PSL UNIVERSITÉ PARIS

THÈSE DE DOCTORAT DE L'UNIVERSITÉ PSL

Préparée à l'École Normale Supérieure de Paris

Level one algebraic automorphic forms of type F_4

Soutenue par
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Le 28 Mai 2025

École doctorale n°574
Ecole Doctorale de Mathématique Hadamard (EDMH)

Spécialité
Mathématiques

Préparée au
Département de Mathématiques et Applications (DMA)

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« Nous étions exacts dans l'exceptionnel qui seul sait se soustraire au caractère alternatif du mystère de vivre. »

Evoûtement à la Renardière, René Char,
Fureur et mystère

Remerciements

Tout d'abord, je souhaite remercier mon directeur de thèse Gaëtan Chenevier, sans ses conseils et les discussions que nous avons eues, rien n'aurait été possible. Il m'a introduite au monde des représentations automorphes, et à un sujet de thèse très intéressant. Son attitude rigoureuse envers la recherche et l'écriture académique a eu un impact profond sur ma vie mathématique. Je le remercie également pour sa gentillesse et ses encouragements, qui ont fait de moi une personne plus confiante.

Je suis très reconnaissante envers Wee Teck Gan et Aaron Pollack d'avoir accepté d'être rapporteurs, ainsi qu'envers Anne-Marie Aubert, Nadya Gurevich et David Renard d'avoir accepté d'être examinateur · rice · s[1].

Pendant ces années, j'ai effectué plusieurs visites de recherche, et je remercie chaleureusement les personnes qui ont rendu cela possible: merci à Wee Teck Gan pour le temps que j'ai passé à Singapour, à Olivier Taïbi pour ma visite à Lyon, à Alberto Mínguez pour celle à Vienne, et à Raphaël Beuzart-Plessis pour celle à Marseille. Un grand merci aux professeurs mentionnés ci-dessus, ainsi qu'à Aaron Pollack et Gordan Savin pour des discussions qui furent très utiles.

Ce fut une expérience magnifique de rédiger cette thèse au DMA de l'ENS. Je remercie toutes les personnes amicales que j'y ai rencontrées, et les moments de déjeuner et de pause café que nous avons partagés me manqueront assurément. Merci en particulier aux professeur · e · s Olivier Benoist, Djalil Chafaï, François Charles, Cyril Houdayer, Cyril Imbert, Muriel Livernet, Julien Marché, Vlerë Mehmeti, Silvain Ridean-Kikuchi et Nicolas Tholozan, ainsi qu'à Christie Barthelus, Marion Peres, Claire Le Poulennec, Fabienne Renia et Yue Teng pour les nombreuses fois où elles m'ont aidée. Je tiens à remercier spécialement Ariane Mézard pour son aide et son soutien inestimables au cours des six dernières années. Je remercie Nicolas Bergeron et Zoé Chatzidakis, que j'aurais aimé avoir encore parmi nous. Parmi les doctorant · e · s et postdoc du DMA, je suis ravie d'avoir fait la connaissance d'Ambre, Arnaud V, Asbjørn, Chuhao, Coline, Eduardo, Georgios, Louise, Matija, Nataniel, Olivier, Paul, Pierre, Romain, Samuel, Stefan, Thomas, Vadim et Yitong.

Merci aux merveilleuses personnes dans mon domaine de recherche que j'ai rencontrées au cours de ces années : Paul Boisseau, Cheng Chen, Juliette Coutens, Rahul Dalal, Mohamed Ez-Zarraq, Edmund Karasiewicz, Nhat Hoang Le, Huajie Li, Chuan Qin, Maxime Sommella,

[1] I am very grateful to Wee Teck Gan and Aaron Pollack for accepting to be referees for my thesis, and to Anne-Marie Aubert, Nadya Gurevich and David Renard for accepting to be part of the jury.

Hongjie Yu, Jialiang Zou et Jiandi Zou. Merci à Rein Janssen Groesbeek, Jiangfan Yuan et Yicheng Zhou d'avoir organisé le séminaire Mathjeunes avec moi.

Je remercie infiniment Juan Chacin et Wenyi Xiang pour leurs soins de santé pendant ma transition. Un grand merci à mes ami · e · s : Simon Alonso, Yuan Gao, Yulin Gong, Chuhao Huang, Yixuan Huang, Yuezhao Li, Yilin Ni, Junhui Qin, Runlei Xiao, Hongyu Wang, Shiyi Wei, Bingyu Zhang et Junyan Zhang. Je remercie Jiangfan Yuan d'avoir toujours été à mes côtés, et pour ses histoires sur la théorie géométrique des représentations que je n'ai jamais vraiment comprises. Je remercie Lan Pan et Jiangfan Shan d'avoir respecté mes choix dans ma carrière mathématique, et j'espère qu'ils pourront me soutenir dans les autres aspects de ma vie. Comme je l'avais écrit dans mon mémoire de licence, merci à la musique métal de m'avoir accompagnée et soutenue pendant toutes ces années.

Enfin, pour toutes les personnes avec qui j'ai partagé des moments jusqu'à présent et qui m'ont offert une vie riche et pleine, je tiens à exprimer ma sincère gratitude.

Abstract

In this thesis, we study level one automorphic representations for the $\mathbb{Q}$-form $\mathbf{F}_4$ of the exceptional compact group of Lie type F_4, The work is divided into the following two parts.

Level one automorphic representations of $\mathbf{F}_4$ with a given weight. First, following the method of Chenevier and Renard, we calculate the number of level one automorphic representations for $\mathbf{F}_4$ with any given archimedean component. More explicitly, we study the automorphism group of the two *Albert $\mathbb{Z}$-algebras* studied by Gross, as well as the dimension of the invariants of these groups in any irreducible representation of $\mathbf{F}_4(\mathbb{R})$.

Next, assuming standard conjectures by Arthur and Langlands on automorphic representations, we refine this counting by studying the contribution of the representations whose *global Arthur parameter* has any possible image. This includes a detailed description of all those images, and precise statements for the Arthur's multiplicity formula in each case. Our result provides in particular a conjectural but explicit formula for the number of algebraic, cuspidal, level one automorphic representations of $\mathbf{GL}_{26}$ over $\mathbb{Q}$ with any given "F_4-*regular*" weight and of Sato-Tate group $\mathbf{F}_4(\mathbb{R})$.

Exceptional theta correspondence for $\mathbf{F}_4 \times \mathbf{PGL}_2$. We study the global exceptional theta correspondence for the reductive dual pair $\mathbf{F}_4 \times \mathbf{PGL}_2$. Our main result states that for any automorphic representation of $\mathbf{PGL}_2$ associated with a cuspidal Hecke eigenform for $\mathbf{SL}_2(\mathbb{Z})$, its global theta lift to $\mathbf{F}_4$ is a non-zero irreducible automorphic representation. This verifies a conjectural calculation made in the previous part. Motivated by Pollack's work, our main tool is to construct a family of *exceptional theta series*, which are holomorphic cusp forms of $\mathbf{SL}_2(\mathbb{Z})$, and we show that this family spans the entire space of level one cusp forms.

Keywords : Automorphic forms, Exceptional groups, Langlands program, Theta correspondence

Résumé

Dans cette thèse, nous étudions les représentations automorphes de niveau un pour la $\mathbb{Q}$-forme $\mathbf{F}_4$ du groupe exceptionnel compact de type de Lie F_4. Ce travail est divisé en les deux parties suivantes.

Représentations automorphes de niveau un de $\mathbf{F}_4$ avec un poids donné. D'abord, en suivant la méthode de Chenevier et Renard, nous calculons le nombre de représentations automorphes de niveau un pour $\mathbf{F}_4$ avec une composante archimédienne donnée. Plus précisément, nous étudions le groupe d'automorphismes des deux *algèbres d'Albert sur* $\mathbb{Z}$ étudiées par Gross, ainsi que la dimension des invariants de ces groupes dans toute représentation irréductible de $\mathbf{F}_4(\mathbb{R})$.

Ensuite, en admettant les conjectures standards d'Arthur et Langlands sur les représentations automorphes, nous affinons ce comptage en étudiant la contribution des représentations dont le *paramètre global d'Arthur* a n'importe quelle image possible. Cela inclut une description détaillée de toutes ces images, et des énoncés précis pour la formule de multiplicité d'Arthur dans chaque cas. Notre résultat fournit en particulier une formule conjecturale mais explicite pour le nombre de représentations automorphes algébriques, cuspidales, de niveau un de $\mathbf{GL}_{26}$ sur $\mathbb{Q}$ ayant un poids « F_4-*régulier* » donné, et pour groupe de Sato-Tate $\mathbf{F}_4(\mathbb{R})$ tout entier.

Correspondance thêta exceptionnelle pour $\mathbf{F}_4 \times \mathbf{PGL}_2$. Nous étudions la correspondance thêta exceptionnelle globale pour la paire duale réductive $\mathbf{F}_4 \times \mathbf{PGL}_2$. Notre résultat principal affirme que pour toute représentation automorphe de $\mathbf{PGL}_2$ associée à une forme parabolique propre de Hecke pour $\mathbf{SL}_2(\mathbb{Z})$, son Θ-lift global est une représentation automorphe irréductible non nulle de $\mathbf{F}_4$. Cela vérifie un calcul conjectural effectué dans la partie précédente. Motivés par les travaux de Pollack, notre principal outil consiste à construire une famille de *séries thêta exceptionnelles*, qui sont des formes paraboliques holomorphes de $\mathbf{SL}_2(\mathbb{Z})$, et nous montrons que cette famille engendre tout l'espace des formes paraboliques de niveau un.

Mots clés : Formes automorphes, Groupes exceptionnels, Programme de Langlands, Correspondance thêta

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Chapter 1

Introduction

The work developed in this thesis belongs to the area of *automorphic representations*, especially those with level one for an anisotropic exceptional algebraic $\mathbb{Q}$-group $\mathbf{F}_4$. The first part (from Chapter 3 to Chapter 7) corresponds to [Shan, 2024], in which we study the number of level one automorphic representations for $\mathbf{F}_4$ with a given arbitrary weight, and (*conjecturally*) classifies their global Arthur parameters. The second part (Chapter 8) corresponds to [Shan, 2025], and we consider the global exceptional theta correspondence for $\mathbf{F}_4 \times \mathbf{PGL}_2$ in this part.

1.1 A motivation: geometric Galois representations with given image

The absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ encodes a lot of arithmetic information about number fields, and a natural way to study $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is to consider its representations, especially those arising from algebraic geometry. Motivated by the inverse Galois problem, the following question has been studied by a lot of mathematicians:

Problem 1. *Let ℓ be a prime number and $\mathbf{H}$ a connected reductive algebraic group over $\overline{\mathbb{Q}_\ell}$. Is there an ℓ-adic Galois representation $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbf{H}(\overline{\mathbb{Q}_\ell})$ such that it is semisimple and geometric (in the sense of Fontaine-Mazur [Taylor, 2004, Conjecture 1.1]), and whose image is Zariski dense in $\mathbf{H}(\overline{\mathbb{Q}_\ell})$?*

In the case $\mathbf{H} = \mathbf{GL}_2 \simeq \mathbf{GSp}_2$ or $\mathbf{GSp}_4$, or more generally, a (similitude) classical group, there are many well-known constructions and examples. For instance, one can use the Poincaré pairing on ℓ-adic cohomologies of algebraic varieties to construct Galois representations with images in classical groups. The case of exceptional groups, *i.e.* groups with Lie types G_2, F_4, E_6, E_7 and E_8, is harder, but we still have some examples in [DettweilerReiter, 2010; GrossSavin, 1998; Yun, 2014; Patrikis, 2016; BoxerCalegariEmertonLevinMadapusi PeraPatrikis, 2019]. Notice that when $\mathbf{H}$ has Lie type G_2 or E_8, this question is related to Serre's question on motives [Serre, 1994, Question 8.8, §1].

Composing $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbf{H}(\overline{\mathbb{Q}_\ell})$ with an algebraic representation $\mathbf{H} \to \mathbf{GL}_n$, we obtain an n-dimensional geometric ℓ-adic representation. One can associate two invariants with a geometric

ℓ-adic Galois representation $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbf{GL}_n(\overline{\mathbb{Q}_\ell})$: the *(Artin) conductor* $\mathrm{N}(\rho) \in \mathbb{N}$, and the *Hodge-Tate weights* $\mathrm{HT}(\rho)$, a multiset of n integers (see, for example, [Taylor, 2004]). In the aforementioned works, the conductors of the geometric ℓ-adic representations that they construct are usually not controlled. One may refine Problem 1 naturally by fixing these two invariants:

Problem 2. *Let ℓ be a prime number, $n \geq 1$ and H a connected reductive subgroup of $\mathbf{GL}_n$ over $\overline{\mathbb{Q}_\ell}$. What is the number (up to equivalence) of geometric ℓ-adic Galois representations $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbf{GL}_n(\overline{\mathbb{Q}_\ell})$ of given conductor and Hodge-Tate weights such that the Zariski closure of $\mathrm{Im}(\rho)$ is $\mathbf{H}(\overline{\mathbb{Q}_\ell})$?*

For $(\mathbf{H}, n) = (\mathbf{GL}_2, 2)$ or $(\mathbf{SO}_{2g+1}, 2g+1)$, this question is for instance related to the dimension of spaces of classical or Siegel modular forms. We have less knowledge of the cases of other groups $\mathbf{H}$. When the conductor $N = 1$, Problem 2 is solved *conjecturally* by Chenevier and Renard in [ChenevierRenard, 2015] for the following groups (n is chosen to be the dimension of the standard representation when $\mathbf{H}$ is a (similitude) classical group, and to be 7 when $\mathbf{H}$ has type G_2):

$$\mathbf{GL}_2 \simeq \mathbf{GSp}_2,\ \mathbf{GSp}_4,\ \mathbf{SO}_4,\ \mathbf{SO}_5,\ \mathbf{GSp}_6,\ \mathbf{GSp}_8,\ \mathbf{SO}_8,\ \mathbf{G}_2,$$

via the conjectural connection between n-dimensional geometric ℓ-adic representations and cuspidal automorphic representations of $\mathbf{GL}_n$. See also [Taïbi, 2017; ChenevierTaïbi, 2020] for higher dimensions. In [Lachaussée, 2020], Lachaussée extends the results for $\mathbf{GSp}_{2g}$, $1 \leq g \leq 4$ to the case of Artin conductor $N = 2$. Now we concentrate on the case of conductor one (see Remark 1.6.3 for more explanations about this assumption).

In this thesis, following [ChenevierRenard, 2015], we give a *conjectural* solution to Problem 2 when $N = 1$, $\mathbf{H}$ has Lie type F_4, and $n = 26$. For a 26-dimensional geometric ℓ-adic Galois representation ρ such that $\overline{\mathrm{Im}(\rho)}$ has type F_4, its multiset of Hodge-Tate weights only depends on 4 variables $a, b, c, d \in \mathbb{N}$, and has the form

$$\mathrm{HT}(a,b,c,d) := \left\{ \begin{array}{c} 0, 0, \pm a, \pm b, \pm(a+b), \pm(b+c), \pm(a+b+c), \pm(b+c+d), \pm(a+b+c+d), \pm(a+2b+c), \\ \pm(a+2b+c+d), \pm(a+2b+2c+d), \pm(a+3b+2c+d), \pm(2a+3b+2c+d). \end{array} \right\}$$

As a conjectural corollary of our results in this thesis, we propose the following conjecture on F_4-type geometric ℓ-adic representations:

Conjecture A. *The number of equivalence classes of* 26*-dimensional conductor one geometric ℓ-adic Galois representations ρ such that*

- *the Zariski closure of* $\mathrm{Im}(\rho)$ *is a connected reductive group of type* F_4,
- *and* $\mathrm{HT}(\rho) = \mathrm{HT}(a,b,c,d)$*,* $a, b, c, d \geq 1$*,*

is $\mathrm{F}_4(a-1, b-1, c-1, d-1)$*, where* $\mathrm{F}_4(\lambda)$ *is the computable function on* $\mathbb{N}^4$ *given by Proposition 7.4.1.*

Remark 1.1.1. The formula for $\mathrm{F}_4(\lambda)$ has so many terms that we will not write down the full formula in this paper. However, under some hypothesis this formula becomes much simpler. For instance, when $a > b+c+d+3$, $b, c, d > 0$ and c, d are both odd, a short formula for $\mathrm{F}_4(a,b,c,d)$ is given in Remark 7.4.2.

1.2 An automorphic variant of the counting problem

Now we send Problem 2 to the automorphic side. Let $\mathbf{G}$ be a connected reductive group over $\mathbb{Q}$ with a reductive $\mathbb{Z}$-model (see Section 3.2). As we will talk about Galois representations, it will be convenient to assume that its Langlands dual group $\widehat{\mathbf{G}}$ is defined over $\overline{\mathbb{Q}}$, and we fix two embeddings: $\mathbb{C} \overset{\iota_\infty}{\hookleftarrow} \overline{\mathbb{Q}} \overset{\iota_\ell}{\hookrightarrow} \overline{\mathbb{Q}_\ell}$. We also fix a maximal compact subgroup G_c of $\widehat{\mathbf{G}}(\mathbb{C})$.

Let π be an *L-algebraic*[1] level one automorphic representation of $\mathbf{G}$. By a conjecture of Buzzard and Gee [BuzzardGee, 2014, Conjecture 3.2.1], one should be able to associate with π a compatible conductor one geometric ℓ-adic representation $\rho_{\pi,\iota} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \widehat{\mathbf{G}}(\overline{\mathbb{Q}_\ell})$, which depends on the choice of embeddings $\iota = (\iota_\infty, \iota_\ell)$. By the standard conjectures of Fontaine-Mazur and Langlands, every conductor one geometric ℓ-adic representation into $\widehat{\mathbf{G}}(\overline{\mathbb{Q}_\ell})$ should arise in this way. If any two element-conjugate homomorphisms from a connected compact Lie group into G_c are conjugate (see Section 5.1 for a detailed explanation), the following question gives an automorphic variant of Problem 2 for $\mathbf{H} = \widehat{\mathbf{G}} \times_{\iota_\ell} \overline{\mathbb{Q}_\ell}$:

Problem 3. *Let* $\mathbf{G}$ *be a connected reductive group over* $\mathbb{Q}$ *admitting a reductive* $\mathbb{Z}$*-model.*

*(1) (*Counting*) Count the number (up to equivalence) of level one algebraic*[2] *discrete automorphic representations for* $\mathbf{G}$ *with an arbitrary given archimedean component.*

*(2) (*Refinement*) Refine this counting by* "Sato-Tate groups" *of automorphic representations.*

Remark 1.2.1 ("Sato-Tate groups"). In the above question, the "Sato-Tate group" $\mathrm{H}(\pi)$ of a level one automorphic representation π for $\mathbf{G}$ is a certain conjugacy class of subgroups of G_c that we will explain carefully in Section 6.3.1, and we can briefly introduce it as follows. Based on Arthur's parametrization of automorphic representations, one can *conjecturally* associate with π a group homomorphism

$$\psi_\pi : \mathcal{L}_\mathbb{Z} \times \mathrm{SU}(2) \to G_c,$$

where $\mathcal{L}_\mathbb{Z}$ is the *hypothetical Langlands group of* $\mathbb{Z}$, which is connected and compact (see Section 6.3). We define $\mathrm{H}(\pi)$ to be the conjugacy class of the image of ψ_π in G_c. When the restriction of ψ_π to $1 \times \mathrm{SU}(2) \subset \mathcal{L}_\mathbb{Z} \times \mathrm{SU}(2)$ is trivial, this notion $\mathrm{H}(\pi)$ coincides with the usual notion of Sato-Tate groups. In general, we decided to include the $\mathrm{SU}(2)$ factor in the definition as it provides convenience for stating some of our results.

The point of the refinement part in Problem 3 is that in general many level one discrete automorphic representations π for $\mathbf{G}$, for example the *endoscopic* ones, will have a Sato-Tate group strictly smaller than G_c. For these π, $\overline{\mathrm{Im}(\rho_{\pi,\iota})}$ should be a proper subgroup of $\widehat{\mathbf{G}}(\overline{\mathbb{Q}_\ell})$. Hence we have to find a way to exclude these representations to obtain the desired number in Problem 2.

In [ChenevierRenard, 2015], Chenevier and Renard solve the part (1) of Problem 3 for a number of classical groups of small ranks, namely, $\mathbf{G}$ is one of the following groups:

$$\mathbf{SL}_2 = \mathbf{Sp}_2,\ \mathbf{Sp}_4,\ \mathbf{SO}_{2,2},\ \mathbf{SO}_{3,2},\ \mathbf{SO}_7,\ \mathbf{SO}_8 \text{ and } \mathbf{SO}_9,$$

[1] For the definition of L-algebraicity, see [BuzzardGee, 2014, Definition 2.3.1]. For a representation which is algebraic in the sense of Definition 6.4.3 but not L-algebraic, one should replace $\widehat{\mathbf{G}}$ by some "similitude" group.

[2] One can remove this algebraicity condition by restricting to semisimple $\mathbb{Q}$-groups.

and also for a connected semisimple $\mathbb{Q}$-group of type G_2 with compact real points. For the part (2) of Problem 3, their method relies in an important way on Arthur's classification of automorphic representations [Arthur, 1989; Arthur, 2013]. Their results for $\mathbf{SO}_7, \mathbf{SO}_8, \mathbf{SO}_9$ and $\mathbf{G}_2$ are conditional to Arthur's conjectures for these groups, since $\mathbf{SO}_7, \mathbf{SO}_8$ and $\mathbf{SO}_9$ are not quasi-split, and $\mathbf{G}_2$ is not covered by Arthur's results. In [Taïbi, 2017; Taïbi, 2019], Taïbi makes these results unconditional (except for $\mathbf{G}_2$), and he also extends them to the following split classical groups:

$$\mathbf{Sp}_{2g} \text{ with } g \leq 7,\ \mathbf{SO}_{n+1,n} \text{ with } n \leq 8 \text{ and } \mathbf{SO}_{2m,2m} \text{ with } m \leq 4.$$

In particular, Taïbi's solution to Problem 3 for $\mathbf{Sp}_8$ will be important in our work.

In the first part of this thesis, we apply the method of [ChenevierRenard, 2015] to $\mathbf{F}_4$, the unique (up to isomorphism) connected semisimple algebraic group over $\mathbb{Q}$ of type F_4, with compact real points and split over $\mathbb{Q}_p$ for every prime p (see Section 3.1). For this group, automorphic representations are automatically L-algebraic. Moreover, it turns our that there is no local-global conjugacy problems for connected subgroups of $(\mathbf{F}_4)_c = \mathbf{F}_4(\mathbb{R})$ (see Proposition 5.1.5). As a consequence, Conjecture A follows from standard conjectures and our answer to Problem 3 for $\mathbf{F}_4$.

Remark 1.2.2. The automorphic representations for $\mathbf{F}_4$ (and their local components) have been studied in [Savin, 1994; MagaardSavin, 1997; Gan, 2000; Pollack, 2023; KarasiewiczSavin, 2023] via exceptional theta correspondences, and we will explain some links between these correspondences with our work in Section 7.5. Let us mention also that automorphic representations for $\mathbf{F}_4$ have also been studied in the past by Seth Padowitz in [Padowitz, 1998, §9]. Padowitz rather considers the automorphic representations which are Steinberg at a fixed *non-empty* set of primes and unramified elsewhere, and he tries to enumerate them using the stable trace formula, in the spirit of works of Gross-Pollack [GrossPollack, 2005]. The results are only partial, as several stable local orbital integrals there are not determined[3], and we hope to go back to this question in the future.

1.3 Counting level one automorphic representations

In [Gross, 1996], Gross proves the following result for $\mathbf{F}_4$, which is important in our solution to the part (1) of Problem 3 for $\mathbf{F}_4$:

Theorem B. *(Proposition 3.3.6) Up to $\mathbb{Z}$-isomorphism, there are two smooth affine group schemes over $\mathbb{Z}$ with generic fiber isomorphic to $\mathbf{F}_4$, whose special fiber over $\mathbb{Z}/p\mathbb{Z}$ is reductive for all primes p.*

The $\mathbb{Z}$-group schemes in Theorem B are reductive $\mathbb{Z}$-models of $\mathbf{F}_4$. Their constructions are related to integral structures of the 27-dimensional definite exceptional Jordan algebra over $\mathbb{Q}$. Gross proves this result via the mass formula for $\mathbf{F}_4$ and some results in [ATLAS]. The goal

[3] Another minor problem is that the author asserts on [Padowitz, 1998, P.42] that the 26-dimensional irreducible representation of $\mathbf{F}_4$ is "excellent" in his sense, which is not correct. See Remark 4.5.5 for a counterexample.

of Chapter 3 is to recall the construction of $\mathbf{F}_4$ and to give a new proof of Theorem B without using [ATLAS].

Since the method of counting in [ChenevierRenard, 2015] can be applied to any algebraic $\mathbb{Q}$-group that has compact real points and admits a reductive $\mathbb{Z}$-model, we recall and apply this method to $\mathbf{F}_4$ in Chapter 4. One important input is the structure (*e.g.* generators, conjugacy classes) of the finite subgroup $\mathscr{G}(\mathbb{Z})$ of $\mathbf{F}_4(\mathbb{R})$, where $\mathscr{G}$ is one of the two reductive $\mathbb{Z}$-models in Theorem B. This input is given by our analysis in the proof of Theorem B. We obtain the answer for the part (1) of Problem 3 for $\mathbf{F}_4$:

Theorem C. *(Theorem 4.6.1 and Corollary 6.1.8)*

(1) For an irreducible representation V_λ of $\mathbf{F}_4(\mathbb{R})$ with highest weight λ, we have an explicit and computable formula for the number $d(\lambda)$ of equivalence classes of level one automorphic representations π with $\pi_\infty \simeq \mathrm{V}_\lambda$.

(2) For dominant weights $\lambda = \sum_{i=1}^{4} \lambda_i \varpi_i$[4] satisfying $2\lambda_1 + 3\lambda_2 + 2\lambda_3 + \lambda_4 \leq 13$, we list the numbers $d(\lambda)$ in Table A.3, Appendix A.

1.4 Candidates for Sato-Tate groups

The part (2) of Problem 3 involves a classification of all possible Sato-Tate groups for level one automorphic representations of $\mathbf{F}_4$. For this $\mathbb{Q}$-group, its Langlands dual group $\widehat{\mathbf{F}_4}$ is isomorphic to $\mathbf{F}_4 \times_{\mathbb{Q}} \mathbb{C}$, and as mentioned in Remark 1.2.1, Sato-Tate groups in this case are conjugacy classes of subgroups of the compact Lie group $\mathbf{F}_4(\mathbb{R})$. Our goal of Chapter 5 is to exclude some subgroups of $\mathbf{F}_4(\mathbb{R})$, and to give a list of candidates for Sato-Tate groups in this case:

Theorem D. *(Theorem 5.6.7) There are 13 conjugacy classes of proper connected subgroups H of $\mathbf{F}_4(\mathbb{R})$ such that:*

- *the centralizer of H in $\mathbf{F}_4(\mathbb{R})$ is isomorphic to the product of finitely many copies of $\mathbb{Z}/2\mathbb{Z}$;*
- *the zero weight appears twice in the restriction of the 26-dimensional irreducible representation of $\mathbf{F}_4(\mathbb{R})$ to H.*

We will prove this classification result step by step, following Dynkin's strategy in [Dynkin, 1952]. It is worth mentioning two important ingredients in the proof:

- A local-global conjugacy result (Proposition 5.1.5) for $\mathbf{F}_4(\mathbb{R})$, which we have already mentioned in the end of Section 1.2. This relies on a result about Lie algebras (Theorem 5.1.3) proved by Losev in [Losev, 2010].
- A useful criterion (Proposition 5.2.1) given in Section 5.2 for the conjugacy of two homomorphisms from a connected compact Lie group into $\mathbf{F}_4(\mathbb{R})$.

Example 1.4.1. Among the conjugacy classes of subgroups classified in Theorem D, we have

$$\mathrm{Spin}(9),\ \mathrm{Spin}(8),\ \mathrm{G}_2 \times \mathrm{SO}(3),\ (\mathrm{Sp}(3)\times\mathrm{SU}(2))/\mu_2^{\Delta},\ (\mathrm{Sp}(2)\times\mathrm{SU}(2)\times\mathrm{SU}(2))/\mu_2^{\Delta},$$

[4] Here we follow the notations in [Bourbaki, 2002, §VI.4.9].

where the notations will be explained in Notation 5.3.1 and Notation 5.3.3. The remaining subgroups are all centrally isogenous to products of n copies of SU(2), $n \leq 4$. Note that among the subgroups listed above, only Spin(9) and $(\mathrm{Sp}(3) \times \mathrm{SU}(2))/\mu_2^{\Delta}$ are maximal proper connected regular subgroups of $\mathbf{F}_4(\mathbb{R})$.

1.5 Arthur's conjectures

As in [ChenevierRenard, 2015], for the part (2) of Problem 3, we need some conjectures on automorphic representations. For a connected reductive algebraic group $\mathbf{G}$ over $\mathbb{Q}$, Arthur introduces in [Arthur, 1989] a conjectural parametrization of discrete automorphic representations, via *discrete global Arthur parameters* for $\mathbf{G}$. In the level one case, these parameters are $\widehat{\mathbf{G}}(\mathbb{C})$-conjugacy classes of admissible morphisms

$$\psi : \mathcal{L}_{\mathbb{Z}} \times \mathbf{SL}_2(\mathbb{C}) \to \widehat{\mathbf{G}}(\mathbb{C}),$$

where $\mathcal{L}_{\mathbb{Z}}$ is the hypothetical Langlands group of $\mathbb{Z}$ (see Section 6.3 for more details), and $\widehat{\mathbf{G}}$ is the Langlands dual group of $\mathbf{G}$. Arthur proposes a conjectural formula for the multiplicity of an irreducible $\mathbf{G}(\mathbb{A})$-representation in the discrete automorphic spectrum of $\mathbf{G}$, in terms of the associated global Arthur parameters.

In [Arthur, 2013], Arthur reformulates his conjectures for any quasi-split classical group $\mathbf{G}$, avoiding the appearance of the hypothetical Langlands group $\mathcal{L}_{\mathbb{Z}}$. In this case, he relates the global Arthur parameters for $\mathbf{G}$ to cuspidal automorphic representations of linear groups, and proves the endoscopic classifications, relying in particular on the works of Mœglin-Waldspurger [MoeglinWaldspurger, 2014], Ngô [Ngô, 2010] and many others. We refer to [ChenevierLannes, 2019, §8] for precise statements of Arthur's results in [Arthur, 2013] in the case of level one cohomological automorphic representations of classical groups.

Of course $\mathbf{F}_4$ is not a classical group, and Arthur's general conjectures [Arthur, 1989] are still open in this case. Nevertheless, they can still be formulated quite precisely if we admit the existence of $\mathcal{L}_{\mathbb{Z}}$. See also [ChenevierLannes, 2019, §6.4] for some general forms of Arthur's conjectures in the level one case.

Notation 1.5.1. In the rest of the thesis, we will mark any result conditional to the existence of $\mathcal{L}_{\mathbb{Z}}$ and Arthur's multiplicity formula (Conjecture 6.6.5) with a star $*$.

Now we briefly explain Arthur's conjectures for $\mathbf{F}_4$, and a more precise description in the general case for simply-connected anisotropic groups admitting reductive $\mathbb{Z}$-models will be provided in Chapter 6. For a level one automorphic representation π of $\mathbf{F}_4$ with global Arthur parameter $\psi : \mathcal{L}_{\mathbb{Z}} \times \mathbf{SL}_2(\mathbb{C}) \to \mathbf{F}_4(\mathbb{C})$, we may compose ψ with the 26-dimensional irreducible representation $\mathrm{r} : \mathbf{F}_4(\mathbb{C}) \to \mathbf{GL}_{26}(\mathbb{C})$[5], and thus obtain a representation of $\mathcal{L}_{\mathbb{Z}} \times \mathbf{SL}_2(\mathbb{C})$. This representation is decomposed as:

$$\mathrm{r} \circ \psi \simeq \pi_1[d_1] \oplus \cdots \oplus \pi_k[d_k], \qquad (\star)$$

[5]The image of r is even inside $\mathbf{SO}_{26}(\mathbb{C}) \subset \mathbf{SL}_{26}(\mathbb{C}) \subset \mathbf{GL}_{26}(\mathbb{C})$.

where π_i is an n_i-dimensional irreducible representation of $\mathcal{L}_{\mathbb{Z}}$ and $[d_i]$ stands for the irreducible d_i-dimensional representation of $\mathbf{SL}_2(\mathbb{C})$, and $\sum_{i=1}^{k} n_i d_i = 26$. We identify π_i as a level one cuspidal representations of $\mathbf{PGL}_{n_i}$, and observe that it is always self-dual and algebraic in this case (see Section 6.4). In a similar way as in [Arthur, 2013], we view the global Arthur parameter ψ as a formal sum of $\pi_i[d_i]$'s.

We derive from Theorem D that the Sato-Tate group of any π_i appearing in the decomposition ($\star$) is one of the following compact Lie groups:

$$\mathrm{SU}(2),\ \mathrm{Sp}(2),\ \mathrm{Sp}(3),\ \mathrm{SO}(8),\ \mathrm{SO}(9),\ \mathrm{G}_2, \mathbf{F}_4(\mathbb{R}). \qquad (\star\star)$$

Cuspidal representations with Sato-Tate group $\mathbf{F}_4(\mathbb{R})$ conjecturally correspond to the desired ℓ-adic representations in Problem 2, and those with other Sato-Tate groups in ($\star\star$) are related to level one automorphic representations for the following $\mathbb{Q}$-groups:

$$\mathbf{PGL}_2, \mathbf{SO}_{3,2}, \mathbf{SO}_7, \mathbf{SO}_8, \mathbf{Sp}_8, \mathbf{G}_2,$$

which have already been studied in [ChenevierRenard, 2015; Taïbi, 2017; ChenevierTaïbi, 2020].

Conversely, for a global Arthur parameter $\psi : \mathcal{L}_{\mathbb{Z}} \times \mathbf{SL}_2(\mathbb{C}) \to \mathbf{F}_4(\mathbb{C})$ whose "archimedean component" is an *Adams-Johnson parameter* (see Definition 6.6.1 and Remark 6.6.2), the multiplicity of its corresponding irreducible $\mathbf{F}_4(\mathbb{A})$-representation in the automorphic spectrum can be calculated via Arthur's formula in [Arthur, 1989], and an explicit formula for $\mathbf{F}_4$ will be given in Section 7.2.

1.6 Refinement of the counting

The goal of Chapter 7 is to refine the counting in Theorem C. For a global Arthur parameter $\psi : \mathcal{L}_{\mathbb{Z}} \times \mathbf{SL}_2(\mathbb{C}) \to \mathbf{F}_4(\mathbb{C})$, one can associate two invariants:

- its Sato-Tate group $\mathrm{H}(\psi) := \psi(\mathcal{L}_{\mathbb{Z}} \times \mathrm{SU}(2))$, viewed as a conjugacy of subgroups in the compact group $\mathbf{F}_4(\mathbb{R})$;
- its "weights", *i.e.* eigenvalues of its infinitesimal character under the 26-dimensional irreducible representation $\mathrm{r} : \mathbf{F}_4 \to \mathbf{SL}_{26}$.

Given any conjugacy class of proper subgroups H of $\mathbf{F}_4(\mathbb{R})$ classified in Theorem D, in Section 7.3 we classify all the possible decompositions ($\star$) of $\mathrm{r} \circ \psi$ for global Arthur parameters ψ with $\mathrm{H}(\psi) = H$. If ψ corresponds to an irreducible level one $\mathbf{F}_4(\mathbb{A})$-representation π, an important part of our work is to give an exact formula for the multiplicity of π, for each case of Sato-Tate groups. Roughly speaking, the multiplicity depends on how the weights of ψ are distributed in the summands $\pi_i[d_i]$'s of ($\star$). In conclusion, we have the following result:

Theorem* E. *(Theorem 7.3.1)*

(a) The Sato-Tate group of a level one automorphic representation for $\mathbf{F}_4$ is either $\mathbf{F}_4(\mathbb{R})$ or one of the proper subgroups of $\mathbf{F}_4(\mathbb{R})$ classified in Theorem D except $\mathrm{Spin}(8)$.

(b) *For global Arthur parameters of $\mathbf{F}_4$ with a given Sato-Tate group, the multiplicity of its corresponding irreducible level one $\mathbf{F}_4(\mathbb{A})$-representation* (0 *or* 1) *is given explicitly by the formulas in Proposition 7.3.4 to Proposition 7.3.18.*

Remark 1.6.1. We observe that not all subgroups in Theorem D come from *endoscopic groups* of $\mathbf{F}_4$, in the sense of [Arthur, 2013]. For example, the subgroup $\mathrm{G}_2 \times \mathrm{SO}(3)$ has trivial centralizer in $\mathbf{F}_4(\mathbb{R})$, thus it can not be the centralizer of any element in $\mathbf{F}_4(\mathbb{R})$. As a result, our conjectural refinement is finer than Arthur's endoscopic classification.

Given an irreducible representation V_λ of $\mathbf{F}_4(\mathbb{R})$, from Theorem C we know the number of equivalence classes of level one automorphic representations π for $\mathbf{F}_4$ with $\pi_\infty \simeq \mathrm{V}_\lambda$. The weights of the global Arthur parameter ψ_π of π are determined by V_λ. We can enumerate all the possible global Arthur parameters with these weights, and then use the multiplicity formulas in Theorem E to determine their multiplicities. In this way, we obtain a *conjectural* refinement of the counting in Theorem C. As a consequence, we obtain a conjectural solution to Problem 2, stated in terms of automorphic representations:

Theorem* F. *(Proposition 7.4.1 and Proposition 7.4.3) The number of algebraic, cuspidal, level one automorphic representations of $\mathbf{GL}_{26}$ over $\mathbb{Q}$ satisfying:*

- *the Sato-Tate group is $\mathbf{F}_4(\mathbb{R})$,*
- *and the multiset of weights*[6] *is $\mathrm{HT}(a,b,c,d)$ for $a,b,c,d \geq 1$,*

is $\mathrm{F}_4(a-1,b-1,c-1,d-1)$, where $\mathrm{F}_4(\lambda)$ is an explicit function on $\mathbb{N}^4$ given by Proposition 7.4.1.

Example 1.6.2. The quadruples $(a,b,c,d) \in \mathbb{N}^4$ such that

- the largest weight $2a+3b+2c+d+8$ in the multiset $\mathrm{HT}(a+1,b+1,c+1,d+1)$ is not larger than 22,
- and $\mathrm{F}_4(a,b,c,d) \neq 0$,

are listed in Table A.8, Appendix A. We also list the values of $\mathrm{F}_4(a,b,c,d)$ for these quadruples.

Remark 1.6.3. One may want to remove the level one condition, like in [Lachaussée, 2020]. For the part (1) of Problem 3 for $\mathbf{F}_4$, one can probably calculate the dimension of invariants under other congruence subgroups, and obtain results similar to Theorem C for higher levels. However, for the part (2) of Problem 3 for $\mathbf{F}_4$, what we use is a simplified version of Arthur's recipe in [Arthur, 1989]. When allowing ramifications at some finite place p, one needs some properties of *local Arthur packets* for $\mathbf{F}_4(\mathbb{Q}_p)$, which are still unknown to us.

1.7 Connection with exceptional theta correspondences

Roughly speaking, for a *reductive dual pair* $\mathbf{G} \times \mathbf{H}$ inside $\mathbf{E}$, where $\mathbf{E}$ is an algebraic $\mathbb{Q}$-group admitting a *minimal representation*, the *local* (resp. *global*) *theta correspondence* studies the "restriction" of a minimal representation of $\mathbf{E}(F)$, F being a local field (*resp.* $\mathbf{E}(\mathbb{A})$) to

[6]See Section 6.4 for the precise definition of weights for an algebraic cuspidal level one automorphic representation of $\mathbf{GL}_n$.

$\mathbf{G}(F) \times \mathbf{H}(F)$ (*resp.* $\mathbf{G}(\mathbb{A}) \times \mathbf{H}(\mathbb{A})$), and gives a correspondence between representations of $\mathbf{G}$ and $\mathbf{H}$. In the second part (Chapter 8), we study the global theta correspondence for the dual pair $\mathbf{F}_4 \times \mathbf{PGL}_2$ inside $\mathbf{E}_7$, an exceptional group of type E_7 and real rank 3, and the main goal is to prove the following theorem:

Theorem G. *(Theorem 8.6.12) Let π be the level one algebraic automorphic representation of $\mathbf{PGL}_2$ associated to a non-zero cuspidal Hecke eigenform for $\mathbf{SL}_2(\mathbb{Z})$. Under the global theta correspondence for $\mathbf{F}_4 \times \mathbf{PGL}_2$, the global theta lift $\Theta(\pi)$ is a non-zero irreducible automorphic representation of $\mathbf{F}_4$.*

We have a detailed introduction in Chapter 8 for this global exceptional theta correspondence, here we present a motivation arising from our conjectural computation in Theorem E.

By Flath's theorem, the automorphic representation π in Theorem G can be factorized as a restricted tensor product $\otimes'_v \pi_v$, where π_v is an irreducible representation of $\mathbf{PGL}_2(\mathbb{Q}_v)$. The results of the local exceptional theta correspondence for $\mathbf{F}_4 \times \mathbf{PGL}_2$ [GrossSavin, 1998, Proposition 3.2; Savin, 1994; KarasiewiczSavin, 2023] show that for any place $v = p$ or ∞, the (big) local theta lift $\Theta(\pi_v)$ is a non-zero irreducible representation of $\mathbf{F}_4(\mathbb{Q}_v)$. We take Π to be the irreducible representation $\otimes'_v \Theta(\pi_v)$ of $\mathbf{F}_4(\mathbb{A})$. Using the explicit (conjectural) multiplicity formula in Proposition 7.3.6, we find that the multiplicity of Π in the automorphic spectrum of $\mathbf{F}_4$ is always 1, no matter the choice of π. It is natural to expect the global theta lift $\Theta(\pi)$ to be non-zero for any π associated to some cuspidal Hecke eigenform of level $\mathbf{SL}_2(\mathbb{Z})$.

Remark 1.7.1. Another exceptional theta correspondence related to Theorem E in a similar way is the that for the dual pair $\mathbf{F}_4 \times \mathbf{G}_2^{\mathrm{s}}$, where $\mathbf{G}_2^{\mathrm{s}}$ is the generic fiber of the split Chevalley group of Lie type G_2. In [Pollack, 2023], Pollack shows that any level one cuspidal automorphic representation associated to some quaternionic modular form of $\mathbf{G}_2^{\mathrm{s}}$ has non-zero global theta lift to $\mathbf{F}_4$.

1.8 Exceptional theta series

Our main tool for proving Theorem G is to develop a notion of "exceptional theta series", motivated by [Pollack, 2023]. This is a variant of the classical weighted theta series associated with an even unimodular lattice L inside the Euclidean space $\mathbb{R}^n$ and a homogeneous harmonic polynomial P on $\mathbb{R}^n$:

$$\vartheta_{L,P} = \sum_{v \in L} P(v) q^{\frac{v.v}{2}}, \text{ where } q = e^{2\pi i z}, z \in \mathcal{H} = \{x + iy \,|\, x, y \in \mathbb{R}, y > 0\}.$$

This (classical) theta series is a modular form of level $\mathbf{SL}_2(\mathbb{Z})$ and weight $n/2 + \deg P$, and is cuspidal if P is not constant. In [Waldspurger, 1979], Waldspurger shows that for any fixed pair of natural numbers (n, d), where $8|n$, the space $\mathrm{S}_{n/2+d}(\mathbf{SL}_2(\mathbb{Z}))$ of weight $n/2 + d$ cusp forms is spanned by $\vartheta_{L,P}$, L varying over even unimodular lattices in the Euclidean space $\mathbb{R}^n$ and P varying over homogeneous harmonic polynomial of degree d on $\mathbb{R}^n$.

In the exceptional case, we replace the classical setting by the corresponding objects in the following table:

classical case	exceptional case
Euclidean space $\mathbb{R}^n$	Exceptional Jordan algebra $\mathbb{R}$-algebra $\mathrm{J}_{\mathbb{R}}$ (Definition 3.1.3)
even unimodular lattice	*Albert lattice* inside $\mathrm{J}_{\mathbb{R}}$ (Definition 8.2.12)
harmonic polynomials	"F_4-*harmonic*" polynomials (Definition 8.4.5)

Table 1.1: Comparison between classical and exceptional cases

The starting point of the exceptional theta series is the work of Elkies and Gross [ElkiesGross, 1996]. For any Albert lattice J inside $\mathrm{J}_{\mathbb{R}}$, they construct the following theta series:

$$\vartheta_J = 1 + 240 \sum_{\substack{J \ni T \geq 0, \\ \operatorname{rank} T = 1}} \sigma_3(\mathrm{c}_J(T)) q^{\mathrm{Tr}(T)} \in \mathrm{M}_{12}(\mathbf{SL}_2(\mathbb{Z})),$$

where $\mathrm{c}_J(T)$ is the largest integer c such that $T/c \in J$, and $\sigma_3(n) = \sum_{d|n} d^3$. We extend the construction of Elkies-Gross by weighting this exceptional theta series:

Theorem H. *(Theorem 8.5.2 and Corollary 8.5.5) For any Albert lattice J inside $\mathrm{J}_{\mathbb{R}}$ and any homogeneous F_4-harmonic polynomial P on $\mathrm{J}_{\mathbb{R}}$, the theta series:*

$$\vartheta_{J,P} := \sum_{\substack{J \ni T \geq 0, \\ \operatorname{rank} T = 1}} \sigma_3(\mathrm{c}_J(T)) P(T) q^{\mathrm{Tr}(T)}$$

is a modular form of weight $2 \deg P + 12$ for $\mathbf{SL}_2(\mathbb{Z})$, and it is a cusp form if P is not constant.

As a consequence of Theorem G, we prove the following analogue of [Waldspurger, 1979]:

Theorem I. *(Corollary 8.6.13) For any $d > 0$, the space $\mathrm{S}_{2d+12}(\mathbf{SL}_2(\mathbb{Z}))$ is spanned by the set of weighted theta series $\vartheta_{J,P}$, as J varies over Albert lattices inside $\mathrm{J}_{\mathbb{R}}$ and P varies over F_4-harmonic polynomial of degree d over $\mathrm{J}_{\mathbb{R}}$.*

Organization

Chapter 3 recalls the definition of $\mathbf{F}_4$ and some results of Gross [Gross, 1996] on reductive $\mathbb{Z}$-models of $\mathbf{F}_4$, and we also give a new proof for Theorem B. We prove Theorem C in Chapter 4. In Chapter 5, we study the subgroups of the compact Lie group $\mathbf{F}_4(\mathbb{R})$ and prove Theorem D. In Chapter 6, we recall the theory of level one automorphic representations and the conjectures by Arthur and Langlands, mainly following [ChenevierRenard, 2015; ChenevierLannes, 2019]. Then we apply these conjectures to $\mathbf{F}_4$ and prove Theorem E and Theorem F in Chapter 7. Finally, Chapter 8, a reproduction of [Shan, 2025], studies the exceptional theta correspondence for the dual pair $\mathbf{PGL}_2 \times \mathbf{F}_4$, and proves Theorem G, Theorem H and Theorem I. Some figures and tables used in this thesis are provided in Appendix A.

Chapter 2

Introduction en français

Les travaux développés dans cette thèse appartiennent au domaine des *représentations automorphes*, en particulier de celles de niveau un pour un groupe algébrique exceptionnel anisotrope $\mathbf{F}_4$ défini sur $\mathbb{Q}$. La première partie (du Chapitre 3 au Chapitre 7) correspond à [Shan, 2024], dans lequel nous étudions le nombre de représentations automorphes de niveau un pour $\mathbf{F}_4$ ayant un poids arbitraire donné, et (*conjecturalement*) classifions leurs paramètres d'Arthur globaux. La second partie (le Chapitre 8) correspond à [Shan, 2025], et nous y considérons la correspondance thêta globale pour la paire duale $\mathbf{F}_4 \times \mathbf{PGL}_2$.

