mathbf{Spin}_9}(\Theta_\phi(\varphi)) = \int_{\mathbf{PGL}_2(\mathbb{A})} \widetilde{\phi}(\gamma_0 g^\Delta)\overline{W_{\varphi,\psi}(g)}dg = \prod_v I_v(\widetilde{\phi}_v, \varphi_v, \psi_v),$$

where the local zeta integral $I_v(\widetilde{\phi}_v, \varphi_v, \psi_v)$ is defined by:

$$I_v(\widetilde{\phi}_v, \varphi_v, \psi_v) := \int_{\mathbf{PGL}_2(\mathbb{Q}_v)} \widetilde{\phi}_v(\gamma_{0,v} g_v^\Delta)\overline{W_{\varphi_v,\psi_v}(g_v)}dg_v.$$

6.3 Unramified calculations

The goal of this section is to calculate the local zeta integral $I_p(\widetilde{\phi}_p, \varphi_p, \psi_p)$:

Proposition 6.3.1. *Let φ_p be the normalized spherical vector of the unramified principal series π_p of $\mathbf{PGL}_2(\mathbb{Q}_p)$ whose Satake parameter is $\begin{pmatrix} \alpha_p & \\ & \alpha_p^{-1}\end{pmatrix} \in \mathbf{SL}_2(\mathbb{C})_{\mathrm{ss}}$, and $\widetilde{\phi}_p = \mathrm{Res}(\Phi_p)$ the normalized spherical section of $\mathrm{I}(3,7)_p$, then we have:*

$$I_p(\widetilde{\phi}_p, \varphi_p, \psi_p) = \frac{(1-p^{-4})(1-p^{-8})}{(1-p^{-\frac{5}{2}}\alpha_p)(1-p^{-\frac{5}{2}}\alpha_p^{-1})(1-p^{-\frac{11}{2}}\alpha_p)(1-p^{-\frac{11}{2}}\alpha_p^{-1})}.$$

Proof. With the choice of measures in Notation 6.2.1, we write I_p as a double integral:

$$\begin{aligned} &I_p(\widetilde{\phi}_p, \varphi_p, \psi_p) \\ &= \int_{\mathbf{B_0}(\mathbb{Q}_p)}\int_{\mathbf{PGL}_2(\mathbb{Z}_p)} \widetilde{\phi}_p(\gamma_0 b^\Delta k^\Delta)\overline{W_{\varphi_p,\psi_p}(bk)}dbdk \\ &= \int_{\mathbb{Q}_p^\times}\int_{\mathbb{Q}_p} \widetilde{\phi}_p\left(\gamma_0 \begin{pmatrix} t & \\ & 1\end{pmatrix}^\Delta \begin{pmatrix} 1 & x \\ & 1\end{pmatrix}^\Delta\right)\overline{W_{\varphi_p,\psi_p}\left(\begin{pmatrix} t & \\ & 1\end{pmatrix}\begin{pmatrix} 1 & x \\ & 1\end{pmatrix}\right)}d^\times t dx \end{aligned} \tag{6.4}$$

As the normalized spherical section of $\mathrm{I}(3,7)_p$, $\widetilde{\phi}_p$ satisfies that:

$$\widetilde{\phi}_p\left(\gamma_0 \begin{pmatrix} t & \\ & 1\end{pmatrix}^\Delta \begin{pmatrix} 1 & x \\ & 1\end{pmatrix}^\Delta\right) = \begin{cases} |t|^2 & , x \in \mathbb{Z}_p \\ |t|^2 \cdot |x|^{-4} & , x \notin \mathbb{Z}_p \end{cases} \tag{6.5}$$

On the other hand, the values of the spherical Whittaker function W_{φ_p,ψ_p} comes from a standard result [Cog04, Proposition 7.4]:

$$W_{\varphi_p,\psi_p}\left(\begin{pmatrix} t & \\ & 1\end{pmatrix}\begin{pmatrix} 1 & x \\ & 1\end{pmatrix}\right) = \begin{cases} 0 & , t \notin \mathbb{Z}_p \\ p^{-n/2}\psi_p(tx)\cdot \frac{\alpha_p^{n+1}-\alpha_p^{-n-1}}{\alpha_p-\alpha_p^{-1}} & , t \in p^n\mathbb{Z}_p^\times \text{ for some } n \geq 0 \end{cases} \tag{6.6}$$