2.1 Une motivation : Représentations galoisiennes géométriques avec image donnée

Le groupe de Galois absolu $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ encode beaucoup d'informations arithmétiques sur les corps de nombres, et une manière naturelle d'étudier $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ consiste à considérer ses représentations, notamment celles provenant de la géométrie algébrique. Motivée par la théorie de Galois inverse, la question suivante a été étudiée par de nombreux mathématiciens :

Problème 1. *Soit ℓ un nombre premier et $\mathbf{H}$ un groupe algébrique réductif connexe défini sur $\overline{\mathbb{Q}_\ell}$. Existe-t-il une représentation galoisienne ℓ-adique $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbf{H}(\overline{\mathbb{Q}_\ell})$ qui soit semisimple et géométrique (au sens de Fontaine-Mazur [Taylor, 2004, Conjecture 1.1]), et dont l'image soit dense dans $\mathbf{H}(\overline{\mathbb{Q}_\ell})$ pour la topologie de Zariski?*

Dans le cas où $\mathbf{H} = \mathbf{GL}_2 \simeq \mathbf{GSp}_2$ ou $\mathbf{GSp}_4$, ou plus généralement un groupe classique (des similitudes), il existe de nombreuses constructions et exemples bien connus. Par exemple, on peut utiliser l'accouplement de Poincaré sur la cohomologie ℓ-adique des variétés algébriques pour construire des représentations galoisiennes dont l'image tombe dans un groupe classique. Le cas des groupes exceptionnels, c'est-à-dire les groupes de types de Lie G_2, F_4, E_6, E_7 et E_8, est plus difficile, mais nous avons encore quelques exemples dans [DettweilerReiter, 2010; GrossSavin, 1998; Yun, 2014; Patrikis, 2016; BoxerCalegariEmertonLevinMadapusi PeraPatrikis, 2019]. Remarquons que lorsque $\mathbf{H}$ est de type de Lie G_2 ou E_8, cette question est liée à la question de Serre sur les motifs [Serre, 1994, Question 8.8, §1].

Composant $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbf{H}(\overline{\mathbb{Q}_\ell})$ avec une représentation algébrique $\mathbf{H} \to \mathbf{GL}_n$, on obtient une représentation galoisienne ℓ-adique géométrique de dimension n. On peut associer deux invariants à une représentation galoisienne ℓ-adique géométrique $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbf{GL}_n(\overline{\mathbb{Q}_\ell})$: le *conducteur (d'Artin)* $\mathrm{N}(\rho) \in \mathbb{N}$ et les *poids de Hodge-Tate* $\mathrm{HT}(\rho)$, un multiensemble de n entiers (voir, par exemple, [Taylor, 2004]). Dans les travaux susmentionnés, les conducteurs des représentations galoisiennes ℓ-adiques géométriques construites ne sont généralement pas contrôlés. On peut affiner naturellement le Problème 1 en fixant ces deux invariants :

Problème 2. *Soit ℓ un nombre premier, $n \geq 1$ et H un sous-groupe réductif connexe de $\mathbf{GL}_n$ défini sur $\overline{\mathbb{Q}_\ell}$. Quel est le nombre (à équivalence près) des représentations galoisiennes ℓ-adiques géométriques $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbf{GL}_n(\overline{\mathbb{Q}_\ell})$ de conducteur et de poids de Hodge-Tate donnés, telles que l'adhérence de Zariski de $\mathrm{Im}(\rho)$ soit $\mathbf{H}(\overline{\mathbb{Q}_\ell})$?*

Pour $(\mathbf{H}, n) = (\mathbf{GL}_2, 2)$ ou $(\mathbf{SO}_{2g+1}, 2g+1)$, cette question est, par exemple, liée à la dimension des espaces de formes modulaires classiques ou de Siegel. Nous avons moins de connaissances sur les cas concernant d'autres groupes $\mathbf{H}$. Lorsque le conducteur $N = 1$, le Problème 2 est résolu *conjecturalement* par Chenevier et Renard dans [ChenevierRenard, 2015] pour les groupes suivants (n est choisi comme étant la dimension de la représentation standard lorsque $\mathbf{H}$ est un groupe classique (ou de similitudes), et $n = 7$ lorsque $\mathbf{H}$ est de type G_2) :

$$\mathbf{GL}_2 \simeq \mathbf{GSp}_2,\ \mathbf{GSp}_4,\ \mathbf{SO}_4,\ \mathbf{SO}_5,\ \mathbf{GSp}_6,\ \mathbf{GSp}_8,\ \mathbf{SO}_8,\ \mathbf{G}_2,$$

via la connexion conjecturale entre les représentations galoisiennes ℓ-adiques géométriques de dimension n et les représentations automorphes cuspidales de $\mathbf{GL}_n$. Voir également [Taïbi, 2017; ChenevierTaïbi, 2020] pour les dimensions supérieures. Dans [Lachaussée, 2020], Lachaussée étend les résultats pour $\mathbf{GSp}_{2g}$, $1 \leq g \leq 4$ au cas du conducteur d'Artin $N = 2$. Nous nous concentrons maintenant sur le cas du conducteur un (voir la Remarque 2.6.3 pour plus d'explications sur cette hypothèse).

Dans cette thèse, à la suite de [ChenevierRenard, 2015], nous donnons une solution *conjecturale* au Problème 2 dans le cas où $N = 1$, $\mathbf{H}$ est de type de Lie F_4, et $n = 26$. Pour une représentation galoisienne ℓ-adique géométrique de dimension 26, ρ, telle que $\overline{\mathrm{Im}(\rho)}$ est de type F_4, son multiensemble de poids de Hodge-Tate ne dépend que de 4 variables $a, b, c, d \in \mathbb{N}$ et a la forme:

$$\mathrm{HT}(a,b,c,d) := \left\{ \begin{array}{c} 0, 0, \pm a, \pm b, \pm(a+b), \pm(b+c), \pm(a+b+c), \pm(b+c+d), \pm(a+b+c+d), \pm(a+2b+c), \\ \pm(a+2b+c+d), \pm(a+2b+2c+d), \pm(a+3b+2c+d), \pm(2a+3b+2c+d). \end{array} \right\}$$

Comme corollaire conjectural de nos résultats dans cette thèse, nous proposons la conjecture suivante sur les représentations ℓ-adiques géométriques de type F_4 :

Conjecture A. *Le nombre de classes d'équivalence de représentations galoisiennes ℓ-adiques géométriques de dimension 26 et de conducteur un, ρ, telles que :*

- *l'adhérence de Zariski de $\mathrm{Im}(\rho)$ est un groupe réductif connexe de type F_4,*
- *et $\mathrm{HT}(\rho) = \mathrm{HT}(a,b,c,d)$, avec $a, b, c, d \geq 1$,*

est $\mathrm{F}_4(a-1,b-1,c-1,d-1)$*, où* $\mathrm{F}_4(\lambda)$ *est la fonction calculable sur* $\mathbb{N}^4$ *donnée par la Proposition 7.4.1.*

Remarque 2.1.1. La formule pour $\mathrm{F}_4(\lambda)$ contient tellement de termes que nous ne donnerons pas la formule complète dans cet article. Cependant, sous certaines hypothèses, cette formule devient beaucoup plus simple. Par exemple, lorsque $a > b+c+d+3$, $b,c,d > 0$ et c,d sont impairs, une formule simplifiée pour $\mathrm{F}_4(a,b,c,d)$ est donnée dans la Remarque 7.4.2.

2.2 Une variante automorphe du problème de comptage

Nous transférons maintenant le Problème 2 du côté automorphe. Soit $\mathbf{G}$ un groupe réductif connexe sur $\mathbb{Q}$ avec un modèle réductif sur $\mathbb{Z}$ (voir la Section 3.2). Comme nous allons parler de représentations galoisiennes, il sera pratique de supposer que son groupe dual de Langlands $\widehat{\mathbf{G}}$ est défini sur $\overline{\mathbb{Q}}$, et nous fixons deux plongements : $\mathbb{C} \overset{\iota_\infty}{\hookleftarrow} \overline{\mathbb{Q}} \overset{\iota_\ell}{\hookrightarrow} \overline{\mathbb{Q}_\ell}$. Nous fixons également un sous-groupe compact maximal G_c de $\widehat{\mathbf{G}}(\mathbb{C})$.

Soit π une représentation automorphe L-*algébrique*[1] de niveau un pour $\mathbf{G}$. D'après une conjecture de Buzzard et Gee [BuzzardGee, 2014, Conjecture 3.2.1], on devrait pouvoir associer à π une représentation galoisienne ℓ-adique géométrique compatible de conducteur un, $\rho_{\pi,\iota} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \widehat{\mathbf{G}}(\overline{\mathbb{Q}_\ell})$, dépendant du choix de plongements $\iota = (\iota_\infty, \iota_\ell)$. Selon ces conjectures standards de Fontaine-Mazur et Langlands, toute représentation galoisienne ℓ-adique géométrique de conducteur un vers $\widehat{\mathbf{G}}(\overline{\mathbb{Q}_\ell})$ devrait provenir de cette manière. Si deux morphismes élément-conjugués d'un groupe de Lie compact connexe dans G_c sont conjugués (voir la Section 5.1 pour une explication détaillée), la question suivante fournit une variante automorphe du Problème 2 pour $\mathbf{H} = \widehat{\mathbf{G}} \times_{\iota_\ell} \overline{\mathbb{Q}_\ell}$:

Problème 3. *Soit* $\mathbf{G}$ *un groupe réductif connexe sur* $\mathbb{Q}$ *admettant un modèle réductif sur* $\mathbb{Z}$*.*

(1) *(*Comptage*) Compter le nombre (à équivalence près) de représentations automorphes discrètes algébriques*[2] *de niveau un pour* $\mathbf{G}$ *avec une composante archimédienne donnée arbitrairement.*

(2) *(*Raffinement*) Raffiner ce comptage par les « *groupes de Sato-Tate* » des représentations automorphes.*

Remarque 2.2.1 (« Groupes de Sato-Tate »)*.* Dans la question ci-dessus, le « groupe de Sato-Tate » $\mathrm{H}(\pi)$ d'une représentation automorphe de niveau un π pour $\mathbf{G}$ est une certaine classe de conjugaison de sous-groupes de G_c que nous expliquerons en détail dans la Section 6.3.1. Nous pouvons l'introduire brièvement comme suit. En se basant sur la paramétrisation d'Arthur des représentations automorphes, on peut *conjecturalement* associer à π un morphisme de groupes

$$\psi_\pi : \mathcal{L}_\mathbb{Z} \times \mathrm{SU}(2) \to G_c,$$

[1]Pour la définition de L-algébricité, voir [BuzzardGee, 2014, Definition 2.3.1]. Pour une représentation qui est algébrique au sens de la Définition 6.4.3 mais pas L-algébrique, il faut remplacer $\widehat{\mathbf{G}}$ par un certain groupe de « similitude ».

[2]On peut enlever cette condition d'algébricité en se restreignant aux $\mathbb{Q}$-groupes semi-simples.

où $\mathcal{L}_{\mathbb{Z}}$ est *l'hypothétique groupe de Langlands de* $\mathbb{Z}$, qui est connexe et compact (voir la Section 6.3). Nous définissons $\mathrm{H}(\pi)$ comme la classe de conjugaison de l'image de ψ_π dans G_c. Lorsque la restriction de ψ_π à $1 \times \mathrm{SU}(2) \subseteq \mathcal{L}_{\mathbb{Z}} \times \mathrm{SU}(2)$ est triviale, cette notion $\mathrm{H}(\pi)$ coïncide avec la notion usuelle des groupes de Sato-Tate. En général, nous avons décidé d'inclure le facteur $\mathrm{SU}(2)$ dans la définition car il permet d'énoncer plus facilement certains de nos résultats.

Le point de la partie « raffinement » dans le Problème 3 est que, en général, de nombreuses représentations automorphes discrètes de niveau un π pour $\mathbf{G}$, par exemple les représentations *endoscopiques*, auront un groupe de Sato-Tate strictement plus petit que G_c. Pour ces π, $\overline{\mathrm{Im}(\rho_{\pi,\iota})}$ devrait être un sous-groupe propre de $\widehat{\mathbf{G}}(\overline{\mathbb{Q}_\ell})$. Nous devons donc trouver un moyen d'exclure ces représentations pour obtenir le nombre souhaité dans le Problème 2.

Dans [ChenevierRenard, 2015], Chenevier et Renard résolvent la partie (1) du Problème 3 pour plusieurs groupes classiques de petits rangs, à savoir, $\mathbf{G}$ est l'un des groupes suivants :

$$\mathbf{SL}_2 = \mathbf{Sp}_2,\ \mathbf{Sp}_4,\ \mathbf{SO}_{2,2},\ \mathbf{SO}_{3,2},\ \mathbf{SO}_7,\ \mathbf{SO}_8 \text{ et } \mathbf{SO}_9,$$

ainsi que pour un groupe semisimple connexe de type G_2 sur $\mathbb{Q}$ avec des points réels compacts. Pour la partie (2) du Problème 3, leur méthode repose de manière essentielle sur la classification d'Arthur des représentations automorphes [Arthur, 1989; Arthur, 2013]. Leurs résultats pour $\mathbf{SO}_7, \mathbf{SO}_8, \mathbf{SO}_9$ et $\mathbf{G}_2$ sont conditionnels aux conjectures d'Arthur pour ces groupes, puisque $\mathbf{SO}_7, \mathbf{SO}_8$ et $\mathbf{SO}_9$ ne sont pas quasi-déployés, et $\mathbf{G}_2$ n'est pas couvert par les résultats d'Arthur.

Dans [Taïbi, 2017; Taïbi, 2019], Taïbi rend ces résultats inconditionnels (sauf pour $\mathbf{G}_2$), et il les étend également aux groupes classiques déployés suivants :

$$\mathbf{Sp}_{2g} \text{ avec } g \leq 7,\ \mathbf{SO}_{n+1,n} \text{ avec } n \leq 8 \text{ et } \mathbf{SO}_{2m,2m} \text{ avec } m \leq 4.$$

En particulier, la solution de Taïbi au Problème 3 pour $\mathbf{Sp}_8$ sera importante dans notre travail.

Dans la première partie de cette thèse, nous appliquons la méthode de [ChenevierRenard, 2015] à $\mathbf{F}_4$, le groupe algébrique semisimple connexe unique (à isomorphisme près) sur $\mathbb{Q}$ de type F_4, avec des points réels compacts et déployé sur $\mathbb{Q}_p$ pour chaque premier p (voir la Section 3.1). Pour ce groupe, les représentations automorphes sont automatiquement L-algébriques. De plus, il s'avère qu'il n'y a pas de problèmes de conjugaison local-global pour les sous-groupes connexes de $(\mathbf{F}_4)_c = \mathbf{F}_4(\mathbb{R})$ (voir la Proposition 5.1.5). En conséquence, la Conjecture A découle des conjectures standards et de notre réponse au Problème 3 pour $\mathbf{F}_4$.

Remarque 2.2.2. Les représentations automorphes de $\mathbf{F}_4$ (et leurs composantes locales) ont été étudiées dans [Savin, 1994; MagaardSavin, 1997; Gan, 2000; Pollack, 2023; KarasiewiczSavin, 2023] via les correspondances thêta exceptionnelles, et nous expliquerons certains liens entre ces correspondances et notre travail dans la Section 7.5. Nous mentionnons également que les représentations automorphes pour $\mathbf{F}_4$ ont été étudiées dans le passé par Padowitz [Padowitz, 1998, §9]. Padowitz considère plutôt les représentations automorphes qui sont Steinberg pour un ensemble fixe *non vide* de nombres premiers et non ramifiées ailleurs, et il tente de les énumérer en utilisant la formule de trace stable, dans l'esprit des travaux de Gross-Pollack [GrossPollack, 2005]. Les résultats sont partiels, car plusieurs intégrales orbitales locales stables ne sont pas déterminées[3], et nous espérons revenir sur cette question à l'avenir.

2.3 Comptage des représentations automorphes de niveau un

Dans [Gross, 1996], Gross prouve le résultat suivant pour $\mathbf{F}_4$, qui est important pour notre solution à la partie (1) du Problème 3 pour $\mathbf{F}_4$:

Théorème B. *(Proposition 3.3.6) À isomorphisme près sur $\mathbb{Z}$, il existe deux schémas en groupes affines lisses sur $\mathbb{Z}$ dont la fibre générique est isomorphe à $\mathbf{F}_4$, et dont la fibre spéciale sur $\mathbb{Z}/p\mathbb{Z}$ est réductive pour tous les nombres premiers p.*

Les schémas en groupes sur $\mathbb{Z}$ dans le Théorème B sont des modèles réductifs de $\mathbf{F}_4$. Leurs constructions sont liées aux structures intégrales de l'algèbre de Jordan exceptionnelle définie de dimension 27 sur $\mathbb{Q}$. Gross démontre ce résultat en utilisant la formule de masse pour $\mathbf{F}_4$ et certains résultats dans [ATLAS]. L'objectif du Chapitre 3 est de rappeler la construction de $\mathbf{F}_4$ et de donner une nouvelle preuve du Théorème B sans utiliser [ATLAS].

Puisque la méthode de comptage dans [ChenevierRenard, 2015] peut être appliquée à tout groupe algébrique défini sur $\mathbb{Q}$ qui a des points réels compacts et qui admet un modèle réductif sur $\mathbb{Z}$, nous rappelons et appliquons cette méthode à $\mathbf{F}_4$ dans le Chapitre 4. Une donnée importante est la structure (par exemple les générateurs, les classes de conjugaison) du sous-groupe fini $\mathscr{G}(\mathbb{Z})$ de $\mathbf{F}_4(\mathbb{R})$, où $\mathscr{G}$ est l'un des deux modèles réductifs sur $\mathbb{Z}$ dans le Théorème B. Cette donnée est fournie par notre analyse dans la démonstration du Théorème B. Nous obtenons la réponse pour la partie (1) du Problème 3 pour $\mathbf{F}_4$:

Théorème C. *(Théorème 4.6.1 et Corollaire 6.1.8)*

(1) Pour une représentation irréductible V_λ de $\mathbf{F}_4(\mathbb{R})$ de plus haut poids λ, nous avons une formule explicite et calculable pour le nombre $d(\lambda)$ de classes d'équivalence de représentations automorphes de niveau un π avec $\pi_\infty \simeq \mathrm{V}_\lambda$.

(2) Pour les poids dominants $\lambda = \sum_{i=1}^4 \lambda_i \varpi_i$[4] satisfaisant $2\lambda_1 + 3\lambda_2 + 2\lambda_3 + \lambda_4 \leq 13$, nous listons les nombres $d(\lambda)$ dans la Table A.3.

2.4 Candidats pour les groupes de Sato-Tate

La partie (2) du Problème 3 implique une classification de tous les groupes de Sato-Tate possibles pour les représentations automorphes de niveau un de $\mathbf{F}_4$. Pour ce groupe défini sur $\mathbb{Q}$, son groupe dual de Langlands $\widehat{\mathbf{F}_4}$ est isomorphe à $\mathbf{F}_4 \times_{\mathbb{Q}} \mathbb{C}$, et comme mentionné dans la Remarque 2.2.1, les groupes de Sato-Tate dans ce cas sont des classes de conjugaison de sous-groupes du groupe de Lie compact $\mathbf{F}_4(\mathbb{R})$. Le but du Chapitre 5 est d'exclure certains sous-groupes de $\mathbf{F}_4(\mathbb{R})$, et de donner une liste de candidats pour les groupes de Sato-Tate :

[3] Un autre problème mineur est que l'auteur affirme dans [Padowitz, 1998, P.42] que la représentation irréductible de 26 dimensions de $\mathbf{F}_4$ est « excellente » dans son sens, ce qui n'est pas correct. Voir la Remarque 4.5.5 pour un contre-exemple.

[4] Nous suivons ici les notations de [Bourbaki, 2002, §VI.4.9].

Théorème D. *(Théorème 5.6.7) Il existe* 13 *classes de conjugaison de sous-groupes propres et connexes H de $\mathbf{F}_4(\mathbb{R})$ tels que :*

- *le centralisateur de H dans $\mathbf{F}_4(\mathbb{R})$ est isomorphe au produit d'un nombre fini de copies de $\mathbb{Z}/2\mathbb{Z}$;*
- *le poids nul apparaît deux fois dans la restriction de la représentation irréductible de dimension* 26 *de $\mathbf{F}_4(\mathbb{R})$ à H.*

Nous prouverons ce résultat de classification étape par étape, en suivant la stratégie de Dynkin dans [Dynkin, 1952]. Il convient de mentionner deux ingrédients importants dans la démonstration :

- Un résultat de conjugaison local-global (Proposition 5.1.5) pour $\mathbf{F}_4(\mathbb{R})$, que nous avons déjà mentionné à la fin de la Section 2.2. Cela repose sur un résultat concernant les algèbres de Lie (Théorème 5.1.3) prouvé par Losev dans [Losev, 2010].
- Un critère utile (Proposition 5.2.1) donné dans la Section 5.2 pour la conjugaison de deux morphismes d'un groupe de Lie compact connexe dans $\mathbf{F}_4(\mathbb{R})$.

Exemple 2.4.1. Parmi les classes de conjugaison des sous-groupes classifiés dans le Théorème D, nous avons

$$\mathrm{Spin}(9),\ \mathrm{Spin}(8),\ \mathrm{G}_2 \times \mathrm{SO}(3),\ (\mathrm{Sp}(3) \times \mathrm{SU}(2))\,/\mu_2^{\Delta},\ (\mathrm{Sp}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2))\,/\mu_2^{\Delta},$$

où les notations seront expliquées dans les Notations 5.3.1 et 5.3.3. Les sous-groupes restants sont tous centralement isogènes à des produits de n copies de $\mathrm{SU}(2)$, $n \leq 4$. Notons que parmi les sous-groupes listés ci-dessus, seuls $\mathrm{Spin}(9)$ et $(\mathrm{Sp}(3) \times \mathrm{SU}(2))\,/\mu_2^{\Delta}$ sont des sous-groupes réguliers connexes propres maximaux de $\mathbf{F}_4(\mathbb{R})$.

2.5 Les conjectures d'Arthur

Comme dans [ChenevierRenard, 2015], pour la partie (2) du Problème 3, nous avons besoin de quelques conjectures sur les représentations automorphes. Pour un groupe algébrique réductif connexe $\mathbf{G}$ sur $\mathbb{Q}$, Arthur introduit dans [Arthur, 1989] une paramétrisation conjecturale des représentations automorphes discrètes, via les *paramètres d'Arthur globaux discrets* pour $\mathbf{G}$. Dans le cas du niveau un, ces paramètres sont des classes de conjugaison par $\widehat{\mathbf{G}}(\mathbb{C})$ de morphismes admissibles

$$\psi : \mathcal{L}_{\mathbb{Z}} \times \mathbf{SL}_2(\mathbb{C}) \to \widehat{\mathbf{G}}(\mathbb{C}),$$

où $\mathcal{L}_{\mathbb{Z}}$ est l'hypothétique groupe de Langlands de $\mathbb{Z}$ (voir la Section 6.3 pour plus de détails), et $\widehat{\mathbf{G}}$ est le groupe dual de Langlands de $\mathbf{G}$. Arthur propose une formule conjecturale pour la multiplicité d'une représentation irréductible de $\mathbf{G}(\mathbb{A})$ dans le spectre automorphe discret de $\mathbf{G}$, en termes des paramètres d'Arthur globaux associés.

Dans [Arthur, 2013], Arthur reformule ses conjectures pour tout groupe classique quasi-déployé $\mathbf{G}$, évitant l'apparition du groupe de Langlands $\mathcal{L}_{\mathbb{Z}}$. Dans ce cas, il relie les paramètres globaux d'Arthur pour $\mathbf{G}$ aux représentations automorphes cuspidales des groupes linéaires, et il démontre les classifications endoscopiques, en s'appuyant notamment sur les travaux de Mœglin-Waldspurger [MoeglinWaldspurger, 2014], Ngô [Ngô, 2010] et bien d'autres. Nous renvoyons à [ChenevierLannes, 2019, §8] pour des énoncés précis des résultats d'Arthur dans le cas des représentations automorphes cohomologiques de niveau un pour les groupes classiques.

Bien entendu, $\mathbf{F}_4$ n'est pas un groupe classique, et les conjectures générales d'Arthur [Arthur, 1989] restent ouvertes dans ce cas. Néanmoins, elles peuvent encore être formulées assez précisément si l'on admet l'existence de $\mathcal{L}_{\mathbb{Z}}$. Voir aussi [ChenevierLannes, 2019, §6.4] pour quelques formes générales des conjectures d'Arthur dans le cas du niveau un.

Notation 2.5.1. Dans le reste de la thèse, nous marquerons tout résultat conditionnel à l'existence de $\mathcal{L}_{\mathbb{Z}}$ et à la formule des multiplicités d'Arthur (Conjecture 6.6.5) par $*$.

Nous expliquons brièvement les conjectures d'Arthur pour $\mathbf{F}_4$, et une description plus précise dans le cas général des groupes simplement connexes anisotropes admettant des $\mathbb{Z}$-modèles réductifs sera donnée au Chapitre 6. Pour une représentation automorphe de niveau un π de $\mathbf{F}_4$, avec un paramètre d'Arthur global $\psi : \mathcal{L}_{\mathbb{Z}} \times \mathbf{SL}_2(\mathbb{C}) \to \mathbf{F}_4(\mathbb{C})$, nous pouvons composer ψ avec la représentation irréductible $\mathrm{r} : \mathbf{F}_4(\mathbb{C}) \to \mathbf{GL}_{26}(\mathbb{C})$[5] de dimension 26, et nous obtenons ainsi une représentation de $\mathcal{L}_{\mathbb{Z}} \times \mathbf{SL}_2(\mathbb{C})$. Cette représentation se décompose comme suit :

$$\mathrm{r} \circ \psi \simeq \pi_1[d_1] \oplus \cdots \oplus \pi_k[d_k], \qquad (\star)$$

où π_i est une représentation irréductible de $\mathcal{L}_{\mathbb{Z}}$ de dimension n_i, et $[d_i]$ désigne la représentation irréductible de $\mathbf{SL}_2(\mathbb{C})$ de dimension d_i, et $\sum_{i=1}^{k} n_i d_i = 26$. Nous identifions π_i comme une représentation cuspidale de niveau un de $\mathbf{PGL}_{n_i}$, et observons qu'elle est toujours auto-duale et algébrique dans ce cas (voir la Section 6.4). De manière similaire à [Arthur, 2013], nous considérons le paramètre d'Arthur global ψ comme une somme formelle des $\pi_i[d_i]$.

Nous déduisons du Théorème D que le groupe de Sato-Tate de tout π_i apparaissant dans la décomposition $(\star)$ est l'un des groupes de Lie compacts suivants :

$$\mathrm{SU}(2),\ \mathrm{Sp}(2),\ \mathrm{Sp}(3),\ \mathrm{SO}(8),\ \mathrm{SO}(9),\ \mathrm{G}_2, \mathbf{F}_4(\mathbb{R}). \qquad (\star\star)$$

Les représentations cuspidales avec le groupe de Sato-Tate $\mathbf{F}_4(\mathbb{R})$ correspondent conjecturalement aux représentations ℓ-adiques souhaitées dans le Problème 2, et celles ayant d'autres groupes de Sato-Tate dans $(\star\star)$ sont liées aux représentations automorphes de niveau un des groupes suivants :

$$\mathbf{PGL}_2,\ \mathbf{SO}_{3,2},\ \mathbf{SO}_7,\ \mathbf{SO}_8,\ \mathbf{Sp}_8, \mathbf{G}_2,$$

qui ont déjà été étudiés dans [ChenevierRenard, 2015; Taïbi, 2017; ChenevierTaïbi, 2020].

Réciproquement, étant donné un paramètre d'Arthur global $\psi : \mathcal{L}_{\mathbb{Z}} \times \mathbf{SL}_2(\mathbb{C}) \to \mathbf{F}_4(\mathbb{C})$ dont la « composante archimédienne » est un *paramètre d'Adams-Johnson* (voir la Définition 6.6.1 et la Remarque 6.6.2), la multiplicité de sa représentation irréductible correspondante de $\mathbf{F}_4(\mathbb{A})$ dans le spectre automorphe peut être calculée via la formule d'Arthur dans la [Arthur, 1989], et une formule explicite pour $\mathbf{F}_4$ sera donnée dans la Section 7.2.

[5] L'image de r est même incluse dans $\mathbf{SO}_{26}(\mathbb{C}) \subset \mathbf{SL}_{26}(\mathbb{C}) \subset \mathbf{GL}_{26}(\mathbb{C})$.

2.6 Raffinement du comptage

Le but du Chapitre 7 est de raffiner le comptage dans le Théorème C. Pour un paramètre d'Arthur global $\psi : \mathcal{L}_{\mathbb{Z}} \times \mathbf{SL}_2(\mathbb{C}) \to \mathbf{F}_4(\mathbb{C})$, on peut associer deux invariants :

- son groupe de Sato-Tate $\mathrm{H}(\psi) := \psi(\mathcal{L}_{\mathbb{Z}} \times \mathrm{SU}(2))$, vu comme une classe de conjugaison de sous-groupes dans le groupe compact $\mathbf{F}_4(\mathbb{R})$;
- ses « poids », c'est-à-dire les valeurs propres de son caractère infinitésimal sous la représentation irréductible de dimension 26, $\mathrm{r} : \mathbf{F}_4 \to \mathbf{SL}_{26}$.

Étant donné une classe de conjugaison de sous-groupes propres H de $\mathbf{F}_4(\mathbb{R})$ apparaissant dans le Théorème D, nous classifions toutes les décompositions possibles $(\star)$ de $\mathrm{r} \circ \psi$ pour les paramètres d'Arthur globaux ψ avec $\mathrm{H}(\psi) = H$ dans la Section 7.3. Si ψ correspond à une représentation irréductible de niveau un de $\mathbf{F}_4(\mathbb{A})$, une partie importante de notre travail consiste à donner une formule exacte pour la multiplicité de π, pour chaque cas de groupes de Sato-Tate. Grossièrement, la multiplicité dépend de la façon dont les poids de ψ sont répartis dans les termes $\pi_i[d_i]$ de $(\star)$. En conclusion, nous obtenons le résultat suivant :

Théorème* E. *(Théorème 7.3.1)*

(a) Le groupe de Sato-Tate d'une représentation automorphe de niveau un pour $\mathbf{F}_4$ est soit $\mathbf{F}_4(\mathbb{R})$, soit l'un des sous-groupes propres de $\mathbf{F}_4(\mathbb{R})$ apparaissant dans le Théorème D à l'exception de $\mathrm{Spin}(8)$.

(b) Pour les paramètres d'Arthur globaux de $\mathbf{F}_4$ ayant un groupe de Sato-Tate donné, la multiplicité de sa représentation irréductible de niveau un de $\mathbf{F}_4(\mathbb{A})$ correspondante (0 ou 1) est donnée explicitement par les formules des Proposition 7.3.4 à Proposition 7.3.18.

Remarque 2.6.1. Nous observons que tous les sous-groupes du Théorème D ne proviennent pas de *groupes endoscopiques* de $\mathbf{F}_4$, au sens de [Arthur, 2013]. Par exemple, le sous-groupe $\mathrm{G}_2 \times \mathrm{SO}(3)$ a un centralisateur trivial dans $\mathbf{F}_4(\mathbb{R})$, il ne peut donc pas être le centralisateur d'un élément de $\mathbf{F}_4(\mathbb{R})$. En conséquence, notre raffinement conjectural est plus fin que la classification endoscopique d'Arthur.

Étant donné une représentation irréductible V_λ de $\mathbf{F}_4(\mathbb{R})$, d'après le Théorème C, nous connaissons le nombre de classes d'équivalence de représentations automorphes de niveau un π pour $\mathbf{F}_4$ telles que $\pi_\infty \simeq \mathrm{V}_\lambda$. Les poids du paramètre d'Arthur global ψ_π de π sont déterminés par V_λ. Nous pouvons énumérer tous les paramètres d'Arthur globaux possibles avec ces poids, puis utiliser les formules de multiplicité dans le Théorème E pour déterminer leurs multiplicités. De cette manière, nous obtenons un raffinement *conjectural* du comptage dans le Théorème C. Comme conséquence, nous obtenons une solution conjecturale au Problème 2, énoncée en termes de représentations automorphes :

Théorème* F. *(Proposition 7.4.1 et Proposition 7.4.3) Le nombre de représentations automorphes cuspidales, algébriques, de niveau un pour $\mathbf{GL}_{26}$, satisfaisant :*

- *le groupe de Sato-Tate est $\mathbf{F}_4(\mathbb{R})$,*

- *et le multiensemble de poids*[6] *est* $\mathrm{HT}(a,b,c,d)$ *pour* $a,b,c,d \geq 1$,

est $\mathrm{F}_4(a-1,b-1,c-1,d-1)$, *où* $\mathrm{F}_4(\lambda)$ *est une fonction explicite sur* $\mathbb{N}^4$ *donnée par la Proposition 7.4.1.*

Exemple 2.6.2. Les quadruplets $(a,b,c,d) \in \mathbb{N}^4$ tels que :

- le plus grand poids $2a+3b+2c+d+8$ dans le multiensemble $\mathrm{HT}(a+1,b+1,c+1,d+1)$ n'est pas plus grand que 22,
- et $\mathrm{F}_4(a,b,c,d) \neq 0$,

sont listés dans la Table A.8. Nous listons également les valeurs de $\mathrm{F}_4(a,b,c,d)$ pour ces quadruplets.

Remarque 2.6.3. On pourrait vouloir enlever la condition de niveau un, comme dans [Lachaussée, 2020]. Pour la partie (1) du Problème 3 pour $\mathbf{F}_4$, il est possible de calculer la dimension des invariants sous d'autres sous-groupes de congruence, et d'obtenir des résultats similaires au Théorème C pour des niveaux supérieurs. Cependant, pour la partie (2) du Problème 3 pour $\mathbf{F}_4$, ce que nous utilisons est une version simplifiée de la recette d'Arthur dans [Arthur, 1989]. En autorisant les ramifications en un certain nombre premier p, on a besoin de certaines propriétés des *paquets d'Arthur locaux* pour $\mathbf{F}_4(\mathbb{Q}_p)$, qui nous restent encore inconnues.

2.7 Lien avec les correspondances thêta exceptionnelles

En gros, pour une *paire duale réductive* $\mathbf{G} \times \mathbf{H}$ à l'intérieur de $\mathbf{E}$, où $\mathbf{E}$ est un groupe algébrique défini sur $\mathbb{Q}$ admettant une *représentation minimale*, la *correspondance thêta locale (*resp. *globale)* étudie la « restriction » d'une représentation minimale de $\mathbf{E}(F)$, F étant un corps local (*resp.* $\mathbf{E}(\mathbb{A})$), à $\mathbf{G}(F) \times \mathbf{H}(F)$ (*resp.* $\mathbf{G}(\mathbb{A}) \times \mathbf{H}(\mathbb{A})$), et donne une correspondance entre les représentations de $\mathbf{G}$ et de $\mathbf{H}$. Dans la deuxième partie (Chapitre 8), nous étudions la correspondance thêta globale pour la paire duale $\mathbf{F}_4 \times \mathbf{PGL}_2$ à l'intérieur de $\mathbf{E}_7$, un groupe exceptionnel de type E_7 et de rang réel 3, et l'objectif principal est de démontrer le théorème suivant :

Théorème G. *(Théorème 8.6.12) Soit* π *la représentation automorphe algébrique de niveau un de* $\mathbf{PGL}_2$ *associée à une forme parabolique propre de Hecke pour* $\mathbf{SL}_2(\mathbb{Z})$*. Sous la correspondance thêta globale pour* $\mathbf{F}_4 \times \mathbf{PGL}_2$*, le* Θ*-lift global* $\Theta(\pi)$ *est une représentation automorphe irréductible non nulle de* $\mathbf{F}_4$*.*

Nous renvoyons à l'introduction détaillée du Chapitre 8 concernant cette correspondance thêta exceptionnelle globale. Ici, nous présentons une motivation issue de notre calcul conjectural dans le Théorème E.

D'après le théorème de Flath, la représentation automorphe π dans le Théorème G est factorisée commme un produit restreint $\otimes'_v \pi_v$, où π_v est une représentation irréductible de $\mathbf{PGL}_2(\mathbb{Q}_v)$. Les résultats de la correspondance thêta exceptionnelle locale pour $\mathbf{F}_4 \times \mathbf{PGL}_2$

[6] Voir la Section 6.4 pour la définition précise des poids pour une représentation automorphe cuspidale algébrique de niveau un de $\mathbf{GL}_n$.

[GrossSavin, 1998, Proposition 3.2; Savin, 1994; KarasiewiczSavin, 2023] montrent que, pour toute place $v = p$ ou ∞, le (grand) relèvement thêta local $\Theta(\pi_v)$ est une représentation irréductible non nulle de $\mathbf{F}_4(\mathbb{Q}_v)$. Nous prenons pour Π la représentation irréductible $\otimes'_v\Theta(\pi_v)$ de $\mathbf{F}_4(\mathbb{A})$. En utilisant la formule explicite (conjecturale) de multiplicité dans la Proposition 7.3.6, nous trouvons que la multiplicité de Π dans le spectre automorphe de $\mathbf{F}_4$ est toujours égale à 1, quel que soit le choix de π. Il est donc naturel de s'attendre que le Θ-lift global $\Theta(\pi)$ soit non nul pour tout π associé à une forme parabolique propre de niveau $\mathbf{SL}_2(\mathbb{Z})$.

Remarque 2.7.1. Une autre correspondance thêta exceptionnelle liée au Théorème E de manière similaire est celle de la paire $\mathbf{F}_4 \times \mathbf{G}_2^{\mathrm{s}}$, où $\mathbf{G}_2^{\mathrm{s}}$ est le groupe déployé de type G_2 sur $\mathbb{Q}$. Dans [Pollack, 2023], Pollack montre que toute représentation automorphe cuspidale de niveau un associée à une forme modulaire quaternionique de $\mathbf{G}_2^{\mathrm{s}}$ possède un Θ-lift global non nul vers $\mathbf{F}_4$.

2.8 Séries thêta exceptionnelles

Notre principal outil pour démontrer le Théorème G est de développer une notion de « séries thêta exceptionnelles », motivée par [Pollack, 2023]. Il s'agit d'une variante de la séries thêta (pondérée) classique associée à un réseau unimodulaire pair L dans l'espace euclidien $\mathbb{R}^n$ et un polynôme harmonique homogène P sur $\mathbb{R}^n$:

$$\vartheta_{L,P} = \sum_{v\in L} P(v) q^{\frac{v.v}{2}},\ \text{où } q = e^{2\pi i z}, z \in \mathcal{H} = \{x+iy \,|\, x,y\in\mathbb{R}, y>0\}\,.$$

Cette série thêta est une forme modulaire de niveau $\mathbf{SL}_2(\mathbb{Z})$ et de poids $n/2 + \deg P$, et est parabolique si P n'est pas constant. Dans [Waldspurger, 1979], Waldspurger montre que, pour toute paire fixée de nombres naturels (n,d), où $8|n$, l'espace $\mathrm{S}_{n/2+d}(\mathbf{SL}_2(\mathbb{Z}))$ des formes paraboliques de poids $n/2+d$ est engendré par les $\vartheta_{L,P}$, où L varie sur les réseaux unimodulaires pairs dans l'espace euclidien $\mathbb{R}^n$ et P varie sur les polynômes harmoniques homogènes de degré d sur $\mathbb{R}^n$.

Dans le cas exceptionnel, nous remplaçons le cadre classique par les objets correspondants dans le tableau suivant :

Cas classique	Cas exceptionnel
Espace euclidien $\mathbb{R}^n$	Algèbre de Jordan exceptionnelle $\mathrm{J}_{\mathbb{R}}$ (Définition 3.1.3)
Réseau unimodulaire pair	*Réseau d'Albert* dans $\mathrm{J}_{\mathbb{R}}$ (Définition 8.2.12)
Polynômes harmoniques	Polynômes « F_4-*harmoniques* » (Définition 8.4.5)

Table 2.1: Comparaison entre les cas classique et exceptionnel

Le point de départ de la séries thêta exceptionnelle est le travail d'Elkies et Gross [Elkies-Gross, 1996]. Pour tout réseau d'Albert J dans $\mathrm{J}_{\mathbb{R}}$, ils construisent la série thêta suivante :

$$\vartheta_J = 1 + 240 \sum_{\substack{J\ni T\geq 0,\\ \mathrm{rang}\, T=1}} \sigma_3(\mathrm{c}_J(T)) q^{\mathrm{Tr}(T)} \in \mathrm{M}_{12}(\mathbf{SL}_2(\mathbb{Z})),$$

où $c_J(T)$ est le plus grand entier c tel que $T/c \in J$, et $\sigma_3(n) = \sum_{d|n} d^3$. Nous étendons la construction d'Elkies-Gross en pondérant cette série thêta exceptionnelle :

Théorème H. *(Théorème 8.5.2 et Corollaire 8.5.5) Pour tout réseau d'Albert J dans $\mathrm{J}_{\mathbb{R}}$ et tout polynôme homogène F_4-harmonique P sur $\mathrm{J}_{\mathbb{R}}$, la série thêta :*

$$\vartheta_{J,P} := \sum_{\substack{J \ni T \geq 0, \\ \mathrm{rang}\, T = 1}} \sigma_3(\mathrm{c}_J(T)) P(T) q^{\mathrm{Tr}(T)}$$

est une forme modulaire de poids $2 \deg P + 12$ pour $\mathbf{SL}_2(\mathbb{Z})$, et c'est une forme parabolique si P n'est pas constant.

En conséquence du Théorème G, nous prouvons l'analogue suivant de [Waldspurger, 1979] :

Théorème I. *(Corollaire 8.6.13) Pour tout $d > 0$, l'espace $\mathrm{S}_{2d+12}(\mathbf{SL}_2(\mathbb{Z}))$ est engendré par l'ensemble des séries thêta pondérées $\vartheta_{J,P}$, où J varie sur les réseaux d'Albert dans $\mathrm{J}_{\mathbb{R}}$ et P varie sur les polynômes F_4-harmoniques de degré d sur $\mathrm{J}_{\mathbb{R}}$.*

Organisation

Le Chapitre 3 rappelle la définition de $\mathbf{F}_4$ et certains résultats de Gross [Gross, 1996] sur les modèles réductifs de $\mathbf{F}_4$ sur $\mathbb{Z}$. Nous y donnons également une nouvelle démonstration du Théorème B. Nous prouvons le Théorème C dans le Chapitre 4. Dans le Chapitre 5, nous étudions les sous-groupes du groupe de Lie compact $\mathbf{F}_4(\mathbb{R})$ et prouvons le Théorème D. Dans le Chapitre 6, nous rappelons la théorie des représentations automorphes de niveau un et les conjectures d'Arthur et Langlands, principalement en suivant [ChenevierRenard, 2015; Chenevier-Lannes, 2019]. Nous appliquons ensuite ces conjectures à $\mathbf{F}_4$ et prouvons les Théorème E et Théorème F dans le Chapitre 7. Enfin, le Chapitre 8, qui est une reproduction de [Shan, 2025], étudie la correspondance thêta exceptionnelle pour la paire duale $\mathbf{PGL}_2 \times \mathbf{F}_4$, et prouve les Théorème G, Théorème H et Théorème I. Certaines figures et tables utilisées dans la thèse sont fournies dans l'annexe.

Chapter 3

Exceptional group $\mathbf{F}_4$ and its reductive $\mathbb{Z}$-models

This chapter introduces the algebraic group $\mathbf{F}_4$ that we will discuss in this thesis, with a focus on its reductive $\mathbb{Z}$-models.

3.1 The compact group F_4 and its rational structure

To construct Lie groups of exceptional types, we need to recall the notion of octonions, and our main reference is [Conrad, 2015, §5].

Definition 3.1.1. An *octonion algebra* C over a field k is a non-associative k-algebra of k-dimension 8 with 2-sided identity element e such that there exists a non-degenerate quadratic form N on C satisfying $N(xy) = N(x)N(y), x, y \in C$. The quadratic form N is referred as the *norm* on C.

When considering octonion algebras over $\mathbb{R}$, we have the following classification result:

Proposition 3.1.2. *[Adams, 1996, Theorem 15.1] Up to $\mathbb{R}$-algebra isomorphism, there is a unique octonion algebra $\mathbb{O}_{\mathbb{R}}$ over $\mathbb{R}$ whose norm* N *is positive definite, which is named as the* real octonion division algebra.

The multiplication law $\mathbb{O}_{\mathbb{R}} \times \mathbb{O}_{\mathbb{R}} \to \mathbb{O}_{\mathbb{R}}$ can be given as follows: as a vector space $\mathbb{O}_{\mathbb{R}}$ admits a basis $\{\mathrm{e}, \mathrm{e}_1, \ldots, \mathrm{e}_7\}$ such that e is the identity element and as an $\mathbb{R}$-algebra $\mathbb{O}_{\mathbb{R}}$ is generated by $\{\mathrm{e}_1, \ldots, \mathrm{e}_7\}$ subject to the relations

- for all i, $\mathrm{e}_i^2 = -\mathrm{e}$;
- viewing subscripts as elements in $\mathbb{Z}/7\mathbb{Z}$, the subspace of $\mathbb{O}_{\mathbb{R}}$ generated by $\{\mathrm{e}, \mathrm{e}_i, \mathrm{e}_{i+1}, \mathrm{e}_{i+3}\}$ is an associative algebra with relations

$$\mathrm{e}_i^2 = \mathrm{e}_{i+1}^2 = \mathrm{e}_{i+3}^2 = -\mathrm{e}, \mathrm{e}_i\mathrm{e}_{i+1} = -\mathrm{e}_{i+1}\mathrm{e}_i = \mathrm{e}_{i+3}.$$

We identify the real numbers $\mathbb{R}$ with the subalgebra $\mathbb{R}\mathrm{e}$ of $\mathbb{O}_\mathbb{R}$ and the identity element of $\mathbb{O}_\mathbb{R}$ will be denoted as 1. Now we recall some basic properties of $\mathbb{O}_\mathbb{R}$, for which we refer to [Conrad, 2015, §5]. There is an anti-involution of algebra $x \mapsto \overline{x}$ called the *conjugation* on $\mathbb{O}_\mathbb{R}$, defined by $\overline{1} = 1$ and $\overline{\mathrm{e}_i} = -\mathrm{e}_i$ for each i. The *trace* and *norm* on $\mathbb{O}_\mathbb{R}$ are defined as:

$$\mathrm{Tr}(x) = x + \overline{x},\ \mathrm{N}(x) = x \cdot \overline{x} = \overline{x} \cdot x.$$

The multiplication law on $\mathbb{O}_\mathbb{R}$ implies that

$$\mathrm{Tr}(xy) = \mathrm{Tr}(yx) = \mathrm{Tr}(\overline{x} \cdot \overline{y}) \text{ for all } x, y \in \mathbb{O}_\mathbb{R}. \tag{3.1}$$

For an element $x = x_0 + \sum\limits_{i=1}^{7} x_i \mathrm{e}_i \in \mathbb{O}_\mathbb{R}$, its norm $\mathrm{N}(x)$ equals $\sum_{i=0}^{7} x_i^2$, from which we can see that N is a positive definite quadratic form. Its associated symmetric bilinear form is $\langle x, y\rangle := \mathrm{N}(x+y) - \mathrm{N}(x) - \mathrm{N}(y) = x \cdot \overline{y} + y \cdot \overline{x} = \mathrm{Tr}(x \cdot \overline{y})$.

Although the multiplication law of $\mathbb{O}_\mathbb{R}$ is not associative, it is still trace-associative in the sense that

$$\mathrm{Tr}((x \cdot y) \cdot z) = \mathrm{Tr}(x \cdot (y \cdot z)) \text{ for all } x, y, z \in \mathbb{O}_\mathbb{R},$$

and we can define $\mathrm{Tr}(xyz) := \mathrm{Tr}((x \cdot y) \cdot z) = \mathrm{Tr}(x \cdot (y \cdot z))$.

We also recall the exceptional Jordan algebra over $\mathbb{R}$, following [Conrad, 2015, §6]:

Definition 3.1.3. The *(positive definite) real exceptional Jordan algebra*, denoted by $\mathrm{J}_\mathbb{R}$, is the 27-dimensional $\mathbb{R}$-vector space consisting of "Hermitian" matrices in $\mathrm{M}_3(\mathbb{O}_\mathbb{R})$, *i.e.* matrices of the form

$$\begin{pmatrix} a & z & \overline{y} \\ \overline{z} & b & x \\ y & \overline{x} & c \end{pmatrix},\ a, b, c \in \mathbb{R},\ x, y, z \in \mathbb{O}_\mathbb{R},$$

equipped with the $\mathbb{R}$-bilinear multiplication law

$$\mathrm{J}_\mathbb{R} \times \mathrm{J}_\mathbb{R} \to \mathrm{J}_\mathbb{R},\ A \circ B := \frac{1}{2}(AB + BA),$$

where AB and BA denote the usual product of octonionic matrices, and with 2-sided identity element I given by the standard matrix identity element $\mathrm{diag}(1, 1, 1)$.

As an $\mathbb{R}$-algebra, $\mathrm{J}_\mathbb{R}$ is commutative but not associative.

Notation 3.1.4. To compress the space, when we do not need to emphasize the matrix structure of elements in $\mathrm{J}_\mathbb{R}$, we denote the element

$$\begin{pmatrix} a & z & \overline{y} \\ \overline{z} & b & x \\ y & \overline{x} & c \end{pmatrix},\ a, b, c \in \mathbb{R},\ x, y, z \in \mathbb{O}_\mathbb{R}$$

by $[a, b, c\,; x, y, z]$ for short.

The *trace* of $A = [a, b, c\,; x, y, z] \in \mathrm{J}_{\mathbb{R}}$ is defined as $\mathrm{Tr}(A) := a + b + c$. The underlying vector space of $\mathrm{J}_{\mathbb{R}}$ is equipped with the non-degenerate positive definite quadratic form:

$$\mathrm{Q}(A) := \mathrm{Tr}(A \circ A)/2 = \frac{1}{2}(a^2 + b^2 + c^2) + \mathrm{N}(x) + \mathrm{N}(y) + \mathrm{N}(z). \tag{3.2}$$

Its associated bilinear form is $\mathrm{B_Q}(A, B) := \mathrm{Q}(A + B) - \mathrm{Q}(A) - \mathrm{Q}(B) = \mathrm{Tr}(A \circ B)$. The *determinant* of the matrix A is defined by

$$\det(A) := abc + \mathrm{Tr}(xyz) - a\mathrm{N}(x) - b\mathrm{N}(y) - c\mathrm{N}(z). \tag{3.3}$$

It defines a cubic form on $\mathrm{J}_{\mathbb{R}}$.

We denote by F_4 the subgroup $\mathrm{Aut}(\mathrm{J}_{\mathbb{R}}, \circ)$ of $\mathrm{GL}(\mathrm{J}_{\mathbb{R}})$ consisting of elements $g \in \mathrm{GL}(\mathrm{J}_{\mathbb{R}})$ such that for all $A, B \in \mathrm{J}_{\mathbb{R}}$, $g(A \circ B) = g(A) \circ g(B)$. It is a compact Lie group of type F_4 [Adams, 1996, Theorem 16.7].

In this paper, we deal with automorphic forms so we want a reductive group over $\mathbb{Q}$ whose real points is isomorphic to F_4. For this purpose, we first define the following $\mathbb{Q}$-algebras:

Definition 3.1.5. *Cayley's definite octonion algebra* $\mathbb{O}_{\mathbb{Q}}$ is the sub-$\mathbb{Q}$-algebra of $\mathbb{O}_{\mathbb{R}}$ generated by $\{\mathrm{e}_1, \ldots, \mathrm{e}_7\}$. The *(positive definite) rational exceptional Jordan algebra* $\mathrm{J}_{\mathbb{Q}}$ is the sub-$\mathbb{Q}$-space of $\mathrm{J}_{\mathbb{R}}$ consisting of $[a, b, c\,; x, y, z]$, $a, b, c \in \mathbb{Q}$, $x, y, z \in \mathbb{O}_{\mathbb{Q}}$ equipped with the multiplication $\circ$.

The main object considered in this paper is the following algebraic group:

Definition 3.1.6. We define $\mathbf{F}_4$ to be the closed subgroup of the algebraic $\mathbb{Q}$-group $\mathbf{GL}_{\mathrm{J}_{\mathbb{Q}}}$, which as a functor sends a commutative unital $\mathbb{Q}$-algebra R to the group

$$\mathbf{F}_4(R) := \mathrm{Aut}(\mathrm{J}_{\mathbb{Q}} \otimes_{\mathbb{Q}} R, \circ) = \{g \in \mathrm{GL}(\mathrm{J}_{\mathbb{Q}} \otimes_{\mathbb{Q}} R) \,|\, g(A \circ B) = g(A) \circ g(B), \forall A, B \in \mathrm{J}_{\mathbb{Q}} \otimes_{\mathbb{Q}} R\}.$$

From the definition we have $\mathbf{F}_4(\mathbb{R}) = \mathrm{F}_4$. By [SpringerVeldkamp, 2000, Theorem 7.2.1], $\mathbf{F}_4$ is a semisimple and simply-connected group over $\mathbb{Q}$.

Remark 3.1.7. We have an alternative description of $\mathbf{F}_4$ that will be used later: the closed subgroup $\mathbf{Aut}_{(\mathrm{J}_{\mathbb{Q}}, \det, \mathrm{I})/\mathbb{Q}}$ of $\mathbf{GL}_{\mathrm{J}_{\mathbb{Q}}}$ consisting of linear automorphisms that preserve both the cubic form det and the identity element I. The closed subgroups $\mathbf{F}_4 = \mathbf{Aut}_{(\mathrm{J}_{\mathbb{Q}}, \circ)/\mathbb{Q}}$ and $\mathbf{Aut}_{(\mathrm{J}_{\mathbb{Q}}, \det, \mathrm{I})/\mathbb{Q}}$ inside $\mathbf{GL}_{\mathrm{J}_{\mathbb{Q}}}$ are both smooth and they have the same geometric points according to [SpringerVeldkamp, 2000, Proposition 5.9.4], so they coincide.

3.2 Reductive $\mathbb{Z}$-models of reductive $\mathbb{Q}$-groups

Now we recall some results in [Gross, 1996; Gross, 1999b]. In this section, let $\mathbf{G}$ be a connected reductive algebraic group over $\mathbb{Q}$. Denote the product $\prod_p \mathbb{Z}_p$ by $\widehat{\mathbb{Z}}$ and let $\mathbb{A}_f = \widehat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$ be the ring of finite adèles, and $\mathbb{A} = \mathbb{R} \times \mathbb{A}_f$.

Definition 3.2.1. A *reductive $\mathbb{Z}$-model* of $\mathbf{G}$ is a pair $(\mathscr{G}, \iota)$ consisting of:

- an affine smooth group scheme $\mathscr{G}$ of finite type over $\mathbb{Z}$ such that $\mathscr{G} \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z}$ is reductive over $\mathbb{Z}/p\mathbb{Z}$ for each prime number p,
- an isomorphism $\iota : \mathscr{G} \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathbf{G}$ of algebraic groups over $\mathbb{Q}$.

Two reductive $\mathbb{Z}$-models $(\mathscr{G}_1, \iota_1)$ and $(\mathscr{G}_2, \iota_2)$ are said to be isomorphic if there exists an isomorphism $f : \mathscr{G}_1 \to \mathscr{G}_2$ over $\mathbb{Z}$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathscr{G}_1 \otimes_{\mathbb{Z}} \mathbb{Q} & \xrightarrow{f_{\mathbb{Q}}} & \mathscr{G}_2 \otimes_{\mathbb{Z}} \mathbb{Q} \\ & {\scriptstyle \iota_1} \searrow \quad \swarrow {\scriptstyle \iota_2} & \\ & \mathbf{G} & \end{array}$$

Remark 3.2.2. When there is no confusion about ι, we simply say that $\mathscr{G}$ is a reductive $\mathbb{Z}$-model of $\mathbf{G}$.

From the theory of *Chevalley groups* in [SGA3, §XXV], every group $\mathbf{G}$ split over $\mathbb{Q}$ admits a reductive $\mathbb{Z}$-model. Indeed, we can take the Chevalley group with the same root datum of $\mathbf{G}$ to be its reductive $\mathbb{Z}$-model.

When $\mathbf{G}$ is not split, in general the existence of reductive $\mathbb{Z}$-models of $\mathbf{G}$ is no longer ensured. Now we consider the case when $\mathbf{G}$ is *anisotropic*, *i.e.* $\mathbf{G}$ does not contain any non-trivial split $\mathbb{Q}$-torus. When $\mathbf{G}$ has a reductive $\mathbb{Z}$-model, being anisotropic is equivalent to that $\mathbf{G}(\mathbb{R})$ is compact, which is due to [PlatonovRapinchuk, 1994, Theorem 5.5(1)] and [Gross, 1996, Proposition 2.1]. In [Gross, 1996, §1], Gross proves the following result:

Theorem 3.2.3. *Let $\mathbf{G}$ be an anisotropic semisimple simply-connected $\mathbb{Q}$-group such that the root system of $G_{\mathbb{C}}$ is irreducible, then $\mathbf{G}$ admits a reductive $\mathbb{Z}$-model if and only if the Lie type of $\mathbf{G}$ is among:*

$$\mathrm{B}_{(d-1)/2}\,(d \equiv \pm 1 \bmod 8), \mathrm{D}_{d/2}\,(d \equiv 0 \bmod 8), \mathrm{G}_2, \mathrm{F}_4, \mathrm{E}_8\,.$$

The next question is to classify reductive $\mathbb{Z}$-models of a given anisotropic group $\mathbf{G}$ up to some equivalence relation.

Definition 3.2.4. Let $(\mathscr{G}, \mathrm{id})$ be a reductive $\mathbb{Z}$-model of its generic fiber $\mathbf{G} := \mathscr{G} \otimes_{\mathbb{Z}} \mathbb{Q}$. A reductive $\mathbb{Z}$-model $(\mathscr{G}', \iota')$ of $\mathbf{G}$ is said to be in the same *genus* as $\mathscr{G}$, if $\iota'(\mathscr{G}'(\widehat{\mathbb{Z}}))$ and $\mathscr{G}(\widehat{\mathbb{Z}})$ are conjugate in $\mathbf{G}(\mathbb{A}_f)$.

Remark 3.2.5. This condition is equivalent to that for each prime p, $\iota'(\mathscr{G}'(\mathbb{Z}_p))$ is conjugate to $\mathscr{G}(\mathbb{Z}_p)$ in $\mathbf{G}(\mathbb{Q}_p)$, and $\iota'(\mathscr{G}'(\mathbb{Z}_p)) = \mathscr{G}(\mathbb{Z}_p)$ for almost all p.

By [Gross, 1999b, Proposition 1.4], the equivalence classes of reductive $\mathbb{Z}$-models in the genus of $\mathscr{G}$ can be identified with the coset space $\mathbf{G}(\mathbb{A}_f)/\mathscr{G}(\widehat{\mathbb{Z}})$.

The group $\mathbf{G}(\mathbb{Q})$ acts on reductive $\mathbb{Z}$-models in the genus of $\mathscr{G}$ by the formula:

$$g(\mathscr{G}', \iota') = (\mathscr{G}', \mathrm{ad}(g) \circ \iota'),$$

where $\mathrm{ad}(g)$ is the conjugation by g. This induces an action of $G(\mathbb{Q})$ on the equivalence classes of reductive $\mathbb{Z}$-models in the genus of $\mathscr{G}$. We say two reductive $\mathbb{Z}$-models in the genus of $\mathscr{G}$ are $\mathbf{G}(\mathbb{Q})$-*conjugate* if their equivalence classes are in the same $\mathbf{G}(\mathbb{Q})$-orbit.

Now the set of $\mathbf{G}(\mathbb{Q})$-orbits on the equivalence classes of reductive $\mathbb{Z}$-models in the genus of $\mathscr{G}$ can be identified with the double coset space $\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A}_f)/\mathscr{G}(\widehat{\mathbb{Z}})$, which is finite by Borel's famous result [Borel, 1963].

3.3 Reductive $\mathbb{Z}$-models of F_4

For our $\mathbb{Q}$-group $\mathbf{F}_4$, the $\mathbf{F}_4(\mathbb{Q})$-orbits of equivalence classes of reductive $\mathbb{Z}$-models of $\mathbf{F}_4$ in some genus is determined by Gross in [Gross, 1996, Proposition 5.3], using the mass formula [Gross, 1996, Proposition 2.2]. In this section we provide an alternative proof for his result, which will be helpful for our computations in Chapter 4.

3.3.1 Integral structures of $\mathbb{O}_\mathbb{Q}$ and $\mathrm{J}_\mathbb{Q}$

Parallel to the construction of $\mathbf{F}_4$ in Section 3.1, we want to define integral structures of $\mathbb{O}_\mathbb{Q}$ and $\mathrm{J}_\mathbb{Q}$ and then use them to construct reductive $\mathbb{Z}$-models of $\mathbf{F}_4$.

Definition 3.3.1. *Coxeter's integral order* $\mathbb{O}_\mathbb{Z}$ is the $\mathbb{Z}$-lattice of rank 8 inside $\mathbb{O}_\mathbb{Q}$ spanned by the lattice $\mathbb{Z}\oplus\mathbb{Z}\mathrm{e}_1\oplus\cdots\oplus\mathbb{Z}\mathrm{e}_7$ and the four elements

$$\mathrm{h}_1=(1+\mathrm{e}_1+\mathrm{e}_2+\mathrm{e}_4)/2, \mathrm{h}_2=(1+\mathrm{e}_1+\mathrm{e}_3+\mathrm{e}_7)/2,$$
$$\mathrm{h}_3=(1+\mathrm{e}_1+\mathrm{e}_5+\mathrm{e}_6)/2, \mathrm{h}_4=(\mathrm{e}_1+\mathrm{e}_2+\mathrm{e}_3+\mathrm{e}_5)/2,$$

equipped with the multiplication of $\mathbb{O}_\mathbb{Q}$. This lattice contains the identity element of $\mathbb{O}_\mathbb{Q}$ and is stable under the multiplication, *i.e.* is an *order* in $\mathbb{O}_\mathbb{Q}$.

Remark 3.3.2. The underlying lattice of $\mathbb{O}_\mathbb{Z}$ equipped with the quadratic form $\mathrm{N}|_{\mathbb{O}_\mathbb{Z}}$ is isometric to the even unimodular lattice

$$\mathrm{E}_8=\left\{(x_i)\in\mathbb{Z}^8\cup(\mathbb{Z}+\frac{1}{2})^8\,\middle|\,\sum_i x_i\equiv 0 \bmod 2\right\}.$$

Let $\mathrm{J}_\mathbb{Z}$ be the unimodular lattice

$$\{[a,b,c\,;x,y,z]\in\mathrm{J}_\mathbb{Q}\,|\,a,b,c\in\mathbb{Z},x,y,z\in\mathbb{O}_\mathbb{Z}\}$$

of rank 27 inside the $\mathbb{Q}$-vector space $\mathrm{J}_\mathbb{Q}$.

Remark 3.3.3. This lattice is not stable under the Jordan multiplication $\circ$ defined on $\mathrm{J}_\mathbb{Q}$, since $[1,0,0\,;0,0,0]\circ[0,0,0\,;0,1,0]=\frac{1}{2}[0,0,0\,;0,1,0]\notin\mathrm{J}_\mathbb{Z}$.

As in Remark 3.1.7, the $\mathbb{Q}$-group $\mathbf{F}_4$ coincides with the group $\mathbf{Aut}_{(\mathrm{J}_\mathbb{Q},\det,\mathrm{I})/\mathbb{Q}}$. The restriction of the cubic form det to $\mathrm{J}_\mathbb{Z}$ has integral values, and it is a *polynomial law* in the sense of [Roby, 1963]. The triple $(\mathrm{J}_\mathbb{Q},\det,\mathrm{I})$ has a natural integral structure $(\mathrm{J}_\mathbb{Z},\det,\mathrm{I})$, and the $\mathbb{Z}$-group scheme $\mathbf{Aut}_{(\mathrm{J}_\mathbb{Z},\det,\mathrm{I})/\mathbb{Z}}$, sending any commutative $\mathbb{Z}$-algebra R to

$$\{g\in\mathrm{GL}(\mathrm{J}_\mathbb{Z}\otimes_\mathbb{Z}R)\,|\,g\mathrm{I}=\mathrm{I},\det(gX)=\det(X)\text{ for any }X\in\mathrm{J}_\mathbb{Z}\otimes_\mathbb{Z}R\},$$

is expected to be a reductive $\mathbb{Z}$-model of $\mathbf{F}_4$. We are going to consider the $\mathbb{Z}$-group scheme $\mathbf{Aut}_{(\mathrm{J}_{\mathbb{Z}},\det,e)/\mathbb{Z}}$ for any $e \in \mathrm{J}_{\mathbb{Z}}$ satisfying certain conditions, in order to produce several reductive $\mathbb{Z}$-models of $\mathbf{F}_4$ uniformly.

Definition 3.3.4. An element

$$A = \begin{pmatrix} a & z & \overline{y} \\ \overline{z} & b & x \\ y & \overline{x} & c \end{pmatrix} \in \mathrm{J}_{\mathbb{R}}$$

is said to be *positive definite* if its seven "minor determinants"

$$a, b, c, ab - \mathrm{N}(z), bc - \mathrm{N}(x), ca - \mathrm{N}(y), \det(A) \in \mathbb{R}$$

are all positive. A positive definite element e in $\mathrm{J}_{\mathbb{R}}$ with $\det e = 1$ is called a *polarization*.

Given a polarization e contained in the lattice $\mathrm{J}_{\mathbb{Z}}$, one constructs a $\mathbb{Z}$-group scheme $\mathcal{F}_{4,e} := \mathbf{Aut}_{(\mathrm{J}_{\mathbb{Z}},\det,e)/\mathbb{Z}}$ in the same way as $\mathbf{Aut}_{(\mathrm{J}_{\mathbb{Z}},\det,\mathrm{I})/\mathbb{Z}}$. The following result shows that this group scheme is a reductive $\mathbb{Z}$-model of $\mathbf{F}_4$.

Proposition 3.3.5. *[Conrad, 2015, Proposition 6.6, Example 6.7] For any choice of polarization $e \in \mathrm{J}_{\mathbb{Z}}$, the fiber $\mathcal{F}_{4,e} \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z}$ is semisimple for every prime number p, and $\mathcal{F}_{4,e}(\mathbb{R})$ is a compact Lie group of type F_4.*

Taking e to be the identity element I, the generic fiber of $\mathcal{F}_{4,\mathrm{I}}$ is $\mathbf{Aut}_{(\mathrm{J}_{\mathbb{Q}},\det,\mathrm{I})/\mathbb{Q}} = \mathbf{F}_4$, thus $\mathcal{F}_{4,\mathrm{I}}$ is a reductive $\mathbb{Z}$-model of $\mathbf{F}_4$.

If we take e to be

$$\mathrm{E} := [2,2,2\,;\beta,\beta,\beta], \beta = \frac{1}{2}(-1 + \mathrm{e}_1 + \mathrm{e}_2 + \cdots + \mathrm{e}_7) \in \mathrm{J}_{\mathbb{Z}},$$

as in [ElkiesGross, 1996, (5.4)], by [Conrad, 2015, Example 6.7] the generic fiber of $\mathcal{F}_{4,\mathrm{E}}$ is also isomorphic to $\mathbf{F}_4$. We denote the natural isomorphism $\mathcal{F}_{4,\mathrm{E}} \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbf{F}_4$ by ι. Actually ι can be given as the conjugation by an element in $\mathrm{Aut}(\mathrm{J}_{\mathbb{Q}}, \det)$ that sends E to I.

In [Gross, 1996, Proposition 5.3], Gross proves the following result:

Proposition 3.3.6. *There are two $\mathbf{F}_4(\mathbb{Q})$-orbits on the equivalence classes of reductive $\mathbb{Z}$-models of $\mathbf{F}_4$ in the genus of $\mathcal{F}_{4,\mathrm{I}}$, whose representatives are given by $(\mathcal{F}_{4,\mathrm{I}}, \mathrm{id})$ and $(\mathcal{F}_{4,\mathrm{E}}, \iota)$ respectively.*

Applying the mass formula [Gross, 1996, Proposition 2.2] to $\mathbf{F}_4$, we have

$$\sum_{(\mathscr{G},\iota)} \frac{1}{|\mathscr{G}(\mathbb{Z})|} = \frac{1}{2^4}\zeta(-1)\zeta(-5)\zeta(-7)\zeta(-11) = \frac{691}{2^{15}\cdot 3^6 \cdot 5^2 \cdot 7^2 \cdot 13}, \tag{3.4}$$

where $(\mathscr{G}, \iota)$ varies over the $\mathbf{F}_4(\mathbb{Q})$-conjugacy classes of reductive $\mathbb{Z}$-models of $\mathbf{F}_4$ in the genus of $\mathcal{F}_{4,\mathrm{I}}$. As

$$\frac{691}{2^{15}\cdot 3^6 \cdot 5^2 \cdot 7^2 \cdot 13} = \frac{1}{2^{15}\cdot 3^6 \cdot 5^2 \cdot 7} + \frac{1}{2^{12}\cdot 3^5 \cdot 7^2 \cdot 13}, \tag{3.5}$$

in order to prove Proposition 3.3.6 it suffices to prove the following two things:

- $\mathcal{F}_{4,\mathrm{I}}$ and $\mathcal{F}_{4,\mathrm{E}}$ are not $\mathbf{F}_4(\mathbb{Q})$-conjugate.
- $|\mathcal{F}_{4,\mathrm{I}}(\mathbb{Z})| \leq 2^{15} \cdot 3^6 \cdot 5^2 \cdot 7$ and $|\mathcal{F}_{4,\mathrm{E}}(\mathbb{Z})| \leq 2^{12} \cdot 3^5 \cdot 7^2 \cdot 13$.

In his proof, Gross cites some results from [ATLAS], We are going to give another proof of Proposition 3.3.6, which avoids using results in [ATLAS].

3.3.2 $\mathcal{F}_{4,\mathrm{E}}(\mathbb{Z})$

Now we deal with the finite group $\mathcal{F}_{4,\mathrm{E}}(\mathbb{Z})$. Our goal is to prove:

Proposition 3.3.7. $|\mathcal{F}_{4,\mathrm{E}}(\mathbb{Z})| \leq 2^{12} \cdot 3^5 \cdot 7^2 \cdot 13$.

With the choice of polarization E, we can define a new bilinear form on $\mathrm{J}_{\mathbb{Q}}$:

$$\langle A, B\rangle_{\mathrm{E}} = (A, \mathrm{E}, \mathrm{E})(B, \mathrm{E}, \mathrm{E}) - 2(A, B, \mathrm{E}),$$

where the trilinear form $(\ ,\ ,\ ) : \mathrm{J}_{\mathbb{Q}}^3 \to \mathbb{Q}$ is defined by

$$\begin{aligned}(A, B, C) =& \frac{1}{2}[\det(A + B + C) - \det(A + B) - \det(B + C) - \det(C + A) \\ &+ \det(A) + \det(B) + \det(C)].\end{aligned}$$

This bilinear form is positive definite and integral on $\mathrm{J}_{\mathbb{Z}}$ by [ElkiesGross, 1996, Proposition 7.2].

Notation 3.3.8. Here we give some notations for elements in $\mathrm{J}_{\mathbb{R}}$: we write

$$\mathrm{E}_1 := [1, 0, 0\,; 0, 0, 0], \mathrm{E}_2 := [0, 1, 0\,; 0, 0, 0], \mathrm{E}_3 := [0, 0, 1\,; 0, 0, 0]$$

and for any $x \in \mathbb{O}_{\mathbb{R}}$,

$$\mathrm{F}_1(x) := [0, 0, 0\,; x, 0, 0], \mathrm{F}_2(x) := [0, 0, 0\,; 0, x, 0], \mathrm{F}_3(x) := [0, 0, 0\,; 0, 0, x].$$

Note that $1, \mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3, \mathrm{h}_1, \mathrm{h}_2, \mathrm{h}_3, \mathrm{h}_4$ is a basis of the lattice $\mathbb{O}_{\mathbb{Z}}$, thus we have the following basis of $\mathrm{J}_{\mathbb{Z}}$:

$$\mathcal{B} := \begin{pmatrix} \mathrm{E}_1, \mathrm{E}_2, \mathrm{E}_3, \mathrm{F}_1(1), \mathrm{F}_1(\mathrm{e}_1), \mathrm{F}_1(\mathrm{e}_2), \mathrm{F}_1(\mathrm{e}_3), \mathrm{F}_1(\mathrm{h}_1), \mathrm{F}_1(\mathrm{h}_2), \mathrm{F}_1(\mathrm{h}_3), \mathrm{F}_1(\mathrm{h}_4), \mathrm{F}_2(1), \mathrm{F}_2(\mathrm{e}_1), \mathrm{F}_2(\mathrm{e}_2), \\ \mathrm{F}_2(\mathrm{e}_3), \mathrm{F}_2(\mathrm{h}_1), \mathrm{F}_2(\mathrm{h}_2), \mathrm{F}_2(\mathrm{h}_3), \mathrm{F}_2(\mathrm{h}_4), \mathrm{F}_3(1), \mathrm{F}_3(\mathrm{e}_1), \mathrm{F}_3(\mathrm{e}_2), \mathrm{F}_3(\mathrm{e}_3), \mathrm{F}_3(\mathrm{h}_1), \mathrm{F}_3(\mathrm{h}_2), \mathrm{F}_3(\mathrm{h}_3), \mathrm{F}_3(\mathrm{h}_4) \end{pmatrix}. \quad (3.6)$$

In the basis $\mathcal{B}$, we give the Gram matrix of the quadratic lattice $(\mathrm{J}_{\mathbb{Z}}, \langle\ ,\ \rangle_{\mathrm{E}})$ in Fig. A.1, Appendix A.

Proof of Proposition 3.3.7. Each element in $\mathcal{F}_{4,\mathrm{E}}(\mathbb{Z}) = \mathrm{Aut}(\mathrm{J}_{\mathbb{Z}}, \det, \mathrm{E})$ preserves the bilinear form $\langle\ ,\ \rangle_{\mathrm{E}}$ by the definition, thus this finite group is a subgroup of the isometry group $\mathrm{O}(\mathrm{J}_{\mathbb{Z}}, \langle\ ,\ \rangle_{\mathrm{E}})$ of the quadratic lattice $(\mathrm{J}_{\mathbb{Z}}, \langle\ ,\ \rangle_{\mathrm{E}})$.

The order of $\mathrm{O}(\mathrm{J}_{\mathbb{Z}}, \langle\ ,\ \rangle_{\mathrm{E}})$ can be determined with the help of the Plesken-Souvignier algorithm. Concretely, we can apply the `qfauto` function in [PARI/GP] to the Gram matrix Fig. A.1 of $(\mathrm{J}_{\mathbb{Z}}, \langle\ ,\ \rangle_{\mathrm{E}})$, and we find

$$|\mathrm{O}(\mathrm{J}_{\mathbb{Z}}, \langle\ ,\ \rangle_{\mathrm{E}})| = 2^{13} \cdot 3^5 \cdot 7^2 \cdot 13.$$

Notice that the isometry group contains an involution $-\mathrm{id}$, which does not fix E, thus we have

$$|\mathcal{F}_{4,\mathrm{E}}(\mathbb{Z})| \leq \frac{1}{2}\,|\mathrm{O}(\mathrm{J}_{\mathbb{Z}}, \langle\ ,\ \rangle_{\mathrm{E}})| = 2^{12}\cdot 3^5 \cdot 7^2 \cdot 13. \qquad \square$$

Remark 3.3.9. The orthogonal complement of E in $(\mathrm{J}_{\mathbb{Z}}, \langle\ ,\ \rangle_{\mathrm{E}})$ is a 26-dimensional even lattice of determinant 3 and with no roots [ElkiesGross, 1996, Proposition 7.2]. In Borcherds' thesis [Borcherds, 1999, §5.7], he proves that a lattice satisfying these conditions is unique up to isomorphism and calculates the order of its isometry group, giving another proof of Proposition 3.3.7.

Furthermore, the `qfauto` function also give us a set of generators $\{-\mathrm{id}, -\sigma_1, \sigma_2\}$ of the isometry group $\mathrm{O}(\mathrm{J}_{\mathbb{Z}}, \langle\ ,\ \rangle_{\mathrm{E}})$, where the matrices of σ_1, σ_2 in the basis $\mathcal{B}$ chosen in Eq. (3.6) are given in Fig. A.2, Appendix A. Here we write $-\sigma_1$ instead of σ_1 because the second element in the result given by [PARI/GP] sends E to $-$E. The isometry group $\mathrm{O}(\mathrm{J}_{\mathbb{Z}}, \langle\ ,\ \rangle_{\mathrm{E}})$ is the direct product of the subgroup generated by σ_1, σ_2 and the order 2 central subgroup $\pm\mathrm{id}$. In the proof of Proposition 3.3.7, we find that $\mathcal{F}_{4,\mathrm{E}}(\mathbb{Z})$ is a subgroup of the group $\langle \sigma_1, \sigma_2\rangle$.

In the basis $\mathcal{B}$, the cubic form det on $\mathrm{J}_{\mathbb{R}}$ can be written down as a 27-variable polynomial of degree 3, and we give this polynomial function as `MatDet` in our [PARI/GP] program [Codes and tables]. Using [PARI/GP], we verify that σ_1 and σ_2 both preserve the cubic form det and the element E, which implies the following result:

Proposition 3.3.10. *The finite groups $\mathcal{F}_{4,\mathrm{E}}(\mathbb{Z})$ and $\langle\sigma_1, \sigma_2\rangle$ coincide, and*

$$|\mathcal{F}_{4,\mathrm{E}}(\mathbb{Z})| = 2^{12}\cdot 3^5\cdot 7^2\cdot 13.$$

3.3.3 $\mathcal{F}_{4,\mathrm{I}}(\mathbb{Z})$

Now we look at the finite group $\mathcal{F}_{4,\mathrm{I}}(\mathbb{Z}) = \mathrm{Aut}(\mathrm{J}_{\mathbb{Z}}, \det, \mathrm{I})$, and we want to prove the following proposition:

Proposition 3.3.11. *The reductive $\mathbb{Z}$-model $\mathcal{F}_{4,\mathrm{I}}$ of $\mathbf{F}_4$ is not $\mathbf{F}_4(\mathbb{Q})$-conjugate to $\mathcal{F}_{4,\mathrm{E}}$, and $|\mathcal{F}_{4,\mathrm{I}}(\mathbb{Z})| \leq 2^{15}\cdot 3^6\cdot 5^2\cdot 7$.*

Denote the subset of $\mathrm{J}_{\mathbb{Z}}$ consisting of diagonal matrices by D, and the subset of elements whose diagonal entries are zero by D_0. The formula Eq. (3.2) for the quadratic form Q on $\mathrm{J}_{\mathbb{Z}}$ shows that equipped with Q we have $\mathrm{J}_{\mathbb{Z}} = \mathrm{D}_0 \oplus \mathrm{D}$ as quadratic lattices. By Remark 3.3.2, the quadratic lattice $(\mathbb{O}_{\mathbb{Z}}, \mathrm{N})$ is isometric to E_8, thus D_0 is isometric to $\mathrm{E}_8\oplus\mathrm{E}_8\oplus\mathrm{E}_8$. On the other hand, the lattice D is isometric to

$$\mathrm{I}_3 = \mathbb{Z}^3, \mathrm{q} : (x_1, x_2, x_3) \mapsto \frac{1}{2}\left(x_1^2 + x_2^2 + x_3^2\right).$$

Any element of $\mathcal{F}_{4,\mathrm{I}}(\mathbb{Z})$ preserves the quadratic form Q on $\mathrm{J}_{\mathbb{Z}}$, so $\mathcal{F}_{4,\mathrm{I}}(\mathbb{Z})$ is a subgroup of the isometry group $\mathrm{O}(\mathrm{J}_{\mathbb{Z}})$ of the quadratic lattice $\mathrm{J}_{\mathbb{Z}}$. By the theory of root lattices, we have

$$\mathrm{O}(\mathrm{J}_{\mathbb{Z}}) \simeq \mathrm{O}(\mathrm{I}_3) \times (\mathrm{O}(\mathbb{O}_{\mathbb{Z}}) \wr \mathrm{S}_3),$$

where S_3 is the permutation group of three elements and $\wr$ stands for the wreath product. Let p be the restriction map $\mathcal{F}_{4,I}(\mathbb{Z}) \hookrightarrow O(J_{\mathbb{Z}}) \twoheadrightarrow O(D), g \mapsto g|_D$, where $O(D) \simeq O(I_3)$ is isomorphic to $\{\pm 1\}^3 \rtimes S_3$.

Let $O(D\,;I)$ be the group $\{\sigma \in O(D) \mid \sigma(I) = I\}$, which is isomorphic to the permutation group S_3. Since elements in $\mathcal{F}_{4,I}(\mathbb{Z})$ fix I, the image of p is contained in $O(D\,;I)$.

Lemma 3.3.12. *The image of p is $O(D\,;I) \simeq S_3$.*

Proof. For an element $\sigma \in S_3$, we denote by g_σ the element

$$[a_1, a_2, a_3\,; x_1, x_2, x_3] \mapsto [a_{\sigma^{-1}(1)}, a_{\sigma^{-1}(2)}, a_{\sigma^{-1}(3)}\,; \epsilon(\sigma)(x_{\sigma^{-1}(1)}), \epsilon(\sigma)(x_{\sigma^{-1}(2)}), \epsilon(\sigma)(x_{\sigma^{-1}(3)})] \tag{3.7}$$

in $GL(J_{\mathbb{Z}})$, where the map $\epsilon(\sigma) : \mathbb{O}_{\mathbb{Z}} \to \mathbb{O}_{\mathbb{Z}}$ is defined as identity when σ is even, and as the conjugation when σ is odd. In this proof, we write $x^* := \epsilon(\sigma)(x)$ for short.

For any $A = [a_1, a_2, a_3\,; x_1, x_2, x_3] \in J_{\mathbb{Z}}$, by the formula Eq. (3.3) for the cubic form det, we have

$$\begin{aligned} \det(g_\sigma(A)) =& \prod_{i=1}^{3} a_{\sigma^{-1}(i)} + \mathrm{Tr}(x^*_{\sigma^{-1}(1)} x^*_{\sigma^{-1}(2)} x^*_{\sigma^{-1}(3)}) - \sum_{i=1}^{3} a_{\sigma^{-1}(i)} \mathrm{N}(x^*_{\sigma^{-1}(i)}) \\ =& a_1 a_2 a_3 + \mathrm{Tr}(x^*_{\sigma^{-1}(1)} x^*_{\sigma^{-1}(2)} x^*_{\sigma^{-1}(3)}) - \sum_{i=1}^{3} a_i \mathrm{N}(x_i). \end{aligned}$$

The property Eq. (3.1) of Tr implies that for any $x, y, z \in \mathbb{O}_{\mathbb{Z}}$,

$$\mathrm{Tr}(xyz) = \mathrm{Tr}(yzx) = \mathrm{Tr}(zxy) = \mathrm{Tr}(\overline{x} \cdot \overline{z} \cdot \overline{y}) = \mathrm{Tr}(\overline{z} \cdot \overline{y} \cdot \overline{x}) = \mathrm{Tr}(\overline{y} \cdot \overline{x} \cdot \overline{z}),$$

which can also be stated as $\mathrm{Tr}(x^*_{\sigma^{-1}(1)} x^*_{\sigma^{-1}(2)} x^*_{\sigma^{-1}(3)}) = \mathrm{Tr}(x_1 x_2 x_3)$ for any $\sigma \in S_3$. Hence $\det(g_\sigma(A)) = \det(A)$. Since g_σ also fixes I, it is an element in $\mathcal{F}_{4,I}(\mathbb{Z})$ and its restriction $p(g_\sigma) \in O(D\,;I) \simeq S_3$ is σ, thus $\mathrm{Im}(p) = O(D\,;I)$. □

Let $\mathscr{D}$ be the kernel of p, then we have a short exact sequence of finite groups:

$$1 \to \mathscr{D} \to \mathcal{F}_{4,I}(\mathbb{Z}) \to O(D\,;I) \simeq S_3 \to 1. \tag{3.8}$$

Lemma 3.3.13. *The map $\kappa : S_3 \to \mathcal{F}_{4,I}(\mathbb{Z}), \sigma \mapsto g_\sigma$ defined in Eq. (3.7) gives a splitting of the short exact sequence Eq. (3.8).*

Proof. It suffices to show that $\sigma \mapsto g_\sigma$ is a group homomorphism. For $\sigma, \tau \in S_3$, we have

$$\begin{aligned} & g_\tau g_\sigma ([a_1, a_2, a_3\,; x_1, x_2, x_3]) \\ =& g_\tau \left([a_{\sigma^{-1}(1)}, a_{\sigma^{-1}(2)}, a_{\sigma^{-1}(3)}\,; \epsilon(\sigma)(x_{\sigma^{-1}(1)}), \epsilon(\sigma)(x_{\sigma^{-1}(2)}), \epsilon(\sigma)(x_{\sigma^{-1}(3)})] \right) \\ =& \begin{bmatrix} a_{(\tau\sigma)^{-1}(1)}, a_{(\tau\sigma)^{-1}(2)}, a_{(\tau\sigma)^{-1}(3)}\,; \\ \epsilon(\tau)\epsilon(\sigma)(x_{(\tau\sigma)^{-1}(1)}), \epsilon(\tau)\epsilon(\sigma)(x_{(\tau\sigma)^{-1}(2)}), \epsilon(\tau)\epsilon(\sigma)(x_{(\tau\sigma)^{-1}(3)}) \end{bmatrix}. \end{aligned}$$

It can be easily seen that the map $\epsilon : \mathrm{S}_3 \to \mathrm{GL}(\mathbb{O}_\mathbb{Z})$ is a group homomorphism, thus $g_\tau g_\sigma = g_{\tau\sigma}$ and $\sigma \mapsto g_\sigma$ is also a group homomorphism. $\square$

This lemma tells us $\mathcal{F}_{4,\mathrm{I}}(\mathbb{Z}) = \mathscr{D} \rtimes \kappa(\mathrm{S}_3)$ and $|\mathcal{F}_{4,\mathrm{I}}(\mathbb{Z})| = 3! \cdot |\mathscr{D}|$. Now we study the structure of $\mathscr{D}$.

Lemma 3.3.14. *The group $\mathscr{D}$ is isomorphic to the group*

$$\widetilde{\mathrm{SO}(\mathbb{O}_\mathbb{Z})} := \left\{(\alpha,\beta,\gamma) \in \mathrm{SO}(\mathbb{O}_\mathbb{Z})^3 \,\middle|\, \overline{\alpha(x)\beta(y)} = \gamma(\overline{xy}), \forall x, y \in \mathbb{O}_\mathbb{Z}\right\}.$$

Proof. Fix $g \in \mathscr{D}$ and $x \in \mathbb{O}_\mathbb{Z}$, we define $y, z, w \in \mathbb{O}_\mathbb{Z}$ by the formula

$$g.\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & x \\ 0 & \overline{x} & 0 \end{pmatrix} = \begin{pmatrix} 0 & w & \overline{z} \\ \overline{w} & 0 & y \\ z & \overline{y} & 0 \end{pmatrix}.$$

Since $g \in \mathcal{F}_{4,\mathrm{I}}(\mathbb{Z}) \subseteq \mathbf{F}_4(\mathbb{Q})$ preserves the Jordan multiplication $\circ$, we have

$$\begin{aligned}
\mathrm{N}(x)\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} &= g.\left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & x \\ 0 & \overline{x} & 0 \end{pmatrix} \circ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & x \\ 0 & \overline{x} & 0 \end{pmatrix}\right) \\
&= \begin{pmatrix} 0 & w & \overline{z} \\ \overline{w} & 0 & y \\ z & \overline{y} & 0 \end{pmatrix} \circ \begin{pmatrix} 0 & w & \overline{z} \\ \overline{w} & 0 & y \\ z & \overline{y} & 0 \end{pmatrix} \\
&= \begin{pmatrix} \mathrm{N}(z)+\mathrm{N}(w) & \overline{yz} & wy \\ yz & \mathrm{N}(w)+\mathrm{N}(y) & \overline{zw} \\ \overline{wy} & zw & \mathrm{N}(y)+\mathrm{N}(z) \end{pmatrix},
\end{aligned}$$

which implies that $z = w = 0$ and $\mathrm{N}(y) = \mathrm{N}(x)$. This gives us a homomorphism $g \mapsto \alpha_g$ from $\mathscr{D}$ to $\mathrm{O}(\mathbb{O}_\mathbb{Z})$ such that $g[0,0,0\,;x,0,0] = [0,0,0\,;\alpha_g(x),0,0]$ for $\in \mathbb{O}_\mathbb{Z}$.

Symmetrically, we also get $\beta_g, \gamma_g \in \mathrm{O}(\mathbb{O}_\mathbb{Z})$ such that

$$g[0,0,0\,;x,y,z] = [0,0,0\,;\alpha_g(x),\beta_g(x),\gamma_g(x)] \text{ for all } x,y,z \in \mathbb{O}_\mathbb{Z}.$$

Taking determinants of both sides, we get

$$\mathrm{Tr}(xyz) = \mathrm{Tr}(\alpha_g(x)\beta_g(y)\gamma_g(z)) \text{ for all } x,y,z \in \mathbb{O}_\mathbb{Z}.$$

This is equivalent to $\langle\overline{\alpha_g(x)\beta_g(y)}, \gamma_g(z)\rangle = \langle\overline{xy}, z\rangle$. Since $\langle\overline{xy}, z\rangle = \langle\gamma_g(\overline{xy}), \gamma_g(z)\rangle$, we have

$$\langle\overline{\alpha_g(x)\beta_g(y)} - \overline{xy}, \gamma_g(z)\rangle = 0$$

for any $z \in \mathbb{O}_\mathbb{Z}$. The bilinear form $\langle\ ,\ \rangle$ is non-degenerate, so $\overline{\alpha_g(x)\beta_g(y)} = \gamma_g(\overline{xy})$ holds for any $x, y \in \mathbb{O}_\mathbb{Z}$. By [Yokota, 2009, Lemma 1.14.4], we have $\alpha_g, \beta_g, \gamma_g \in \mathrm{SO}(\mathbb{O}_\mathbb{Z})$.

Now we have obtained an injective homomorphism $\mathscr{D} \to \widetilde{\mathrm{SO}(\mathbb{O}_{\mathbb{Z}})}$. Conversely, by the definition of the multiplication $\circ$ and the condition on $(\alpha, \beta, \gamma) \in \widetilde{\mathrm{SO}(\mathbb{O}_{\mathbb{Z}})}$, the morphism

$$[a, b, c\,; x, y, z] \mapsto [a, b, c\,; \alpha(x), \beta(y), \gamma(z)]$$

lies in $\mathscr{D}$, thus $\mathscr{D} \simeq \widetilde{\mathrm{SO}(\mathbb{O}_{\mathbb{Z}})}$. □

Let $\varphi : \widetilde{\mathrm{SO}(\mathbb{O}_{\mathbb{Z}})} \to \mathrm{SO}(\mathbb{O}_{\mathbb{Z}})$ be the homomorphism sending a triple $(\alpha, \beta, \gamma) \in \widetilde{\mathrm{SO}(\mathbb{O}_{\mathbb{Z}})}$ to its third entry $\gamma \in \mathrm{SO}(\mathbb{O}_{\mathbb{Z}})$.

Proof of Proposition 3.3.11. For the bound on $|\mathcal{F}_{4,\mathrm{I}}(\mathbb{Z})|$, it suffices to prove

$$|\widetilde{\mathrm{SO}(\mathbb{O}_{\mathbb{Z}})}| \leq 2^{14} \cdot 3^5 \cdot 5^2 \cdot 7.$$

Let $(\alpha, \beta, \mathrm{id})$ be an element in $\ker \varphi$, so $\alpha(x)\beta(y) = xy$ for all $x, y \in \mathbb{O}_{\mathbb{Z}}$. Set $r = \beta(1)$ and we have $\alpha(x) = xr^{-1}$ and $\beta(y) = ry$. Setting $z = xr^{-1}$, the relation satisfied by $(\alpha, \beta, \mathrm{id})$ becomes:

$$z(ry) = (zr)y, \text{ for all } y, z \in \mathbb{O}_{\mathbb{Z}}.$$

According to [ConwaySmith, 2003, §8, Theorem 1], the octonion r of norm 1 is real, thus $r = \pm 1$ and $\ker \varphi = \{(\mathrm{id}, \mathrm{id}, \mathrm{id}), (-\mathrm{id}, -\mathrm{id}, \mathrm{id})\}$. As a consequence, we have

$$|\widetilde{\mathrm{SO}(\mathbb{O}_{\mathbb{Z}})}| \leq 2 \cdot |\mathrm{SO}(\mathbb{O}_{\mathbb{Z}})| = |\mathrm{O}(\mathbb{O}_{\mathbb{Z}})| = |\mathrm{W}(\mathrm{E}_8)| = 2^{14} \cdot 3^5 \cdot 5^2 \cdot 7,$$

which gives us the desired upper bound for $|\mathcal{F}_{4,\mathrm{I}}(\mathbb{Z})|$.

Suppose that the reductive $\mathbb{Z}$-model $\mathcal{F}_{4,\mathrm{I}}$ of $\mathbf{F}_4$ is $\mathbf{F}_4(\mathbb{Q})$-conjugate to $\mathcal{F}_{4,\mathrm{E}}$, then their $\mathbb{Z}$-points have the same order as finite groups. In the end of Section 3.3.2, we prove that $|\mathcal{F}_{4,\mathrm{E}}(\mathbb{Z})| = 2^{12} \cdot 3^5 \cdot 7^2 \cdot 13$, thus with the same order, the group $\mathcal{F}_{4,\mathrm{I}}(\mathbb{Z})$ contains an element of order 13. However, $\mathcal{F}_{4,\mathrm{I}}(\mathbb{Z})$ is isomorphic to $\widetilde{\mathrm{SO}(\mathbb{O}_{\mathbb{Z}})} \rtimes \mathrm{S}_3$, whose order is not divided by 13. This leads to a contradiction. □

Now Proposition 3.3.7 and Proposition 3.3.11 together imply Proposition 3.3.6, and as a corollary the equality in the upper bound in Proposition 3.3.11 holds:

Corollary 3.3.15. *The finite group $\mathcal{F}_{4,\mathrm{I}}(\mathbb{Z})$ has order $2^{15} \cdot 3^6 \cdot 5^2 \cdot 7$, and φ is surjective.*

Chapter 4

Dimensions of spaces of invariants for F_4

For a finite subgroup Γ and an irreducible representation U of the compact Lie group F_4, an interesting problem is to compute the dimension of the space of invariants U^Γ. In this chapter, we will give an algorithm to compute $\dim U^\Gamma$ for $\Gamma = \mathcal{F}_{4,\mathrm{I}}(\mathbb{Z})$ or $\mathcal{F}_{4,\mathrm{E}}(\mathbb{Z})$. These dimensions will play an important role in our computation of spaces of automorphic forms in Section 6.1.1. The code of the computations in this chapter can be found in [Codes and tables].

4.1 Ideas and obstructions

By the highest weight theory, the isomorphism classes of irreducible $\mathbb{C}$-representations of the compact Lie group F_4 are in natural bijection with dominant weights of the irreducible root system F_4. Using notations in [Bourbaki, 2002, §VI.4.9], we denote the weight $\lambda_1\varpi_1 + \lambda_2\varpi_2 + \lambda_3\varpi_3 + \lambda_4\varpi_4$ by $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$, where $\varpi_1, \varpi_2, \varpi_3, \varpi_4$ are the four fundamental weights of F_4. Let V_λ be a representative of the isomorphism class of irreducible representations of F_4 with highest weight λ. From now on we call V_λ *the* irreducible representation of F_4 with highest weight λ for short.

The starting point of the computation of $\dim \mathrm{V}_\lambda^\Gamma$ for some finite subgroup Γ of F_4 is the following classic lemma:

Lemma 4.1.1. *For a finite subgroup $\Gamma \subset \mathrm{F}_4$, we have*

$$\dim \mathrm{V}_\lambda^\Gamma = \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma} \mathrm{Tr}|_{\mathrm{V}_\lambda}(\gamma) = \frac{1}{|\Gamma|}\sum_{c\in\mathrm{Conj}(\Gamma)} \mathrm{Tr}|_{\mathrm{V}_\lambda}(c)\cdot|c|,$$

where $\mathrm{Conj}(\Gamma)$ is the set of conjugacy classes of Γ and $|c|$ denotes the cardinality of c.