Plugging (6.5) and (6.6) into Equation (6.4), we have:

$$I_p(\widetilde{\phi}_p, \varphi_p, \psi_p) = \sum_{n=0}^{\infty} \int_{p^n\mathbb{Z}_p^\times} p^{-\frac{5}{2}n} \frac{\alpha_p^{n+1}-\alpha_p^{-n-1}}{\alpha_p-\alpha_p^{-1}} I_n(t) d^\times t \tag{6.7}$$

where

$$I_n(t) = \int_{\mathbb{Z}_p} \overline{\psi_p(tx)}dx + \int_{\mathbb{Q}_p\setminus\mathbb{Z}_p} |x|^{-4}\overline{\psi_p(tx)}dx = 1 + \sum_{m=1}^{\infty}\int_{p^{-m}\mathbb{Z}_p^\times} p^{-4m}\overline{\psi_p(tx)}dx.$$

We set $t = p^n t_0$, $t_0 \in \mathbb{Z}_p^\times$ and change the variable of integration by $x = p^{-m}t_0^{-1}y$, which induces that $dx = p^m dy$, then we have:

$$\int_{p^{-m}\mathbb{Z}_p^\times} p^{-4m}\overline{\psi_p(tx)}dx = p^{-3m}\int_{\mathbb{Z}_p^\times} \overline{\psi_p(p^{n-m}y)}dy = \begin{cases} p^{-3m}(1-p^{-1}) & , m \leq n \\ -p^{-3(n+1)}\cdot p^{-1} & , m = n+1 \\ 0 & , m > n+1 \end{cases}$$

Hence the integral $I_n(t)$ is independent of $t \in p^n\mathbb{Z}_p^\times$ and

$$I_n(t) = 1 + \sum_{m=1}^{n} p^{-3m}(1-p^{-1}) - p^{-3(n+1)-1} = \frac{(1-p^{-4})(1-p^{-3n-3})}{1-p^{-3}}.$$

Putting this value in (6.7), we obtain:

$$\begin{aligned} I_p(\widetilde{\phi}_p, \varphi_p, \psi_p) &= \frac{(1-p^{-4})}{(1-p^{-3})(\alpha_p-\alpha_p^{-1})}\sum_{n=0}^{\infty} p^{-\frac{5}{2}n}(\alpha_p^{n+1}-\alpha_p^{-n-1})(1-p^{-3n-3}) \\ &= \frac{(1-p^{-4})}{(1-p^{-3})(\alpha_p-\alpha_p^{-1})}\left(\frac{\alpha_p}{1-p^{-\frac{5}{2}}\alpha_p} - \frac{\alpha_p^{-1}}{1-p^{-\frac{5}{2}}\alpha_p^{-1}} - \frac{p^{-3}\alpha_p}{1-p^{-\frac{11}{2}}\alpha_p} + \frac{p^{-3}\alpha_p^{-1}}{1-p^{-\frac{11}{2}}\alpha_p^{-1}}\right) \\ &= \frac{(1-p^{-4})(1-p^{-8})}{(1-p^{-\frac{5}{2}}\alpha_p)(1-p^{-\frac{5}{2}}\alpha_p^{-1})(1-p^{-\frac{11}{2}}\alpha_p)(1-p^{-\frac{11}{2}}\alpha_p^{-1})}. \end{aligned}$$

□

As a direct consequence of Proposition 6.3.1, we have the following result, which corresponds to Theorem D in the introduction:

Corollary 6.3.2. *(Theorem D in §1) Let $\phi = \phi_\infty \otimes \Phi_f$ be a smooth holomorphic vector in $\Pi_{\min}$ such that $\widetilde{\phi} = \mathrm{Res}(\phi) \neq 0$, and $\varphi \simeq \varphi_\infty \otimes \varphi_f \in \pi$ the automorphic form of $\mathbf{PGL}_2$ associated to a (normalized) Hecke eigenform for $\mathbf{SL}_2(\mathbb{Z})$ of weight $2n+12$, $n > 0$. Then we have:*

$$\mathcal{P}_{\mathbf{Spin}_9}(\Theta_\phi(\varphi)) = \frac{\mathrm{L}(\pi, \frac{5}{2})\mathrm{L}(\pi, \frac{11}{2})}{\zeta(4)\zeta(8)} \cdot I_\infty(\mathrm{Res}(\phi_\infty), \varphi_\infty, \psi_\infty). \tag{6.8}$$