Because of this lemma, it is enough to solve the following two problems to compute $\dim \mathrm{V}_\lambda^\Gamma$:

(i) Find all conjugacy classes of Γ, and choose a representative in a fixed maximal torus $T \subset \mathrm{F}_4$ for each conjugacy class;

(ii) For an element $t \in T$, compute its trace $\mathrm{Tr}|_{\mathrm{V}_\lambda}(t)$.

Problem (ii) can be dealt with the following *degenerate Weyl character formula*:

Proposition 4.1.2. *[ChenevierRenard, 2015, Proposition 2.1] Let G be a connected compact Lie group, T a maximal torus, $X = X^*(T)$ the character group of T, and Φ the root system of (G,T) with Weyl group W. Choose a system of positive roots $\Phi^+ \subset \Phi$ with base Δ and also fix a W-invariant inner product $(\ ,\ )$ on $X \otimes_{\mathbb{Z}} \mathbb{R}$. Let λ be a dominant weight in X and t an element in T. Denote the connected component $C_G(t)^\circ$ of the centralizer of t by M. Set $\Phi_M^+ = \Phi(M,T) \cap \Phi^+$ and $W^M = \{w \in W : w^{-1}\Phi_M^+ \subset \Phi^+\}$. Let ρ and ρ_M be the half-sum of the elements of Φ^+ and Φ_M^+ respectively. We have:*

$$\mathrm{Tr}|_{V_\lambda}(t) = \frac{\sum_{w\in W^M} \varepsilon(w) t^{w(\lambda+\rho)-\rho} \cdot \prod_{\alpha\in\Phi_M^+} \frac{(\alpha, w(\lambda+\rho))}{(\alpha,\rho_M)}}{\prod_{\alpha\in\Phi^+\setminus\Phi_M^+}(1-t^{-\alpha})}, \tag{4.1}$$

where $\varepsilon : W \to \{\pm 1\}$ is the signature and t^x denotes $x(t)$ for convenience.

Using this approach, problem (i) is thus the main difficulty for our computation, and we will solve it in the following sections.

4.2 Generators of $\mathcal{F}_{4,\mathrm{I}}(\mathbb{Z})$ and $\mathcal{F}_{4,\mathrm{E}}(\mathbb{Z})$

The finite groups Γ we are interested in are $\mathcal{F}_{4,\mathrm{I}}(\mathbb{Z})$ and $\mathcal{F}_{4,\mathrm{E}}(\mathbb{Z})$. To find all their conjugacy classes, we first determine generators of these groups in this section.

In the end of Section 3.3.2, we have already showed that the group $\mathcal{F}_{4,\mathrm{E}}(\mathbb{Z})$ is generated by two elements σ_1, σ_2. Their matrices in the basis $\mathcal{B}$, given in Eq. (3.6), are written down in Fig. A.2, Appendix A.

Based on Corollary 3.3.15, we have $\mathcal{F}_{4,\mathrm{I}}(\mathbb{Z}) = \mathscr{D} \rtimes \kappa(\mathrm{S}_3)$, where $\kappa : \mathrm{S}_3 \to \mathcal{F}_{4,\mathrm{I}}(\mathbb{Z})$ is the morphism defined in Eq. (3.7). The group $\mathscr{D}$ is isomorphic to the group $\widetilde{\mathrm{SO}(\mathbb{O}_{\mathbb{Z}})}$, which is a double cover of $\mathrm{SO}(\mathbb{O}_{\mathbb{Z}})$ by Corollary 3.3.15. Therefore it suffices to find generators of $\mathscr{D}$.

Since $\mathrm{O}(\mathbb{O}_{\mathbb{Z}}) \simeq \mathrm{O}(\mathrm{E}_8)$ is equal to the Weyl group of E_8, we can take the following set of generators for $\mathrm{SO}(\mathbb{O}_{\mathbb{Z}})$:

$$\{\mathrm{ref}(\alpha) \circ \mathrm{ref}(1) \mid \alpha \in \mathbb{O}_{\mathbb{Z}}, \mathrm{N}(\alpha) = 1\},$$

where for a root α in $\mathbb{O}_{\mathbb{Z}}$, *i.e.* an element with $\langle \alpha, \alpha \rangle = 2$, the reflection $\mathrm{ref}(\alpha)$ is defined as

$$\mathrm{ref}(\alpha)(x) := x - \langle x, \alpha \rangle \alpha.$$

For a root $\alpha \in \mathbb{O}_{\mathbb{Z}}$, let L_α (*resp.* R_α) be the left (*resp.* right) multiplication on $\mathbb{O}_{\mathbb{Z}}$ by α, and define $\mathrm{B}_\alpha := \mathrm{L}_\alpha \circ \mathrm{R}_\alpha = \mathrm{R}_\alpha \circ \mathrm{L}_\alpha$. These elements are contained in $\mathrm{SO}(\mathbb{O}_{\mathbb{Z}})$. Notice that for a root $\alpha \in \mathbb{O}_{\mathbb{Z}}$, $\mathrm{ref}(\alpha) \circ \mathrm{ref}(1) = \mathrm{B}_\alpha$.

Lemma 4.2.1. *For any root $\alpha \in \mathbb{O}_{\mathbb{Z}}$, the triple $(\mathrm{L}_{\overline{\alpha}}, \mathrm{R}_{\overline{\alpha}}, \mathrm{B}_\alpha)$ is an element in $\widetilde{\mathrm{SO}(\mathbb{O}_{\mathbb{Z}})}$.*

Proof. For any $x, y \in \mathbb{O}_{\mathbb{Z}}$, $\overline{\mathrm{L}_{\overline{\alpha}}(x)\mathrm{R}_{\overline{\alpha}}(y)} = \overline{(\overline{\alpha}x)(y\overline{\alpha})}$. By *Moufang laws* [ConwaySmith, 2003, §6.5],

$$(\overline{\alpha}x)(y\overline{\alpha}) = (\overline{\alpha}(xy))\overline{\alpha} = \mathrm{B}_{\overline{\alpha}}(xy),$$

thus $\overline{\mathrm{L}_{\overline{\alpha}}(x)\mathrm{R}_{\overline{\alpha}}(y)} = \overline{\mathrm{B}_{\overline{\alpha}}(xy)} = \mathrm{B}_\alpha(\overline{xy})$. □

By this lemma, we can take

$$\{(\mathrm{L}_{\overline{\alpha}}, \mathrm{R}_{\overline{\alpha}}, \mathrm{B}_\alpha) \mid \alpha \in \mathbb{O}_\mathbb{Z}, \mathrm{N}(\alpha) = 1\} \cup \{(-\mathrm{id}, -\mathrm{id}, \mathrm{id})\}$$

as generators of $\mathscr{D}$. Together with a set of generators of $\kappa(\mathrm{S}_3)$ we have obtained generators of $\mathcal{F}_{4,\mathrm{I}}(\mathbb{Z})$.

4.3 Enumeration of conjugacy classes

Now with generators of $\mathcal{F}_I(\mathbb{Z})$ and $\mathcal{F}_{4,\mathrm{E}}(\mathbb{Z})$, we can start to enumerate their conjugacy classes. The `ConjugationClasses` function in [GAP] can assist us in enumerating the conjugacy classes of subgroups of permutation groups. Therefore it is enough to realize these two finite groups as permutation groups.

For $\mathcal{F}_{4,\mathrm{I}}(\mathbb{Z})$, we consider its action on the set of vectors $v \in \mathbb{O}_\mathbb{Z}$ with $\mathrm{B}_\mathrm{Q}(v, v) \leq 2$. The function `qfminim` in [PARI/GP] can list all these vectors in the basis $\mathcal{B}$. There are 738 such vectors and they span the vector space $\mathrm{J}_\mathbb{R}$, so the action of $\mathcal{F}_{4,\mathrm{I}}(\mathbb{Z})$ on this set is faithful, which gives us an embedding $\mathcal{F}_{4,\mathrm{I}}(\mathbb{Z}) \hookrightarrow \mathrm{S}_{738}$. We can thus use this embedding to obtain a set of representatives of conjugacy classes of $\mathcal{F}_{4,\mathrm{I}}(\mathbb{Z})$ via the help of [GAP].

For the other group $\mathcal{F}_{4,\mathrm{E}}(\mathbb{Z})$ we use a similar strategy. As mentioned in Remark 3.3.9, the quadratic lattice $(\mathrm{J}_\mathbb{Z}, \langle\ ,\ \rangle_\mathrm{E})$ has no roots, so we consider the set of $v \in \mathrm{J}_\mathbb{Z}$ such that $\langle v, v\rangle_\mathrm{E} = 3$, which has cardinality 1640 and generates $\mathrm{J}_\mathbb{R}$. This gives an embedding $\mathcal{F}_{4,\mathrm{E}}(\mathbb{Z}) \hookrightarrow \mathrm{S}_{1640}$, then we can use [GAP].

Here we present the results, and all the codes are available in [Codes and tables].

Proposition 4.3.1. *There are* 113 *conjugacy classes in* $\mathcal{F}_{4,\mathrm{I}}(\mathbb{Z})$*, while* $\mathcal{F}_{4,\mathrm{E}}(\mathbb{Z})$ *has* 49 *conjugacy classes.*

Furthermore, [GAP] gives the size of each conjugacy class c, and selects a representative for c in the form of permutation. We rewrite these representatives as matrices in the basis $\mathcal{B}$.

4.4 Kac coordinates

In the previous section, for $\Gamma = \mathcal{F}_{4,\mathrm{I}}(\mathbb{Z})$ or $\mathcal{F}_{4,\mathrm{E}}(\mathbb{Z})$, we obtained a list of its conjugacy classes and a representative element $g_c \in \Gamma$ for each conjugacy class c.

However, the representative g_c may not be contained in the fixed maximal torus in Proposition 4.1.2. Notice that in the computation of the trace of g_c for a Γ-conjugacy class c, what really matters is the F_4-conjugacy class containing c. Furthermore, since c is included in the finite group Γ, the F_4-conjugacy class containing it must be torsion.

In [Reeder, 2010], it is shown that we can choose a representative for a torsion F_4-conjugacy class in a fixed maximal torus using its *Kac coordinates*. Here we provide a brief review, and more details can be found in Reeder's paper.

Let G be a simply-connected simple compact Lie group, T a fixed maximal torus, $X := \mathrm{X}^*(T)$ and $Y := \mathrm{X}_*(T)$ the groups of characters and cocharacters respectively, and Φ the root system of (G, T). Denote the natural pairing $X \times Y \to \mathbb{Z}$ by $\langle\ ,\ \rangle$. Let $\Delta = \{\alpha_1, \ldots, \alpha_r\}$ be a set of simple roots of Φ, and $\{\check{\varpi}_1, \ldots, \check{\varpi}_r\}$ its dual basis in Y, *i.e.* $\langle \alpha_i, \check{\varpi}_j \rangle = \delta_{ij}$.

We have a surjective *exponential map* $\exp : Y \otimes_{\mathbb{Z}} \mathbb{R} \to T$ determined uniquely by the property

$$\alpha\left(\exp(y)\right) = e^{2\pi i \langle \alpha, y \rangle}, \forall \alpha \in X, y \in Y \otimes_{\mathbb{Z}} \mathbb{R}.$$

and Y is the kernel of this exponential map. This induces an isomorphism $(Y \otimes_{\mathbb{Z}} \mathbb{R})/Y \simeq T$.

Let $\widetilde{\alpha}_0 = \sum_{i=1}^{r} a_i \alpha_i$ be the highest root with respect to the choice of simple roots Δ, and set $\alpha_0 = 1 - \widetilde{\alpha}_0, a_0 = 1$ and $\check{\varpi}_0 = 0$. Now we have $\sum_{i=0}^{r} a_i \alpha_i = 1$. The *alcove* determined by Δ is the intersection of half-spaces:

$$C = \{x \in Y \otimes_{\mathbb{Z}} \mathbb{R} \,|\, \langle \alpha_i, x \rangle > 0, \forall i = 0, 1, \ldots, r\},$$

or

$$\overline{C} = \left\{ \sum_{i=0}^{r} x_i \check{\varpi}_i \,\middle|\, \sum_{i=0}^{r} a_i x_i = 1, x_i \geq 0, \forall i = 0, 1, \ldots, r \right\}.$$

Each torsion element $s \in G$ is conjugate to $\exp(x)$ for a unique $x \in \overline{C} \cap (Y \otimes_{\mathbb{Z}} \mathbb{Q})$ since the group G is simply-connected. Let m be the order of s, thus

$$x = \frac{1}{m} \sum_{i=1}^{r} s_i \check{\varpi}_i$$

for some non-negative integers $s_1, \ldots, s_r$ satisfying $\gcd\{m, s_1, \ldots, s_r\} = 1$.

Since $x \in \overline{C}$, we set $s_0 := m - \sum_{i=1}^{r} a_i s_i \geq 0$. Now the non-negative integers $s_0, s_1, \ldots, s_r$ satisfy $\gcd\{s_0, \ldots, s_r\} = 1$ and the equation

$$\sum_{i=0}^{r} a_i s_i = m \text{ with } a_0 = 1.$$

The coordinates $(s_0, s_1, \ldots, s_r)$ are called the *Kac coordinates of s*, which are uniquely determined by the G-conjugacy class of s.

In our case, the compact group F_4 is simply-connected and the highest root $\widetilde{\alpha}_0 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$. Here $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are still chosen as in [Bourbaki, 2002, §VI.4]. In conclusion, we have:

Proposition 4.4.1. *Let T be a fixed maximal torus of F_4. Any element of order m in F_4 is conjugate to a unique element $\exp(\frac{\sum_{i=1}^{4} s_i \varpi_i}{m})$ for some non-negative integers s_1, s_2, s_3, s_4 arising from a 5-tuple $(s_0, s_1, s_2, s_3, s_4)$ in*

$$\left\{ (x_0, \ldots, x_4) \in \mathbb{N}^5 \,\middle|\, x_0 + 2x_1 + 3x_2 + 4x_3 + 2x_4 = m, \gcd\{x_0, \ldots, x_4\} = 1 \right\}. \tag{4.2}$$

By solving the equation in Eq. (4.2), we enumerate all the torsion F_4-conjugacy classes of order m.

4.5 Comparison of conjugacy classes

Now we can enumerate F_4-conjugacy classes of a given order, but there are more constraints on the F_4-conjugacy classes containing Γ-conjugacy classes obtained in Section 4.3. So we define the following class of F_4-conjugacy classes:

Definition 4.5.1. Let c be an F_4-conjugacy class, and we say that c is a *rational conjugacy class* if it satisfies:

- its trace $\mathrm{Tr}(c)|_{\mathfrak{f}_4}$ on the adjoint representation $\mathfrak{f}_4$ of F_4 is a rational number;
- its characteristic polynomial $\mathrm{P}_c(X) := \det(X \cdot \mathrm{id} - g|_{\mathrm{J}_\mathbb{C}})$ on $\mathrm{J}_\mathbb{C} := \mathrm{J}_\mathbb{R} \otimes_\mathbb{R} \mathbb{C}$, $g \in \mathrm{F}_4$ being a representative of c, has rational coefficients.

For $\Gamma = \mathcal{F}_{4,\mathrm{I}}(\mathbb{Z})$ or $\mathcal{F}_{4,\mathrm{E}}(\mathbb{Z})$, since Γ is a subgroup of $\mathrm{GL}(\mathrm{J}_\mathbb{Z})$, the F_4-conjugacy class containing a Γ-conjugacy class of Γ must be rational in the sense of Definition 4.5.1.

Our strategy in this section is:

(1) find all rational torsion F_4-conjugacy classes, and for each of them choose a representative in the maximal torus T fixed before in Section 4.4;

(2) determine which F_4-conjugacy class contains a given Γ-conjugacy class by comparing their traces and characteristic polynomials.

Before explaining the algorithm for step (1), we state the following lemma:

Lemma 4.5.2. *If m is the order of an element in F_4 whose characteristic polynomial on $\mathrm{J}_\mathbb{C}$ has rational coefficients, then $m = 66, 70, 72, 78, 84$ or 90, or $m \leq 60$.*

Proof. As a representation of F_4, $\mathrm{J}_\mathbb{C}$ is isomorphic to $\mathrm{V}_{\varpi_4} \oplus \mathbb{C}$, where $\mathbb{C}$ stands for the trivial representation. Since the zero weight appears twice in the weights of V_{ϖ_4}, the characteristic polynomial is divisible by $(X-1)^3$. On the other hand, the roots of this polynomial contain a primitive mth root of unity, thus the polynomial is also divisible by the mth cyclotomic polynomial. Hence we have $\varphi(m) \leq 24$, where φ denotes the Euler function. This implies $m \leq 60$, or $m = 66, 70, 72, 78, 84$ or 90. □

With the help of [PARI/GP], we enumerate all the Kac coordinates $s = (s_0, s_1, s_2, s_3, s_4)$ satisfying the conditions in Eq. (4.2) for each integer m in

$$\{n \leq 60 \,|\, \varphi(n) \leq 24\} \cup \{66, 70, 72, 78, 84, 90\}.$$

For each such s, we compute the trace on $\mathfrak{f}_4$ and the characteristic polynomial on $\mathrm{J}_\mathbb{C}$ of the corresponding element $t = \exp(\frac{\sum_{i=1}^4 s_i \varpi_i}{m}) \in T$. Using this algorithm, we get the Kac coordinates of all rational torsion F_4-conjugacy classes.

Proposition 4.5.3. *There are exactly 102 rational torsion conjugacy classes in F_4, whose Kac coordinates are listed in Table A.1.*

Our result coincides with [Padowitz, 1998, Table 9.1]. In Table A.1, we also list the invariants defined below for all rational torsion F_4-conjugacy class.

For a representative $g \in \mathrm{F}_4$ of a rational torsion conjugacy class c, we can compute its characteristic polynomial on $\mathrm{J}_{\mathbb{C}}$:

$$\mathrm{P}_g(X) = \det\left(X \cdot \mathrm{id} - g|_{\mathrm{J}_{\mathbb{C}}}\right) = \sum_{i=0}^{27} (-1)^{i+1} a_i(g) X^i.$$

Now we assign to g a quadruple

$$\mathrm{i}(g) := \left(a_{26}(g), a_{25}(g), a_{24}(g), \mathrm{Tr}(\mathrm{Ad}(g)|_{\mathfrak{f}_4})\right),$$

and set $\mathrm{i}(c) := \mathrm{i}(g)$.

Corollary 4.5.4. *Let g_1, g_2 be two elements in either $\mathcal{F}_{4,\mathrm{I}}(\mathbb{Z})$ or $\mathcal{F}_{4,\mathrm{E}}(\mathbb{Z})$, then g_1 and g_2 are conjugate in F_4 if and only if $\mathrm{i}(g_1) = \mathrm{i}(g_2)$.*

Proof. This follows from Table A.1. For each rational torsion conjugacy class c, we list its order $o(c)$ and the associated quadruple $i(c)$. We observe that two different classes c have different $i(c)$. □

Remark 4.5.5. There exist examples of two different rational torsion conjugacy classes in F_4 whose characteristic polynomials on $\mathrm{J}_{\mathbb{C}}$ are the same. For instance, the order 12 conjugacy classes c_1 and c_2 represented by the Kac coordinates $(1,1,1,1,1)$ and $(2,1,0,1,2)$ respectively share the same characteristic polynomial on $\mathrm{J}_{\mathbb{C}}$:

$$X^{27} - X^{24} - 2X^{15} + 2X^{12} + X^3 - 1.$$

However, the trace of c_1 on $\mathfrak{f}_4$ is 0, while that of c_2 is 3. This shows that the 26-dimensional irreducible representation of F_4 is not "*excellent*" in the sense of Padowitz. It is also observed in Padowitz's table [Padowitz, 1998, Table 9.1] that the motives attached to the centralizers of these two conjugacy classes, in the sense of Gross, are different.

Now we explain our algorithm for step (2). For each Γ-conjugacy class c and its representative g_c chosen in Section 4.3, we compute the quadruple $\mathrm{i}(g_c)$ and compare it with Table A.1. By Corollary 4.5.4 we can determine the F_4-conjugacy class containing c. In Table A.2 we list all the Kac coordinates s whose corresponding rational conjugacy class c_s in F_4 satisfies that $c_s \cap \mathcal{F}_{4,\mathrm{I}}(\mathbb{Z})$ or $c_s \cap \mathcal{F}_{4,\mathrm{E}}(\mathbb{Z})$ is non-empty, as well as the cardinalities of intersections $n_1(s) = |c_s \cap \mathcal{F}_{4,\mathrm{I}}(\mathbb{Z})|$ and $n_2(s) = |c_s \cap \mathcal{F}_{4,\mathrm{E}}(\mathbb{Z})|$.

4.6 The formula for $\dim \mathrm{V}_\lambda^\Gamma$

Now we can deduce the formula for $d_i(\lambda) := \dim \mathrm{V}_\lambda^{\Gamma_i}, i = 1,2$, where $\Gamma_1 := \mathcal{F}_{4,\mathrm{I}}(\mathbb{Z})$ and $\Gamma_2 := \mathcal{F}_{4,\mathrm{E}}(\mathbb{Z})$, for a given dominant weight λ:

$$\dim \mathrm{V}_\lambda^{\Gamma_i} = \frac{1}{|\Gamma_i|} \sum_{c \in \mathrm{Conj}(\Gamma_i)} \mathrm{Tr}\,|_{\mathrm{V}_\lambda}(c) \cdot |c| = \frac{1}{|\Gamma_i|} \sum_{c \in \mathrm{Conj}(\mathrm{F}_4)} \mathrm{Tr}\,|_{\mathrm{V}_\lambda}(c) \cdot |c \cap \Gamma_i|.$$

For each rational conjugacy class c whose contribution to this formula is nonzero, we have already given $|c \cap \Gamma_i|$ in Table A.2, and according to Proposition 4.1.2 the trace $\mathrm{Tr}\,|_{V_\lambda}(c')$ is an explicit function of $\lambda_1, \lambda_2, \lambda_3, \lambda_4$.

This gives us the following theorem, which is the main computational result of this paper:

Theorem 4.6.1. *For each dominant weight λ of the compact Lie group F_4, we have an explicit formula for*

$$d_i(\lambda) = \dim V_\lambda^{\Gamma_i}, i = 1, 2.$$

For dominant weights $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ with $2\lambda_1 + 3\lambda_2 + 2\lambda_3 + \lambda_4 \leq 13$, we list all the nonzero $d(\lambda) := d_1(\lambda) + d_2(\lambda)$ in Table A.3.

Remark 4.6.2. Later we will see the condition on λ in Theorem 4.6.1 is equivalent to that the maximal eigenvalue of the infinitesimal character associated to V_λ is not larger than 21.

In [Codes and tables], we also provide a larger table of $[\lambda, d_1(\lambda), d_2(\lambda), d(\lambda)]$ for weights with $2\lambda_1 + 3\lambda_2 + 2\lambda_3 + \lambda_4 \leq 40$.

Chapter 5

Subgroups of F_4

In this chapter, we will classify subgroups of the compact Lie group $\mathrm{F}_4 = \mathrm{Aut}(\mathrm{J}_{\mathbb{R}}, \circ)$ satisfying certain conditions and determine their centralizers in F_4. Our results will be used in Chapter 7, but this problem also has its own interest. Our precise aim is to find all the conjugacy classes of closed subgroups H of F_4 such that:

(1) H is connected;

(2) The centralizer of H in F_4 is an elementary finite abelian 2-groups, *i.e.* it is a product of finitely many copies of $\mathbb{Z}/2\mathbb{Z}$.

(3) The multiplicity of zero weight in the restriction of the 26-dimensional irreducible representation V_{ϖ_4} of F_4 to H is 2.

If we only consider the first condition, the problem is equivalent to classifying connected semisimple Lie subalgebras of the complexified Lie algebra $\mathfrak{f}_4$, up to the adjoint action of $\mathbf{F}_4(\mathbb{C})$. This has been studied by Dynkin in [Dynkin, 1952] for all simple complex Lie algebras, without giving full details. So we will give a detailed classification for F_4 in this chapter, following Dynkin's original idea and Losev's result [Losev, 2010, Theorem 7.1].

Briefly, our strategy is to enumerate first all the connected simple subgroups of F_4 inside maximal proper compact subgroups, and to index them by the restrictions of V_{ϖ_4}. Then we compute their centralizers case by case, and combine these results together to get all the connected subgroups satisfying our conditions.

5.1 Element-conjugacy implies conjugacy

To be more precise, what we want to classify, up to F_4-conjugacy, are embeddings from connected compact Lie groups to F_4 satisfying two additional conditions. In this section we will explain why it is enough to consider their element-conjugacy classes, where the notion of element-conjugacy is defined as follows:

Definition 5.1.1. [FangHanSun, 2016, §1] Let G and H be two compact Lie groups and $\phi, \phi' : H \to G$ be two Lie group homomorphisms. We say that ϕ and ϕ' are *conjugate* if there is an element $g \in G$ such that

$$g\phi(h)g^{-1} = \phi'(h), \text{ for all } h \in H.$$

They are said to be *element-conjugate* if for every $h \in H$, there is a $g \in G$ such that

$$g\phi(h)g^{-1} = \phi'(h).$$

The element-conjugacy can be rephrased in the following way:

Lemma 5.1.2. *Let $\phi, \phi' : H \to G$ be two homomorphisms between compact Lie groups, then they are element-conjugate if and only if for each linear representation $\pi : G \to \mathrm{GL}(V)$ the compositions $\pi \circ \phi$ and $\pi \circ \phi'$ are conjugate in $\mathrm{GL}(V)$.*

Proof. It is a consequence of the Peter-Weyl theorem for compact Lie groups, which says that two elements of G are conjugate if and only if they have the same trace on all the irreducible representations of G. □

It is obvious that two conjugate homomorphisms are element-conjugate, but the converse fails in general. Fortunately, the converse holds when $G = \mathrm{F}_4$ and H is connected, due to the following result for Lie algebras:

Theorem 5.1.3. *[Losev, 2010, Proposition 6.2, Theorem 7.1] Let $\mathfrak{f}_4$ be a simple complex Lie algebra of type F_4 and $\mathrm{F}_{4,\mathbb{C}}$ the complexification of F_4. Let $\mathfrak{h}$ be a reductive algebraic Lie algebra,* i.e. *$\mathfrak{h}$ is the Lie algebra of some reductive complex group, and $\phi, \phi' : \mathfrak{h} \to \mathfrak{f}_4$ two injective Lie algebra homomorphisms. If the restrictions of ϕ and ϕ' to a Cartan subalgebra $\mathfrak{s}$ of $\mathfrak{h}$ are conjugate in the sense that $\varphi \circ \phi|_{\mathfrak{s}} = \phi'|_{\mathfrak{s}}$ for an inner automorphism φ of $\mathfrak{f}_4$, then ϕ and ϕ' are conjugate.*

Remark 5.1.4. Actually, in [Losev, 2010] Losev uses the following equivalence relation on Lie algebra homomorphisms: two Lie algebra homomorphisms $\phi, \phi' : \mathfrak{h} \to \mathfrak{g}$ are equivalent if there exist liftings $H \to G$ of ϕ, ϕ' to reductive complex groups which are G-conjugate in the sense of Definition 5.1.1. By Lie group-Lie algebra correspondence this equivalence relation is the same as $\varphi \circ \phi = \phi'$ for an inner automorphism φ of $\mathfrak{f}_4$.

This theorem implies the result we need for F_4:

Proposition 5.1.5. *For any connected compact Lie group H, two element-conjugate homomorphisms from H to F_4 are conjugate.*

Proof. The argument that deduces this result from Theorem 5.1.3 can be found in the proof of [FangHanSun, 2016, Proposition 3.5]. □

5.2 A criterion for element-conjugacy

According to Lemma 5.1.2 and Proposition 5.1.5, to check whether two homomorphism ϕ and ϕ' from a connected compact Lie group H to F_4 are conjugate, it suffices to verify that for every irreducible representation π of F_4, $\pi \circ \phi$ and $\pi \circ \phi'$ are equivalent as H-representations. Moreover, we have the following useful fact:

Proposition 5.2.1. *Let* (π_0, J_0) *be the* 26*-dimensional irreducible representation of* F_4*. Two homomorphisms* ϕ, ϕ' *from a connected compact subgroup* H *to* F_4 *are conjugate if and only if two* H*-representations* $\pi_0 \circ \phi$ *and* $\pi_0 \circ \phi'$ *are equivalent.*

This result is a part of [Dynkin, 1952, Theorem 1.3], but Dynkin only gives a short sketch of the proof, so in this section we will give the proof of Proposition 5.2.1.

We first give a preliminary discussion on orders. Let X be an abelian group and $\ell : X \to \mathbb{R}$ a $\mathbb{Z}$-linear map. This map induces a total preorder $\leq$ on X defined by $x \leq y$ if and only if $\ell(x) \leq \ell(y)$. A preorder on X of this form will be called an *L-preorder*. If the map ℓ is injective, the L-preorder it induces is an order and we call this order an *L-order*. For instance, any free abelian group of finite rank admits L-orders.

Lemma 5.2.2. *Let* $f : X \to Y$ *be a homomorphism between finitely generated free abelian groups* X *and* Y*, with an* L*-order on* Y*, and* S *a finite subset of* $X - \{0\}$*. There exists an* L*-preorder* $\leq$ *on* X *such that for any* $s \in S$ *we have either* $s > 0$ *or* $s < 0$*, and if* $s > 0$ *then* $f(s) \geq 0$ *in* Y*.*

Proof. We choose $\ell : Y \hookrightarrow \mathbb{R}$ such that the L-order on Y is defined by ℓ. Write $S = S_0 \sqcup S_1$, with $S_0 = S \cap \ker f$. If S_0 is empty, then the L-preorder on X defined by $\ell \circ f$ satisfies the conditions.

If S_0 is not empty, we choose an arbitrary injective $\mathbb{Z}$-linear map $j : X \hookrightarrow \mathbb{R}$ and set

$$\varepsilon := \frac{1}{2} \min_{s \in S_1} \frac{|\ell(f(s))|}{|j(s)|}.$$

We claim that the L-preorder on X defined by $j' = \ell \circ f + \varepsilon j$ satisfies the desired conditions. Indeed, for $s \in S_0$, $j'(s) = \varepsilon j(s)$ is nonzero. Also for $s \in S_1$, by our choice of ε, we have $|\varepsilon j(s)| < |\ell(f(s))|$, so $j'(s)$ is nonzero and of the same sign as $\ell(f(s))$. □

The next lemma concerns the partial order $\preceq$ of the weights of the 26-dimensional irreducible representation π_0 of F_4. Recall that for two weights λ and μ of F_4, fixing a positive root system of F_4, we write $\lambda \succeq \mu$ if $\lambda - \mu$ is a finite sum of positive roots.

Lemma 5.2.3. *The* 26*-dimensional irreducible representation* (π_0, J_0) *of* F_4 *has four unique weights* $\lambda_1 \succ \lambda_2 \succ \lambda_3 \succ \lambda_4$ *satisfying that* $\lambda \prec \lambda_4$ *for all other weights* λ*. Moreover, those* 4 *weights* $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ *form a* $\mathbb{Z}$*-basis of the weight lattice of* F_4*.*

Proof. Fix a maximal torus T of F_4, and let $X = \mathrm{X}^*(T)$ be its character lattice and $\Phi^+ \subset X$ a positive root system with respect to (F_4, T). We still use Bourbaki's notations [Bourbaki, 2002, §VI.4.9] for the root system F_4. The simple roots with respect to Φ^+ are given by

$$\alpha_1 = \varepsilon_2 - \varepsilon_3, \alpha_2 = \varepsilon_3 - \varepsilon_4, \alpha_3 = \varepsilon_4, \alpha_4 = \frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4),$$

where $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ is the basis of $X \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathbb{R}^4$ chosen in [Bourbaki, 2002] satisfying

$$X = \mathbb{Z}\varepsilon_1 + \mathbb{Z}\varepsilon_2 + \mathbb{Z}\varepsilon_3 + \mathbb{Z}\varepsilon_4 + \mathbb{Z}\frac{\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4}{2}.$$

The highest weight of π_0 is $\varpi_4 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 = \varepsilon_1$. The orbit of ϖ_4 under the Weyl group consists of $\pm\varepsilon_i$ for $i = 1, 2, 3, 4$ and $\frac{1}{2}(\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4)$. These 24 weights have multiplicity 1, and the zero weight appears with multiplicity 2.

We claim that the weights

$$\lambda_1 = \varepsilon_1, \lambda_2 = \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4),$$
$$\lambda_3 = \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon_4), \lambda_4 = \frac{1}{2}(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 + \varepsilon_4)$$

satisfy the desired properties. Indeed, this follows from the following table:

positive weight λ	relation with $\lambda_1, \lambda_2, \lambda_3, \lambda_4$
ε_1	λ_1
ε_2	$\lambda_4 - \alpha_3 - \alpha_4$
ε_3	$\lambda_4 - \alpha_1 - \alpha_3 - \alpha_4$
ε_4	$\lambda_4 - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4$
$(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4)/2$	$\lambda_2 = \lambda_1 - \alpha_4$
$(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon_4)/2$	$\lambda_3 = \lambda_2 - \alpha_3$
$(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 + \varepsilon_4)/2$	$\lambda_4 = \lambda_3 - \alpha_2$
$(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4)/2$	$\lambda_4 - \alpha_3$
$(\varepsilon_1 - \varepsilon_2 + \varepsilon_3 + \varepsilon_4)/2$	$\lambda_4 - \alpha_1$
$(\varepsilon_1 - \varepsilon_2 + \varepsilon_3 - \varepsilon_4)/2$	$\lambda_4 - \alpha_1 - \alpha_3$
$(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 + \varepsilon_4)/2$	$\lambda_4 - \alpha_1 - \alpha_2 - \alpha_3$
$(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4)/2$	$\lambda_4 - \alpha_1 - \alpha_2 - 2\alpha_3$

Table 5.1: Positive weights of the 26-dimensional irreducible representation V_{ϖ_4} of F_4

and the following identities:

$$\varepsilon_1 = \lambda_1, \varepsilon_2 = -\lambda_1 + \lambda_3 + \lambda_4, \varepsilon_3 = \lambda_2 - \lambda_4, \varepsilon_4 = \lambda_2 - \lambda_3, \frac{\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4}{2} = \lambda_2. \qquad \square$$

Proof of Proposition 5.2.1. By Proposition 5.1.5 it suffices to show that if $\pi_0 \circ \phi$ and $\pi_0 \circ \phi'$ are equivalent as H-representations, then ϕ and ϕ' are element-conjugate. Since any element of H is included in some maximal torus, we may assume that H is a torus.

We fix a maximal torus T of F_4. As all maximal tori are conjugate in F_4, up to replacing ϕ and ϕ' by some F_4-conjugate, we assume that both $\phi(H)$ and $\phi'(H)$ are contained in T. Let $X = X^*(T)$ and $Y = X^*(H)$, then ϕ and ϕ' induce $\mathbb{Z}$-linear maps $\phi^*, \phi'^{,*} : X \to Y$ respectively.

Choose an arbitrary L-order on Y, and denote by $\Phi \subset X$ the root system of (F_4, T). By Lemma 5.2.2, there is an L-preorder $\leq$ (*resp.* $\leq'$) on X such that for any $\alpha \in \Phi$ we have either $\alpha > 0$ or $\alpha < 0$ (*resp.* either $\alpha >' 0$ or $\alpha <' 0$), and the $\mathbb{Z}$-linear map ϕ^* (*resp.* $\phi'^{,*}$) preserves the preorders on X, Y. We denote the positive root system determined by the L-preorder $\leq$ (*resp.* $\leq'$) by Φ^+ (*resp.* $\Phi^{+,\prime}$).

A general fact about root systems is that the Weyl group of (F_4, T) acts transitively on the set of positive root systems of (F_4, T). Up to conjugating ϕ' by a suitable element in the normalizer $\mathrm{N}_{\mathrm{F}_4}(T)$, we may assume that $\Phi^{+,'} = \Phi^+$. Now our aim is to show $\phi = \phi'$, which is equivalent to $\phi^* = \phi'^{,*}$.

Let $\mathcal{W}$ be the multiset of X consisting of the weights appearing in π_0. Let $\lambda_1 \succ \lambda_2 \succ \lambda_3 \succ \lambda_4$ be the 4 weights of π_0 defined in Lemma 5.2.3 and all of them have multiplicity 1 in π_0. For the $\mathbb{Z}$-linear map $f = \phi^*$ or $\phi'^{,*}$, the preorder-preserving property of f and Table 5.1 imply that $f(\lambda_1) \geq f(\lambda_2) \geq f(\lambda_3) \geq f(\lambda_4)$ and $f(\lambda_4) \geq f(\lambda)$ for all other weights λ of π_0. In other words, $f(\lambda_1)$ is the greatest element of $f(\mathcal{W})$, and for $i = 2, 3, 4$, $f(\lambda_i)$ is the greatest element of $f(\mathcal{W}) \backslash \{f(\lambda_1), \ldots, f(\lambda_{i-1})\}$. By the assumption $\pi_0 \circ \phi = \pi_0 \circ \phi'$, the multisets $\phi^*(\mathcal{W})$ and $\phi'^{,*}(\mathcal{W})$ of Y coincide. It follows that we have $\phi^*(\lambda_i) = \phi'^{,*}(\lambda_i)$ for $i = 1, 2, 3, 4$, and as $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ form a basis of X by Lemma 5.2.3, we deduce $\phi^* = \phi'^{,*}$. □

Hence the conjugacy class of a homomorphism from a connected compact Lie group H to F_4 is determined by the restriction of the 26-dimensional irreducible representation to H.

5.3 Maximal proper connected subgroups

Up to conjugacy, the compact group F_4 has five maximal proper connected subgroups by [Dynkin, 1952, Theorem 5.5, Theorem 14.1]. We will recall these five subgroups in this section and show that there are no other maximal proper connected subgroups.

We first introduce the following notations, which will be used a lot of times in this section:

Notation 5.3.1. In this article, we use the following notations of compact Lie groups:

- For $n \geq 2$, denote by $\mathrm{SU}(n)$ the compact special unitary group with respect to the standard Hermitian form on $\mathbb{C}^n$.
- For $n \geq 3$, denote by $\mathrm{SO}(n)$ the compact special orthogonal group with respect to the standard quadratic form on $\mathbb{R}^n$, and by $\mathrm{Spin}(n)$ the compact spin group, which is a double cover of $\mathrm{SO}(n)$.
- For $n \geq 1$, denote by $\mathrm{Sp}(n)$ the *compact* symplectic group: the group of invertible $n \times n$ quaternionic matrices that preserve the standard Hermitian form

$$\langle x, y \rangle = \overline{x_1} y_1 + \cdots + \overline{x_n} y_n$$

  on $\mathbb{H}^n$, where $\mathbb{H}$ is Hamilton's quaternions.
- The group G_2 is defined as $\mathrm{Aut}(\mathbb{O}_{\mathbb{R}}, \circ)$, the automorphism group of the real octonion division algebra, which is simply connected and has trivial center.

Remark 5.3.2. The complexification of the compact symplectic group $\mathrm{Sp}(n)$ is the usual complex symplectic group $\mathrm{Sp}(2n, \mathbb{C}) = \mathbf{Sp}_{2n}(\mathbb{C})$, which is defined as the group of linear transformations of $\mathbb{C}^{2n}$ preserving the standard symplectic bilinear form.

Notation 5.3.3. We denote by μ_n the group of nth roots of unity. If m groups $G_1, \ldots, G_m$ all have a unique central subgroup isomorphic to μ_n with an embedding $\iota_i : \mu_n \hookrightarrow G_i$, we denote by μ_n^Δ the diagonal subgroup

$$\{(\iota_1(g), \ldots, \iota_m(g)) \mid g \in \mu_n\} \subset G_1 \times \cdots \times G_m.$$

Note that when $n = 2$ the embedding ι_i is unique, but when $n \geq 3$ we have to give $\iota_1, \ldots, \iota_m$ for defining μ_n^Δ.

Following Dynkin's definitions of R-subalgebras and S-subalgebras in [Dynkin, 1952, §7], we give the following definition for subgroups:

Definition 5.3.4. Let G be a connected compact Lie group and H a connected closed subgroup. We say that H is a *regular subgroup* if it is normalized by a maximal torus of G. If there is only one regular subgroup of G containing H, namely G itself, we call H an *S-subgroup*, otherwise we call it an *R-subgroup.*

Examples 5.3.5. (1) Subgroups with maximal ranks are regular.

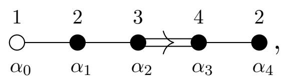
(2) A proper regular subgroup is an R-subgroup.

(3) The principal 3-dimensional subgroups are S-subgroups by [Dynkin, 1952, Theorem 9.1].

(4) A maximal proper regular subgroup has maximal rank.

Let H be a maximal proper regular subgroup of G, *i.e.* if there is another regular subgroup H' of G containing H, then we have $H' = G$. The Borel-de Siebenthal theory tells us the Dynkin diagram of the root system of H is obtained by deleting an ordinary vertex with prime label from the extended Dynkin diagram of the root system of G.

For our compact group F_4, the extended Dynkin diagram is:



The vertex α_1 corresponds to $(\mathrm{Sp}(1) \times \mathrm{Sp}(3))\,/\mu_2^\Delta$, α_2 corresponds to $(\mathrm{SU}(3) \times \mathrm{SU}(3))\,/\mu_3^\Delta$ (we will define this μ_3^Δ in Section 5.3.3), and α_4 corresponds to $\mathrm{Spin}(9)$. The vertex α_3 corresponds to $(\mathrm{SU}(2) \times \mathrm{SU}(4))\,/\mu_2^\Delta$, which is also regular but not maximal since we have the embedding:

$$(\mathrm{SU}(2) \times \mathrm{SU}(4))\,/\mu_2^\Delta \simeq (\mathrm{Spin}(3) \times \mathrm{Spin}(6))\,/\mu_2^\Delta \hookrightarrow \mathrm{Spin}(9).$$

These three maximal proper regular subgroups are also maximal among proper connected subgroups of F_4, because any connected subgroup containing one of them has maximal rank and must be regular.

Besides these three regular subgroups, F_4 also admits other maximal proper connected subgroups that are not regular. A non-regular maximal connected subgroup H of F_4 must be an S-subgroup. As a subgroup of F_4 containing an S-subgroup is also an S-subgroup, it suffices to find all maximal S-subgroups of F_4.

Theorem 5.3.6. *[Dynkin, 1952, Theorem 14.1] Up to conjugacy, there are two maximal S-subgroups in* F_4*: the principal* $\mathrm{PSU}(2)$ *and* $\mathrm{G}_2 \times \mathrm{SO}(3)$*, where* $\mathrm{PSU}(2) := \mathrm{SU}(2)/\{\pm \mathrm{id}\}$ *is the adjoint group of* $\mathrm{SU}(2)$*.*

Putting the Borel-de Siebenthal theory and Theorem 5.3.6 together, we have:

Theorem 5.3.7. *Up to conjugacy, there are five maximal proper connected subgroups of* F_4*. They are respectively isomorphic to*

$$\mathrm{Spin}(9), (\mathrm{Sp}(1)\times\mathrm{Sp}(3))\,/\mu_2^{\Delta}, (\mathrm{SU}(3)\times\mathrm{SU}(3))\,/\mu_3^{\Delta}, \mathrm{G}_2\times\mathrm{SO}(3), \textit{(principal)}\ \mathrm{PSU}(2).$$

In the rest of this section, we will give the explicit embeddings of these five maximal proper connected subgroups into F_4 and compute their centralizers in F_4.

5.3.1 Spin(9)

There is an involution $\sigma\in\mathrm{F}_4$ on $\mathrm{J}_{\mathbb{R}}$ defined by:

$$\sigma\ [a,b,c\,;x,y,z] = [a,b,c\,;x,-y,-z]\,,\ \text{for all}\ a,b,c\in\mathbb{R}, x,y,z\in\mathbb{O}_{\mathbb{R}}.$$

By [Yokota, 2009, Theorem 2.9.1], the centralizer $\mathrm{C}_{\mathrm{F}_4}(\sigma)$ of σ in F_4 is also the stabilizer of $\mathrm{E}_1=\mathrm{diag}(1,0,0)\in\mathrm{J}_{\mathbb{R}}$.

Lemma 5.3.8. *The group* $\mathrm{C}_{\mathrm{F}_4}(\sigma)$ *preserves respectively the subspaces*

$$\mathrm{J}_1 := \{[0,b,-b\,;x,0,0]\,|\,b\in\mathbb{R}, x\in\mathbb{O}_{\mathbb{R}}\}$$

and

$$\mathrm{J}_2 := \{[0,0,0\,;0,y,z]\,|\,y,z\in\mathbb{O}_{\mathbb{R}}\}$$

of $\mathrm{J}_{\mathbb{R}}$.

Proof. The first subspace J_1 is exactly $\{X\in\mathrm{J}_{\mathbb{R}}\,|\,E_1\circ X=0,\mathrm{Tr}(X)=0\}$ and the second subspace is $\{X\in\mathrm{J}_{\mathbb{R}}\,|\,2E_1\circ X=X\}$. The lemma follows from the fact that $\mathrm{C}_{\mathrm{F}_4}(\sigma)$ is the stabilizer of E_1 in F_4. □

This lemma gives the following group homomorphism:

$$\mathrm{C}_{\mathrm{F}_4}(\sigma)\to\mathrm{SO}(\mathrm{J}_1)\simeq\mathrm{SO}(9), g\mapsto g|_{\mathrm{J}_1},$$

which induce an isomorphism $\mathrm{C}_{\mathrm{F}_4}(\sigma)\simeq\mathrm{Spin}(9)$ by [Adams, 1996, Theorem 16.7(ii)]. Since the Borel-de Siebenthal theory shows that the regular connected subgroup of type B_4 is unique up to F_4-conjugacy, so we shall thus refer to this group $\mathrm{C}_{\mathrm{F}_4}(\sigma)$ as Spin(9) in the sequel, by a slight abuse of language.

The restriction of the 26-dimensional irreducible representation (π_0,J_0) to Spin(9) is isomorphic to

$$\mathbf{1}\oplus\mathrm{V}_9\oplus\mathrm{V}_{\mathrm{Spin}}, \tag{5.1}$$

where $\mathbf{1}$ is the trivial representation, V_9 is the standard 9-dimensional representation and V_{Spin} is the 16-dimensional spinor module. These two representations V_9 and V_{Spin} can be realized on J_1 and J_2 respectively.

Notation 5.3.9. To make the restriction of J_0 not too messy when it involves both direct sums and tensor products, we will replace $\oplus$ by $+$ when writing down the decomposition. For example, we write $J_0|_{\mathrm{Spin}(9)}$ as $\mathbf{1} + V_9 + V_{\mathrm{Spin}}$.

The restriction of the adjoint representation $\mathfrak{f}_4$ of F_4 to Spin(9) is isomorphic to:

$$\wedge^2 V_9 + V_{\mathrm{Spin}}, \tag{5.2}$$

where $\wedge^2 V_9$ is the adjoint representation of Spin(9).

Now we compute the centralizer of Spin(9). If an element g centralizes Spin(9), then it must commute with $\sigma \in \mathrm{Spin}(9)$. Hence $C_{F_4}(\mathrm{Spin}(9))$ is contained in $C_{F_4}(\sigma) = \mathrm{Spin}(9)$, thus it is isomorphic to the center of Spin(9), which is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ and generated by σ.

Remark 5.3.10. By symmetry, the stabilizer of $E_2 = \mathrm{diag}(0,1,0)$ (*resp.* $E_3 = \mathrm{diag}(0,0,1)$) is also the centralizer of the map $[a,b,c\,;x,y,z] \mapsto [a,b,c\,;-x,y,-z]$ (*resp.* $[a,b,c\,;-x,-y,z]$) in F_4, and is isomorphic to Spin(9).

5.3.2 $(\mathrm{Sp}(1) \times \mathrm{Sp}(3))/\mu_2^{\Delta}$

The subalgebra of $\mathbb{O}_{\mathbb{R}}$ generated by $1, e_1, e_2, e_4$ is isomorphic to the quaternion division algebra $\mathbb{H}$, and as a real vector space $\mathbb{O}_{\mathbb{R}}$ can be decomposed as $\mathbb{H} \oplus \mathbb{H}e_5$. Using this decomposition, the conjugation on $\mathbb{O}_{\mathbb{R}}$ becomes

$$x + ye_5 \mapsto \overline{x} - ye_5, \text{ for all } x, y \in \mathbb{H}.$$

As $J_{\mathbb{R}} = \mathrm{Herm}_3(\mathbb{O}_{\mathbb{R}})$ is the space of "Hermitian" matrices in $M_3(\mathbb{O}_{\mathbb{R}})$, we embed the space $\mathrm{Herm}_3(\mathbb{H})$ of "Hermitian" matrices in $M_3(\mathbb{H})$ into $J_{\mathbb{R}}$ via our identification of $\mathbb{H}$ as a subalgebra of $\mathbb{O}_{\mathbb{R}}$. Then we have the following isomorphism of vector spaces:

$$\begin{aligned} \mathrm{Herm}_3(\mathbb{H}) \oplus \mathbb{H}^3 &\to J_{\mathbb{R}}, \\ (M, a = (a_1, a_2, a_3)) &\mapsto M + [0,0,0\,; a_1e_5, a_2e_5, a_3e_5]. \end{aligned}$$

With this identification, we have an involution γ in F_4 defined as

$$\gamma(M, a) = (M, -a).$$

Proposition 5.3.11. *[Yokota, 2009, Theorem 2.11.2] Let* $\varphi : \mathrm{Sp}(1) \times \mathrm{Sp}(3) \to \mathrm{GL}(J_{\mathbb{R}})$ *be the morphism defined as*

$$\varphi(p, A)\,(M, a) = \left(AMA^{-1}, paA^{-1}\right), \text{ for } M \in \mathrm{Herm}_3(\mathbb{H}), a \in \mathbb{H}^3.$$

Then the kernel of φ is the diagonal subgroup μ_2^Δ generated by γ, and the image of φ is $C_{F_4}(\gamma)$. In particular, φ induces an isomorphism:

$$(\mathrm{Sp}(1) \times \mathrm{Sp}(3))\,/\mu_2^\Delta \simeq C_{F_4}(\gamma).$$

From now on we refer to the regular connected subgroup $C_{F_4}(\gamma)$ as $(\mathrm{Sp}(1) \times \mathrm{Sp}(3))\,/\mu_2^\Delta$.

The restriction of the irreducible representation J_0 of F_4 to this subgroup is isomorphic to

$$\mathrm{St} \otimes V_6 + \mathbf{1} \otimes V_{14}, \tag{5.3}$$

where St is the 2-dimensional standard representation of $\mathrm{Sp}(1) \simeq \mathrm{SU}(2)$, V_6 is the standard 6-dimensional representation of Sp(3) and V_{14} is the 14-dimensional irreducible representation of Sp(3) which satisfies $\wedge^2 V_6 \simeq V_{14} \oplus \mathbf{1}$. The first component $\mathrm{St} \otimes V_6$ is realized on $\mathbb{H}^3$ and the second component $\mathbf{1} \otimes V_{14}$ is realized on the trace-zero part of $\mathrm{Herm}_3(\mathbb{H})$.

The restriction of the adjoint representation $\mathfrak{f}_4$ of F_4 to $(\mathrm{Sp}(1) \times \mathrm{Sp}(3))\,/\mu_2^\Delta$ is isomorphic to

$$\mathrm{Sym}^2\,\mathrm{St} \otimes \mathbf{1} + \mathrm{St} \otimes V'_{14} + \mathbf{1} \otimes \mathrm{Sym}^2\,V_6, \tag{5.4}$$

where V'_{14} is another 14-dimensional irreducible representation of Sp(3).

By a similar argument in the case of Spin(9), the centralizer of $(\mathrm{Sp}(1) \times \mathrm{Sp}(3))\,/\mu_2^\Delta$ in F_4 is isomorphic to $Z((\mathrm{Sp}(1) \times \mathrm{Sp}(3))\,/\mu_2^\Delta) \simeq \mathbb{Z}/2\mathbb{Z}$. It is generated by the involution γ, which corresponds to $(-1, 1)$ in $Z(\mathrm{Sp}(1) \times \mathrm{Sp}(3)) \simeq \mu_2 \times \mu_2$.

Remark 5.3.12. It may help to notice that there are exactly two conjugacy classes of involutions in F_4, whose centralizers in F_4 are Spin(9) and $(\mathrm{Sp}(1) \times \mathrm{Sp}(3))\,/\mu_2^\Delta$ respectively.

5.3.3 $(\mathrm{SU}(3) \times \mathrm{SU}(3))\,/\mu_3^\Delta$

Take $\omega = \frac{-1+\sqrt{-3}}{2}$ and identify the center of SU(3) with μ_3 by identifying ω with the scalar matrix ωI_3. Then the diagonal subgroup $\mu_3^\Delta \subset \mathrm{SU}(3) \times \mathrm{SU}(3)$ is generated by (ω, ω).

By [Yokota, 2009, Theorem 2.12.2], the centralizer in F_4 of an order 3 element in F_4 is isomorphic to $(\mathrm{SU}(3) \times \mathrm{SU}(3))\,/\mu_3^\Delta$. As before, by an abuse of language we will refer to this subgroup as $(\mathrm{SU}(3) \times \mathrm{SU}(3))\,/\mu_3^\Delta$. Notice that the roots of the first copy of SU(3) are short roots of F_4, and those of the second copy are long roots of F_4.

Since SU(3) admits an outer automorphism, this unique (up to conjugacy) $2\,A_2$-type subgroup $(\mathrm{SU}(3) \times \mathrm{SU}(3))\,/\mu_3^\Delta$ of F_4 has two embeddings into F_4 which are not conjugate. The restrictions of the irreducible representation J_0 along those embeddings are isomorphic to

$$\mathfrak{sl}_3 \otimes \mathbf{1} + V_3 \otimes V'_3 + V'_3 \otimes V_3 \tag{5.5}$$

and

$$\mathfrak{sl}_3 \otimes \mathbf{1} + V_3 \otimes V_3 + V'_3 \otimes V'_3 \tag{5.6}$$

respectively. Here V_3 is the standard 3-dimensional representation of SU(3), V_3' is the dual representation of V_3, and $\mathfrak{sl}_3$ is the adjoint representation of SU(3).

The restriction of the adjoint representation $\mathfrak{f}_4$ of F_4 to $(\mathrm{SU}(3)\times\mathrm{SU}(3))/\mu_3^{\Delta}$ is isomorphic to

$$\mathfrak{sl}_3\otimes\mathbf{1}+\mathbf{1}\otimes\mathfrak{sl}_3+\mathrm{Sym}^2\mathrm{V}_3\otimes\mathrm{V}_3'+\mathrm{Sym}^2\mathrm{V}_3'\otimes\mathrm{V}_3 \tag{5.7}$$

or

$$\mathfrak{sl}_3\otimes\mathbf{1}+\mathbf{1}\otimes\mathfrak{sl}_3+\mathrm{Sym}^2\mathrm{V}_3\otimes\mathrm{V}_3+\mathrm{Sym}^2\mathrm{V}_3'\otimes\mathrm{V}_3'. \tag{5.8}$$

Again, we have an isomorphism $\mathrm{C}_{\mathrm{F}_4}((\mathrm{SU}(3)\times\mathrm{SU}(3))/\mu_3^{\Delta})\simeq\mathbb{Z}/3\mathbb{Z}$.

5.3.4 $\mathrm{G}_2\times\mathrm{SO}(3)$

We define an injective morphism $\iota:\mathrm{G}_2\times\mathrm{SO}(3)\hookrightarrow\mathrm{GL}(\mathrm{J}_{\mathbb{R}})$ by

$$\iota(g,O)[a,b,c\,;x,y,z]=O[a,b,c\,;g(x),g(y),g(z)]O^{-1},\ \text{ for all } a,b,c\in\mathbb{R},x,y,z\in\mathbb{O}_{\mathbb{R}}, \tag{5.9}$$

by viewing $O\in\mathrm{SO}(3)$ as an element in $\mathrm{GL}_3(\mathbb{O}_{\mathbb{R}})$ with entries in $\mathbb{R}$. This morphism is well-defined since real numbers $\mathbb{R}$ is the center of the octonion division algebra $\mathbb{O}_{\mathbb{R}}$. For any $g\in\mathrm{G}_2$ and $O\in\mathrm{SO}(3)$, the linear automorphism $\iota(g,O)$ preserves the cubic form det and the polarization I, thus ι induces an embedding of $\mathrm{G}_2\times\mathrm{SO}(3)$ into F_4. In the sequel we will refer to the image of ι as $\mathrm{G}_2\times\mathrm{SO}(3)$.

The restriction of the irreducible representation J_0 to $\mathrm{G}_2\times\mathrm{SO}(3)$ is isomorphic to

$$\mathrm{V}_7\otimes\mathrm{Sym}^2\mathrm{St}+\mathbf{1}\otimes\mathrm{Sym}^4\mathrm{St}, \tag{5.10}$$

where V_7 is the fundamental 7-dimensional representation of G_2 (the trace-zero part of $\mathbb{O}_{\mathbb{C}}$) and St denotes the standard 2-dimensional representation of SU(2). Here we use the exceptional isomorphism $\mathrm{SO}(3)\simeq\mathrm{PSU}(2)=\mathrm{SU}(2)/\mu_2$ to view odd dimensional irreducible representations $\mathrm{Sym}^{2n}\mathrm{St},n\in\mathbb{N}$ of SU(2) as irreducible representations of SO(3). The first component $\mathrm{V}_7\otimes\mathrm{Sym}^2\mathrm{St}$ is realized on the space

$$\{[0,0,0\,;x,y,z]\,|\,x,y,z\in\mathbb{O}_{\mathbb{R}},\mathrm{Tr}(x)=\mathrm{Tr}(y)=\mathrm{Tr}(z)=0\},$$

and the second component $\mathbf{1}\otimes\mathrm{Sym}^4\mathrm{St}$ is realized on the space

$$\{[a,b,c\,;x,y,z]\,|\,a,b,c,x,y,z\in\mathbb{R},a+b+c=0\}.$$

The restriction of the adjoint representation $\mathfrak{f}_4$ of F_4 to $\mathrm{G}_2\times\mathrm{SO}(3)$ is isomorphic to

$$\mathfrak{g}_2\otimes\mathbf{1}+\mathrm{V}_7\otimes\mathrm{Sym}^4\mathrm{St}+\mathbf{1}\otimes\mathrm{Sym}^2\mathrm{St}, \tag{5.11}$$

where $\mathfrak{g}_2$ is the adjoint representation of G_2.

Proposition 5.3.13. *The centralizer of* $G_2 \times SO(3)$ *in* F_4 *is trivial.*

Proof. Let g be an element in $C_{F_4}(G_2 \times SO(3))$. Because the image of $\mathrm{diag}(1,-1,-1) \in SO(3)$ in F_4 is the involution σ defined in Section 5.3.1, g lies in $C_{F_4}(\sigma)$, thus it stabilizes E_1. By Remark 5.3.10, we also have g stabilizes E_2 and E_3 respectively. According to [Adams, 1996, Theorem 16.7(iii), Lemma 15.15], g is an element of the form

$$[a,b,c\,;x,y,z] \mapsto [a,b,c\,;\alpha(x),\beta(y),\gamma(z)], \text{ for all } a,b,c \in \mathbb{R}, x,y,z \in \mathbb{O}_{\mathbb{R}},$$

where $\alpha, \beta, \gamma \in SO(\mathbb{O}_{\mathbb{R}})$ satisfy

$$\overline{\alpha(x)\beta(y)} = \gamma(\overline{xy}) \text{ for all } x,y \in \mathbb{O}_{\mathbb{R}}. \tag{5.12}$$

The image of $\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{smallmatrix}\right) \in SO(3)$ in F_4 is the map

$$[a,b,c\,;x,y,z] \mapsto [a,c,b\,;-\overline{x},-\overline{z},\overline{y}].$$

The fact that it commutes with g implies that $\alpha(\overline{x}) = \overline{\alpha(x)}$ and $\beta(\overline{x}) = \overline{\gamma(x)}$ for all $x \in \mathbb{O}_{\mathbb{R}}$. By symmetry we get $\alpha = \beta = \gamma$ and Eq. (5.12) shows that

$$\alpha(x)\alpha(y) = \overline{\alpha(\overline{xy})} = \alpha(\overline{\overline{xy}}) = \alpha(xy), \text{ for all } x,y \in \mathbb{O}_{\mathbb{R}}.$$

Hence $\alpha \in G_2$ and we have proved that $C_{F_4}(SO(3)) = G_2$, thus the centralizer of $G_2 \times SO(3)$ in F_4 is the center of G_2, which is trivial. □

5.3.5 The principal PSU(2)

The image of the *principal embedding* from SU(2) into F_4, in the sense of [CollingwoodMcGovern, 1993, Theorem 4.1.6], is also a maximal proper connected subgroup of F_4. The restriction of the irreducible representation J_0 to this SU(2) is isomorphic to

$$\mathrm{Sym}^8\,\mathrm{St} + \mathrm{Sym}^{16}\,\mathrm{St},$$

where St is the standard 2-dimensional representation of SU(2). This implies that the image is isomorphic to PSU(2), and we call it the principal PSU(2) of F_4.

By the general property of principal embeddings, its centralizer is the center of F_4. It is well-known that the center of F_4 is trivial.

5.4 Classification of A_1-subgroups

In this section we will classify A_1-*subgroups* of F_4, *i.e.* subgroups that are isomorphic to SU(2) or PSU(2). By [Dynkin, 1952, Theorem 9.3] every A_1-subgroup X of F_4 is either the principal PSU(2) or an R-subgroup, *i.e.* X is contained in some proper regular subgroup of F_4. When X

is an R-subgroup, up to conjugacy it is contained in one of the three regular maximal proper connected subgroups of F_4 we have found in Section 5.3. All these three regular subgroups arise from classical groups, thus their A_1-subgroups are well-known.

By Proposition 5.2.1, a conjugacy class of A_1-groups of F_4 is determined uniquely by the restriction of the 26-dimensional representation J_0 to it.

Notation 5.4.1. An isomorphism class of n-dimensional representation of SU(2) gives a partition of the integer n. We will use the notation $[N^{k_N}, (N-1)^{k_{N-1}}, \dots, 2^{k_2}, 1^{k_1}]$, where $k_N \neq 0$ and $\sum_{i=1}^{N} ik_i = n$, for a partition of n. For example, the restriction of J_0 to the principal PSU(2) is isomorphic to $\mathrm{Sym}^8\,\mathrm{St} + \mathrm{Sym}^{16}\,\mathrm{St}$, thus we index this A_1-subgroup by the partition $[17, 9]$ of $\dim J_0 = 26$.

5.4.1 A_1-subgroups of Spin(9)

We start from A_1-subgroups of SO(9). According to [CollingwoodMcGovern, 1993, Theorem 5.1.2], the conjugacy classes of morphisms $SU(2) \to SO(9)$ are in bijection with partitions of 9 in which each even number appears even times.

Lemma 5.4.2. *(1) There are* 12 *different conjugacy classes of* A_1*-subgroups of* Spin(9)*, which correspond to the following partitions of* 9*:*

$$[9], [7, 1^2], [5, 3, 1], [5, 2^2], [5, 1^4], [4^2, 1], [3^3], [3^2, 1^3], [3, 2^2, 1^2], [3, 1^6], [2^4, 1], [2^2, 1^5].$$

(2) There are 10 *different conjugacy classes of* A_1*-subgroups of* F_4 *that are contained in the subgroup* Spin(9) *given in Section 5.3.1. The restrictions of the* 26*-dimensional irreducible representation* J_0 *of* F_4 *to these* A_1*-subgroups correspond to the following partitions of* 26*:*

$$\begin{aligned} &[11, 9, 5, 1], [7^3, 1^5], [5^3, 3^3, 1^2], [3^6, 1^8], \\ &[5^2, 4^2, 3, 2^2, 1], [5, 4^4, 1^5], [4^2, 3^3, 2^4, 1], [3^3, 2^6, 1^5], [3, 2^8, 1^7], [2^6, 1^{14}]. \end{aligned} \tag{5.13}$$

Proof. By the lifting property of covering maps and the fact that SU(2) is simply connected, every A_1-subgroup of SO(9) is lifted uniquely to an A_1-subgroup of Spin(9). The assertion (1) follows directly from [CollingwoodMcGovern, 1993, Theorem 5.1.2], and the assertion (2) follows from the equivalence Eq. (5.1). □

The A_1-subgroups in the first row of Eq. (5.13) are isomorphic to PSU(2) and the A_1-subgroups in the second row are isomorphic to SU(2).

5.4.2 A_1-subgroups of $(\mathrm{Sp}(1) \times \mathrm{Sp}(3))/\mu_2^{\Delta}$

We apply the same argument for A_1-subgroups of $(\mathrm{Sp}(1) \times \mathrm{Sp}(3))/\mu_2^{\Delta}$. By [CollingwoodMcGovern, 1993, Theorem 5.1.3], the set of conjugacy classes of morphisms $SU(2) \to Sp(3)$ are in bijection with partitions of 6 in which each odd number appears even times.

Lemma 5.4.3. *(1) There are* 7 *different conjugacy classes of* A_1*-subgroups of* $Sp(3)$*, which correspond to the following partitions of* 6*:*

$$[6], [4,2], [4,1^2], [3^2], [2^3], [2^2,1^2], [2,1^4].$$

(2) There are 11 *different conjugacy classes of* A_1*-subgroups of* F_4 *that are contained in the subgroup* $(Sp(1) \times Sp(3)) / \mu_2^{\Delta}$ *given in Section 5.4.2. The restrictions of the* 26*-dimensional irreducible representation* J_0 *of* F_4 *to these* A_1*-subgroups correspond to the following partitions of* 26*:*

$$\begin{gathered} [9,7,5^2], [5^3,3^3,1^2], [5,3^7], [3^6,1^8], \\ [9,6^2,5], [5^2,4^2,3,2^2,1], [5,4^4,1^5], [5,4^2,3^3,2^2], [3^3,2^6,1^5], [3,2^8,1^7], [2^6,1^{14}]. \end{gathered} \tag{5.14}$$

Proof. The assertion (1) follows directly from [CollingwoodMcGovern, 1993, Theorem 5.1.3]. A morphism from $SU(2)$ to $(Sp(1) \times Sp(3)) / \mu_2^{\Delta}$ arises from the product of two morphisms $SU(2) \to Sp(1)$ and $SU(2) \to Sp(3)$. The assertion (2) follows from the equivalence Eq. (5.3). □

The A_1-subgroups in the first row of Eq. (5.14) are isomorphic to $PSU(2)$ and the A_1-subgroups in the second row are isomorphic to $SU(2)$.

5.4.3 A_1 subgroups of $(SU(3) \times SU(3)) / \mu_3^{\Delta}$

The restriction of the standard representation V_3 of $SU(3)$ to an A_1-subgroup of $SU(3)$ can only be [3] or [2, 1]. By the equivalences Eq. (5.5) and Eq. (5.8), we have the following result:

Lemma 5.4.4. *There are* 8 *different conjugacy classes of* A_1*-subgroups of* F_4 *that are contained in the subgroup* $(SU(3) \times SU(3)) / \mu_3^{\Delta}$ *given in Section 5.3.3. The restrictions of the* 26*-dimensional irreducible representation* J_0 *of* F_4 *to these* A_1*-subgroups correspond to the following partitions of* 26*:*

$$\begin{gathered} [5^3,3^3,1^2], [5,3^7], [3^6,1^8] \\ [5,4^2,3^3,2^2], [4^2,3^3,2^4,1], [3^3,2^6,1^5], [3,2^8,1^7], [2^6,1^{14}]. \end{gathered} \tag{5.15}$$

The A_1-subgroups in the first row of Eq. (5.15) are isomorphic to $PSU(2)$ and subgroups in the second row are isomorphic to $SU(2)$.

5.4.4 Conclusion

Now we have enumerated (up to conjugacy) all A_1-subgroups of F_4 and indexed them by the restriction of the 26-dimensional irreducible representation J_0 of F_4.

Proposition 5.4.5. *(1) There are* 7 *conjugacy classes of subgroups of* F_4 *that are isomorphic to* $PSU(2)$*, corresponding to the following partitions of* 26*:*

$$[17,9], [11,9,5,1], [9,7,5^2], [7^3,1^5], [5^3,3^3,1^2], [5,3^7], [3^6,1^8].$$

(2) There are 7 *conjugacy classes of subgroups of* F_4 *that are isomorphic to* $\mathrm{SU}(2)$*, corresponding to the following partitions of* 26*:*

$$[9,6^2,5],[5^2,4^2,3,2^2,1],[5,4^4,1^5],[5,4^2,3^3,2^2],[4^2,3^3,2^4,1],[3^3,2^6,1^5],[3,2^8,1^7],[2^6,1^{14}].$$

The theory of Jacobson-Morozov shows that the set of conjugacy classes of morphisms $\mathrm{SU}(2)\to\mathrm{F}_4$ is in bijection with the set of nilpotent orbits of the semisimple Lie algebra $\mathfrak{f}_4$. The nilpotent orbits of $\mathfrak{f}_4$ are labeled in [CollingwoodMcGovern, 1993, §8.4], and we will use the same labelings for A_1-subgroups of F_4:

Label	Restriction of J_0	Label	Restriction of J_0	Label	Restriction of J_0
A_1	$[2^6,1^{14}]$	$\mathrm{A}_2+\widetilde{\mathrm{A}_1}$	$[4^2,3^3,2^4,1]$	B_3	$[7^3,1^5]$
$\widetilde{\mathrm{A}_1}$	$[3,2^8,1^7]$	B_2	$[5,4^4,1^5]$	C_3	$[9,6^2,5]$
$\mathrm{A}_1+\widetilde{\mathrm{A}_1}$	$[3^3,2^6,1^5]$	$\widetilde{\mathrm{A}_2}+\mathrm{A}_1$	$[5,4^2,3^3,2^2]$	$\mathrm{F}_4(a_2)$	$[9,7,5^2]$
A_2	$[3^6,1^8]$	$\mathrm{C}_3(a_1)$	$[5^2,4^2,3,2^2,1]$	$\mathrm{F}_4(a_1)$	$[11,9,5,1]$
$\widetilde{\mathrm{A}_2}$	$[5,3^7]$	$\mathrm{F}_4(a_3)$	$[5^3,3^3,1^2]$	F_4	$[17,9]$

Table 5.2: Labels of A_1-subgroups of F_4

Notation 5.4.6. With Table 5.2, for a conjugacy class of A_1-subgroups of F_4, we have two ways to refer to it. For example, for the conjugacy class of principal $\mathrm{PSU}(2)$, we call it the class $[17,9]$ or the class with label F_4.

5.4.5 Centralizers

The next thing we are going to do is to compute the centralizer, or the neutral component of the centralizer, of each A_1-subgroup of F_4. In the rest of this section, we will choose a representative $\mathrm{SU}(2)\to\mathrm{F}_4$ for each conjugacy class of A_1-subgroups, whose image is denoted by X, and then determine $\mathrm{C}_{\mathrm{F}_4}(X)$ or $\mathrm{C}_{\mathrm{F}_4}(X)^\circ$.

The following lemma will be used when computing the centralizer of a subgroup in F_4:

Lemma 5.4.7. *Let G be the quotient of a Lie group G_0 by a finite central subgroup Γ. If H_0 is a connected subgroup of G_0, whose image in G is denoted by H, then the inverse image of $\mathrm{C}_G(H)$ in G_0 is $\mathrm{C}_{G_0}(H_0)$ and $\mathrm{C}_G(H)\simeq\mathrm{C}_{G_0}(H_0)/\Gamma$.*

Proof. It suffices to prove that any $g_0\in G_0$ whose image g lies in $\mathrm{C}_G(H)$ centralizes H_0. For any $h_0\in H_0$ with image h in H, we have $ghg^{-1}h^{-1}=1$ in G, thus $g_0h_0g_0^{-1}h_0^{-1}\in\Gamma$. The continuous map $\varphi:H_0\to\Gamma, h_0\mapsto g_0h_0g_0^{-1}h_0^{-1}$ for $h_0\in H_0$ must be constant, because H_0 is connected and Γ is discrete as a finite group. The map φ sends $1\in H_0$ to $1\in\Gamma$, thus $\varphi\equiv 1$, which implies that g_0 centralizes H_0 in G_0. □

In some cases we can not compute the centralizer $\mathrm{C}_{\mathrm{F}_4}(X)$ easily, then we use the following lemma to determine its neutral component $\mathrm{C}_{\mathrm{F}_4}(X)^\circ$:

Lemma 5.4.8. *Let H be a connected subgroup of a compact Lie group G, and d the multiplicity of $\mathbf{1}$ in the restriction of the adjoint representation $\mathfrak{g}$ of G to H. If there is a d-dimensional connected subgroup C of $\mathrm{C}_G(H)$, then we have $\mathrm{C}_G(H)^\circ = C$. In particular, the centralizer $\mathrm{C}_G(H)$ is a finite group when $d = 0$.*

Proof. As subalgebras of $\mathfrak{g}$, the Lie algebra $\mathrm{Lie}(\mathrm{C}_G(H)^\circ)$ of $\mathrm{C}_G(H)^\circ$ is contained in

$$\mathrm{C}_{\mathfrak{g}}(H) := \{X \in \mathfrak{g} \mid \mathrm{Ad}(g)X = X \text{ for all } g \in H_{\mathbb{C}}\},$$

where $H_{\mathbb{C}}$ is the complexification of H. The dimension of $\mathrm{C}_{\mathfrak{g}}(H)$ equals the multiplicity d of $\mathbf{1}$ in $\mathfrak{g}|_H$.

Let $\mathfrak{c}$ be the complexified Lie algebra of C. We have the inclusions $\mathfrak{c} \subset \mathrm{Lie}(\mathrm{C}_G(H)^\circ) \subset \mathrm{C}_{\mathfrak{g}}(H)$. Since $\dim \mathfrak{c} = d = \dim \mathrm{C}_{\mathfrak{g}}(H)$, these three subspaces of $\mathfrak{g}$ are equal. It is well known that a connected Lie group is generated by a neighborhood of the identity element, thus the connected subgroups C and $\mathrm{C}_G(H)^\circ$ of G coincide. □

5.4.5.1 [17, 9]

We choose X to be the principal PSU(2) in F_4, whose centralizer in F_4 is trivial.

5.4.5.2 [11, 9, 5, 1]

We choose X to be the principal PSU(2) of the Spin(9) given in Section 5.3.1. The restriction of the adjoint representation $\mathfrak{f}_4$ of F_4 to X corresponds to the partition $[15, 11^2, 7, 5, 3]$ of 52, which implies that $\mathrm{C}_{\mathrm{F}_4}(X)$ is a finite group by Lemma 5.4.8.

5.4.5.3 $[9, 7, 5^2]$

We choose X to be the principal PSU(2) of the $(\mathrm{Sp}(1) \times \mathrm{Sp}(3))/\mu_2^{\Delta}$ given in Section 5.3.2. The restriction of the adjoint representation $\mathfrak{f}_4$ to X corresponds to the partition $[11^2, 9, 7, 5, 3^3]$ of 52, thus $\mathrm{C}_{\mathrm{F}_4}(X)$ is a finite group by Lemma 5.4.8.

5.4.5.4 $[7^3, 1^5]$

We choose X to be the principal PSU(2) of the factor G_2 in the subgroup $\mathrm{G}_2 \times \mathrm{SO}(3)$ given in Section 5.3.4. The other factor SO(3) of $\mathrm{G}_2 \times \mathrm{SO}(3)$ centralizes this A_1-subgroup X. The restriction of the adjoint representation $\mathfrak{f}_4$ of F_4 to X corresponds to the partition $[11, 7^5, 3, 1^3]$ of 52, thus $\mathrm{C}_{\mathrm{F}_4}(X)^\circ$ is the SO(3) in $\mathrm{G}_2 \times \mathrm{SO}(3)$ by Lemma 5.4.8, which is in the class $[5, 3^7]$ and labeled by $\widetilde{\mathrm{A}_2}$.

5.4.5.5 $[5^3, 3^3, 1^2]$

We choose X to be the principal PSU(2) of the $(\mathrm{SU}(3) \times \mathrm{SU}(3))/\mu_3^{\Delta}$ given in Section 5.3.3. The restriction of the adjoint representation $\mathfrak{f}_4$ of F_4 to X corresponds to the partition $[7^2, 5^4, 3^6]$ of 52, thus $\mathrm{C}_{\mathrm{F}_4}(X)$ is a finite group by Lemma 5.4.8. The center of $(\mathrm{SU}(3) \times \mathrm{SU}(3))/\mu_3^{\Delta}$, which is a cyclic group of order 3, is contained in $\mathrm{C}_{\mathrm{F}_4}(X)$.

5.4.5.6 $[5, 3^7]$

We choose X to be the factor SO(3) in the subgroup $\mathrm{G}_2 \times \mathrm{SO}(3)$ of F_4 given in Section 5.3.4. In the proof of Proposition 5.3.13, we have shown that the centralizer $\mathrm{C}_{\mathrm{F}_4}(X)$ is the other factor G_2.

5.4.5.7 $[3^6, 1^8]$

We choose X to be the principal PSU(2) of the second copy of SU(3) in the subgroup $(\mathrm{SU}(3) \times \mathrm{SU}(3))/\mu_3^{\Delta}$ given in Section 5.3.3. The first copy of SU(3) centralizes X and has dimension 8. The restriction of the adjoint representation $\mathfrak{f}_4$ of F_4 to X corresponds to the partition $[5, 3^{13}, 1^8]$ of 52, thus $\mathrm{C}_{\mathrm{F}_4}(X)^{\circ}$ is the first copy of SU(3) in $(\mathrm{SU}(3) \times \mathrm{SU}(3))/\mu_3^{\Delta}$ by Lemma 5.4.8, whose roots are short roots of F_4.

5.4.5.8 $[9, 6^2, 5]$

We choose X_0 to be the principal SU(2) of Sp(3), and X to be the image of X_0 in the subgroup $(\mathrm{Sp}(1) \times \mathrm{Sp}(3))/\mu_2^{\Delta}$ given in Section 5.3.2. The group $(\mathrm{Sp}(1) \times \mathrm{Sp}(3))/\mu_2^{\Delta}$ is defined as $\mathrm{C}_{\mathrm{F}_4}(\gamma)$, where γ is an involution in F_4 and is the image of $(1, -\mathrm{I}_3) \in \mathrm{Sp}(1) \times \mathrm{Sp}(3)$ in the quotient group.

Since X contains the element γ, the centralizer of X in F_4 is contained in $\mathrm{C}_{\mathrm{F}_4}(\gamma) = (\mathrm{Sp}(1) \times \mathrm{Sp}(3))/\mu_2^{\Delta}$, thus $\mathrm{C}_{\mathrm{F}_4}(X) = \mathrm{C}_{(\mathrm{Sp}(1)\times\mathrm{Sp}(3))/\mu_2^{\Delta}}(X)$. By Lemma 5.4.7, we have:

$$\mathrm{C}_{(\mathrm{Sp}(1)\times\mathrm{Sp}(3))/\mu_2^{\Delta}}(X) = \mathrm{C}_{\mathrm{Sp}(1)\times\mathrm{Sp}(3)}(1 \times X_0)/\mu_2^{\Delta} = (\mathrm{Sp}(1) \times \mathrm{Z}(\mathrm{Sp}(3)))/\mu_2^{\Delta} \simeq \mathrm{Sp}(1).$$

Hence $\mathrm{C}_{\mathrm{F}_4}(X)$ is an A_1-subgroup in the class $[2^6, 1^{14}]$ and labeled by A_1.

5.4.5.9 $[5^2, 4^2, 3, 2^2, 1]$

We choose X_0 to be the image of

$$\mathrm{SU}(2) \hookrightarrow \mathrm{Sp}(1) \times \mathrm{Sp}(2) \hookrightarrow \mathrm{Sp}(3),$$

where the first arrow is the principal morphism , and the second is defined as $(x, A) \mapsto \left(\begin{smallmatrix} x & 0 \\ 0 & A \end{smallmatrix}\right)$, for any $x \in \mathrm{Sp}(1), A \in \mathrm{Sp}(2)$. Let X be the image of X_0 in $(\mathrm{Sp}(1) \times \mathrm{Sp}(3))/\mu_2^{\Delta} = \mathrm{C}_{\mathrm{F}_4}(\gamma)$.

The element γ corresponds to $(1, -\mathrm{I}_3)$ in $\mathrm{Sp}(1)\times\mathrm{Sp}(3)$, thus it is contained in X, so $\mathrm{C}_{\mathrm{F}_4}(X) \subset$

$C_{F_4}(\gamma)$ and $C_{F_4}(X) = C_{(Sp(1)\times Sp(3))/\mu_2^\Delta}(X)$. Again by Lemma 5.4.7, we have:

$$C_{(Sp(1)\times Sp(3))/\mu_2^\Delta}(X) = C_{Sp(1)\times Sp(3)}(1\times X_0)/\mu_2^\Delta = (Sp(1)\times\langle\gamma_1\rangle\times\langle\gamma_2\rangle)/\mu_2^\Delta,$$

where $\gamma_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $\gamma_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ are two order 2 elements in $Sp(3)$. Hence $C_{F_4}(X)$ is the product of $Sp(1)$ and an order 2 group, and this A_1-subgroup $Sp(1)$ is in the class $[2^6, 1^{14}]$ and labeled by A_1.

5.4.5.10 $[5, 4^4, 1^5]$

We choose a morphism:

$$SU(2) \hookrightarrow Spin(5) \hookrightarrow Spin(5)\times Spin(4) \to Spin(9) \hookrightarrow F_4,$$

where the first arrow is the principal morphism of $Spin(5)$, and the subgroup $Spin(9)$ of F_4 is defined as $C_{F_4}(\sigma)$ in Section 5.3.1. This morphism is injective since the factor $Spin(5)$ has zero intersection with the kernel of $Spin(5)\times Spin(4)\to Spin(9)$, and we denote its image by X.

The element σ defined in Section 5.3.1 is contained in X, hence the centralizer of X in F_4 is contained in $Spin(9)$, thus $C_{F_4}(X) = C_{Spin(9)}(X)$. Denote the natural projection $Spin(9)\to SO(9)$ by p. The centralizer of $p(X)$ in $SO(9)$ is $SO(4)$, the image of $Spin(4)$ under p. By Lemma 5.4.7, we have

$$C_{Spin(9)}(X) = p^{-1}(SO(4)) = Spin(4) \simeq SU(2)\times SU(2),$$

and as a result $C_{F_4}(X)$ is the product of two A_1-subgroups in the class $[2^6, 1^{14}]$.

5.4.5.11 $[5, 4^2, 3^3, 2^2]$

We choose an embedding:

$$SU(2) \hookrightarrow Sp(1)\times SO(3) \hookrightarrow Sp(1)\times Sp(3),$$

where the first arrow is the principal morphism of $Sp(1)\times SO(3)$, and the embedding $SO(3)\to Sp(3)$ is given by viewing an orthogonal 3×3 matrix as an matrix in $GL(3,\mathbb{H})$ preserving the standard Hermitian form on $\mathbb{H}^3$. Let X_0 be the image of this embedding, and X the image of X in the subgroup $(Sp(1)\times Sp(3))/\mu_2^\Delta = C_{F_4}(\gamma)$ of F_4 given in Section 5.3.2.

The group X_0 contains $(-1, I_3)$, thus the element γ is contained in X. So the centralizer $C_{F_4}(X)$ is contained in $C_{F_4}(\gamma)$ and $C_{F_4}(X) = C_{(Sp(1)\times Sp(3))/\mu_2^\Delta}(X)$. By Lemma 5.4.7, we have

$$C_{(Sp(1)\times Sp(3))/\mu_2^\Delta}(X) = \Big(Z(Sp(1))\times C_{Sp(3)}(SO(3))\Big)/\mu_2^\Delta \simeq C_{Sp(3)}(SO(3)).$$

A 3×3 matrix in $Sp(3)$ commutes with all elements in $SO(3)$ if and only if it is a scalar matrix, thus it must be of the form $h\cdot I_3$ for some norm 1 element $h\in\mathbb{H}$. Hence $C_{F_4}(X)\simeq Sp(1)$ is an A_1-subgroup in the class $[3^3, 2^6, 1^5]$ and labeled by $A_1 + \widetilde{A_1}$.

5.4.5.12 $[4^2, 3^3, 2^4, 1]$

We choose a morphism:

$$\mathrm{Spin}(3) \hookrightarrow \mathrm{Spin}(3) \times \mathrm{Spin}(3) \times \mathrm{Spin}(3) \to \mathrm{Spin}(9) = \mathrm{C}_{\mathrm{F}_4}(\sigma) \hookrightarrow \mathrm{F}_4,$$

where the first arrow is the diagonal embedding. This is also an embedding and we denote its image in F_4 by X.

Again we have $\mathrm{C}_{\mathrm{F}_4}(X) = \mathrm{C}_{\mathrm{Spin}(9)}(X)$, and by Lemma 5.4.7, the centralizer of X in Spin(9) is the inverse image in Spin(9) of the subgroup

$$\left\{ \begin{pmatrix} a_{11}\mathrm{I}_3 & a_{12}\mathrm{I}_3 & a_{13}\mathrm{I}_3 \\ a_{21}\mathrm{I}_3 & a_{22}\mathrm{I}_3 & a_{23}\mathrm{I}_3 \\ a_{31}\mathrm{I}_3 & a_{32}\mathrm{I}_3 & a_{33}\mathrm{I}_3 \end{pmatrix} \middle| \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \in \mathrm{SO}(3) \right\}$$

of SO(9). Hence $\mathrm{C}_{\mathrm{F}_4}(X) \simeq \mathrm{Spin}(3)$ is also an A_1-subgroup in the class $[4^2, 3^3, 2^4, 1]$.

5.4.5.13 $[3^3, 2^6, 1^5]$

We denote by X_0 the image of $\mathrm{Sp}(1) \hookrightarrow \mathrm{Sp}(3)$ given by $h \mapsto h\mathrm{I}_3$, and by X the image of X_0 under the embedding of Sp(3) into the group $(\mathrm{Sp}(1) \times \mathrm{Sp}(3))/\mu_2^\Delta = \mathrm{C}_{\mathrm{F}_4}(\gamma)$ given in Section 5.3.2.

The element $\gamma = (1, -\mathrm{I}_3)$ (modulo μ_2^Δ) is contained in X, so the centralizer $\mathrm{C}_{\mathrm{F}_4}(X)$ equals $\mathrm{C}_{(\mathrm{Sp}(1)\times\mathrm{Sp}(3))/\mu_2^\Delta}(X)$. By Lemma 5.4.7, we have

$$\mathrm{C}_{(\mathrm{Sp}(1)\times\mathrm{Sp}(3))/\mu_2^\Delta}(X) = \mathrm{C}_{\mathrm{Sp}(1)\times\mathrm{Sp}(3)}(1 \times X_0)/\mu_2^\Delta = \left(\mathrm{Sp}(1) \times \mathrm{C}_{\mathrm{Sp}(3)}(X_0)\right)/\mu_2^\Delta.$$

A 3×3 matrix $A \in \mathrm{Sp}(3)$ commutes with $h\mathrm{I}_3$ for all norm 1 quaternions h, if and only if all entries of A are real. Hence $\mathrm{C}_{\mathrm{Sp}(3)}(X_0) = \mathrm{GL}(3,\mathbb{R}) \cap \mathrm{Sp}(3) = \mathrm{O}(3)$, and as a result $\mathrm{C}_{\mathrm{F}_4}(X) \simeq \mathrm{Sp}(1) \times \mathrm{SO}(3)$ is the product of two A_1-subgroups in the classes $[2^6, 1^{14}]$ and $[5, 3^7]$ respectively. These two A_1-subgroups are labeled by A_1 and $\widetilde{\mathrm{A}_2}$ respectively.

5.4.5.14 $[3, 2^8, 1^7]$

We choose a morphism:

$$\mathrm{Spin}(3) \hookrightarrow \mathrm{Spin}(3) \times \mathrm{Spin}(6) \to \mathrm{Spin}(9) = \mathrm{C}_{\mathrm{F}_4}(\sigma) \hookrightarrow \mathrm{F}_4,$$

which is injective, and denote by X its image in F_4.

The element σ is contained in X, thus $\mathrm{C}_{\mathrm{F}_4}(X) = \mathrm{C}_{\mathrm{Spin}(9)}(X)$. Again by Lemma 5.4.7, this centralizer is the group Spin(6) in the morphism we choose.

5.4.5.15 $[2^6, 1^{14}]$

We choose X to be the factor $\mathrm{Sp}(1)$ in the $(\mathrm{Sp}(1) \times \mathrm{Sp}(3))/\mu_2^{\Delta}$ given in Section 5.3.2. Using Lemma 5.4.7, we obtain that the centralizer $\mathrm{C}_{\mathrm{F}_4}(X)$ is the other factor $\mathrm{Sp}(3)$.

5.5 Connected simple subgroups

In this section, we will classify connected simple subgroups of F_4 whose ranks are larger than 1, and then determine their centralizers in F_4.

Let H be a proper connected simple subgroup of F_4 whose rank is larger than 1. It is (up to conjugacy) contained in one of the following four maximal proper connected subgroups classified in Section 5.3:

$$\mathrm{Spin}(9), (\mathrm{Sp}(1) \times \mathrm{Sp}(3))/\mu_2^{\Delta}, (\mathrm{SU}(3) \times \mathrm{SU}(3))/\mu_3^{\Delta}, \mathrm{G}_2 \times \mathrm{SO}(3).$$

Moreover, by [Dynkin, 1952, Theorem 14.2] the group F_4 has no simple S-subgroup except the principal $\mathrm{PSU}(2)$, so we have:

Lemma 5.5.1. *Let H be a proper connected simple subgroup of F_4 with $\mathrm{rank}\, H \geq 2$, then up to conjugacy H is contained in one of the following fixed subgroups of F_4:*

$$\mathrm{Spin}(9), \mathrm{Sp}(3), (\mathrm{SU}(3) \times \mathrm{SU}(3))/\mu_3^{\Delta}.$$

The possible Lie types for H are:

$$\mathrm{A}_2, \mathrm{A}_3, \mathrm{A}_4, \mathrm{B}_2, \mathrm{B}_3, \mathrm{B}_4, \mathrm{C}_3, \mathrm{C}_4, \mathrm{D}_4, \mathrm{G}_2.$$

Proposition 5.5.2. *There are no connected subgroups of F_4 whose Lie type is A_4 or C_4.*

Proof. Suppose that F_4 admits a connected subgroup H with type A_4 or C_4. Since $\mathrm{rank}(H) = 4$, by Lemma 5.5.1 there exists an embedding of H into $\mathrm{Spin}(9)$.

The case that H is of type C_4 is impossible, because $\dim H = 36 = \dim \mathrm{Spin}(9)$ but H and $\mathrm{Spin}(9)$ have different Lie types. Hence H has type A_4. The morphism $H \hookrightarrow \mathrm{Spin}(9) \rightarrow \mathrm{SO}(9)$ gives H a self-dual 9-dimensional representation of H, which leads to contradiction since the A_4-type group H does not admit such a representation. □

5.5.1 Cases except A_2

In the remaining possible Lie types for connected simple subgroups of F_4, the type A_2 is more complicated. So we first look at the other types:

Proposition 5.5.3. *(1) For each type among*

$$\mathrm{A}_3, \mathrm{B}_2, \mathrm{B}_3, \mathrm{B}_4, \mathrm{C}_3, \mathrm{D}_4, \mathrm{G}_2,$$

there exists a simply-connected subgroup of F_4 *with this type.*
(2) Let H *be a connected compact Lie group such that it admits an embedding into* F_4 *and its Lie type is among*

$$A_3, B_2, B_3, B_4, C_3, D_4, G_2.$$

Then H *is simply-connected and the embedding* $H \hookrightarrow F_4$ *is unique up to conjugacy.*

Before proving this proposition case by case, we explain our strategy. Fixing a Lie type, we first construct an embedding ϕ_0 from the simply-connected compact Lie group H_0 of the given type into F_4. We claim that to prove Proposition 5.5.3(2) for this Lie type, it suffices to show that for any connected simple compact Lie group H of the same type with H_0, *i.e.* H is isomorphic to the quotient of H_0 by a finite central subgroup, and any embedding $\phi : H \to F_4$, the restriction of the 26-dimensional irreducible representation J_0 along ϕ is unique, up to equivalence of H_0-representations. Here we view the restriction of J_0 along $\phi : H \to F_4$ as a representation of H_0 by the composition with a central isogeny $H_0 \to H$.

Proof of the claim. For a connected compact Lie group H of the same Lie type as H_0 and an embedding $\phi : H \hookrightarrow F_4$, we can lift ϕ to a morphism $\phi \circ i : H_0 \to F_4$ via a central isogeny $i : H_0 \to H$. This morphism $\phi \circ i$ is conjugate to ϕ_0 by the uniqueness of $J_0|_{H_0}$ and Proposition 5.2.1, thus i is injective, which implies that H is also simply-connected. For any two embeddings $\phi, \phi' : H \hookrightarrow F_4$, applying Proposition 5.2.1 to $\phi \circ i$ and $\phi' \circ i$, we have $\phi \circ i$ and $\phi' \circ i$ are conjugate in F_4, thus ϕ and ϕ' are conjugate. □

5.5.1.1 B_4

In this case $H_0 \simeq \mathrm{Spin}(9)$ and we take ϕ_0 to be $H_0 \simeq \mathrm{Spin}(9) \hookrightarrow F_4$, where $\mathrm{Spin}(9) \hookrightarrow F_4$ is constructed in Section 5.3.1.

For any embedding ϕ from a B_4-type connected compact Lie group H into F_4, by Lemma 5.5.1 the image $\mathrm{Im}(\phi)$ (up to conjugate) is a subgroup of the $\mathrm{Spin}(9)$ in F_4, thus ϕ factors through an embedding $H \to \mathrm{Spin}(9)$. This embedding must be an isomorphism, so the restrictions of J_0 along ϕ_0 and ϕ are equivalent as H_0-representations.

5.5.1.2 D_4

In this case $H_0 \simeq \mathrm{Spin}(8)$ and we take ϕ_0 to be the composition of the natural embedding $\mathrm{Spin}_8 \hookrightarrow \mathrm{Spin}(9)$ with $\mathrm{Spin}(9) \hookrightarrow F_4$.

For any embedding ϕ from a D_4-type connected compact Lie group H into F_4, ϕ (up to conjugacy) factors through an embedding $H \to \mathrm{Spin}(9)$ by Lemma 5.5.1. The restriction of the 9-dimensional irreducible representation V_9 to H is isomorphic to either $\mathbf{1} + V_8$ or $\mathbf{1} + V_{\mathrm{Spin}}^+$ or $\mathbf{1} + V_{\mathrm{Spin}}^-$, where V_8 is the standard 8-dimensional representation of $\mathrm{Spin}(8)$, and $V_{\mathrm{Spin}}^{\pm}$ are two 8-dimensional spinor representations of $\mathrm{Spin}(8)$. For those three possibilities, we obtain the same equivalence class of $J_0|_H$, which is equivalent to $\mathbf{1}^{\oplus 2} + V_8 + V_{\mathrm{Spin}}^+ + V_{\mathrm{Spin}}^-$ as H_0-representations. This representation is stable under the outer automorphisms of H_0, so the restriction of J_0 along ϕ is unique, up to equivalence of H_0-representations.

5.5.1.3 A_3

In this case $H_0 \simeq \mathrm{SU}(4)$, and we take ϕ_0 to be the composition of the natural embedding $\mathrm{SU}(4) \simeq \mathrm{Spin}(6) \hookrightarrow \mathrm{Spin}(9)$ with $\mathrm{Spin}(9) \hookrightarrow \mathrm{F}_4$.

For any embedding ϕ from a A_3-type connected compact Lie group H into F_4, ϕ (up to conjugacy) factors through an embedding from H to $\mathrm{Sp}(3)$ or $\mathrm{Spin}(9)$ by Lemma 5.5.1.

If ϕ factors through $\mathrm{Sp}(3)$, then the image of ϕ gives a A_3-type subgroup of $\mathrm{Sp}(3)$. This subgroup of $\mathrm{Sp}(3)$ must be regular, but this contradicts with the Borel-de Siebenthal theory.