The L*-function* $\mathrm{L}(\pi, s)$ *appearing in* (6.8) *is the standard automorphic* L*-function of* π*, defined as the Euler product* $\prod_p (1-p^{-s}\alpha_p)(1-p^{-s}\alpha_p^{-1})$*, where the* $\mathbf{SL}_2(\mathbb{C})$*-conjugacy class of* $\mathrm{diag}(\alpha_p, \alpha_p^{-1})$ *is the Satake parameter of* π_p*.*

Remark 6.3.3. The L-factor $\mathrm{L}(\pi, \frac{5}{2})\mathrm{L}(\pi, \frac{11}{2})$ appearing in (6.8) agrees with the prediction of the global conjecture [SV17, §17; Sak21, Table 1] of Sakellaridis-Venkatesh for the spherical variety $\mathbf{Spin}_9\backslash\mathbf{F}_4$.

The Rankin-Selberg theory shows that the standard automorphic L-function $\mathrm{L}(\pi, s)$ has no zero at $s = \frac{5}{2}$ or $\frac{11}{2}$. As a consequence, the non-vanishing of $\mathcal{P}_{\mathbf{Spin}_9}(\Theta_\phi(\varphi))$ is equivalent to that of the archimedean zeta integral $I_\infty(\mathrm{Res}(\phi_\infty), \varphi_\infty, \psi_\infty)$.

6.4 Non-vanishing of $\Theta_\phi(\varphi)$

By Corollary 6.3.2, for the non-vanishing of $\Theta(\pi)$, it suffices to find some smooth vector $\phi_\infty \in \Pi^+ \subseteq \Pi_{\mathrm{min},\infty}$ such that $I_\infty(\mathrm{Res}(\phi_\infty), \varphi_\infty, \psi_\infty) \neq 0$. Notice that for the cuspidal automorphic form φ associated to *any* Hecke eigenform of weight $2n + 12$, its archimedean component φ_∞ is the unique (up to some scalar) holomorphic lowest weight vector in $d_{hol}(2n+12) \subseteq \mathcal{D}(2n + 12)$, thus we only need to prove the following:

Proposition 6.4.1. *For any $n > 1$, there exist an automorphic form $\varphi_n \in \mathcal{A}_{\mathrm{cusp}}(\mathbf{PGL}_2)$ associated to some Hecke eigenform in $\mathrm{S}_{2n+12}(\mathbf{SL}_2(\mathbb{Z}))$, and a smooth vector $\phi_n \in \Pi^+ \subseteq \Pi_{\mathrm{min},\infty}$, such that $I_\infty(\mathrm{Res}(\phi_n), \varphi_{n,\infty}, \psi_\infty) \neq 0$, or equivalently, $\mathcal{P}_{\mathbf{Spin}_9}(\Theta_{\phi_n\otimes\Phi_f}(\varphi_n)) \neq 0$.*

Proof. For each $n > 1$, Theorem 5.2.1 shows that there exists a non-zero $\mathbf{Spin}_9(\mathbb{R})$-invariant polynomial P_n in $\mathrm{V}_n(\mathrm{J}_\mathbb{C})$ such that the weighted theta series $\vartheta_{\mathrm{J}_\mathbb{Z},P_n}$ defined as (5.3) is non-zero. Let $\alpha_n \in \mathcal{A}_{\mathrm{V}_{n\varpi_4}}(\mathbf{F}_4)$ to be the vector-valued automorphic form such that $\alpha_n(1) = \sum_{\gamma\in\Gamma_\mathrm{I}} \gamma.P_n \in \mathrm{V}_n(\mathrm{J}_\mathbb{C})^{\Gamma_\mathrm{I}}$ and $\alpha_n(\gamma_\mathrm{E}) = 0$, then the global theta lift $\Theta(\alpha_n)$ is a non-zero holomorphic cuspidal automorphic form of $\mathbf{PGL}_2$. Hence there exists an automorphic form $\varphi_n \in \mathcal{A}_{\mathrm{cusp}}(\mathbf{PGL}_2)$ associated to some Hecke eigenform in $\mathrm{S}_{2n+12}(\mathbf{SL}_2(\mathbb{Z}))$, such that the Petersson inner product

$$\int_{[\mathbf{PGL}_2]} \Theta(\alpha_n)(g)\overline{\varphi_n(g)}dg \tag{6.9}$$

is non-zero. Putting the definition of $\Theta(\alpha_n)$ into (6.9), we have:

$$0 \neq \frac{1}{|\Gamma_\mathrm{I}|}\int_{[\mathbf{PGL}_2]} \left\{\Theta_n(g), \sum_{\gamma\in\Gamma_\mathrm{I}} \gamma.P_n\right\}\overline{\varphi_n(g)}dg = \int_{[\mathbf{PGL}_2]} \{\Theta_n(g), P_n\}\,\overline{\varphi_n(g)}dg. \tag{6.10}$$