If ϕ factors through $\mathrm{Spin}(9)$, the standard representation V_9 of $\mathrm{Spin}(9)$ gives a self-dual 9-dimensional representation of H. Up to equivalence, there are two possibilities for the restriction of V_9 to H:

$$\mathbf{1}^{\oplus 3} + \wedge^2 \mathrm{V}_4 \text{ or } \mathbf{1} + \mathrm{V}_4 + \mathrm{V}_4',$$

where V_4 is the standard 4-dimensional representation of $\mathrm{SU}(4)$ and V_4' is its dual. For both cases, the restriction of the irreducible representation J_0 of F_4 along ϕ is isomorphic to

$$\mathbf{1}^{\oplus 4} + \mathrm{V}_4^{\oplus 2} + (\mathrm{V}_4')^{\oplus 2} + \wedge^2 \mathrm{V}_4.$$

This representation is stable under the outer automorphism of H_0, so the restriction of J_0 along ϕ is unique, up to equivalence of H_0-representations.

5.5.1.4 B_3

In this case $H_0 \simeq \mathrm{Spin}(7)$, and we take ϕ_0 to be the composition of the natural embedding $\mathrm{Spin}(7) \hookrightarrow \mathrm{Spin}(9)$ with $\mathrm{Spin}(9) \hookrightarrow \mathrm{F}_4$.

For any embedding ϕ from a B_3-type connected compact Lie group H into F_4, by Lemma 5.5.1 and the Borel-de Siebenthal theory, ϕ (up to conjugacy) factors through an embedding from H to $\mathrm{Spin}(9)$. The restriction of the standard representation V_9 of $\mathrm{Spin}(9)$ to H must be isomorphic to either $\mathbf{1}^{\oplus 2} + \mathrm{V}_7$ or $\mathbf{1} + \mathrm{V}_{\mathrm{Spin}}$, where V_7 is the standard 7-dimensional representation of $\mathrm{Spin}(7)$, and $\mathrm{V}_{\mathrm{Spin}}$ is the 8-dimensional spinor representation of $\mathrm{Spin}(7)$. For both cases, the restriction of the irreducible representation J_0 of F_4 along ϕ is isomorphic to

$$\mathbf{1}^{\oplus 3} + \mathrm{V}_7 + \mathrm{V}_{\mathrm{Spin}}^{\oplus 2}.$$

Hence the restriction of J_0 along ϕ is unique, up to equivalence of H_0-representations.

5.5.1.5 C_3

In this case $H_0 \simeq \mathrm{Sp}(3)$, and we take ϕ_0 to be $\mathrm{Sp}(3) \hookrightarrow (\mathrm{Sp}(1) \times \mathrm{Sp}(3))/\mu_2^{\Delta} \hookrightarrow \mathrm{F}_4$, where the subgroup $(\mathrm{Sp}(1) \times \mathrm{Sp}(3))/\mu_2^{\Delta}$ is given in Section 5.3.2.

For any embedding ϕ from a C_3-type connected compact Lie group H into F_4, ϕ (up to conjugacy) factors through a central-kernel morphism from H_0 to $\mathrm{Sp}(3)$ or $\mathrm{Spin}(9)$ by Lemma 5.5.1.

If ϕ factors through $\mathrm{Spin}(9)$, then the standard representation V_9 of $\mathrm{Spin}(9)$ induces an orthogonal 9-dimensional representation of $\mathrm{Sp}(3)$. However, each non-trivial irreducible orthogonal representation of Sp(3) has dimension larger than 9, which leads to a contradiction.

If ϕ factors through Sp(3), then the embedding $H \to \mathrm{Sp}(3)$ must be an isomorphism. This implies that the restriction of the irreducible representation J_0 of F_4 along ϕ is isomorphic to $V_6^{\oplus 2} + V_{14}$, where V_6 and V_{14} stand for the same representations in Eq. (5.3). Hence the restriction of J_0 along ϕ is unique, up to equivalence of H_0-representations.

5.5.1.6 B_2

In this case $H_0 \simeq \mathrm{Sp}(2) \simeq \mathrm{Spin}(5)$, and we take ϕ_0 to be the composition of the natural embedding $\mathrm{Sp}(2) \hookrightarrow \mathrm{Sp}(3) \hookrightarrow (\mathrm{Sp}(1) \times \mathrm{Sp}(3))/\mu_2^{\Delta}$ with the embedding $(\mathrm{Sp}(1) \times \mathrm{Sp}(3))/\mu_2^{\Delta} \hookrightarrow F_4$ given in Section 5.3.2.

For any embedding ϕ from a B_2-type connected compact Lie group H into F_4, by Lemma 5.5.1 and the Borel-de Siebenthal theory, ϕ (up to conjugacy) factors through an embedding from H to Sp(3) or Spin(9).

If ϕ factors through Sp(3), then the restriction of the standard representation V_6 of Sp(3) to H must be isomorphic to $\mathbf{1}^{\oplus 2} + V_4$, where V_4 is the standard 4-dimensional symplectic representation of Sp(2). The restriction of the irreducible representation J_0 along ϕ is isomorphic to $\mathbf{1}^{\oplus 5} + V_4^{\oplus 4} + V_5$, where V_5 is the standard 5-dimensional orthogonal representation of Spin(5).

If ϕ factors through Spin(9), then the restriction of the standard representation V_9 to H must be isomorphic to $\mathbf{1}^{\oplus 4} + V_5$ or $\mathbf{1} + V_4^{\oplus 2}$. For these two possibilities, the restriction of J_0 along ϕ is isomorphic to $\mathbf{1}^{\oplus 5} + V_4^{\oplus 4} + V_5$. Hence the restriction of J_0 along ϕ is unique, up to equivalence of H_0-representations.

5.5.1.7 G_2

In this case $H_0 \simeq G_2$, and we take ϕ_0 to be the embedding $G_2 \hookrightarrow G_2 \times \mathrm{SO}(3) \hookrightarrow F_4$, as given in Section 5.3.4.

Combining Lemma 5.5.1 and the fact that all non-trivial representations of G_2 have dimension larger than 6, any embedding ϕ from a G_2-type connected compact Lie group H into F_4 (up to conjugacy) factors through an embedding from H to Spin(9). The restriction of the standard representation V_9 of Spin(9) to H must be isomorphic to $\mathbf{1}^{\oplus 2} + V_7$, where V_7 is the same as in Eq. (5.10). So the restriction of the representation J_0 of F_4 along ϕ must be isomorphic to $\mathbf{1}^{\oplus 5} + V_7^{\oplus 3}$. Hence the restriction of J_0 along ϕ is unique, up to equivalence of H_0-representations.

5.5.2 The case A_2

For the Lie type A_2, our idea is the same with the proof of Proposition 5.5.3, but this time we have several conjugacy classes of embeddings from a A_2-type group to F_4.

Proposition 5.5.4. *(1) There are* 3 *conjugacy classes of embeddings from* SU(3) *to* F_4,
(2) There is a unique conjugacy class of embeddings from $\mathrm{PSU}(3) = \mathrm{SU}(3)/\mathrm{Z}(\mathrm{SU}(3))$ *to* F_4.

Proof. By Lemma 5.5.1, any embedding ϕ from a connected A_2-type compact Lie group H to F_4 (up to conjugacy) factors through Spin(9) or Sp(3) or $(\mathrm{SU}(3) \times \mathrm{SU}(3))/\mu_2^{\Delta}$.

We start from the case that ϕ factors through $(\mathrm{SU}(3) \times \mathrm{SU}(3))/\mu_3^{\Delta}$. Fix an embedding $\iota : (\mathrm{SU}(3) \times \mathrm{SU}(3))/\mu_3^{\Delta} \hookrightarrow \mathrm{F}_4$ such that the restriction of the irreducible representation J_0 of F_4 along this embedding is isomorphic to Eq. (5.6). We denote the outer automorphism of SU(3) by θ. It is easy to classify the conjugacy classes of embeddings $\psi : H \hookrightarrow (\mathrm{SU}(3) \times \mathrm{SU}(3))/\mu_3^{\Delta}$, where H is a connected A_2-type compact Lie group, *i.e.* $H \simeq \mathrm{SU}(3)$ or PSU(3). We list the conjugacy classes as follows:

Index	H	ψ	The restriction of J_0 along $\phi = \iota \circ \psi$
1	SU(3)	$g \mapsto (g, 1)$	$(\mathrm{V}_3 + \mathrm{V}_3')^{\oplus 3} + \mathfrak{sl}_3$
2	SU(3)	$g \mapsto (1, g)$	$\mathbf{1}^{\oplus 8} + (\mathrm{V}_3 + \mathrm{V}_3')^{\oplus 3}$
3	PSU(3)	$g \mapsto (g, g)$	$\mathbf{1}^{\oplus 2} + \mathfrak{sl}_3^{\oplus 3}$
4	SU(3)	$g \mapsto (g, \theta(g))$	$\mathrm{V}_3 + \mathrm{V}_3' + \mathrm{Sym}^2\,\mathrm{V}_3 + \mathrm{Sym}^2\,\mathrm{V}_3' + \mathfrak{sl}_3$

Table 5.3: Embeddings from A_2-type connected compact Lie groups to $(\mathrm{SU}(3) \times \mathrm{SU}(3))/\mu_3^{\Delta}$

The representations of SU(3) appearing in this table have been explained in Section 5.3.3. If we choose the embedding ι to be the one corresponding to Eq. (5.5), then by Proposition 5.2.1 we get the same conjugacy classes of embeddings.

If ϕ factors through Sp(3), the standard representation V_6 of Sp(3) gives a self-dual 6-dimensional representation of H, thus the restriction of V_6 to H must be isomorphic to $\mathrm{V}_3 + \mathrm{V}_3'$. So the restriction of J_0 to H is isomorphic to $(\mathrm{V}_3 + \mathrm{V}_3')^{\oplus 3} + \mathfrak{sl}_3$.

If ϕ factors through Spin(9), the standard representation V_9 of Spin(9) gives a self-dual 9-dimensional representation of H, thus the restriction of V_9 to H must be isomorphic to $\mathbf{1}^{\oplus 3} + \mathrm{V}_3 + \mathrm{V}_3'$ or $\mathbf{1} + \mathfrak{sl}_3$. For the first case, the restriction of J_0 to H is isomorphic to $\mathbf{1}^{\oplus 8} + (\mathrm{V}_3 + \mathrm{V}_3')^{\oplus 3}$, and for the second case, the restriction of J_0 to H is isomorphic to $\mathbf{1}^{\oplus 2} + \mathfrak{sl}_3^{\oplus 3}$.

In conclusion, combining Proposition 5.2.1 with our analysis on the restriction of J_0, we get that every embedding from a connected A_2-type compact Lie group to F_4 is conjugate to one of the embeddings $\phi = \iota \circ \psi$ in Table 5.3. □

5.5.3 Centralizers

Similarly with the arguments in Section 5.4.5, using Lemma 5.4.7 and Lemma 5.4.8, for each conjugacy class of embeddings from a connected simple compact Lie group to F_4, we can determine its centralizer in F_4:

- Type B_4: the centralizer is a cyclic group of order 2.
- Type D_4: the centralizer is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.
- Type A_3: the centralizer is an A_1-subgroup in the class $[3, 2^8, 1^7]$, which is labeled by $\widetilde{\mathrm{A}_1}$.
- Type B_3: the centralizer is the product of a rank 1 torus with a cyclic group of order 2.
- Type C_3: the centralizer is an A_1-subgroup in the class $[2^6, 1^{14}]$, which is labeled by A_1.
- Type B_2: the centralizer is the direct product of two A_1-subgroups in the class $[2^6, 1^{14}]$.
- Type G_2: the centralizer is an A_1-subgroup in the class $[5, 3^7]$, which is labeled by $\widetilde{\mathrm{A}_2}$.

- Type A_2: Let $\phi : H \hookrightarrow F_4$ be a representative of a conjugacy class of embeddings listed in Table 5.3, which is indexed by a number from 1 to 4.
  (1) If ϕ is indexed by 1, then its centralizer is conjugate to the SU(3) indexed by 2.
  (2) If ϕ is indexed by 2, then its centralizer is conjugate to the SU(3) indexed by 1.
  (3) If ϕ is indexed by 3, then its centralizer is finite and contains an order 3 element.
  (4) If ϕ is indexed by 4, then its centralizer is a cyclic group of order 3.

5.6 Connected subgroups satisfying certain conditions

After a long journey of classifying conjugacy classes of connected simple subgroups of F_4 and computing their centralizers in F_4, we are finally able to enumerate all the connected subgroups H of F_4 satisfying our three conditions listed in the beginning of Chapter 5.

We first classify all the connected subgroups H of F_4 such that $C_{F_4}(H)$ is an elementary finite abelian 2-group, via our classifications in Section 5.4 and Section 5.5.

Notation 5.6.1. From now on, for an A_1-subgroup of F_4, if its conjugacy class corresponds to the partition p of 26, we will simply denote this A_1-subgroup by A_1^p. For example, we will denote the principal PSU(2) of F_4 by $A_1^{[17,9]}$. For an A_2-type subgroup of F_4, if its conjugacy class is indexed by $n \in \{1,2,3,4\}$ in Table 5.3, then we denote it simply by $A_2^{(n)}$.

Now let H be a connected subgroup of F_4 whose centralizer in F_4 is an elementary finite abelian 2-group. Let Φ be the root system of H, and we can write it as a disjoint union of irreducible root systems:

$$\Phi = \Phi_1 \sqcup \cdots \sqcup \Phi_s.$$

We denote by m the number of $i \in \{1,2,\ldots,s\}$ such that $\Phi_i \simeq A_1$.

Lemma 5.6.2. *If $s = 1$, i.e. H is simple, then H is conjugate to one of the following subgroups of F_4:*

$$F_4, \mathrm{Spin}(9), \mathrm{Spin}(8), A_1^{[17,9]}, A_1^{[11,9,5,1]}, A_1^{[9,7,5^2]}.$$

Proof. By our computations in Section 5.4.5 and Section 5.5.3, we have if the centralizer of H in F_4 is finite, then it must be conjugate to one of the following subgroups of F_4:

$$F_4, \mathrm{Spin}(9), \mathrm{Spin}(8), A_2^{(3)}, A_2^{(4)}, A_1^{[17,9]}, A_1^{[11,9,5,1]}, A_1^{[9,7,5^2]}, A_1^{[5^3,3^3,1^2]}.$$

According to Section 5.4.5.5 and Section 5.5.3, if H is in the conjugacy class of $A_2^{(3)}, A_2^{(4)}$ or $A_1^{[5^3,3^3,1^2]}$, then the centralizer of H in F_4 contains an element of order 3. □

Lemma 5.6.3. *If $s > 1$ and $m = 0$, then there is no such H satisfying $C_{F_4}(H)$ is an elementary finite abelian 2-group.*

Proof. Since $s > 1$ and $m = 0$, the irreducible root systems Φ_1 and Φ_2 both have rank 2 and $s = 2$. Hence H must be isomorphic to the quotient of $\mathrm{SU}(3)\times\mathrm{SU}(3)$ by a finite central subgroup. By our classification in Section 5.5.2, H is conjugate to the subgroup $(\mathrm{SU}(3) \times \mathrm{SU}(3))/\mu_3^{\Delta}$

constructed in Section 5.3.3. However, the centralizer of this subgroup contains its center, which is a cyclic group of order 3, so in this case there is no H whose centralizer in F_4 is an elementary finite abelian 2-group. □

Lemma 5.6.4. *If $s = 2$ and $m \geq 1$, then H is conjugate to one of the following subgroups of F_4:*

$$\left(\mathrm{A}_1^{[2^6,1^{14}]} \times \mathrm{Sp}(3)\right)/\mu_2^{\Delta}, \left(\mathrm{A}_1^{[3,2^8,1^7]} \times \mathrm{Spin}(5)\right)/\mu_2^{\Delta}, \mathrm{A}_1^{[5,3^7]} \times \mathrm{G}_2,$$
$$\mathrm{A}_1^{[7^3,1^5]} \times \mathrm{A}_1^{[5,3^7]}, \left(\mathrm{A}_1^{[9,6^2,5]} \times \mathrm{A}_1^{[2^6,1^{14}]}\right)/\mu_2^{\Delta}, \left(\mathrm{A}_1^{[5^2,4^2,3,2^2,1]} \times \mathrm{A}_1^{[2^6,1^{14}]}\right)/\mu_2^{\Delta},$$
$$\left(\mathrm{A}_1^{[5,4^4,1^5]} \times \mathrm{A}_1^{[3,2^8,1^7]}\right)/\mu_2^{\Delta}, \left(\mathrm{A}_1^{[5,4^2,3^3,2^2]} \times \mathrm{A}_1^{[3^3,2^6,1^5]}\right)/\mu_2^{\Delta}, \left(\mathrm{A}_1^{[4^2,3^3,2^4,1]} \times \mathrm{A}_1^{[4^2,3^3,2^4,1]}\right)/\mu_2^{\Delta}.$$

Proof. Since $s = 2$ and $m \geq 1$, up to conjugacy H is of the form $(X \times H_0)/\Gamma$, where X is an A_1-subgroup of F_4, H_0 is a connected simple subgroup of F_4, and Γ is either trivial or the subgroup μ_2^{Δ} of $X \times H_0$. Since the centralizer of H in F_4 is an elementary finite abelian 2-groups, the centralizer of H_0 in $\mathrm{C}_{\mathrm{F}_4}(X)$ and the centralizer of X in $\mathrm{C}_{\mathrm{F}_4}(X)$ are both elementary finite abelian 2-groups.

If $\mathrm{rank}(H_0) > 1$, by Section 5.5.3 we have the following possibilities for the conjugacy class of H:

$$\left(\mathrm{A}_1^{[2^6,1^{14}]} \times \mathrm{Sp}(3)\right)/\mu_2^{\Delta}, \left(\mathrm{A}_1^{[3,2^8,1^7]} \times \mathrm{Spin}(5)\right)/\mu_2^{\Delta}, \mathrm{A}_1^{[5,3^7]} \times \mathrm{G}_2\,.$$

If H_0 is also an A_1-subgroup of F_4, by Section 5.4.5 we have the following possibilities for the conjugacy class of H:

$$\mathrm{A}_1^{[7^3,1^5]} \times \mathrm{A}_1^{[5,3^7]}, \left(\mathrm{A}_1^{[9,6^2,5]} \times \mathrm{A}_1^{[2^6,1^{14}]}\right)/\mu_2^{\Delta}, \left(\mathrm{A}_1^{[5^2,4^2,3,2^2,1]} \times \mathrm{A}_1^{[2^6,1^{14}]}\right)/\mu_2^{\Delta},$$
$$\left(\mathrm{A}_1^{[5,4^4,1^5]} \times \mathrm{A}_1^{[3,2^8,1^7]}\right)/\mu_2^{\Delta}, \left(\mathrm{A}_1^{[5,4^2,3^3,2^2]} \times \mathrm{A}_1^{[3^3,2^6,1^5]}\right)/\mu_2^{\Delta}, \left(\mathrm{A}_1^{[4^2,3^3,2^4,1]} \times \mathrm{A}_1^{[4^2,3^3,2^4,1]}\right)/\mu_2^{\Delta}.$$ □

Lemma 5.6.5. *If $s > 2$, then H is conjugate to one of the following subgroups of F_4:*

$$\left(\mathrm{A}_1^{[2^6,1^{14}]} \times \mathrm{A}_1^{[2^6,1^{14}]} \times \mathrm{Sp}(2)\right)/\mu_2^{\Delta},$$
$$\mathrm{A}_1^{[5,3^7]} \times \left(\mathrm{A}_1^{[3^3,2^6,1^5]} \times \mathrm{A}_1^{[2^6,1^{14}]}\right)/\mu_2^{\Delta},$$
$$\left(\mathrm{A}_1^{[5,4^4,1^5]} \times \mathrm{A}_1^{[2^6,1^{14}]} \times \mathrm{A}_1^{[2^6,1^{14}]}\right)/\mu_2^{\Delta},$$
$$\left(\mathrm{A}_1^{[3,2^8,1^7]} \times \mathrm{A}_1^{[3,2^8,1^7]} \times \mathrm{A}_1^{[3,2^8,1^7]}\right)/\langle(1,-1,-1),(-1,-1,1)\rangle,$$
$$\prod_{i=1}^{4} \mathrm{A}_1^{[2^6,1^{14}]}/\mu_2^{\Delta} := \left(\mathrm{A}_1^{[2^6,1^{14}]} \times \mathrm{A}_1^{[2^6,1^{14}]} \times \mathrm{A}_1^{[2^6,1^{14}]} \times \mathrm{A}_1^{[2^6,1^{14}]}\right)/\mu_2^{\Delta}.$$

Proof. This follows from a similar argument as in the proof of Lemma 5.6.4 and the results in Section 5.4.5 and Section 5.5.3. □

In Lemma 5.6.2, Lemma 5.6.3, Lemma 5.6.4 and Lemma 5.6.5, we have enumerated all the conjugacy classes of connected subgroups H of F_4 such that the centralizer of H in F_4 is an elementary finite abelian 2-group. There are 20 such conjugacy classes, but some of them do not satisfy the third condition given in the beginning of Chapter 5:

Lemma 5.6.6. *If a subgroup H of F_4 is conjugate to one of the following subgroups:*

$$A_1^{[11,9,5,1]}, A_1^{[9,7,5^2]}, \left(A_1^{[3,2^8,1^7]} \times \mathrm{Spin}(5)\right)/\mu_2^\Delta, \left(A_1^{[5^2,4^2,3,2^2,1]} \times A_1^{[2^6,1^{14}]}\right)/\mu_2^\Delta,$$
$$\left(A_1^{[5,4^4,1^5]} \times A_1^{[3,2^8,1^7]}\right)/\mu_2^\Delta, A_1^{[3,2^8,1^7]} \times A_1^{[3,2^8,1^7]} \times A_1^{[3,2^8,1^7]}/\langle(1,-1,-1),(-1,-1,1)\rangle,$$

then the zero weight appears 4 times in the restriction of the 26-dimensional irreducible representation J_0 of F_4 to H.

Proof. The restrictions of the representation J_0 of F_4 to the two A_1-subgroups in the list above can be read from their corresponding partitions. In both cases, the multiplicity of the zero weight in $J_0|_H$ is 4.

If H is conjugate to $\left(A_1^{[3,2^8,1^7]} \times \mathrm{Spin}(5)\right)/\mu_2^\Delta$, then the restriction $J_0|_H$ is isomorphic to

$$\left(\mathbf{1}^{\oplus 2} + \mathrm{Sym}^2\,\mathrm{St}\right) \otimes \mathbf{1} + \mathrm{St}^{\oplus 2} \otimes V_4 + \mathbf{1} \otimes V_5,$$

in which the zero weight appears 4 times.

If H is conjugate to $\left(A_1^{[5^2,4^2,3,2^2,1]} \times A_1^{[2^6,1^{14}]}\right)/\mu_2^\Delta$, then the restriction $J_0|_H$ is isomorphic to

$$\left((\mathrm{Sym}^4\,\mathrm{St})^{\oplus 2} + \mathrm{Sym}^2\,\mathrm{St} + \mathbf{1}\right) \otimes \mathbf{1} + \left(\mathrm{Sym}^3\,\mathrm{St} + \mathrm{St}\right) \otimes \mathrm{St},$$

in which the zero weight appears 4 times.

If H is conjugate to $\left(A_1^{[5,4^4,1^5]} \times A_1^{[3,2^8,1^7]}\right)/\mu_2^\Delta$, then the restriction $J_0|_H$ is isomorphic to

$$\mathbf{1} \otimes \left(\mathbf{1}^{\oplus 2} + \mathrm{Sym}^2\,\mathrm{St}\right) + \left(\mathrm{Sym}^3\,\mathrm{St} \otimes \mathrm{St}\right)^{\oplus 2} + \mathrm{Sym}^4\,\mathrm{St} \otimes \mathbf{1},$$

in which the zero weight appears 4 times.

If H is conjugate to $A_1^{[3,2^8,1^7]} \times A_1^{[3,2^8,1^7]} \times A_1^{[3,2^8,1^7]}/\langle(1,-1,-1),(-1,-1,1)\rangle$, then the restriction $J_0|_H$ is isomorphic to

$$\mathbf{1} + (\mathrm{St} \otimes \mathrm{St} \otimes \mathrm{St})^{\oplus 2} + \mathrm{Sym}^2\,\mathrm{St} \otimes \mathbf{1} \otimes \mathbf{1} + \mathbf{1} \otimes \mathrm{Sym}^2\,\mathrm{St} \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{1} \otimes \mathrm{Sym}^2\,\mathrm{St},$$

in which the zero weight appears 4 times. □

In conclusion, we have proved the following theorem:

Theorem 5.6.7. *There are 13 conjugacy classes of proper connected subgroups H of F_4 satisfying the following conditions:*

(1) The centralizer of H in F_4 is an elementary finite abelian 2-group.

(2) The zero weight appears twice in the restriction of the 26-dimensional irreducible representation J_0 of F_4 to H.

These 13 *subgroups are:*

$$\mathrm{A}_1^{[17,9]}, \mathrm{Spin}(9), \mathrm{Spin}(8), \mathrm{A}_1^{[5,3^7]} \times \mathrm{G}_2, \mathrm{A}_1^{[7^3,1^5]} \times \mathrm{A}_1^{[5,3^7]}, \left(\mathrm{A}_1^{[2^6,1^{14}]} \times \mathrm{Sp}(3)\right)/\mu_2^{\Delta},$$
$$\left(\mathrm{A}_1^{[2^6,1^{14}]} \times \mathrm{A}_1^{[2^6,1^{14}]} \times \mathrm{Sp}(2)\right)/\mu_2^{\Delta}, \left(\mathrm{A}_1^{9,6^2,5} \times \mathrm{A}_1^{[2^6,1^{14}]}\right)/\mu_2^{\Delta}, \left(\mathrm{A}_1^{[5,4^2,3^3,2^2]} \times \mathrm{A}_1^{[3^3,2^6,1^5]}\right)/\mu_2^{\Delta},$$
$$\left(\mathrm{A}_1^{[4^2,3^3,2^4,1]} \times \mathrm{A}_1^{[4^2,3^3,2^4,1]}\right)/\mu_2^{\Delta}, \mathrm{A}_1^{[5,3^7]} \times \left(\mathrm{A}_1^{[3^3,2^6,1^5]} \times \mathrm{A}_1^{[2^6,1^{14}]}\right)/\mu_2^{\Delta},$$
$$\left(\mathrm{A}_1^{[5,4^4,1^5]} \times \mathrm{A}_1^{[2^6,1^{14}]} \times \mathrm{A}_1^{[2^6,1^{14}]}\right)/\mu_2^{\Delta}, \prod_{i=1}^{4} \mathrm{A}_1^{[2^6,1^{14}]}/\mu_2^{\Delta}.$$

For the 13 conjugacy classes of subgroups H in Theorem 5.6.7, in the rest of this section we are going to list some information will be used in Chapter 7:

- the centralizer $\mathrm{C}_{\mathrm{F}_4}(H)$ of H in F_4,
- the restriction of the 26-dimensional irreducible representation J_0 to H,
- and the restriction of the adjoint representation $\mathfrak{f}_4$ of F_4 to H.

5.6.1 $\mathrm{A}_1^{[17,9]}$

This is the principal PSU(2) of F_4, whose centralizer in F_4 is trivial. The restriction of J_0 to H corresponds to the partition $[17, 9]$ of 26, and the restriction of $\mathfrak{f}_4$ to H corresponds to the partition $[23, 15, 11, 3]$ of 52.

5.6.2 Spin(9)

The centralizer of H in F_4 is the center of H, which is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

The restriction of J_0 to H is isomorphic to

$$\mathbf{1} + \mathrm{V}_9 + \mathrm{V}_{\mathrm{Spin}},$$

and the restriction of $\mathfrak{f}_4$ to H is isomorphic to

$$\wedge^2 \mathrm{V}_9 + \mathrm{V}_{\mathrm{Spin}},$$

where V_9 is the standard representation of Spin(9) and $\mathrm{V}_{\mathrm{Spin}}$ is the 16-dimensional spinor representation.

5.6.3 $\left(\mathrm{A}_1^{[2^6,1^{14}]} \times \mathrm{Sp}(3)\right)/\mu_2^{\Delta}$

The centralizer of H in F_4 is the center of H, which is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

The restriction of J_0 to H is isomorphic to

$$\mathrm{St} \otimes \mathrm{V}_6 + \mathbf{1} \otimes \mathrm{V}_{14},$$

and the restriction of $\mathfrak{f}_4$ to H is isomorphic to

$$\mathrm{Sym}^2\, \mathrm{St} \otimes \mathbf{1} + \mathrm{St} \otimes \mathrm{V}'_{14} + \mathbf{1} \otimes \mathrm{Sym}^2\, \mathrm{V}_6,$$

where V_6 is the standard 6-dimensional representation of Sp(3), V_{14} is the 14-dimensional irreducible representation of Sp(3) that is a sub-representation of $\wedge^2 V_6$, and V'_{14} is another 14-dimensional irreducible representation of Sp(3) that is not equivalent to V_{14}. From now on, we will denote V_{14} by $\wedge^* V_6$, and similarly for the 5-dimensional irreducible representation of Sp(2).

5.6.4 $A_1^{[5,3^7]} \times G_2$

The centralizer of H in F_4 is trivial.

The restriction of J_0 to H is isomorphic to

$$\mathrm{Sym}^2\,\mathrm{St} \otimes V_7 + \mathrm{Sym}^4\,\mathrm{St} \otimes \mathbf{1},$$

and the restriction of $\mathfrak{f}_4$ to this subgroup is isomorphic to

$$\mathbf{1} \otimes \mathfrak{g}_2 + \mathrm{Sym}^2\,\mathrm{St} \otimes \mathbf{1} + \mathrm{Sym}^4\,\mathrm{St} \otimes V_7,$$

where V_7 is the 7-dimensional irreducible representation of G_2, and $\mathfrak{g}_2$ is the adjoint representation of G_2.

5.6.5 Spin(8)

The centralizer of H in F_4 is the center of H, which is isomorphic to $Z(\mathrm{Spin}(8)) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$.

The restriction of J_0 to H is isomorphic to

$$\mathbf{1}^{\oplus 2} + V_8 + V_{\mathrm{Spin}}^{+} + V_{\mathrm{Spin}}^{-},$$

and the restriction of $\mathfrak{f}_4$ to H is isomorphic to

$$\wedge^2 V_8 + V_8 + V_{\mathrm{Spin}}^{+} + V_{\mathrm{Spin}}^{-},$$

where V_8 is the 8-dimensional vector representation of Spin(8), *i.e.* the composition of $\mathrm{Spin}(8) \to \mathrm{SO}(8)$ with the standard 8-dimensional representation of SO(8), and $V_{\mathrm{Spin}}^{\pm}$ are two 8-dimensional spinor representations.

5.6.6 $\left(A_1^{[2^6,1^{14}]} \times A_1^{[2^6,1^{14}]} \times \mathrm{Sp}(2)\right)/\mu_2^{\Delta}$

The centralizer of H in F_4 is the center of H, which is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

The restriction of J_0 to H is isomorphic to

$$\mathbf{1} + \mathrm{St} \otimes \mathrm{St} \otimes \mathbf{1} + \mathrm{St} \otimes \mathbf{1} \otimes V_4 + \mathbf{1} \otimes \mathrm{St} \otimes V_4 + \mathbf{1} \otimes \mathbf{1} \otimes \wedge^* V_4,$$

and the restriction of $\mathfrak{f}_4$ to H is isomorphic to

$$\left(\mathrm{Sym}^2\,\mathrm{St} \otimes \mathbf{1} + \mathbf{1} \otimes \mathrm{Sym}^2\,\mathrm{St}\right) \otimes \mathbf{1} + (\mathrm{St} \otimes \mathbf{1} + \mathbf{1} \otimes \mathrm{St}) \otimes V_4$$
$$+\mathrm{St} \otimes \mathrm{St} \otimes \wedge^* V_4 + \mathbf{1} \otimes \mathbf{1} \otimes \mathrm{Sym}^2\,V_4,$$

where V_4 is the standard representation of Sp(2) and $\wedge^* V_4$ is the 5-dimensional irreducible representation of Sp(2).

5.6.7 $A_1^{[7^3,1^5]} \times A_1^{[5,3^7]}$

The centralizer of H in F_4 is trivial.

The restriction of J_0 to H is isomorphic to

$$\mathrm{Sym}^6\,\mathrm{St} \otimes \mathrm{Sym}^2\,\mathrm{St} + \mathbf{1} \otimes \mathrm{Sym}^4\,\mathrm{St},$$

and the restriction of $\mathfrak{f}_4$ to H is isomorphic to

$$\left(\mathrm{Sym}^{10}\,\mathrm{St} + \mathrm{Sym}^2\,\mathrm{St}\right) \otimes \mathbf{1} + \mathbf{1} \otimes \mathrm{Sym}^2\,\mathrm{St} + \mathrm{Sym}^6\,\mathrm{St} \otimes \mathrm{Sym}^4\,\mathrm{St}.$$

5.6.8 $A_1^{[5,3^7]} \times \left(A_1^{[3^3,2^6,1^5]} \times A_1^{[2^6,1^{14}]}\right)/\mu_2^{\Delta}$

The centralizer of H in F_4 is the center of H, which is a cyclic group of order 2.

The restriction of J_0 to H is isomorphic to

$$\mathrm{Sym}^4\,\mathrm{St} \otimes \mathbf{1} \otimes \mathbf{1} + \mathrm{Sym}^2\,\mathrm{St} \otimes \left(\mathrm{St} \otimes \mathrm{St} + \mathrm{Sym}^2\,\mathrm{St} \otimes \mathbf{1}\right),$$

and the restriction of $\mathfrak{f}_4$ to H is isomorphic to

$$\begin{gathered}\mathrm{Sym}^4\,\mathrm{St} \otimes \left(\mathrm{St} \otimes \mathrm{St} + \mathrm{Sym}^2\,\mathrm{St} \otimes \mathbf{1}\right) + \mathrm{Sym}^2\,\mathrm{St} \otimes \mathbf{1} \otimes \mathbf{1} \\ + \mathbf{1} \otimes \left(\mathrm{Sym}^2\,\mathrm{St} \otimes \mathbf{1} + \mathbf{1} \otimes \mathrm{Sym}^2\,\mathrm{St} + \mathrm{Sym}^3\,\mathrm{St} \otimes \mathrm{St}\right).\end{gathered}$$

5.6.9 $\left(A_1^{[5,4^4,1^5]} \times A_1^{[2^6,1^{14}]} \times A_1^{[2^6,1^{14}]}\right)/\mu_2^{\Delta}$

The centralizer of H in F_4 is the center of H, which is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

The restriction of J_0 to H is isomorphic to

$$\mathbf{1} + \mathbf{1} \otimes \mathrm{St} \otimes \mathrm{St} + \mathrm{Sym}^3\,\mathrm{St} \otimes (\mathrm{St} \otimes \mathbf{1} + \mathbf{1} \otimes \mathrm{St}) + \mathrm{Sym}^4\,\mathrm{St} \otimes \mathbf{1} \otimes \mathbf{1},$$

and the restriction of $\mathfrak{f}_4$ to H is isomorphic to

$$\begin{gathered}\mathbf{1} \otimes \left(\mathrm{Sym}^2\,\mathrm{St} \otimes \mathbf{1} + \mathbf{1} \otimes \mathrm{Sym}^2\,\mathrm{St}\right) + \mathrm{Sym}^2\,\mathrm{St} \otimes \mathbf{1} \otimes \mathbf{1} + \mathrm{Sym}^3\,\mathrm{St} \otimes (\mathrm{St} \otimes \mathbf{1} + \mathbf{1} \otimes \mathrm{St}) \\ + \mathrm{Sym}^4\,\mathrm{St} \otimes \mathrm{St} \otimes \mathrm{St} + \mathrm{Sym}^6\,\mathrm{St} \otimes \mathbf{1} \otimes \mathbf{1}.\end{gathered}$$

5.6.10 $\left(A_1^{[9,6^2,5]} \times A_1^{[2^6,1^{14}]}\right)/\mu_2^{\Delta}$

The centralizer of H in F_4 is the center of H, which is a cyclic group of order 2.

The restriction of J_0 to H is isomorphic to

$$\mathrm{Sym}^5\,\mathrm{St} \otimes \mathrm{St} + \left(\mathrm{Sym}^8\,\mathrm{St} + \mathrm{Sym}^4\,\mathrm{St}\right) \otimes \mathbf{1},$$

and the restriction of $\mathfrak{f}_4$ to H is isomorphic to

$$\mathbf{1} \otimes \mathrm{Sym}^2 \mathrm{St} + \left(\mathrm{Sym}^9 \mathrm{St} + \mathrm{Sym}^3 \mathrm{St}\right) \otimes \mathrm{St} + \left(\mathrm{Sym}^{10} \mathrm{St} + \mathrm{Sym}^6 \mathrm{St} + \mathrm{Sym}^2 \mathrm{St}\right) \otimes \mathbf{1}.$$

5.6.11 $\left(\mathrm{A}_1^{[5,4^2,3^3,2^2]} \times \mathrm{A}_1^{[3^3,2^6,1^5]}\right) / \mu_2^{\Delta}$

The centralizer of H in F_4 is the center of H, which is a cyclic group of order 2.

The restriction of J_0 to H is isomorphic to

$$\mathrm{Sym}^4 \mathrm{St} \otimes \mathbf{1} + \left(\mathrm{Sym}^3 \mathrm{St} + \mathrm{St}\right) \otimes \mathrm{St} + \mathrm{Sym}^2 \mathrm{St} \otimes \mathrm{Sym}^2 \mathrm{St},$$

and the restriction of $\mathfrak{f}_4$ to H is isomorphic to

$$\mathrm{St} \otimes \mathrm{Sym}^3 \mathrm{St} + \left(\mathrm{Sym}^4 \mathrm{St} + \mathbf{1}\right) \otimes \mathrm{Sym}^2 \mathrm{St} + \left(\mathrm{Sym}^5 \mathrm{St} + \mathrm{Sym}^3 \mathrm{St}\right) \otimes \mathrm{St} + \left(\mathrm{Sym}^2 \mathrm{St}\right)^{\oplus 2} \otimes \mathbf{1}.$$

5.6.12 $\left(\mathrm{A}_1^{[4^2,3^3,2^4,1]} \times \mathrm{A}_1^{[4^2,3^3,2^4,1]}\right) / \mu_2^{\Delta}$

The centralizer of H in F_4 is the center of H, which is a cyclic group of order 2.

The restriction of J_0 to H is isomorphic to

$$\mathbf{1} + \mathrm{Sym}^3 \mathrm{St} \otimes \mathrm{St} + \mathrm{Sym}^2 \mathrm{St} \otimes \mathrm{Sym}^2 \mathrm{St} + \mathrm{St} \otimes \mathrm{Sym}^3 \mathrm{St},$$

and the restriction of $\mathfrak{f}_4$ to H is isomorphic to

$$\left(\mathrm{Sym}^4 \mathrm{St} + \mathbf{1}\right) \otimes \mathrm{Sym}^2 \mathrm{St} + \mathrm{Sym}^2 \mathrm{St} \otimes \left(\mathrm{Sym}^4 \mathrm{St} + \mathbf{1}\right) + \mathrm{Sym}^3 \mathrm{St} \otimes \mathrm{St} + \mathrm{St} \otimes \mathrm{Sym}^3 \mathrm{St}.$$

5.6.13 $\prod_{i=1}^4 \mathrm{A}_1^{[2^6,1^{14}]} / \mu_2^{\Delta}$

The centralizer of H in F_4 is the center of H, which is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

The restriction of J_0 to H is isomorphic to

$$\mathbf{1}^{\oplus 2} + \sum_{\mathrm{Sym}} \mathrm{St} \otimes \mathrm{St} \otimes \mathbf{1} \otimes \mathbf{1},$$

where the second term stands for the direct sum of tensor products of standard representations at every two copies of $\mathrm{A}_1^{[2^6,1^{14}]}$ in H. The restriction of $\mathfrak{f}_4$ to H is isomorphic to

$$\sum_{\mathrm{Sym}} \mathrm{Sym}^2 \mathrm{St} \otimes \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} + \sum_{\mathrm{Sym}} \mathrm{St} \otimes \mathrm{St} \otimes \mathbf{1} \otimes \mathbf{1} + \mathrm{St} \otimes \mathrm{St} \otimes \mathrm{St} \otimes \mathrm{St}.$$

Chapter 6

Arthur's conjectures on automorphic representations

In this chapter, we are going to review the theory of automorphic representations and Arthur's conjectures on discrete automorphic representations. For our purposes, it is enough to restrict to the special case of level 1 algebraic automorphic forms of a reductive group G over $\mathbb{Q}$ admitting a reductive $\mathbb{Z}$-model, as in [ChenevierRenard, 2015; ChenevierLannes, 2019]. We mainly follow these two references.

6.1 A brief review of automorphic representations

In this section we give a quick review on automorphic representations, following [Chenevier-Lannes, 2019, §4.3]. Let $\mathbf{G}$ be a connected reductive group over $\mathbb{Q}$ with a reductive $\mathbb{Z}$-model $(\mathscr{G}, \mathrm{id})$, and $\mathbf{A_G}$ be the maximal $\mathbb{Q}$-split torus of the center $\mathbf{Z}(\mathbf{G})$ of $\mathbf{G}$. Denote by $\mathbf{G}(\mathbb{A})^1$ the quotient of $\mathbf{G}(\mathbb{A})$ by the neutral component of $\mathbf{A_G}(\mathbb{R})$, and consider the adelic quotient

$$[\mathbf{G}] := \mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})^1 = \mathbf{G}(\mathbb{Q})\mathbf{A_G}(\mathbb{R})^\circ\backslash\mathbf{G}(\mathbb{A}).$$

We have a left $\mathbf{G}(\mathbb{Q})$-invariant right Haar measure μ on $\mathbf{G}(\mathbb{A})$ by [Weil, 1940, §II.9], and the volume of $[\mathbf{G}]$ is finite with respect to this measure. The topological group $\mathbf{G}(\mathbb{A})$ acts on the space $\mathcal{L}(\mathbf{G}) := \mathrm{L}^2([\mathbf{G}])$ of square-integrable functions on $[\mathbf{G}]$ by right translations. Equipped with the *Petersson inner product* defined as

$$\langle f, f'\rangle := \int \overline{f} f' d\mu,$$

the space $\mathcal{L}(\mathbf{G})$ becomes a unitary representation of $\mathbf{G}(\mathbb{A})$. We denote the closure of the sum of all closed and topologically irreducible subrepresentations of $\mathcal{L}(\mathbf{G})$ by $\mathcal{L}_{\mathrm{disc}}(\mathbf{G})$.

Denote by $\Pi(\mathbf{G})$ the set of equivalence classes of irreducible unitary complex representations π of $\mathbf{G}(\mathbb{A})$ such that $\pi = \pi_\infty \otimes \pi_f$, where π_∞ is an irreducible unitary representation of $\mathbf{G}(\mathbb{R})$, and π_f is a smooth irreducible representation of $\mathbf{G}(\mathbb{A}_f)$ satisfying $\pi_f^{\mathscr{G}(\widehat{\mathbb{Z}})} \neq 0$. We have the following decomposition:

$$\mathcal{L}_{\mathrm{disc}}(\mathbf{G})^{\mathscr{G}(\widehat{\mathbb{Z}})} = \overline{\bigoplus_{\pi\in\Pi(\mathbf{G})}} \mathrm{m}(\pi)\, \pi^{\mathscr{G}(\widehat{\mathbb{Z}})} = \overline{\bigoplus_{\pi\in\Pi(\mathbf{G})}} \mathrm{m}(\pi)\, \pi_\infty \otimes \pi_f^{\mathscr{G}(\widehat{\mathbb{Z}})}, \tag{6.1}$$

where the integers $\mathrm{m}(\pi) \geq 0$ are finite due to a fundamental result of Harish-Chandra [Harish-Chandra, 1968, §I.2, Theorem 1]. We call the integer $\mathrm{m}(\pi)$ the *multiplicity* of π in $\mathcal{L}_{\mathrm{disc}}(\mathbf{G})$.

Now we give the definition of level one discrete automorphic representations, and refer to [BorelJacquet, 1979, §4] for the general definition of automorphic representations.

Definition 6.1.1. A *level one discrete automorphic representation* is a representation π of $\mathbf{G}(\mathbb{A})$ in $\Pi(\mathbf{G})$ such that its multiplicity $\mathrm{m}(\pi)$ in Eq. (6.1) is nonzero. We denote the subset of $\Pi(\mathbf{G})$ consisting of level one discrete automorphic representations by $\Pi_{\mathrm{disc}}(\mathbf{G})$.

Notation 6.1.2. Since in this paper we only deal with level one automorphic representations, so we will always omit "level one" from now on.

Definition 6.1.3. A square-integrable Borel function $f : [\mathbf{G}] \to \mathbb{C}$ is a *cusp form* if for the unipotent radical $\mathbf{U}$ of each proper parabolic subgroup of $\mathbf{G}$, we have

$$\int_{\mathbf{U}(\mathbb{Q})\backslash\mathbf{U}(\mathbb{A})} f(ug)du = 0$$

for almost all $g \in \mathbf{G}(\mathbb{A})$. We denote the subspace of $\mathcal{L}(\mathbf{G})$ consisting of the classes of cusp forms by $\mathcal{L}_{\mathrm{cusp}}(\mathbf{G})$. A discrete automorphic representation is *cuspidal* if it is a subrepresentation of $\mathcal{L}_{\mathrm{cusp}}(\mathbf{G})$, and we denote by $\Pi_{\mathrm{cusp}}(\mathbf{G})$ the subset of $\Pi(\mathbf{G})$ consisting of cuspidal representations.

Remark 6.1.4. It is well-known that [GelfandGraevPyatetskii-Shapiro, 1969]:

$$\mathcal{L}_{\mathrm{cusp}}(\mathbf{G}) \subset \mathcal{L}_{\mathrm{disc}}(\mathbf{G}) \text{ and } \Pi_{\mathrm{cusp}}(\mathbf{G}) \subset \Pi_{\mathrm{disc}}(\mathbf{G}).$$

When $\mathbf{G}(\mathbb{R})$ is compact, every automorphic representation of $\mathbf{G}$ is discrete by the Peter-Weyl theorem.

Denote by $\mathrm{H}(\mathbf{G}) = \bigotimes_p \mathrm{H}_p(\mathbf{G})$ the *spherical Hecke algebra* of the pair $\left(\mathbf{G}(\mathbb{A}_f), \mathscr{G}(\widehat{\mathbb{Z}})\right)$. For any representation $\pi = \pi_\infty \otimes \pi_f \in \Pi(\mathbf{G})$, the space $\pi_f^{\mathscr{G}(\widehat{\mathbb{Z}})}$ is an irreducible representation of the spherical Hecke algebra $\mathrm{H}(\mathbf{G})$. Since $\mathrm{H}(\mathbf{G})$ is commutative [Gross, 1998, Proposition 2.10], the dimension of $\pi_f^{\mathscr{G}(\widehat{\mathbb{Z}})}$ is 1. Hence the $\mathscr{G}(\widehat{\mathbb{Z}})$-invariant space of the π-isotypic subspace $\mathcal{L}_{\mathrm{disc}}(\mathbf{G})_\pi$ of $\mathcal{L}_{\mathrm{disc}}(\mathbf{G})$, as a $\mathbf{G}(\mathbb{R})$-representation, is the direct sum of $\mathrm{m}(\pi)$ copies of π_∞. This implies the following result:

Lemma 6.1.5. *Let V be an irreducible unitary representation of the Lie group $\mathbf{G}(\mathbb{R})$, and $\mathcal{A}_V(\mathbf{G})$ the space of $\mathbf{G}(\mathbb{R})$-equivariant linear maps from V to $\mathcal{L}_{\mathrm{disc}}(\mathbf{G})^{\mathscr{G}(\widehat{\mathbb{Z}})}$. Then we have the following equality:*

$$\dim \mathcal{A}_V(\mathbf{G}) = \sum_{\pi\in\Pi(\mathbf{G}),\, \pi_\infty\simeq V} \mathrm{m}(\pi). \tag{6.2}$$

Remark 6.1.6. The space $\mathcal{A}_V(\mathbf{G}) = \mathrm{Hom}_{\mathbf{G}(\mathbb{R})}(V, \mathcal{L}_{\mathrm{disc}}(\mathbf{G})^{\mathscr{G}(\widehat{\mathbb{Z}})})$ can be viewed as the multiplicity space of V in Eq. (6.1).

6.1.1 Automorphic representations for $\mathbf{F}_4$

When the reductive group $\mathbf{G}$ has compact real points, due to [Gross, 1999a] we can describe the multiplicity space $\mathcal{A}_V(\mathbf{G})$ of V in $\mathcal{L}_{\mathrm{disc}}(\mathbf{G})^{\mathscr{G}(\widehat{\mathbb{Z}})}$ in a more computable manner, which is explained in [ChenevierLannes, 2019, §4.4.1]. Applying [ChenevierLannes, 2019, Lemma 4.4.2] to $\mathbf{F}_4$ and using the fact that every irreducible representation of F_4 is self-dual, we get:

Proposition 6.1.7. *Let (ρ, V) be an irreducible representation of $\mathrm{F}_4 = \mathbf{F}_4(\mathbb{R})$. The vector space $\mathcal{A}_V(\mathbf{F}_4)$ is canonically isomorphic to the following space:*

$$\mathrm{M}_V(\mathbf{F}_4) := \left\{ f : \mathbf{F}_4(\mathbb{A}_f)/\mathcal{F}_{4,\mathrm{I}}(\widehat{\mathbb{Z}}) \to V \,\middle|\, f(\gamma g) = \rho(\gamma) f(g) \text{ for all } \gamma \in \mathbf{F}_4(\mathbb{Q}), g \in \mathbf{F}_4(\mathbb{A}_f) \right\}.$$

We choose a set of representatives $\{1, g_{\mathrm{E}}\}$ of $\mathbf{F}_4(\mathbb{Q})\backslash\mathbf{F}_4(\mathbb{A}_f)/\mathcal{F}_{4,\mathrm{I}}(\widehat{\mathbb{Z}})$ corresponding to the two reductive $\mathbb{Z}$-models $(\mathcal{F}_{4,\mathrm{I}}, \mathrm{id})$ and $(\mathcal{F}_{4,\mathrm{E}}, \iota)$ of $\mathbf{F}_4$ in Proposition 3.3.6. By [ChenevierLannes, 2019, Equation (4.4.1)] the evaluation map $f \mapsto (f(1), f(g_{\mathrm{E}}))$ induces a bijection:

$$\mathrm{M}_V(\mathbf{F}_4) \simeq V^{\mathcal{F}_{4,\mathrm{I}}(\mathbb{Z})} \oplus V^{\mathcal{F}_{4,\mathrm{E}}(\mathbb{Z})}.$$

Combining the results in this section with Theorem 4.6.1, we have the following computational result:

Corollary 6.1.8. *For any dominant weight λ of F_4, we have an explicit formula for the dimension of $\mathcal{A}_{\mathrm{V}_\lambda}(\mathbf{F}_4)$, where V_λ is the irreducible representation of $\mathrm{F}_4 = \mathbf{F}_4(\mathbb{R})$ with highest weight λ. For $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ with $2\lambda_1 + 3\lambda_2 + 2\lambda_3 + \lambda_4 \leq 13$, the dimension $\dim \mathcal{A}_{\mathrm{V}_\lambda}(\mathbf{F}_4)$ equals the $d(\lambda)$ in Table A.3.*

6.2 Local parametrization of $\Pi(\mathbf{G})$

Let $\mathbf{G}$ be a connected reductive group over $\mathbb{Q}$ with a fixed reductive $\mathbb{Z}$-model $(\mathscr{G}, \mathrm{id})$. Let $\widehat{\mathbf{G}}$ be its complex Langlands dual group, *i.e.* the root datum of $\widehat{\mathbf{G}}$ is the dual root datum of $\mathbf{G}$. A representation $\pi \in \Pi(\mathbf{G})$ can be decomposed as $\pi = \pi_\infty \otimes \left(\bigotimes_p \pi_p\right)$, where π_p is a *spherical* irreducible smooth representation of $\mathbf{G}(\mathbb{Q}_p)$ for each p, *i.e.* $\pi_p^{\mathscr{G}(\mathbb{Z}_p)} \neq 0$, and π_∞ is an irreducible unitary representation of the Lie group $\mathbf{G}(\mathbb{R})$.

In this section, we will recall the parametrizations for spherical irreducible smooth representations of $\mathbf{G}(\mathbb{Q}_p)$ and for irreducible unitary representations of $\mathbf{G}(\mathbb{R})$. Our main reference is [ChenevierLannes, 2019, §6.2, §6.3].

6.2.1 Satake parameter

For each prime number p, a spherical irreducible smooth representation π of $\mathbf{G}(\mathbb{Q}_p)$ is determined by the action of the spherical Hecke algebra $\mathrm{H}_p(\mathbf{G})$ for the pair $(\mathbf{G}(\mathbb{Q}_p), \mathscr{G}(\mathbb{Z}_p))$ on the subspace of invariants $\pi^{\mathscr{G}(\mathbb{Z}_p)}$. Since $\dim \pi^{\mathscr{G}(\mathbb{Z}_p)} = 1$, the equivalence class of π is determined uniquely by the ring homomorphism $\mathrm{H}_p(\mathbf{G}) \to \mathbb{C}$ given by the $\mathrm{H}_p(\mathbf{G})$-action on $\pi^{\mathscr{G}(\mathbb{Z}_p)}$.

By [ChenevierLannes, 2019, Scholium 6.2.2], the *Satake isomorphism* gives a canonical bijection between the set of ring homomorphisms $\mathrm{H}_p(\mathbf{G}) \to \mathbb{C}$ and the set $\widehat{\mathbf{G}}(\mathbb{C})_{\mathrm{ss}}$ of semisimple conjugacy classes in $\widehat{\mathbf{G}}(\mathbb{C})$. This induces a bijection $\pi \mapsto \mathrm{c}_p(\pi)$ between the set of equivalence classes of spherical irreducible smooth representations of $\mathbf{G}(\mathbb{Q}_p)$ and the set $\widehat{\mathbf{G}}(\mathbb{C})_{\mathrm{ss}}$. The conjugacy class $\mathrm{c}_p(\pi)$ is called the *Satake parameter* of π_p.

6.2.2 Infinitesimal character

Let $\mathfrak{g}$ be the Lie algebra of $\mathbf{G}(\mathbb{C})$, and $\widehat{\mathfrak{g}}$ the Lie algebra of $\widehat{\mathbf{G}}(\mathbb{C})$. We fix a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ and a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ containing $\mathfrak{t}$, and denote the Weyl group of $\mathfrak{g}$ with respect to $\mathfrak{t}$ by W.

As explained in [ChenevierLannes, 2019, §6.3.4], we can associate a character $\mathrm{Z}(\mathrm{U}(\mathfrak{g})) \to \mathbb{C}$ to an irreducible unitary representation (π, V) of $\mathbf{G}(\mathbb{R})$, where $\mathrm{Z}(\mathrm{U}(\mathfrak{g}))$ is the center of the universal enveloping algebra of $\mathfrak{g}$. By [ChenevierLannes, 2019, Scholium 6.3.2 and Equation (6.3.1)], the *Harish-Chandra isomorphism* induces the following canonical bijections:

$$\mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathrm{Z}(\mathrm{U}(\mathfrak{g})), \mathbb{C}) \simeq \widehat{\mathfrak{g}}_{\mathrm{ss}} \simeq \left(X^*(\mathfrak{t}) \otimes_{\mathbb{Z}} \mathbb{C}\right)/W, \tag{6.3}$$

where $\widehat{\mathfrak{g}}_{\mathrm{ss}}$ is the set of semisimple conjugacy classes in $\widehat{\mathfrak{g}}$. Hence we associate to (π, V) a semisimple conjugacy class $\mathrm{c}_\infty(\pi) \in \widehat{\mathfrak{g}}_{\mathrm{ss}}$, called the *infinitesimal character* of π.

As proved by Harish-Chandra [Knapp, 1986, Corollary 10.37], up to isomorphism there are only a finite number of irreducible unitary representations of $\mathbf{G}(\mathbb{R})$ with a given infinitesimal character. When $\mathbf{G}(\mathbb{R})$ is compact, the situation is much simpler due to the following result:

Proposition 6.2.1. *[Dixmier, 1977, §7.4.6] Let $\mathbf{G}(\mathbb{R})$ be a compact group, and $\rho \in X^*(\mathfrak{t}) \otimes \mathbb{C}$ the half-sum of positive roots with respect to $(\mathfrak{g}, \mathfrak{b}, \mathfrak{t})$. For a dominant weight λ of $\mathbf{G}(\mathbb{R})$, the infinitesimal character of the highest weight representation V_λ of $\mathbf{G}(\mathbb{R})$ is $\lambda + \rho$, viewed as an element in $\widehat{\mathfrak{g}}_{\mathrm{ss}}$ via Eq. (6.3). In particular, the infinitesimal character $\lambda + \rho$ determines V_λ uniquely.*

6.2.3 Langlands parametrization

Now we recall Langlands parametrization of $\Pi(\mathbf{G})$, following [ChenevierLannes, 2019, §6.4.2].

Definition 6.2.2. Let $\mathbf{H}$ be a connected reductive $\mathbb{C}$-group with complex Lie algebra $\mathfrak{h}$. We denote by $\mathbf{H}(\mathbb{C})_{\mathrm{ss}}$ (*resp.* $\mathfrak{h}_{\mathrm{ss}}$) the set of $\mathbf{H}(\mathbb{C})$-conjugacy classes of semisimple elements of $\mathbf{H}(\mathbb{C})$ (*resp.* $\mathfrak{h}$). Denote by $\mathscr{X}(\mathbf{H})$ the set of families $(\mathrm{c}_\infty, \mathrm{c}_2, \mathrm{c}_3, \mathrm{c}_5, \ldots)$, where $\mathrm{c}_\infty \in \mathfrak{h}_{\mathrm{ss}}$ and $\mathrm{c}_p \in \mathbf{H}(\mathbb{C})_{\mathrm{ss}}$ for all primes p.

By results in Section 6.2.1 and Section 6.2.2, we associate to a representation $\pi = \pi_\infty \otimes \left(\bigotimes_p \pi_p\right) \in \Pi(\mathbf{G})$ a conjugacy class $\mathrm{c}_p(\pi) := \mathrm{c}_p(\pi_p)$ in $\widehat{\mathbf{G}}(\mathbb{C})_{\mathrm{ss}}$ for each p, and a conjugacy class

$c_\infty(\pi) := c_\infty(\pi_\infty)$ in $\widehat{\mathfrak{g}}_{ss}$. Hence we have a canonical map $\Pi(\mathbf{G}) \to \mathcal{X}(\widehat{\mathbf{G}})$ defined as

$$\pi = \pi_\infty \otimes \left(\bigotimes_p \pi_p\right) \mapsto c(\pi) = (c_\infty(\pi), c_2(\pi), c_3(\pi), \cdots) \in \mathcal{X}(\widehat{\mathbf{G}}).$$

The family of conjugacy classes $c(\pi)$ determines π_f and the infinitesimal character of π_∞, and the map c has finite fibers. When $\mathbf{G}(\mathbb{R})$ is compact, the fiber of c is either empty or a singleton.

Definition 6.2.3. Let $\mathbf{G}$ be a semisimple $\mathbb{Q}$-group admitting a reductive $\mathbb{Z}$-model, and $r : \widehat{\mathbf{G}} \to \mathbf{SL}_n$ an algebraic representation of its dual group, which induces a map $\mathcal{X}(\widehat{\mathbf{G}}) \to \mathcal{X}(\mathbf{SL}_n)$. For any $\pi \in \Pi(\mathscr{G})$, we define the following family of conjugacy classes:

$$\psi(\pi, r) := r\,(c(\pi)) \in \mathcal{X}(\mathbf{SL}_n),$$

and refer to it as the *Langlands parameter of the pair* (π, r).

6.3 Global parametrization and the Langlands group

For the global parametrization of level one discrete automorphic representations, now we need to use a *conjectural* group $\mathcal{L}_\mathbb{Z}$, the so-called *Langlands group of* $\mathbb{Z}$, to formulate the global Arthur-Langlands conjecture. In Arthur's work [Arthur, 1989], he uses another group $\mathcal{L}_\mathbb{Q}$. However, since we only consider level one discrete automorphic representations in this paper, it is more convenient to use the group $\mathcal{L}_\mathbb{Z}$ that we are going to recall, following [ChenevierRenard, 2015, Appendix B; ChenevierLannes, 2019, Preface].

We assume that $\mathcal{L}_\mathbb{Z}$ is a compact Hausdorff topological group equipped with

- A conjugacy class Frob_p in $\mathcal{L}_\mathbb{Z}$, for each prime p,
- A conjugacy class of continuous homomorphisms $h : W_\mathbb{R} \to \mathcal{L}_\mathbb{Z}$, called the *Hodge morphism*. Here $W_\mathbb{R}$ is the *Weil group of* $\mathbb{R}$, which is a non-split extension of $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \{1, j\}$ by $W_\mathbb{C} = \mathbb{C}^\times$, for the natural action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ on $\mathbb{C}^\times$. It is generated by its open subgroup $\mathbb{C}^\times$ together with an element j, with relations $j^2 = -1$ and $jzj^{-1} = \overline{z}$ for every $z \in \mathbb{C}^\times$.

This group $\mathcal{L}_\mathbb{Z}$ satisfies three axioms that we will introduce one by one.

Axiom 1. *(*Cebotarev property*) The union of conjugacy classes* Frob_p *is dense in* $\mathcal{L}_\mathbb{Z}$.

Remark 6.3.1. In [ChenevierRenard, 2015, Appendix B], the axiom they use is the *general Sato-Tate conjecture*: the conjugacy classes Frob_p are equidistributed in the compact group $\mathcal{L}_\mathbb{Z}$ equipped with its Haar measure of mass 1. This is a universal form of the Sato-Tate conjecture for automorphic representations and it implies the Cebotarev property, but Axiom 1 is enough for us in this article.

This axiom tells us for two homomorphisms ψ, ψ' from $\mathcal{L}_\mathbb{Z}$ to some topological group H, if $\psi(\mathrm{Frob}_p)$ and $\psi'(\mathrm{Frob}_p)$ are conjugate in H for each prime p, then ψ and ψ' are element-conjugate. An important type of homomorphisms involving $\mathcal{L}_\mathbb{Z}$ is:

Definition 6.3.2. Let $\mathbf{G}$ be a reductive $\mathbb{Q}$-group admitting a reductive $\mathbb{Z}$-model. A *discrete global Arthur parameter (of level one)* of $\mathbf{G}$ is a $\widehat{\mathbf{G}}(\mathbb{C})$-conjugacy class of continuous group homomorphisms

$$\psi : \mathcal{L}_{\mathbb{Z}} \times \mathrm{SL}_2(\mathbb{C}) \to \widehat{\mathbf{G}}(\mathbb{C})$$

such that $\psi|_{\mathbf{SL}_2(\mathbb{C})}$ is algebraic and the centralizer C_ψ of $\mathrm{Im}(\psi)$ in $\widehat{\mathbf{G}}(\mathbb{C})$ is finite modulo the center of $\widehat{\mathbf{G}}(\mathbb{C})$. We call C_ψ the *(global) component group* of ψ, and denote the set of discrete global Arthur parameters of $\mathbf{G}$ by $\Psi_{\mathrm{disc}}(\mathbf{G})$.

Remark 6.3.3. The condition on C_ψ in Definition 6.3.2 implies that a discrete global Arthur parameter for $\mathbf{G} = \mathbf{GL}_n$ is an equivalence class of n-dimensional irreducible representations of $\mathcal{L}_{\mathbb{Z}} \times \mathbf{SL}_2(\mathbb{C})$.

In parallel with Langlands parametrization in Section 6.2.3, we can also associate to any $\psi \in \Psi_{\mathrm{disc}}(\mathbf{G})$ a collection of conjugacy classes $\mathrm{c}(\psi) = (\mathrm{c}_\infty(\psi), \mathrm{c}_2(\psi), \mathrm{c}_3(\psi), \cdots) \in \mathcal{X}(\widehat{\mathbf{G}})$. For each prime p, the conjugacy class $\mathrm{c}_p(\psi)$ is defined by:

$$\mathrm{c}_p(\psi) := \psi(\mathrm{Frob}_p, e_p),\ e_p = \begin{pmatrix} p^{-1/2} & 0 \\ 0 & p^{1/2} \end{pmatrix} \in \mathbf{SL}_2(\mathbb{C}).$$

The infinitesimal character $\mathrm{c}_\infty(\psi)$ of ψ is defined to be the infinitesimal character of the archimedean Arthur parameter $\psi \circ (\mathrm{h} \times \mathrm{id}) : \mathrm{W}_{\mathbb{R}} \times \mathbf{SL}_2(\mathbb{C}) \to \widehat{\mathbf{G}}(\mathbb{C})$, which is explained in [ChenevierRenard, 2015, §A.2].

The following axiom connects the collection of conjugacy classes attached to a discrete automorphic representation and that attached to a discrete global Arthur parameter.

Axiom 2. *(Arthur-Langlands conjecture for* $\mathbf{GL}_n$*) For every integer* $n \geq 1$*, there is a unique bijection*

$$\Pi_{\mathrm{disc}}(\mathbf{GL}_n) \xrightarrow{\sim} \Psi_{\mathrm{disc}}(\mathbf{GL}_n),\ \pi \mapsto \psi_\pi$$

such that $c(\pi) = c(\psi_\pi)$ *for all discrete automorphic representations* π *of* $\mathbf{GL}_n$*. Moreover, the discrete global Arthur parameter* ψ_π *is trivial on* $\mathbf{SL}_2(\mathbb{C})$ *if and only if we have* $\pi \in \Pi_{\mathrm{cusp}}(\mathbf{GL}_n)$*.*

Remark 6.3.4. This axiom and the compactness of $\mathcal{L}_{\mathbb{Z}}$ imply the so-called *generalized Ramanujan conjecture*: for any $\pi \in \Pi_{\mathrm{cusp}}(\mathbf{GL}_n)$ and any prime p, the eigenvalues of $\mathrm{c}_p(\pi)$ all have absolute value 1.

For general reductive groups, we have the following third axiom:

Axiom 3. *Let* $\mathbf{G}$ *be a reductive group admitting a reductive* $\mathbb{Z}$*-model* $(\mathscr{G}, \mathrm{id})$*, then there exists a decomposition*

$$\mathcal{L}_{\mathrm{disc}}(\mathbf{G})^{\mathscr{G}(\widehat{\mathbb{Z}})} = \overset{\perp}{\bigoplus}_{\psi \in \Psi_{\mathrm{disc}}(\mathbf{G})} \mathcal{A}_\psi(\mathbf{G}), \tag{6.4}$$

stable under the actions of $\mathbf{G}(\mathbb{R})$ *and* $\mathrm{H}(\mathbf{G})$*, and satisfying the following property: for* $\pi \in \Pi(\mathbf{G})$*, if* $\pi^{\mathscr{G}(\widehat{\mathbb{Z}})}$ *appears in* $\mathcal{A}_\psi(\mathbf{G})$*, then we have* $\mathrm{c}(\pi) = \mathrm{c}(\psi)$*.*

This axiom tells us for any level one discrete automorphic representation $\pi \in \Pi_{\mathrm{disc}}(\mathbf{G})$, there exists a discrete global Arthur parameter ψ of $\mathbf{G}$ such that $\mathrm{c}(\psi) = \mathrm{c}(\pi)$. In general, this discrete global Arthur parameter is not unique since two element-conjugate embeddings into $\widehat{\mathbf{G}}(\mathbb{C})$ may not be conjugate. Conversely, given a discrete global Arthur parameter ψ of $\mathbf{G}$, there are finitely many (possibly zero) adelic representations $\pi \in \Pi(\mathbf{G})$ satisfying $\mathrm{c}(\pi) = \mathrm{c}(\psi)$, and we denote the subset of $\Pi(\mathbf{G})$ consisting of such representations by $\Pi_\psi(\mathbf{G})$.

In other words, discrete global Arthur parameters are the objects parametrizing discrete automorphic representations, but a natural problem that we need to deal with is that which representations in $\Pi_\psi(\mathbf{G})$ for a given ψ appear in the discrete spectrum $\mathcal{L}(\mathbf{G})_{\mathrm{disc}}$. We will see the (conjectural) answer in Section 6.6.

Another property about $\mathcal{L}_{\mathbb{Z}}$ that we will use is that it is connected:

Proposition 6.3.5. *[ChenevierLannes, 2019, Proposition 9.3.4] Suppose that $\mathcal{L}_{\mathbb{Z}}$ is a compact topological group satisfying the axioms above, then it is connected.*

6.3.1 Sato-Tate group

For a discrete global Arthur parameter $\psi \in \Psi_{\mathrm{disc}}(\mathbf{G})$, we pick a representative $\mathcal{L}_{\mathbb{Z}} \times \mathbf{SL}_2(\mathbb{C}) \to \widehat{\mathbf{G}}(\mathbb{C})$ and consider its restriction to a maximal compact subgroup:

$$\psi_{\mathrm{c}} : \mathcal{L}_{\mathbb{Z}} \times \mathrm{SU}(2) \to \widehat{\mathbf{G}}(\mathbb{C}).$$

The image of this morphism is contained in some maximal compact subgroup of $\widehat{\mathbf{G}}(\mathbb{C})$. Fix a maximal connected compact subgroup K of $\widehat{\mathbf{G}}(\mathbb{C})$, and without loss of generality we assume that ψ_{c} is a morphism from $\mathcal{L}_{\mathbb{Z}} \times \mathrm{SU}(2) \to K$.

Definition 6.3.6. For any $\psi \in \Psi_{\mathrm{disc}}(\mathbf{G})$, we define $\mathrm{H}(\psi)$ to be the K-conjugacy class of the image of its associated morphism $\mathcal{L}_{\mathbb{Z}} \times \mathrm{SU}(2) \to K$. For any $\pi \in \Pi_{\mathrm{disc}}(\mathbf{G})$, if there exists a unique global Arthur parameter $\psi_\pi \in \Psi_{\mathrm{disc}}(\mathbf{G})$ such that $\mathrm{c}(\pi) = \mathrm{c}(\psi_\pi)$, we define $\mathrm{H}(\pi)$ to be $\mathrm{H}(\psi_\pi)$.

Remark 6.3.7. Since maximal connected compact subgroups of $\mathbf{SL}_2(\mathbb{C})$ are unique up to conjugacy, the $\widehat{\mathbf{G}}(\mathbb{C})$-conjugacy class of the image of $\mathcal{L}_{\mathbb{Z}} \times \mathrm{SU}(2) \to K$ is well-defined. Combining with [FangHanSun, 2016, Lemma 2.4], the K-conjugacy class $\mathrm{H}(\psi)$ is well-defined.

Remark 6.3.8. The conjugacy class $\mathrm{H}(\psi)$, or $\mathrm{H}(\pi)$, of subgroups of K is called the *"Sato-Tate group"* in the introduction Chapter 1, although it coincides with the usual Sato-Tate group (see [ChenevierRenard, 2015, Proposition-Definition B.1]) if and only if the restriction of ψ to $\mathbf{SL}_2(\mathbb{C})$ is trivial.

A cuspidal representation π of $\mathbf{PGL}_n$ can be viewed as an element of $\Pi_{\mathrm{cusp}}(\mathbf{GL}_n)$ with trivial central character, and the global Arthur parameter ψ_π associated to π via Axiom 2 takes value in $\mathbf{SL}_n(\mathbb{C}) = \widehat{\mathbf{PGL}_n}(\mathbb{C})$. In this case, the global Arthur parameter ψ_π is trivial on $\mathbf{SL}_2(\mathbb{C})$, and the conjugacy class $\mathrm{H}(\pi)$ of subgroups of $\mathrm{SU}(n)$ coincides with the usual Sato-Tate group of π.

6.4 Cuspidal representations of $\mathbf{GL}_n$

Arthur's classification of automorphic representations involves self-dual cuspidal representations of $\mathbf{GL}_n, n \geq 1$. Moreover, these representations of $\mathbf{GL}_n$ are trivial on the center of $\mathbf{GL}_n$ when they have level one, thus we can replace $\mathbf{GL}_n$ by $\mathbf{PGL}_n$. In this section we will say more about this class of automorphic representations.

Definition 6.4.1. A representation $\pi \in \Pi_{\mathrm{cusp}}(\mathbf{PGL}_n)$ is *self-dual* if it is isomorphic to its dual representation $\pi^\vee$, and we denote the subset of $\Pi_{\mathrm{cusp}}(\mathbf{PGL}_n)$ consisting of self-dual representations by $\Pi_{\mathrm{cusp}}^{\perp}(\mathbf{PGL}_n)$.

Remark 6.4.2. By the multiplicity one theorem of Jacquet-Shalika, this self-dual condition is equivalent to that $\mathrm{c}_p(\pi) = \mathrm{c}_p(\pi)^{-1}$ for each prime p and $\mathrm{c}_\infty(\pi) = -\mathrm{c}_\infty(\pi)$.

For a representation $\pi \in \Pi_{\mathrm{cusp}}(\mathbf{PGL}_n)$, its infinitesimal character $\mathrm{c}_\infty(\pi)$ is a conjugacy class in $\mathfrak{sl}_n$. Denote by $\mathrm{Weights}(\pi)$ the multiset of eigenvalues of $\mathrm{c}_\infty(\pi)$.

Definition 6.4.3. A cuspidal automorphic representation π of $\mathbf{PGL}_n$ is

- *algebraic*[1] if $\mathrm{Weights}(\pi) \subset \frac{1}{2}\mathbb{Z}$ and for any $w, w' \in \mathrm{Weights}(\pi)$ we have $w - w' \in \mathbb{Z}$;
- *regular* if $|\mathrm{Weights}(\pi)| = n$.

We denote by $\Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_n)$ the subset of $\Pi_{\mathrm{cusp}}^{\perp}(\mathbf{PGL}_n)$ consisting of algebraic representations, and by $\Pi_{\mathrm{alg,reg}}^{\perp}(\mathbf{PGL}_n)$ the subset consisting of algebraic regular representations.

For an algebraic self-dual cuspidal representation π of $\mathbf{PGL}_n$, let $k_1 \geq k_2 \geq \cdots \geq k_n$ be the weights of π (counted with multiplicity). Since π is self-dual, we have $k_i = -k_{n+1-i}$ for $i = 1, 2, \ldots, n$. Following [ChenevierRenard, 2015, §1.5], we call the integers

$$w_i = 2k_i,\ i = 1, 2, \ldots, [n/2]$$

the *Hodge weights* of π and call the maximal Hodge weight $\mathrm{w}(\pi) := w_1$ the *motivic weight* of π.

6.4.1 Arthur's orthogonal-symplectic alternative

We can divide the set of self-dual cuspidal representations of $\mathbf{PGL}_n$ into two parts, by Arthur's *symplectic-orthogonal alternative*. Our reference is [ChenevierLannes, 2019, §8.3.1].

The classical groups over $\mathbb{Z}$ that are Chevalley groups are therefore $\mathbf{Sp}_{2g}$ for $g \geq 1$, $\mathbf{SO}_{r,r}$ for $r \geq 2$, and $\mathbf{SO}_{r+1,r}$ for $r \geq 1$. For one of these groups $\mathbf{G}$, we denote the standard representation of $\widehat{\mathbf{G}}(\mathbb{C})$ by $\mathrm{St} : \widehat{\mathbf{G}}(\mathbb{C}) \hookrightarrow \mathbf{SL}_{\mathrm{n}(\mathbf{G})}(\mathbb{C})$. For instance, $\mathrm{n}(\mathbf{Sp}_{2g}) = 2g+1$, $\mathrm{n}(\mathbf{SO}_{r,r}) = 2r$ and $\mathrm{n}(\mathbf{SO}_{r+1,r}) = 2r$. This map St also induces a natural map from $\mathcal{X}(\widehat{\mathbf{G}})$ to $\mathcal{X}(\mathbf{SL}_{\mathrm{n}(\mathbf{G})})$. We have the following theorem by Arthur:

Theorem 6.4.4. *[Arthur, 2013, Theorem 1.4.1] For any $n \geq 1$ and a self-dual cuspidal representation π of $\mathbf{PGL}_n$, there exists a classical Chevalley group $\mathbf{G}^\pi$, unique up to isomorphism, with the following properties:*

[1] The term *algebraic* is in the sense of Borel [Borel, 1979, §18.2].

(i) We have $\mathrm{n}(\mathbf{G}^\pi) = n$.

(ii) There exists a representation $\pi' \in \Pi_{\mathrm{disc}}(\mathbf{G}^\pi)$ *such that* $\psi(\pi', \mathrm{St}) = \mathrm{c}(\pi)$.

Definition 6.4.5. A representation $\pi \in \Pi^\perp_{\mathrm{cusp}}(\mathbf{PGL}_n)$ is called *orthogonal* if $\widehat{\mathbf{G}^\pi}(\mathbb{C}) \simeq \mathbf{SO}_n(\mathbb{C})$ and *symplectic* otherwise. We denote the subset of $\Pi^\perp_{\mathrm{cusp}}(\mathbf{PGL}_n)$ consisting of orthogonal representations by $\Pi^{\mathrm{o}}_{\mathrm{cusp}}(\mathbf{PGL}_n)$, and the subset consisting of symplectic representations by $\Pi^{\mathrm{s}}_{\mathrm{cusp}}(\mathbf{PGL}_n)$.

For $* = \mathrm{alg}$ or $\mathrm{alg}, \mathrm{reg}$, we define $\Pi^{\mathrm{o}}_*(\mathbf{PGL}_n) = \Pi^{\mathrm{o}}_{\mathrm{cusp}}(\mathbf{PGL}_n) \cap \Pi^\perp_*(\mathbf{PGL}_n)$ and $\Pi^{\mathrm{s}}_*(\mathbf{PGL}_n) = \Pi^{\mathrm{s}}_{\mathrm{cusp}}(\mathbf{PGL}_n) \cap \Pi^\perp_*(\mathbf{PGL}_n)$. We define the subset $\Pi^{\mathbf{Sp}_{2n}}_{\mathrm{alg}}(\mathbf{PGL}_{2n}) \subset \Pi^{\mathrm{s}}_{\mathrm{alg,reg}}(\mathbf{PGL}_{2n})$ as:

$$\left\{ \pi \in \Pi^{\mathrm{s}}_{\mathrm{alg,reg}}(\mathbf{PGL}_{2n}) \,\middle|\, \mathrm{Im}(\psi_\pi) \simeq \mathrm{Sp}(n) \right\},$$

and similarly define

$$\Pi^{\mathbf{SO}_n}_{\mathrm{alg}}(\mathbf{PGL}_n) = \left\{ \pi \in \Pi^{\mathrm{o}}_{\mathrm{alg,reg}}(\mathbf{PGL}_n) \,\middle|\, \mathrm{Im}(\psi_\pi) \simeq \mathrm{SO}(n) \right\}.$$

Example 6.4.6. A representation $\pi \in \Pi_{\mathrm{cusp}}(\mathbf{PGL}_2)$ is necessarily self-dual and symplectic, thus $\Pi_{\mathrm{cusp}}(\mathbf{PGL}_2) = \Pi^\perp_{\mathrm{cusp}}(\mathbf{PGL}_2) = \Pi^{\mathrm{s}}_{\mathrm{cusp}}(\mathbf{PGL}_2)$. Moreover, for each positive integer w we have a bijection between the set of level one normalized Hecke eigenforms of weight $w+1$ and the set of $\pi \in \Pi^\perp_{\mathrm{alg}}(\mathbf{PGL}_2)$ with Hodge weight w. In particular, level one algebraic cuspidal representations with Hodge weight w exist only when $w \geq 11$.

6.4.2 Global ε-factor

An important factor related to a cuspidal representation π is its *global ε-factor* $\varepsilon(\pi)$. We briefly give its definition as follows: for two level one cuspidal representations $\pi \in \Pi_{\mathrm{cusp}}(\mathbf{PGL}_n)$ and $\pi' \in \Pi_{\mathrm{cusp}}(\mathbf{PGL}_{n'})$, Jacquet, Shalika and Piatetski-Shapiro define a factor $\varepsilon(\pi \times \pi')$ when studying the meromorphic continuation and functional equation of the Rankin-Selberg L-function $\mathrm{L}(s, \pi \times \pi')$ [Cogdell, 2004, §9].

Definition 6.4.7. The *global ε-factor* of $\pi \in \Pi_{\mathrm{cusp}}(\mathbf{PGL}_n)$ is defined as $\varepsilon(\pi) := \varepsilon(\pi \times \mathbf{1})$.

For orthogonal algebraic representations, we have the following result by Arthur:

Theorem 6.4.8. *[Arthur, 2013, Theorem 1.5.3] If* $\pi \in \Pi^{\mathrm{o}}_{\mathrm{alg}}(\mathbf{PGL}_n)$, *then* $\varepsilon(\pi) = 1$.

In [ChenevierLannes, 2019, §8.2.21], a method to compute $\varepsilon(\pi)$ for $\pi \in \Pi^{\mathrm{s}}_{\mathrm{alg}}(\mathbf{PGL}_n)$ is explained. To recall that method, we review first the archimedean Local Langlands correspondence [Langlands, 1989]. We can associate with each irreducible unitary representation U of $\mathbf{GL}_n(\mathbb{R})$ a unique (up to conjugacy) semisimple representation $\mathrm{L}(U) : \mathrm{W}_\mathbb{R} \to \mathbf{GL}_n(\mathbb{C})$. By Clozel's purity lemma [Clozel, 1990, Lemma 4.9], for a representation $\pi \in \Pi^\perp_{\mathrm{alg}}(\mathbf{PGL}_n)$, the associated representation $\mathrm{L}(\pi_\infty)$ is a direct sum of the following types of irreducible representations:

- the trivial representation $\mathbf{1}$,
- the sign character $\epsilon_{\mathbb{C}/\mathbb{R}} = \eta/|\eta|$,

- and the 2-dimensional induced representation $\mathbf{I}_w := \mathrm{Ind}_{\mathrm{W}_\mathbb{C}}^{\mathrm{W}_\mathbb{R}}\left(z \mapsto z^{w/2}\overline{z}^{-w/2}\right)$ for some positive integer w, where $z \mapsto z^{w/2}\overline{z}^{-w/2}$ stands for the character $z \mapsto (z/|z|)^w$ by an abuse of notation.

There is a unique way to associate a fourth root of unity $\varepsilon(\rho)$ with each $\rho : \mathrm{W}_\mathbb{R} \to \mathbf{GL}_n(\mathbb{C})$ of the above forms such that $\varepsilon(\rho \oplus \rho') = \varepsilon(\rho)\varepsilon(\rho')$ and

$$\varepsilon(\mathbf{1}) = 1,\ \varepsilon(\epsilon_{\mathbb{C}/\mathbb{R}}) = i,\ \varepsilon(\mathbf{I}_w) = i^{w+1} \text{ for any integer } w > 0.$$

There is a connection between this factor $\varepsilon\left(\mathrm{L}(\pi_\infty)\right)$ and the global ε-factor of π:

Proposition 6.4.9. *[ChenevierLannes, 2019, Proposition 8.2.22] For $\pi \in \Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_n)$, we have*

$$\varepsilon(\pi) = \varepsilon(\mathrm{L}(\pi_\infty)).$$

As a consequence, we can calculate the global ε-factor of π provided we know the representation $\mathrm{L}(\pi_\infty)$ of $\mathrm{W}_\mathbb{R}$ corresponding to π_∞. Actually, one has the following result:

Proposition 6.4.10. *[ChenevierLannes, 2019, Proposition 8.2.13] Let $\pi \in \Pi_{\mathrm{alg}}^{\mathrm{s}}(\mathbf{PGL}_n)$ and $w_1 \geq w_2 \geq \cdots \geq w_{n/2}$ its Hodge weights, then*

$$\mathrm{L}(\pi_\infty) \simeq \mathbf{I}_{w_1} \oplus \mathbf{I}_{w_2} \oplus \cdots \oplus \mathbf{I}_{w_{n/2}}.$$

6.5 Arthur-Langlands conjecture

Assuming the existence of the Langlands group $\mathcal{L}_\mathbb{Z}$ described in Section 6.3. Axiom 3 says that for any reductive group $\mathbf{G}$ admitting a reductive $\mathbb{Z}$-model and any discrete automorphic representation π of $\mathbf{G}$, there exists a discrete global Arthur parameter $\psi : \mathcal{L}_\mathbb{Z} \times \mathbf{SL}_2(\mathbb{C}) \to \widehat{\mathbf{G}}(\mathbb{C})$ such that $\mathrm{c}(\pi) = \mathrm{c}(\psi)$.

Remark 6.5.1. When the group $\widehat{\mathbf{G}}(\mathbb{C})$ satisfies the "element-conjugacy implies conjugacy" property as in Proposition 5.1.5, the discrete global Arthur parameter ψ satisfying $\mathrm{c}(\psi) = \mathrm{c}(\pi)$, as a conjugacy class of homomorphisms $\mathcal{L}_\mathbb{Z} \times \mathbf{SL}_2(\mathbb{C}) \to \widehat{\mathbf{G}}(\mathbb{C})$, is unique.

Let $\mathbf{G}$ be semisimple, and fix an irreducible algebraic representation $r : \widehat{\mathbf{G}} \to \mathbf{SL}_{n,\mathbb{C}}$. Following [ChenevierLannes, 2019, §6.4.4], we are going to see what the Langlands parameter $\psi(\pi, r)$ defined in Definition 6.2.3 looks like for a discrete automorphic representation π of $\mathbf{G}$.

Composing r with a discrete global Arthur parameter $\psi : \mathcal{L}_\mathbb{Z} \times \mathbf{SL}_2(\mathbb{C}) \to \widehat{\mathbf{G}}(\mathbb{C})$ corresponding to π, we get an n-dimensional representation $r \circ \psi$ of $\mathcal{L}_\mathbb{Z} \times \mathbf{SL}_2(\mathbb{C})$. This representation can be decomposed as

$$\bigoplus_{i=1}^{k} r_i \otimes \mathrm{Sym}^{d_i - 1}\,\mathrm{St}$$

for some irreducible representations $r_i : \mathcal{L}_\mathbb{Z} \to \mathbf{SL}_{n_i}$ and certain integers $d_i \geq 1$, where St denotes the standard 2-dimensional representation of $\mathbf{SL}_2(\mathbb{C})$.

By Arthur-Langlands conjecture for general linear groups, *i.e.* Axiom 2 in Section 6.3, every irreducible representation $r_i : \mathcal{L}_\mathbb{Z} \to \mathbf{GL}_{n_i}(\mathbb{C})$ corresponds to a unique cuspidal representation

π_i of $\mathbf{PGL}_{n_i}$. For $v = p$ or ∞, we have an identity between conjugacy classes:

$$r(\mathrm{c}_v(\pi)) = \bigoplus_i^k \mathrm{c}_v(\pi_i) \otimes \mathrm{Sym}^{d_i-1}(e_v).$$

To formulate a global identity, we introduce the following notations:

- Define $e \in \mathcal{X}(\mathbf{SL}_2)$ to be $(e_\infty, e_2, e_3, \cdots)$ and denote $\mathrm{Sym}^{d-1}(e) \in \mathcal{X}(\mathbf{SL}_d)$ by $[d]$.
- Denote by $(c, c') \mapsto c \oplus c'$ the map $\mathcal{X}(\mathbf{SL}_a) \times \mathcal{X}(\mathbf{SL}_b) \to \mathcal{X}(\mathbf{SL}_{a+b})$ induced by the direct sum, and by $(c, c') \mapsto c \otimes c'$ the map $\mathcal{X}(\mathbf{SL}_a) \times \mathcal{X}(\mathbf{SL}_b) \to \mathcal{X}(\mathbf{SL}_{ab})$ induced by the tensor product. We write $c \otimes [d]$ as $c[d]$ for short.
- For $\pi \in \Pi_{\mathrm{cusp}}(\mathbf{PGL}_m)$, the element $\mathrm{c}(\pi) \in \mathcal{X}(\mathbf{SL}_m)$ will simply be denoted by π.

With these notations, we can combine the identities for $r(\mathrm{c}_v(\pi))$ together into one:

$$\psi(\pi, r) = r(\mathrm{c}(\pi)) = \bigoplus_{i=1}^k \pi_i[d_i],\ \pi_i \in \Pi_{\mathrm{cusp}}(\mathbf{PGL}_{n_i}).$$

Now we state Arthur-Langlands conjecture for semisimple groups:

Conjecture 6.5.2. *(*Arthur-Langlands conjecture*) Let $\mathbf{G}$ be a semisimple $\mathbb{Q}$-group admitting a reductive $\mathbb{Z}$-model. For any $\pi \in \Pi_{\mathrm{disc}}(\mathbf{G})$ and every algebraic representation $r : \widehat{\mathbf{G}} \to \mathbf{SL}_{n,\mathbb{C}}$, there exists a collection of triples $(n_i, \pi_i, d_i)_{i=1,\dots,k}$ with $d_i, n_i \geq 1$ integers satisfying $n = \sum_i n_i d_i$ and $\pi_i \in \Pi_{\mathrm{cusp}}(\mathbf{PGL}_{n_i})$ such that*

$$\psi(\pi, r) = \pi_1[d_1] \oplus \cdots \oplus \pi_k[d_k].$$

This conjecture was proved by Arthur in [Arthur, 2013] when $\mathbf{G}$ is a split classical group and r is the standard representation of $\widehat{\mathbf{G}}$. Moreover, the collection of triples (n_i, π_i, d_i) in the conjecture is necessarily unique up to permutation by a result of Jacquet and Shalika [JacquetShalika, 1981]:

Proposition 6.5.3. *[ChenevierLannes, 2019, Proposition 6.4.5] Let $k, l \geq 1$ be integers. For $1 \leq i \leq k$ (*resp. $1 \leq j \leq l$*), consider integers $n_i, d_i \geq 1$ (*resp. $n'_j, d'_j \geq 1$*) and a representation π_i (*resp. π'_j*) in $\Pi_{\mathrm{cusp}}(\mathbf{PGL}_{n_i})$ (*resp. $\Pi_{\mathrm{cusp}}(\mathbf{PGL}_{n'_j})$*). Suppose that we have $n := \sum_i n_i d_i = \sum_j n'_j d'_j$ and*

$$\pi_1[d_1] \oplus \cdots \oplus \pi_k[d_k] = \pi'_1[d'_1] \oplus \cdots \oplus \pi'_l[d'_l].$$

Then $k = l$ and there exists a permutation $\sigma \in \mathrm{S}_k$ such that for every $1 \leq i \leq k$ we have $(n'_i, \pi'_i, d'_i) = (n_{\sigma(i)}, \pi_{\sigma(i)}, d_{\sigma(i)})$.

We call the triple $(k, (n_i, d_i)_{1 \leq i \leq k})$, up to permutations of the (n_i, d_i), the *endoscopic type* of $\psi(\pi, r)$. The parameter is called *stable* if $k = 1$ and *endoscopic* otherwise. It is called *tempered* if $d_i = 1$ for all i and *non-tempered* otherwise.

In Conjecture 6.5.2, cuspidal representations of $\mathbf{PGL}_n, n \geq 1$ are building blocks of Langlands parameters $\psi(\pi, r)$. Furthermore, the following result shows that under some conditions, for example when $\mathbf{G}(\mathbb{R})$ is compact, we only need algebraic cuspidal representations:

Proposition 6.5.4. *[ChenevierLannes, 2019, Proposition 8.2.8] Let* $\mathbf{G}$ *be a semisimple* $\mathbb{Q}$*-group admitting a reductive* $\mathbb{Z}$*-model,* $\pi \in \Pi_{\text{disc}}(\mathbf{G})$ *and* $r : \widehat{\mathbf{G}} \to \mathbf{SL}_{n,\mathbb{C}}$ *an* n*-dimensional algebraic representation of* $\widehat{\mathbf{G}}$*. Suppose that*

(i) $\mathrm{c}_\infty(\pi) \in \widehat{\mathfrak{g}}_{\text{ss}}$ *is the infinitesimal character of a finite-dimensional irreducible complex representation of* $G_{\mathbb{C}}$*,*

(ii) and $\psi(\pi, r) = \oplus_{i=1}^{k} \pi_i[d_i]$ *with* $\pi_i \in \Pi_{\text{cusp}}(\mathbf{PGL}_{n_i}), i = 1, \ldots, k$*.*

Then π_i *is algebraic for* $i = 1, \ldots, k$*. Moreover, the class of* $\mathrm{w}(\pi_i) + d_i - 1$ *in* $\mathbb{Z}/2\mathbb{Z}$ *depends only on* r *and not on the integer* i *or even on* π*.*

6.6 Arthur's multiplicity formula

Arthur gives a *conjectural* formula for the multiplicity of an adelic representation $\pi \in \Pi(\mathbf{G})$ in the discrete spectrum $\mathcal{L}_{\text{disc}}(\mathbf{G})$. In this section, we will state this for a simply-connected anisotropic $\mathbb{Q}$-group $\mathbf{G}$ admitting a reductive $\mathbb{Z}$-model, following [Arthur, 1989, §8].

For a representation $\pi \in \Pi(\mathbf{G})$, there are finitely many discrete global Arthur parameters ψ of $\mathbf{G}$ such that $\mathrm{c}(\pi) = \mathrm{c}(\psi)$. According to [Arthur, 1989], the multiplicity $\mathrm{m}(\pi)$ of π in $\mathcal{L}_{\text{disc}}(\mathbf{G})$ should be the sum of m_ψ over the set of all such ψ, where m_ψ is some integer that we are going to introduce. We note that these ψ all belong to the following subset of $\Psi_{\text{disc}}(\mathbf{G})$:

Definition 6.6.1. We define $\Psi_{\text{AJ}}(\mathbf{G})$ to be the subset of $\Psi_{\text{disc}}(\mathbf{G})$ consisting of $\psi \in \Psi_{\text{disc}}(\mathbf{G})$ satisfying that $\mathrm{c}_\infty(\psi)$ is the infinitesimal character of a finite dimensional irreducible representation of $G_{\mathbb{C}}$.

Remark 6.6.2. The subscript AJ stands for *Adams-Johnson*. This means the archimedean Arthur parameter $\mathrm{W}_{\mathbb{R}} \times \mathbf{SL}_2(\mathbb{C}) \to \widehat{\mathbf{G}}(\mathbb{C})$ for $\psi \in \Psi_{\text{disc}}(\mathbf{G})$ is an *Adams-Johnson parameter* in the sense of [ChenevierLannes, 2019, §8.4.14] if and only if $\psi \in \Psi_{\text{AJ}}(\mathbf{G})$. The condition that $\mathrm{c}_\infty(\psi)$ is the infinitesimal character of a finite-dimensional irreducible representation is the condition (AJ1) in [ChenevierLannes, 2019, §8.4.14], and the second condition (AJ2) for Adams-Johnson parameters is automatically satisfied in our case by [Taïbi, 2017, §4.2.2; NairPrasad, 2021, Proposition 6].

Now we let $\psi \in \Psi_{\text{AJ}}(\mathbf{G})$. In Definition 6.3.2, the global component group C_ψ of ψ is defined to be the centralizer of $\mathrm{Im}(\psi)$ in $\widehat{\mathbf{G}}(\mathbb{C})$. When $\mathbf{G}$ is semisimple, this group is finite since the center of $\widehat{\mathbf{G}}$ is finite. Moreover, as explained in [ChenevierLannes, 2019, §8.4.14], C_ψ is an elementary finite abelian 2-group, *i.e.* a product of finitely many copies of $\mathbb{Z}/2\mathbb{Z}$. For any $\psi \in \Psi_{\text{AJ}}(\mathbf{G})$, Arthur's formula for m_ψ involves two quadratic characters of C_ψ.

6.6.1 The character $\rho_\psi^\vee$

The first character of C_ψ is defined as follows.