Take the following smooth vector in $\Pi^+ \subseteq \Pi_{\mathrm{min},\infty}$:

$$\phi_n := \{\mathrm{D}^n\Phi_\infty, P_n\} = \sum_{\alpha_1,\ldots,\alpha_n} \{\mathrm{X}^\vee_{\alpha_1} \otimes \cdots \mathrm{X}^\vee_{\alpha_n}, P_n\} \cdot (\mathrm{X}_{\alpha_n}\cdots\mathrm{X}_{\alpha_1}.\Phi_\infty),$$

where Φ_∞ is the specific vector chosen in §4.2 and D is the operator $\Pi^+ \to \Pi^+ \otimes \mathfrak{p}^-_\mathrm{J}$ sending ϕ to $\sum_\alpha \mathrm{X}_\alpha\phi \otimes \mathrm{X}^\vee_\alpha$, with an arbitrary choice of basis $\{\mathrm{X}_\alpha\}$ of $\mathfrak{p}^+_\mathrm{J}$ and its dual basis $\{\mathrm{X}^\vee_\alpha\}$. By Definition 4.2.6, the automorphic realization $\theta : \Pi_{\mathrm{min}} \hookrightarrow \mathrm{L}^2([\mathbf{E}_7])$ maps $\phi_n \otimes \Phi_f$ to

$$\theta(\phi_n \otimes \Phi_f) = \{\mathrm{D}^n\theta(\Phi_\infty \otimes \Phi_f), P_n\} = \{\mathrm{D}^n\Theta_{Kim}, P_n\} = \{\Theta_n, P_n\}.$$

Use $\theta(\phi_n \otimes \Phi_f)$ as the kernel function to define a global theta lift of φ_n, then we calculate the $\mathbf{Spin}_9$-period integral of this global theta lift:

$$\mathcal{P}_{\mathbf{Spin}_9}(\Theta_{\phi_n\otimes\Phi_f}(\varphi_n)) = \int_{[\mathbf{PGL}_2]\times[\mathbf{Spin}_9]} \{\Theta_n(gh), P_n\}\overline{\varphi_n(g)}dgdh.$$

Since we have the strong approximation property $\mathbf{Spin}_9(\mathbb{A}) = \mathbf{Spin}_9(\mathbb{Q})\mathbf{Spin}_9(\mathbb{R})\mathbf{Spin}_9(\widehat{\mathbb{Z}})$, the $\mathbf{Spin}_9$-period integral is a non-zero multiple of

$$\begin{aligned}\int_{[\mathbf{PGL}_2]}\int_{\mathbf{Spin}_9(\mathbb{R})} \{\Theta_n(gh_\infty), P_n\}\overline{\varphi_n(g)}dgdh_\infty &= \int_{[\mathbf{PGL}_2]}\int_{\mathbf{Spin}_9(\mathbb{R})} \{h_\infty^{-1}.\Theta_n(g), P_n\}\overline{\varphi_n(g)}dgdh_\infty \\ &= \int_{\mathbf{Spin}_9(\mathbb{R})} dh_\infty \cdot \int_{[\mathbf{PGL}_2]} \{\Theta_n(g), P_n\}\overline{\varphi_n(g)}dg,\end{aligned}$$

where we use Lemma 4.2.9 and the $\mathbf{Spin}_9(\mathbb{R})$-invariance of P_n. Combining this with (6.10), we obtain the non-vanishing of $\mathcal{P}_{\mathbf{Spin}_9}(\Theta_{\phi_n\otimes\Phi_f}(\varphi_n))$, which is equivalent to the non-vanishing of $I_\infty(\mathrm{Res}(\phi_n), \varphi_{n,\infty}, \psi_\infty)$ by Corollary 6.3.2. ☐

Our main theorem is a direct consequence of Corollary 6.3.2 and Proposition 6.4.1:

Theorem 6.4.2. *(Theorem B in §1) Let $\pi \in \Pi_{\mathrm{cusp}}^{\mathrm{unr}}(\mathbf{PGL}_2)$ be the automorphic representation associated to a Hecke eigenform in $\mathrm{S}_k(\mathbf{SL}_2(\mathbb{Z}))$, then its global theta lift $\Theta(\pi)$ to $\mathbf{F}_4$ is non-zero. Furthermore, we have the local-global compatibility of theta correspondence,* i.e.

$$\Theta(\pi) \simeq \otimes'_v \theta(\pi_v).$$

Proof. The case when $k \geq 16$ is a corollary of Proposition 6.4.1 and Corollary 6.3.2. When $k = 12$, this is a result in [EG96] (see also Remark 5.1.3). The local-global compatibility of theta correspondence follows from Proposition 3.2.3 and Proposition 3.3.1. ☐

Corollary 6.4.3. *(Theorem E in §1) For $n \geq 2$, the following map is surjective:*

$$\begin{aligned}\mathcal{A}_{\mathrm{V}_{n\varpi_4}}(\mathbf{F}_4) &\to \mathrm{S}_{2n+12}(\mathbf{SL}_2(\mathbb{Z})) \\ (\alpha : \mathcal{J} \to \mathrm{V}_{n\varpi_4}) &\mapsto f_{\Theta(\alpha)} = \frac{1}{|\Gamma_{\mathrm{I}}|}\vartheta_{\mathrm{J}_{\mathbb{Z}},\alpha(\mathrm{J}_{\mathbb{Z}})} + \frac{1}{|\Gamma_{\mathrm{E}}|}\vartheta_{\mathrm{J}_{\mathrm{E}},\alpha(\mathrm{J}_{\mathrm{E}})}\end{aligned}$$

Proof. Suppose that the map $\alpha \mapsto f_{\Theta(\alpha)}$ is not surjective, then there exists a non-zero Hecke eigenform $f \in \mathrm{S}_{2n+12}(\mathbf{SL}_2(\mathbb{Z}))$, such that its associated automorphic form $\varphi \in \mathcal{A}(\mathbf{PGL}_2)$ is orthogonal to $\Theta(\alpha)$ for all $\alpha \in \mathcal{A}_{\mathrm{V}_{n\varpi_4}}(\mathbf{F}_4)$, with respect to the Petersson inner product. In particular, φ is orthogonal to $\Theta(\alpha_n)$, where α_n is the algebraic modular form chosen in the proof of Proposition 6.4.1. Take $\phi_n \in \Pi_{\min}$ to be the one in Proposition 6.4.1, we have:

$$0 = \int_{[\mathbf{PGL}_2]\times[\mathbf{Spin}_9]} \{\Theta_n(gh), P_n\}\overline{\varphi(g)}dgdh = \mathcal{P}_{\mathbf{Spin}_9}(\Theta_{\phi_n\otimes\Phi_f}(\varphi)),$$

which leads to a contradiction. ☐

References

[Ada96] J. F. Adams. *Lectures on exceptional Lie groups*. Chicago Lectures in Mathematics. With a foreword by J. Peter May. University of Chicago Press, Chicago, IL, 1996, pp. xiv+122.

[Art89] James Arthur. "Unipotent automorphic representations: conjectures". In: 171-172. Orbites unipotentes et représentations, II. 1989, pp. 13–71.

[BJ79] A. Borel and H. Jacquet. "Automorphic forms and automorphic representations". In: *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1*. Vol. XXXIII. Proc. Sympos. Pure Math. Amer. Math. Soc., Providence, RI, 1979, pp. 189–207.

[CL19] Gaëtan Chenevier and Jean Lannes. *Automorphic forms and even unimodular lattices*. Vol. 69. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Springer, Cham, 2019, pp. xxi+417.

[Cog04] James W. Cogdell. "Lectures on L-functions, converse theorems, and functoriality for GL_n". In: *Lectures on automorphic L-functions*. Vol. 20. Fields Inst. Monogr. Amer. Math. Soc., Providence, RI, 2004, pp. 1–96.

[Con15] Brian Conrad. "Non-split reductive groups over $\mathbf{Z}$". In: *Autours des schémas en groupes. Vol. II*. Vol. 46. Panor. Synthèses. Soc. Math. France, Paris, 2015, pp. 193–253.

[EG96] Noam D. Elkies and Benedict H. Gross. "The exceptional cone and the Leech lattice". In: *Internat. Math. Res. Notices* 14 (1996), pp. 665–698.

[Fre54] Hans Freudenthal. "Beziehungen der E_7 und E_8 zur Oktavenebene. I". In: *Indag. Math.* 16 (1954). Nederl. Akad. Wetensch. Proc. Ser. A **57**, pp. 218–230.

[GPR23] Skip Garibaldi, Holger P. Petersson, and Michel L. Racine. "Albert algebras over $\mathbb{Z}$ and other rings". In: *Forum Math. Sigma* 11 (2023), Paper No. e18, 38.

[Gro96] Benedict H. Gross. "Groups over $\mathbf{Z}$". In: *Invent. Math.* 124.1-3 (1996), pp. 263–279.

[GS98] Benedict H. Gross and Gordan Savin. "Motives with Galois group of type G_2: an exceptional theta-correspondence". In: *Compositio Math.* 114.2 (1998), pp. 153–217.

[Hec40] E. Hecke. "Analytische Arithmetik der positiven quadratischen Formen". In: *Danske Vid. Selsk. Mat.-Fys. Medd.* 17.12 (1940), p. 134.

[Kim93] Henry H. Kim. "Exceptional modular form of weight 4 on an exceptional domain contained in $\mathbf{C}^{27}$". In: *Rev. Mat. Iberoamericana* 9.1 (1993), pp. 139–200.

[KS23] Edmund Karasiewicz and Gordan Savin. *The Dual Pair* $\mathrm{Aut}(C) \times F_4$ *(p-adic case)*. 2023. arXiv: 2312.02853 [math.RT].

[KY16] Henry H. Kim and Takuya Yamauchi. "Cusp forms on the exceptional group of type E_7". In: *Compos. Math.* 152.2 (2016), pp. 223–254.

[KY23] Henry H. Kim and Takuya Yamauchi. "Higher level cusp forms on exceptional group of type E_7". In: *Kyoto J. Math.* 63.3 (2023), pp. 579–614.

[Lep70] James Lepowsky. "Representations of semisimple Lie groups and an enveloping algebra decomposition". PhD thesis. Massachusetts Institute of Technology, 1970.

[Pol20] Aaron Pollack. "The Fourier expansion of modular forms on quaternionic exceptional groups". In: *Duke Math. J.* 169.7 (2020), pp. 1209–1280.

[Pol23] Aaron Pollack. *Exceptional theta functions and arithmeticity of modular forms on G_2*. 2023. arXiv: 2211.05280 [math.NT].

[Rob63] Norbert Roby. "Lois polynomes et lois formelles en théorie des modules". In: *Ann. Sci. École Norm. Sup. (3)* 80 (1963), pp. 213–348.

[Sah93] Siddhartha Sahi. "Unitary representations on the Shilov boundary of a symmetric tube domain". In: *Representation theory of groups and algebras*. Vol. 145. Contemp. Math. Amer. Math. Soc., Providence, RI, 1993, pp. 275–286.

[Sak21] Yiannis Sakellaridis. "Functorial transfer between relative trace formulas in rank 1". In: *Duke Math. J.* 170.2 (2021), pp. 279–364.

[Sav94] Gordan Savin. "Dual pair $G_{\mathscr{J}} \times \mathrm{PGL}_2$ [where] $G_{\mathscr{J}}$ is the automorphism group of the Jordan algebra $\mathscr{J}$". In: *Invent. Math.* 118.1 (1994), pp. 141–160.

[Sha24] Yi Shan. *Level one automorphic representations of an anisotropic exceptional group over $\mathbb{Q}$ of type F_4*. 2024. arXiv: 2407.05859 [math.NT].

[SV00] Tonny A. Springer and Ferdinand D. Veldkamp. *Octonions, Jordan algebras and exceptional groups*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2000, pp. viii+208.

[SV17] Yiannis Sakellaridis and Akshay Venkatesh. "Periods and harmonic analysis on spherical varieties". In: *Astérisque* 396 (2017), pp. viii+360.

[Wal79a] J.-L. Waldspurger. "Engendrement par des séries thêta de certains espaces de formes modulaires". In: *Invent. Math.* 50.2 (1979), pp. 135–168.

[Wal79b] Nolan R. Wallach. "The analytic continuation of the discrete series. II". In: *Trans. Amer. Math. Soc.* 251 (1979), pp. 19–37.

[Yok09] Ichiro Yokota. *Exceptional Lie groups*. 2009. arXiv: 0902.0431 [math.DG].