By Proposition 6.2.1, the conjugacy class $\mathrm{c}_\infty(\psi)$ for $\psi \in \Psi_{\text{AJ}}(\mathbf{G})$ is regular, viewed as a cocharacter of a maximal torus $\widehat{\mathbf{T}}$ of $\widehat{\mathbf{G}}$ chosen as in [ChenevierLannes, 2019, §8.4.14]. Hence there is a unique Borel subgroup $\widehat{\mathbf{B}} \supset \widehat{\mathbf{T}}$ of $\widehat{\mathbf{G}}$ with respect to whom the infinitesimal character

$\mathrm{c}_\infty(\psi)$ is strictly dominant. Let $\rho_\psi^\vee$ be the half-sum of positive roots with respect to $(\widehat{\mathbf{G}}, \widehat{\mathbf{B}}, \widehat{\mathbf{T}})$. Since $\mathbf{G}$ is simply-connected, $\rho_\psi^\vee \in \frac{1}{2}X^*(\widehat{\mathbf{T}})$ is a character of $\widehat{\mathbf{T}}$. Its restriction to the component group C_ψ is the first character we need, and we denote $\rho^\vee|_{\mathrm{C}_\psi}$ by $\rho_\psi^\vee$ for short.

6.6.2 Arthur's character ε_ψ

A discrete global Arthur parameter $\psi \in \Psi_{\mathrm{AJ}}(\mathbf{G})$ induces a morphism

$$\mathrm{C}_\psi \times \mathcal{L}_\mathbb{Z} \times \mathbf{SL}_2(\mathbb{C}) \to \widehat{\mathbf{G}}(\mathbb{C}).$$

Restricting the adjoint representation $\widehat{\mathfrak{g}}$ of $\widehat{\mathbf{G}}(\mathbb{C})$ along this morphism, it can be decomposed into a direct sum

$$\widehat{\mathfrak{g}}|_{\mathrm{C}_\psi \times \mathcal{L}_\mathbb{Z} \times \mathbf{SL}_2(\mathbb{C})} = \bigoplus_{i=1}^{l} \chi_i \otimes \pi_i[d_i], \tag{6.5}$$

where χ_i is a quadratic character of C_ψ, and π_i is an n_i-dimensional irreducible representation of $\mathcal{L}_\psi$ which is identified as an element in $\Pi_{\mathrm{cusp}}^\perp(\mathbf{PGL}_{n_i})$. Moreover, since ψ belongs to $\Psi_{\mathrm{AJ}}(\mathbf{G})$, according to Proposition 6.5.4 these cuspidal representations π_i are algebraic.

Definition 6.6.3. [Arthur, 1989, Equation 8.4] Let $\psi \in \Psi_{\mathrm{AJ}}(\mathbf{G})$, and I be the subset of $\{1, \dots, l\}$ consisting of i satisfying that in Eq. (6.5) the cuspidal representation π_i is self-dual and $\varepsilon(\pi_i) = -1$. Arthur's character $\varepsilon_\psi : \mathrm{C}_\psi \to \mu_2$ is defined by

$$\varepsilon_\psi(s) := \prod_{i \in I} \chi_i(s), \text{ for every } s \in \mathrm{C}_\psi.$$

The following result shows that it is sufficient to calculate the global epsilon factors $\varepsilon(\pi_i)$ for i in a subset of $\{1, \dots, l\}$:

Proposition 6.6.4. *Let $\psi \in \Psi_{\mathrm{AJ}}(\mathbf{G})$. For any $s \in \mathrm{C}_\psi$, let I_s be the subset of $\{1, \dots, l\}$ consisting of i satisfying that in Eq. (6.5) the representation π_i is self-dual, d_i is even, and $\chi_i(s) = -1$. Then we have:*

$$\varepsilon_\psi(s) = \prod_{i \in I_s} \varepsilon(\pi_i).$$

Proof. When d_i is odd, the d_i-dimensional irreducible representation of $\mathbf{SL}_2(\mathbb{C})$ is orthogonal. Since the adjoint representation is an orthogonal representation, the self-dual representation π_i of $\mathcal{L}_\mathbb{Z}$ must be also orthogonal, which implies $\varepsilon(\pi_i) = 1$ by Theorem 6.4.8. Hence the subset I in Definition 6.6.3 is a subset of $\{i \,|\, d_i \text{ is even}\}$, and for any $s \in \mathrm{C}_\psi$ we have

$$\varepsilon_\psi(s) = \prod_{2|d_i,\, \pi_i = \pi_i^\vee,\, \varepsilon(\pi_i) = -1} \chi_i(s) = \prod_{2|d_i,\, \pi_i = \pi_i^\vee,\, \chi_i(s) = -1} \varepsilon(\pi_i) = \prod_{i \in I_s} \varepsilon(\pi_i). \qquad \square$$

6.6.3 The multiplicity formula

With two characters $\rho_\psi^\vee$ and ε_ψ in hand, we can state Arthur's following conjecture:

Conjecture 6.6.5. *(*Arthur's multiplicity formula*) Let* $\mathbf{G}$ *be a simply-connected anisotropic* $\mathbb{Q}$*-group with a reductive* $\mathbb{Z}$*-model, and* π *a level one adelic representation in* $\Pi(\mathbf{G})$*. We have the following formula for the multiplicity* $\mathrm{m}(\pi)$ *of* π *in the discrete spectrum* $\mathcal{L}_{\mathrm{disc}}(\mathbf{G})$*:*

$$\mathrm{m}(\pi) = \sum_{\psi \in \Psi_{\mathrm{disc}}(\mathbf{G}),\, c(\psi)=c(\pi)} m_\psi, \ \ \textit{where } m_\psi = \begin{cases} 1, & \textit{if } \rho_\psi^\vee = \varepsilon_\psi, \\ 0, & \textit{otherwise.} \end{cases} \tag{6.6}$$

Chapter 7

Classification of global Arthur parameters for $\mathbf{F}_4$

In this chapter, we are going to apply Arthur's conjectures recalled in Section 6.5 and Section 6.6 to the simply-connected anisotropic $\mathbb{Q}$-group $\mathbf{F}_4$ defined in Definition 3.1.6. The dual group $\widehat{\mathbf{F}_4}$ is isomorphic to the extension $\mathbf{F}_{4,\mathbb{C}}$ of $\mathbf{F}_4$ to $\mathbb{C}$. In other words, the complex Lie group $\widehat{\mathbf{F}_4}(\mathbb{C})$ is isomorphic to the complexification $\mathrm{F}_{4,\mathbb{C}}$ of the real compact Lie group F_4.

7.1 Arthur parameters of $\mathbf{F}_4$

The real points $\mathrm{F}_4 = \mathbf{F}_4(\mathbb{R})$ is compact, so an adelic representation $\pi \in \Pi(\mathbf{F}_4)$ is determined uniquely by $\mathrm{c}(\pi)$. On the other hand, by Proposition 5.1.5 and Axiom 1, a discrete global Arthur parameter ψ of $\mathbf{F}_4$ is also determined uniquely by $\mathrm{c}(\psi) \in \mathcal{X}(\widehat{\mathbf{F}_4})$. Moreover, we have the following criterion, which is a direct corollary of Proposition 5.2.1:

Proposition 7.1.1. *Let ψ_1 and ψ_2 be two discrete global Arthur parameters of $\mathbf{F}_4$, and $\mathrm{r}_0 : \widehat{\mathbf{F}_4} \to \mathbf{SL}_{26,\mathbb{C}}$ the 26-dimensional irreducible representation of $\mathbf{F}_4(\mathbb{C})$. Then $\psi_1 = \psi_2$ if and only if $\mathrm{r}_0(\mathrm{c}(\psi_1)) = \mathrm{r}_0(\mathrm{c}(\psi_2))$.*

By this result, we will identify a discrete global Arthur parameter $\psi \in \Psi_{\mathrm{disc}}(\mathbf{F}_4)$ with the corresponding family of conjugacy classes $\mathrm{r}_0(\mathrm{c}(\psi)) \in \mathcal{X}(\mathbf{SL}_{26})$.

For a level one discrete automorphic representation $\pi \in \Pi_{\mathrm{disc}}(\mathbf{F}_4)$, the discrete global Arthur parameter $\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ such that $\mathrm{c}(\psi) = \mathrm{c}(\pi)$ predicted by Axiom 1 is unique. We denote this parameter by ψ_π, which is identified with $\psi(\pi, \mathrm{r}_0) \in \mathcal{X}(\mathbf{SL}_{26})$. Conversely, for $\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4)$, we denote the unique representation $\pi \in \Pi(\pi)$ such that $\mathrm{c}(\pi) = \mathrm{c}(\psi)$ by π_ψ.

The following lemma gives us some constraint on the infinitesimal character $\mathrm{c}_\infty(\psi)$ of $\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4)$:

Lemma 7.1.2. *Let $\mathrm{c}_\infty \in (\mathfrak{f}_4)_{\mathrm{ss}}$ be the infinitesimal character of an irreducible representation of the compact group F_4, then there exists four non-negative integers a, b, c, d such that the* *eigenvalues (counted with multiplicity) of* $\mathrm{r}_0(\mathrm{c}_\infty) \in (\mathfrak{sl}_{26})_{\mathrm{ss}}$ *are:*

$$0, 0, \pm(a+1), \pm(b+1), \pm(a+b+2), \pm(b+c+2), \pm(a+b+c+3), \pm(b+c+d+3),$$
$$\pm(a+b+c+d+4), \pm(a+2b+c+4), \pm(a+2b+c+d+5), \pm(a+2b+2c+d+6),$$
$$\pm(a+3b+2c+d+7), \pm(2a+3b+2c+d+8).$$

Proof. If we write the highest weight λ of this irreducible representation of F_4 as $a\varpi_1 + b\varpi_2 + c\varpi_3 + d\varpi_4$, then by Proposition 6.2.1 the infinitesimal character c_∞ is $\lambda + \rho = (a+1)\varpi_1 + (b+1)\varpi_2 + (c+1)\varpi_3 + (d+1)\varpi_4$. The eigenvalues of $\mathrm{r}_0(\mathrm{c}_\infty)$ are of the form $\langle \lambda + \rho, \alpha^\vee \rangle$, where $\alpha^\vee$ runs over the 26 weights of $\widehat{\mathbf{F}_4}(\mathbb{C})$ appearing in the representation r_0. By an easy calculation, we get the eigenvalues in the lemma. □

As recalled in Section 6.3.1, we associate to $\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ a morphism $\psi_{\mathrm{c}} : \mathcal{L}_{\mathbb{Z}} \times \mathrm{SU}(2) \to \mathrm{F}_4$ between compact Lie groups. This homomorphism inherits the following properties from ψ:

- the image $\mathrm{Im}(\psi_c)$ is connected due to Proposition 6.3.5,
- the centralizer of $\mathrm{Im}(\psi_{\mathrm{c}})$ in $\mathbf{F}_4$ coincides with the global component group C_ψ of ψ, which is an elementary finite abelian 2-group by [ChenevierLannes, 2019, §8.4.14],
- and the zero weight appears exactly twice in the restriction of the 26-dimensional irreducible representation J_0 of F_4 along ψ_{c} by Lemma 7.1.2.

Hence $\mathrm{Im}(\psi_{\mathrm{c}})$ is a subgroup of F_4 satisfying the three conditions in the beginning of Chapter 5, thus the class $\mathrm{H}(\psi)$ defined in Definition 6.3.6 is the conjugacy class of one of the subgroups of F_4 listed in Theorem 5.6.7.

According to Conjecture 6.5.2, the discrete global Arthur parameter $\psi_\pi = \psi(\pi, \mathrm{r}_0)$ corresponding to a discrete automorphic representation $\pi \in \Pi_{\mathrm{disc}}(\mathbf{F}_4)$ should be of the form:

$$\pi_1[d_1] \oplus \cdots \oplus \pi_k[d_k],$$

where $\pi_i \in \Pi_{\mathrm{cusp}}(\mathbf{PGL}_{n_i})$ and $\sum_{i=1}^k n_i d_i = 26$. By Proposition 6.5.4, every π_i is algebraic, and it is also self-dual by the following lemma:

Lemma* 7.1.3. *Let* $\pi \in \Pi_{\mathrm{disc}}(\mathbf{F}_4)$ *and* $\psi_\pi = \pi_1[d_1] \oplus \cdots \oplus \pi_k[d_k]$ *be its corresponding discrete global Arthur parameter, then for each* $i = 1, \ldots, k$, *the representation* $\pi_i \in \Pi_{\mathrm{cusp}}(\mathbf{PGL}_{n_i})$ *is self-dual.*

Proof. By our classification result in Section 5.6, identifying $\pi_i \in \Pi_{\mathrm{cusp}}(\mathbf{PGL}_{n_i})$ as an irreducible representation of $\mathcal{L}_{\mathbb{Z}}$, it must be of the form $\mathcal{L}_{\mathbb{Z}} \twoheadrightarrow H \xrightarrow{r} \mathbf{SL}_{n_i}(\mathbb{C})$, where H is a connected compact subgroup of F_4 and r is a self-dual irreducible representation of H, thus π_i itself is self-dual. □

So a discrete global Arthur parameter $\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ corresponding to some $\pi \in \Pi_{\mathrm{disc}}(\mathbf{F}_4)$ must be of the form

$$\psi = \pi_1[d_1] \oplus \cdots \oplus \pi_k[d_k], \text{ where } \pi_i \in \Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_{n_i}), \sum_{i=1}^{k} n_i d_i = 26. \tag{7.1}$$

The endoscopic types $(k, (n_i, d_i)_{1\le i\le k})$ can be classified by our results in Section 5.6.

Example 7.1.4. If the class $\mathrm{H}(\psi)$ associated to $\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ is the conjugacy class of

$$H = \left(\mathrm{A}_1^{[9,6^2,5]} \times \mathrm{A}_1^{[2^6,1^{14}]}\right)/\mu_2^{\Delta},$$

by Section 5.6.10 the restriction of the 26-dimensional irreducible representation (r_0, J_0) along ψ is isomorphic to

$$\mathrm{Sym}^5\,\mathrm{St}\otimes\mathrm{St} + \mathrm{Sym}^8\,\mathrm{St}\otimes\mathbf{1} + \mathrm{Sym}^4\,\mathrm{St}\otimes\mathbf{1}.$$

Depending on how $\mathcal{L}_{\mathbb{Z}}$ and $\mathrm{SU}(2)$ are mapped to this subgroup $H \subset \mathrm{F}_4$, we have the following three possible endoscopic types for ψ:

- $(3, (2,6), (1,5), (1,9))$, $\psi = \pi[6] \oplus [5] \oplus [9], \pi \in \Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_2)$;
- $(3, (9,1), (5,1), (6,2))$, $\psi = \mathrm{Sym}^8\,\pi \oplus \mathrm{Sym}^4\,\pi \oplus \mathrm{Sym}^5\,\pi[2], \pi \in \Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_2)$;
- $(3, (9,1), (5,1), (12,1))$, $\psi = \mathrm{Sym}^8\,\pi_1 \oplus \mathrm{Sym}^4\,\pi_2 \oplus (\mathrm{Sym}^5\,\pi_1 \otimes \pi_2), \pi_1, \pi_2 \in \Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_2)$.

7.2 The multiplicity formula for $\mathbf{F}_4$

For a discrete global Arthur parameter $\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4)$, Arthur's multiplicity formula Conjecture 6.6.5 predicts that the multiplicity $\mathrm{m}(\pi_\psi)$ of π_ψ in $\mathcal{L}_{\mathrm{disc}}(\mathbf{F}_4)$ equals to m_ψ, the formula for which is given in Eq. (6.6). To calculate m_ψ, it suffices to know two characters of C_ψ: Arthur's character ε_ψ, and $\rho_\psi^\vee$. We have given the formula of ε_ψ in Proposition 6.6.4, and in this section we will give a recipe for the character $\rho_\psi^\vee$ for our $\mathbb{Q}$-group $\mathbf{F}_4$.

We fix a maximal torus $\widehat{\mathbf{T}}$ of $\widehat{\mathbf{F}_4}$ and a Borel subgroup $\widehat{\mathbf{B}} \supset \widehat{\mathbf{T}}$ as in Section 6.6.1 such that the infinitesimal character $\mathrm{c}_\infty(\psi)$, as a cocharacter of $\widehat{\mathbf{T}}$ is strictly dominant with respect to $(\widehat{\mathbf{F}_4}, \widehat{\mathbf{B}}, \widehat{\mathbf{T}})$. We denote the four simple roots of the root system with respect to $(\widehat{\mathbf{F}_4}, \widehat{\mathbf{B}}, \widehat{\mathbf{T}})$ by $\alpha_i^\vee, i = 1,2,3,4$[1].

By Lemma 7.1.2, we can order the eigenvalues (counted with multiplicity) of $\mathrm{c}_\infty(\psi)$ as $\mu_1 > \mu_2 > \mu_3 > \mu_4 > \mu_5 \ge \cdots > \mu_{26}$. The partial order relation of the positive weights of r_0 in Table 5.1 implies that

$$\mu_1 = \langle \mathrm{c}_\infty(\psi), 2\alpha_1^\vee + 3\alpha_2^\vee + 2\alpha_3^\vee + \alpha_4^\vee\rangle,\ \mu_4 = \langle \mathrm{c}_\infty(\psi), \alpha_1^\vee + 2\alpha_2^\vee + \alpha_3^\vee + \alpha_4^\vee\rangle.$$

Notice that

$$(2\alpha_1^\vee + 3\alpha_2^\vee + 2\alpha_3^\vee + \alpha_4^\vee) + (\alpha_1^\vee + 2\alpha_2^\vee + \alpha_3^\vee + \alpha_4^\vee) \equiv \alpha_1^\vee + \alpha_2^\vee + \alpha_3^\vee \equiv \rho_\psi^\vee \bmod 2X^*(\widehat{\mathbf{T}}),$$

thus the character $\rho_\psi^\vee$ of $\mathrm{C}_\psi \subset \widehat{\mathbf{T}}[2]$ is the product of $(2\alpha_1^\vee + 3\alpha_2^\vee + 2\alpha_3^\vee + \alpha_4^\vee)|_{\mathrm{C}_\psi}$ and $(\alpha_1^\vee + 2\alpha_2^\vee + \alpha_3^\vee + \alpha_4^\vee)|_{\mathrm{C}_\psi}$. Hence it suffices to determine these two characters.

If $\psi = \pi_1[d_1] \oplus \cdots \oplus \pi_k[d_k]$ as in Eq. (7.1), the eigenvalues of $\mathrm{r}_0(\mathrm{c}_\infty(\psi)) \in (\mathfrak{sl}_{26})_{\mathrm{ss}}$ are of the form $w + \frac{j}{2}$, where w is a weight of π_i and $j \in \{d_i - 1, d_i - 3, \ldots, -d_i + 3, -d_i + 1\}$. For each

[1] Here we still follow Bourbaki's notation, but since we are considering the root system of the dual group $\widehat{G}$, the simple root $\alpha_i^\vee, 1 \le i \le 4$ corresponds to α_{5-i} in Bourbaki.

$i = 1, \ldots, k$, we define a multiset

$$\mathcal{W}_i := \left\{ w + \frac{j}{2} \,\middle|\, w \in \mathrm{Weights}(\pi_i) \text{ and } j = d_i - 1, d_i - 3, \ldots, -(d_i - 3), -(d_i - 1) \right\}.$$

Proposition 7.2.1. *There exists a unique index i (resp. j) in $\{1, \ldots, k\}$ such that $\mu_1 \in \mathcal{W}_i$ (resp. $\mu_4 \in \mathcal{W}_j$). If we denote respectively by ϵ_i and ϵ_j the characters of C_ψ induced by the C_ψ-actions on $\pi_i[d_i]$ and $\pi_j[d_j]$, then $\rho_\psi^\vee = \epsilon_i \cdot \epsilon_j$.*

Proof. The uniqueness of i and j follows from the fact that μ_1 and μ_4 are different from other eigenvalues of $\mathrm{r}_0(\mathrm{c}_\infty(\psi))$.

For any $s \in \mathrm{C}_\psi$, we have

$$\rho_\psi^\vee(s) = (2\alpha_1^\vee + 3\alpha_2^\vee + 2\alpha_3^\vee + \alpha_4^\vee)(s) \cdot (\alpha_1^\vee + 2\alpha_2^\vee + \alpha_3^\vee + \alpha_4^\vee)(s).$$

Since $\mu_1 \in \mathcal{W}_i$, the value $(2\alpha_1^\vee + 3\alpha_2^\vee + 2\alpha_3^\vee + \alpha_4^\vee)(s)$ is the scalar given by the action of s on the irreducible summand $\pi_i[d_i]$, which equals $\epsilon_i(s)$ by definition. Similarly, we have $(\alpha_1^\vee + 2\alpha_2^\vee + \alpha_3^\vee + \alpha_4^\vee)(s) = \epsilon_j(s)$ and the identity $\rho_\psi^\vee = \epsilon_i \cdot \epsilon_j$. □

7.3 Classification of Arthur parameters

Now we can do (*conjectural*) classification of global Arthur parameters for $\mathbf{F}_4$:

Theorem* 7.3.1. *Admitting the existence of the Langlands group $\mathcal{L}_\mathbb{Z}$ defined in Section 6.3 and Arthur's multiplicity formula Conjecture 6.6.5, a (level one) discrete global Arthur parameter $\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ satisfies $\mathrm{m}(\pi_\psi) = 1$ if and only if it belongs to the parameters described in the following propositions (from Proposition 7.3.4 to Proposition 7.3.18).*

In this section, we will prove Theorem 7.3.1 case by case, depending on the conjugacy class $\mathrm{H}(\psi)$ associated to the discrete global Arthur parameter ψ. For each subgroup H of $\mathrm{F}_4 = \mathbf{F}_4(\mathbb{R})$ listed in Section 5.6, we classify all the endoscopic types of $\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ such that $\mathrm{H}(\psi)$ is the conjugacy class of H like what we have done in Example 7.1.4, then apply Arthur's multiplicity formula Conjecture 6.6.5, Proposition 6.6.4 and Proposition 7.2.1 to ψ and get those with $\mathrm{m}(\pi_\psi) = 1$.

Notation 7.3.2. From now on, when $\mathrm{H}(\psi)$ is the F_4-conjugacy class of H, we say $\mathrm{H}(\psi) = H$ by an abuse of notation.

Remark 7.3.3. Since the proof of Theorem 7.3.1 is long, readers can read first the proof of Proposition 7.4.3 in Section 7.4 to see how Arthur's conjectures are used.

7.3.1 $H = \mathrm{A}_1^{[17,9]}$

The restriction of the 26-dimensional irreducible representation J_0 to H is isomorphic to

$$\mathrm{Sym}^{16}\,\mathrm{St} + \mathrm{Sym}^8\,\mathrm{St}.$$

For $\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ satisfying $\mathrm{H}(\psi) = H$ and $\mathrm{m}(\pi_\psi) = 1$, there are two possible endoscopic types:

(i) $(2,(1,17),(1,9))$, which corresponds to the parameter $[17] \oplus [9]$ of the trivial representation of $\mathbf{F}_4(\mathbb{A})$.

(ii) $(2,(17,1),(9,1))$. The discrete global Arthur parameters ψ with this type are constructed as follows: for a representation $\pi \in \Pi^{\perp}_{\mathrm{alg}}(\mathbf{PGL}_2)$ and a positive integer k, we denote by $\mathrm{Sym}^k \pi$ the representation in $\Pi^{\perp}_{\mathrm{alg,reg}}(\mathbf{PGL}_{k+1})$ corresponding to the irreducible representation given by

$$\mathcal{L}_{\mathbb{Z}} \xrightarrow{\psi_\pi} \mathbf{SL}_2(\mathbb{C}) \to \mathbf{SL}(\mathrm{Sym}^k \mathrm{St}) \simeq \mathbf{SL}_{k+1}(\mathbb{C}).$$

A global Arthur parameter of this type is of the form:

$$\mathrm{Sym}^{16} \pi \oplus \mathrm{Sym}^8 \pi,\ \pi \in \Pi^{\perp}_{\mathrm{alg}}(\mathbf{PGL}_2).$$

Proposition* 7.3.4. *For a discrete global Arthur parameter $\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ satisfying $\mathrm{H}(\psi) = H$, the multiplicity $\mathrm{m}(\pi_\psi) = 1$ if and only if ψ is one of the following parameters:*

- $[17] \oplus [9]$*, which corresponds to the trivial representation of* $\mathbf{F}_4(\mathbb{A})$.
- $\mathrm{Sym}^{16} \pi \oplus \mathrm{Sym}^8 \pi$, $\pi \in \Pi^{\perp}_{\mathrm{alg}}(\mathbf{PGL}_2)$.

Proof. This is because C_ψ is trivial. □

7.3.2 $H = \left(\mathrm{A}_1^{[9,6^2,5]} \times \mathrm{A}_1^{[2^6,1^{14}]}\right)/\mu_2^{\Delta}$

By Section 5.6.10 the restriction of the 26-dimensional irreducible representation J_0 of F_4 to H is isomorphic to

$$\mathrm{Sym}^5 \mathrm{St} \otimes \mathrm{St} + (\mathrm{Sym}^8 \mathrm{St} + \mathrm{Sym}^4 \mathrm{St}) \otimes \mathbf{1},$$

and the centralizer of H in F_4 is $\mathrm{Z}(H) \simeq \mathbb{Z}/2\mathbb{Z}$.

For $\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ satisfying $\mathrm{H}(\psi) = H$ and $\mathrm{m}(\pi_\psi) = 1$, there are three possible endoscopic types:

(i) $(3,(2,6),(1,5),(1,9))$. A global Arthur parameter of this type is of the form:

$$\pi[6] \oplus [5] \oplus [9],\ \pi \in \Pi^{\perp}_{\mathrm{alg}}(\mathbf{PGL}_2).$$

(ii) $(3,(9,1),(5,1),(6,2))$. A global Arthur parameter of this type is of the form:

$$\mathrm{Sym}^8 \pi \oplus \mathrm{Sym}^4 \pi \oplus \mathrm{Sym}^5 \pi[2],\ \pi \in \Pi^{\perp}_{\mathrm{alg}}(\mathbf{PGL}_2).$$

(iii) $(3,(12,1),(9,1),(5,1))$. For two representations $\pi_1, \pi_2 \in \Pi^{\perp}_{\mathrm{alg}}(\mathbf{PGL}_2)$, we can construct the following 12-dimensional irreducible representation of $\mathcal{L}_{\mathbb{Z}}$:

$$\mathcal{L}_{\mathbb{Z}} \xrightarrow{(\psi_{\pi_1},\psi_{\pi_2})} \mathbf{SL}_2(\mathbb{C}) \times \mathbf{SL}_2(\mathbb{C}) \xrightarrow{\mathrm{Sym}^5 \otimes \mathrm{id}} \mathbf{SL}_{12}(\mathbb{C}),$$

which induces a cuspidal representation of $\mathbf{PGL}_{12}$, denoted by $\mathrm{Sym}^5 \pi_1 \otimes \pi_2$. A global

Arthur parameter of this type is of the form:

$$\mathrm{Sym}^8 \pi_1 \oplus \mathrm{Sym}^4 \pi_1 \oplus \left(\mathrm{Sym}^5 \pi_1 \otimes \pi_2\right), \ \pi_1, \pi_2 \in \Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_2).$$

Remark 7.3.5. In fact, for a $(3,(12,1),(9,1),(5,1))$-type parameter

$$\psi = \mathrm{Sym}^8 \pi_1 \oplus \mathrm{Sym}^4 \pi_1 \oplus \left(\mathrm{Sym}^5 \pi_1 \otimes \pi_2\right), \ \pi_1, \pi_2 \in \Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_2),$$

there are some conditions on the motivic weights $\mathrm{w}(\pi_1), \mathrm{w}(\pi_2)$ to make ψ a parameter in $\Psi_{\mathrm{AJ}}(\mathbf{F}_4)$. We will add these conditions for global Arthur parameters ψ with $m_\psi = 1$ when necessary. For example, when $\mathrm{w}(\pi_2) > 9\mathrm{w}(\pi_1)$ the condition for $\psi \in \Psi(\mathbf{F}_4)$ is that $\mathrm{w}(\pi_2) \geq 9\mathrm{w}(\pi_1) + 2$, which is satisfied automatically since $\mathrm{w}(\pi_2)$ and $9\mathrm{w}(\pi_1)$ are two distinct odd numbers.

For this subgroup H of F_4, the restriction of the adjoint representation $\mathfrak{f}_4$ of F_4 to H is isomorphic to

$$\mathbf{1} \otimes \mathrm{Sym}^2 \mathrm{St} + \left(\mathrm{Sym}^9 \mathrm{St} + \mathrm{Sym}^3 \mathrm{St}\right) \otimes \mathrm{St} + \left(\mathrm{Sym}^{10} \mathrm{St} + \mathrm{Sym}^6 \mathrm{St} + \mathrm{Sym}^2 \mathrm{St}\right) \otimes \mathbf{1}.$$

Proposition* 7.3.6. *For a discrete global Arthur parameter $\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ satisfying $\mathrm{H}(\psi) = H$, the multiplicity $\mathrm{m}(\pi_\psi) = 1$ if and only if ψ is one of the following parameters:*

- $\pi[6] \oplus [5] \oplus [9]$, *where* $\pi \in \Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_2)$.
- $\mathrm{Sym}^8 \pi \oplus \mathrm{Sym}^4 \pi \oplus \mathrm{Sym}^5 \pi[2]$, *where* $\pi \in \Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_2)$ *satisfies* $\mathrm{w}(\pi) \equiv 3 \bmod 4$.
- $\mathrm{Sym}^8 \pi_1 \oplus \mathrm{Sym}^4 \pi_1 \oplus \left(\mathrm{Sym}^5 \pi_1 \otimes \pi_2\right)$, *where* $\pi_1, \pi_2 \in \Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_2)$ *have motivic weights* w_1, w_2 *respectively such that* $w_2 > 9w_1$ *or* $5w_1 < w_2 < 7w_1$.

Proof. We denote the generator of $\mathrm{C}_\psi = \mathrm{Z}(H)$ by γ.

Case (i): $\psi = \pi[6] \oplus [5] \oplus [9]$, where $\pi \in \Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_2)$ has motivic weight w. In this case the restriction of $\mathfrak{f}_4$ along ψ is isomorphic to

$$\mathrm{Sym}^2 \pi \oplus \pi[10] \oplus \pi[4] \oplus [11] \oplus [7] \oplus [3].$$

By Proposition 6.6.4, we have:

$$\varepsilon_\psi(\gamma) = \varepsilon(\pi) \cdot \varepsilon(\pi) = \varepsilon(\mathbf{I}_w)^2 = 1.$$

On the other side, since $w \geq 11$ we have $\mu_1 = \frac{w+5}{2}$ and $\mu_4 = \frac{w-1}{2}$. Both of them come from the irreducible summand $\pi[6]$ in ψ, so $\rho_\psi^\vee$ must be the trivial character by Proposition 7.2.1. By Arthur's multiplicity formula, $\mathrm{m}(\pi_\psi) = 1$ for any $\pi \in \Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_2)$.

Case (ii): $\psi = \mathrm{Sym}^8 \pi \oplus \mathrm{Sym}^4 \pi \oplus \mathrm{Sym}^5 \pi[2]$, where $\pi \in \Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_2)$ has motivic weight w. In this case the restriction of $\mathfrak{f}_4$ along ψ is isomorphic to

$$\mathrm{Sym}^{10} \pi \oplus \mathrm{Sym}^9 \pi[2] \oplus \mathrm{Sym}^6 \pi \oplus \mathrm{Sym}^3 \pi[2] \oplus \mathrm{Sym}^2 \pi \oplus [3].$$

By Proposition 6.6.4, we have:

$$\begin{aligned}\varepsilon_\psi(\gamma) &= \varepsilon(\mathrm{Sym}^3\pi)\cdot\varepsilon(\mathrm{Sym}^9\pi)\\ &= \varepsilon(\mathbf{I}_{3w}+\mathbf{I}_w)\cdot\varepsilon(\mathbf{I}_{9w}+\mathbf{I}_{7w}+\mathbf{I}_{5w}+\mathbf{I}_{3w}+\mathbf{I}_w)\\ &= (-1)^{(w+1)/2+(3w+1)/2}\cdot(-1)^{(w+1)/2+(3w+1)/2+(5w+1)/2+(7w+1)/2+(9w+1)/2}\\ &= (-1)^{(w+3)/2}.\end{aligned}$$

On the other side, $\mu_1 = 4w$ comes from $\mathrm{Sym}^8\pi$ and $\mu_4 = \frac{5w-1}{2}$ comes from $\mathrm{Sym}^5\pi[2]$. So $\rho_\psi^\vee(\gamma) = -1$ by Proposition 7.2.1. By Arthur's multiplicity formula, $\mathrm{m}(\pi_\psi) = 1$ if and only if $w \equiv 3 \bmod 4$.

Case (iii): $\psi = \mathrm{Sym}^8\pi_1 \oplus \mathrm{Sym}^4\pi_1 \oplus \left(\mathrm{Sym}^5\pi_1\otimes\pi_2\right)$, where $\pi_1,\pi_2 \in \Pi_{\mathrm{alg}}^\perp(\mathbf{PGL}_2)$ have motivic weight w_1, w_2 respectively. Since this parameter is tempered, the character ε_ψ is always trivial. We only need to find what condition w_1, w_2 should satisfy to make $\rho_\psi^\vee(\gamma) = 1$. In this case, γ acts on $\mathrm{Sym}^8\pi_1$ and $\mathrm{Sym}^4\pi_1$ by 1 and on $\mathrm{Sym}^5\pi_1\otimes\pi_2$ by -1. We can see that $\mu_1 = 4w_1$ or $\frac{5w_1+w_2}{2}$, depending on the values of w_1, w_2.

(1) If $\mu_1 = 4w_1$, which is equivalent to $w_2 < 3w_1$. Now $\rho_\psi^\vee(\gamma) = 1$ if and only if $\mu_4 = 3w_1$ since the other positive weights $w_1, 2w_1$ in $\mathrm{Sym}^4\pi_1 \oplus \mathrm{Sym}^8\pi_1$ both have multiplicity 2. However, $3w_1$ is larger than all the Hodge weights of ψ except $4w_1$ and $\frac{5w_1+w_2}{2}$, which shows that it can only be μ_2 or μ_3. So in this case $\rho_\psi^\vee(\gamma) = -1$.

(2) If $\mu_1 = \frac{5w_1+w_2}{2}$, which is equivalent to $w_2 > 3w_1$. Now $\rho_\psi^\vee(\gamma) = 1$ if and only if $\mu_4 = \frac{w_1+w_2}{2}$ or $\frac{-w_1+w_2}{2}$.

  (a) $\mu_4 = \frac{w_1+w_2}{2}$ is equivalent to $4w_1 > \frac{w_1+w_2}{2} > 3w_1$, thus $5w_1 < w_2 < 7w_1$.

  (b) $\mu_4 = \frac{-w_1+w_2}{2}$ is equivalent to $\frac{-w_1+w_2}{2} > 4w_1$, thus $w_2 > 9w_1$.

By Arthur's multiplicity formula $\mathrm{m}(\pi_\psi) = 1$ if and only if $w_2 > 9w_1$ or $5w_1 < w_2 < 7w_1$. □

7.3.3 $H = \left(\mathrm{A}_1^{[5,4^2,3^3,2^2]} \times \mathrm{A}_1^{[3^3,2^6,1^5]}\right)/\mu_2^\Delta$

By Section 5.6.11 the restriction of the 26-dimensional irreducible representation J_0 of F_4 to H is isomorphic to

$$\mathrm{Sym}^4\mathrm{St}\otimes\mathbf{1} + \left(\mathrm{Sym}^3\mathrm{St}+\mathrm{St}\right)\otimes\mathrm{St} + \mathrm{Sym}^2\mathrm{St}\otimes\mathrm{Sym}^2\mathrm{St},$$

and the centralizer of H in F_4 is $\mathrm{Z}(H) \simeq \mathbb{Z}/2\mathbb{Z}$.

For $\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ satisfying $\mathrm{H}(\psi) = H$ and $\mathrm{m}(\pi_\psi) = 1$, there are three possible endoscopic types:

(i) $(4,(3,3),(2,4),(2,2),(1,5))$. A global Arthur parameter of this type is of the form:

$$\mathrm{Sym}^2\pi[3]\oplus\pi[4]\oplus\pi[2]\oplus[5],\ \pi\in\Pi_{\mathrm{alg}}^\perp(\mathbf{PGL}_2).$$

(ii) $(4,(5,1),(4,2),(3,3),(2,2))$. A global Arthur parameter of this type is of the form:

$$\mathrm{Sym}^4\pi \oplus \mathrm{Sym}^3\pi[2] \oplus \mathrm{Sym}^2\pi[3] \oplus \pi[2],\ \pi \in \Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_2).$$

(iii) $(4,(9,1),(8,1),(5,1),(4,1))$. A global Arthur parameter of this type is of the form:

$$\mathrm{Sym}^4\pi_1 \oplus (\mathrm{Sym}^3\pi_1 \otimes \pi_2) \oplus (\mathrm{Sym}^2\pi_1 \otimes \mathrm{Sym}^2\pi_2) \oplus (\pi_1 \otimes \pi_2),\ \pi_1,\pi_2 \in \Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_2),$$

where the representations $\mathrm{Sym}^k\pi_1 \otimes \mathrm{Sym}^l\pi_2$ are defined similarly as the representation $\mathrm{Sym}^5\pi_1 \otimes \pi_2$ appearing in $[(12,1),(9,1),(5,1)]$-type parameters introduced in Section 7.3.2.

For this subgroup H of F_4, the restriction of the adjoint representation $\mathfrak{f}_4$ of F_4 to H is isomorphic to

$$\mathrm{St}\otimes\mathrm{Sym}^3\mathrm{St} + \left(\mathrm{Sym}^4\mathrm{St}+\mathbf{1}\right)\otimes\mathrm{Sym}^2\mathrm{St} + \left(\mathrm{Sym}^5\mathrm{St}+\mathrm{Sym}^3\mathrm{St}\right)\otimes\mathrm{St} + \left(\mathrm{Sym}^2\mathrm{St}\right)^{\oplus 2}\otimes\mathbf{1}.$$

Proposition* 7.3.7. *For a discrete global Arthur parameter $\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ satisfying $\mathrm{H}(\psi) = H$, the multiplicity $\mathrm{m}(\pi_\psi) = 1$ if and only if ψ is one of the following parameters:*

- $\mathrm{Sym}^2\pi[3] \oplus \pi[4] \oplus \pi[2] \oplus [5]$, *where* $\pi \in \Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_2)$.
- $\mathrm{Sym}^4\pi \oplus \mathrm{Sym}^3\pi[2] \oplus \mathrm{Sym}^2\pi[3] \oplus \pi[2]$, *where* $\pi \in \Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_2)$.
- $\mathrm{Sym}^4\pi_1 \oplus (\mathrm{Sym}^3\pi_1 \otimes \pi_2) \oplus (\mathrm{Sym}^2\pi_1 \otimes \mathrm{Sym}^2\pi_2) \oplus (\pi_1 \otimes \pi_2)$, *where* $\pi_1,\pi_2 \in \Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_2)$ *have motivic weights w_1, w_2 respectively such that*

$$w_1 > 3w_2 \text{ or } w_1 < w_2 < 3w_1 \text{ or } 3w_1 < w_2 < 5w_1.$$

Proof. We denote the generator of $\mathrm{C}_\psi = \mathrm{Z}(H)$ by γ.

Case (i): $\psi = \mathrm{Sym}^2\pi[3] \oplus \pi[4] \oplus \pi[2] \oplus [5]$, where $\pi \in \Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_2)$ has motivic weight w. In this case the restriction of $\mathfrak{f}_4$ along ψ is isomorphic to

$$\mathrm{Sym}^3\pi[2] \oplus \mathrm{Sym}^2\pi[5] \oplus \mathrm{Sym}^2\pi \oplus \pi[6] \oplus \pi[4] \oplus [3] \oplus [3].$$

By Proposition 6.6.4, we have:

$$\varepsilon_\psi(\gamma) = \varepsilon(\mathrm{Sym}^3\pi)\cdot\varepsilon(\pi)\cdot\varepsilon(\pi) = \varepsilon(\mathbf{I}_{3w}+\mathbf{I}_w)\cdot\varepsilon(\mathbf{I}_w)^2 = (-1)^{2w+1} = -1.$$

On the other side, $\mu_1 = w+1$ comes from $\mathrm{Sym}^2\pi[3]$ and $\mu_4 = \frac{w+3}{2}$ comes from $\pi[4]$. Since γ acts on $\mathrm{Sym}^2\pi[3]$ by 1 and on $\pi[4]$ by -1, we have $\rho_\psi^\vee(\gamma) = -1$ by Proposition 7.2.1. By Arthur's multiplicity formula, $\mathrm{m}(\pi_\psi) = 1$ for any $\pi \in \Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_2)$.

Case (ii): $\psi = \mathrm{Sym}^4\pi \oplus \mathrm{Sym}^3\pi[2] \oplus \mathrm{Sym}^2\pi[3] \oplus \pi[2]$, where $\pi \in \Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_2)$ has motivic weight w. In this case the restriction of $\mathfrak{f}_4$ along ψ is isomorphic to

$$\mathrm{Sym}^5\pi[2] \oplus \mathrm{Sym}^4\pi[3] \oplus \mathrm{Sym}^3\pi[2] \oplus (\mathrm{Sym}^2\pi)^{\oplus 2} \oplus \pi[4] \oplus [3].$$

By Proposition 6.6.4, we have:

$$\varepsilon_\psi(\gamma) = \varepsilon(\pi)\cdot\varepsilon(\mathrm{Sym}^3\,\pi)\cdot\varepsilon(\mathrm{Sym}^5\,\pi) = \varepsilon(\mathbf{I}_w)\varepsilon(\mathbf{I}_{3w}+\mathbf{I}_w)\varepsilon(\mathbf{I}_{5w}+\mathbf{I}_{3w}+\mathbf{I}_w) = (-1)^{3w+1} = 1.$$

On the other side, $\mu_1 = 2w$ comes from $\mathrm{Sym}^4\,\pi$ and $\mu_4 = w+1$ comes from $\mathrm{Sym}^2\,\pi[3]$. Since γ acts on $\mathrm{Sym}^4\,\pi$ and $\mathrm{Sym}^2\,\pi[3]$ both by 1, we have $\rho_\psi^\vee(\gamma) = 1$ by Proposition 7.2.1. Arthur's multiplicity formula shows that $\mathrm{m}(\pi_\psi) = 1$ for any $\pi \in \Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_2)$.

Case (iii): $\psi = \mathrm{Sym}^4\,\pi_1 \oplus (\mathrm{Sym}^3\,\pi_1\otimes\pi_2)\oplus(\mathrm{Sym}^2\,\pi_1\otimes\mathrm{Sym}^2\,\pi_2)\oplus(\pi_1\otimes\pi_2)$, where $\pi_1,\pi_2\in\Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_2)$ have motivic weights w_1, w_2 respectively. The motivic weights satisfy $w_2\neq w_1, w_2\neq 3w_1$, otherwise the zero weight appears more than twice and ψ fails to be in $\Psi_{\mathrm{AJ}}(\mathbf{F}_4)$. In this case ε_ψ is trivial. The element γ acts on $\mathrm{Sym}^4\,\pi_1$ and $\mathrm{Sym}^2\,\pi_1\otimes\mathrm{Sym}^2\,\pi_2$ by 1, and on $\mathrm{Sym}^3\,\pi_1\otimes\pi_2, \pi_1\otimes\pi_2$ by -1. The largest weight μ_1 is $2w_1$ or w_1+w_2.

(1) If $w_1 > w_2$, then $\mu_1 = 2w_1$. Now μ_4 equals to $\frac{3w_1-w_2}{2}$ or w_1+w_2. The character $\rho_\psi^\vee$ is trivial if and only if $\mu_4 = w_1+w_2$, which is equivalent to $w_1 > 3w_2$.

(2) If $w_1 < w_2$, then $\mu_1 = w_1+w_2$.

  (a) If $w_2 > 3w_1$, then

$$w_1+w_2 > w_2 > \max(-w_1+w_2, \frac{3w_1+w_2}{2}) > \min(-w_1+w_2, \frac{3w_1+w_2}{2})$$

  and they are larger than other weights, thus $\mu_4 = -w_1+w_2$ or $\frac{3w_1+w_2}{2}$. So $\rho_\psi^\vee(\gamma) = 1$ if and only if $\mu_4 = -w_1+w_2$, thus if and only if $\frac{3w_1+w_2}{2} > w_2-w_1$, which is equivalent to that $3w_1 < w_2 < 5w_1$.

  (b) If $w_2 < 3w_1$, then

$$w_1+w_2 > \frac{3w_1+w_2}{2} > \max(2w_1, w_2) > \min(2w_1, w_2)$$

  and they are larger than other weights. So we always have $\rho_\psi^\vee(\gamma) = 1$.

By Arthur's multiplicity formula, $\mathrm{m}(\pi_\psi) = 1$ if and only if $w_1 > 3w_2$ or $w_1 < w_2 < 5w_1$ and $w_2 \neq 3w_1$. □

7.3.4 $H = \left(\mathrm{A}_1^{[4^2,3^3,2^4,1]}\times\mathrm{A}_1^{[4^2,3^3,2^4,1]}\right)/\mu_2^{\Delta}$

By Section 5.6.12, the restriction of the 26-dimensional irreducible representation J_0 of F_4 to H is isomorphic to

$$\mathbf{1} + \mathrm{Sym}^3\,\mathrm{St}\otimes\mathrm{St} + \mathrm{Sym}^2\,\mathrm{St}\otimes\mathrm{Sym}^2\,\mathrm{St} + \mathrm{St}\otimes\mathrm{Sym}^3\,\mathrm{St},$$

and the centralizer of H in F_4 is $\mathrm{Z}(H)\simeq\mathbb{Z}/2\mathbb{Z}$.

For $\psi\in\Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ satisfying $\mathrm{H}(\psi) = H$ and $\mathrm{m}(\pi_\psi) = 1$, there are two possible endoscopic types:

(i) $(4,(4,2),(3,3),(2,4),(1,1))$. A global Arthur parameter of this type is of the form:

$$\mathrm{Sym}^3\pi[2]\oplus\mathrm{Sym}^2\pi[3]\oplus\pi[4]\oplus[1],\ \pi\in\Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_2).$$

(ii) $(4,(9,1),(8,1),(8,1),(1,1))$. A global Arthur parameter of this type is of the form:

$$(\mathrm{Sym}^3\pi_1\otimes\pi_2)\oplus(\mathrm{Sym}^2\pi_1\otimes\mathrm{Sym}^2\pi_2)\oplus(\pi_1\otimes\mathrm{Sym}^3\pi_2)\oplus[1],\ \pi_1,\pi_2\in\Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_2).$$

For this subgroup H of F_4, the restriction of the adjoint representation $\mathfrak{f}_4$ of F_4 to H is isomorphic to

$$\left(\mathrm{Sym}^4\mathrm{St}+\mathbf{1}\right)\otimes\mathrm{Sym}^2\mathrm{St}+\mathrm{Sym}^2\mathrm{St}\otimes\left(\mathrm{Sym}^4\mathrm{St}+\mathbf{1}\right)+\mathrm{Sym}^3\mathrm{St}\otimes\mathrm{St}+\mathrm{St}\otimes\mathrm{Sym}^3\mathrm{St}.$$

Proposition* 7.3.8. *A discrete global Arthur parameter $\psi\in\Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ satisfying $\mathrm{H}(\psi)=H$ and $\mathrm{m}(\pi_\psi)=1$ must be of one of the following parameters:*

- $\mathrm{Sym}^3\pi[2]\oplus\mathrm{Sym}^2\pi[3]\oplus\pi[4]\oplus[1]$, *where $\pi\in\Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL_2})$ satisfies $\mathrm{w}(\pi)\equiv 3\bmod 4$.*
- $(\mathrm{Sym}^3\pi_1\otimes\pi_2)\oplus(\mathrm{Sym}^2\pi_1\otimes\mathrm{Sym}^2\pi_2)\oplus(\pi_1\otimes\mathrm{Sym}^3\pi_2)\oplus[1]$, *where π_1,π_2 have motivic weights w_1,w_2 respectively such that $w_2<w_1<3w_2$.*

Proof. We denote the generator of $\mathrm{C}_\psi=\mathrm{Z}(H)$ by σ.

Case (i): $\psi=\mathrm{Sym}^3\pi[2]\oplus\mathrm{Sym}^2\pi[3]\oplus\pi[4]\oplus[1]$, where $\pi\in\Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_2)$ has motivic weight w. In this case the restriction of $\mathfrak{f}_4$ along ψ is isomorphic to

$$\mathrm{Sym}^4\pi[3]\oplus\mathrm{Sym}^3\pi[2]\oplus\mathrm{Sym}^2\pi[5]\oplus\mathrm{Sym}^2\pi\oplus\pi[4]\oplus[3].$$

By Proposition 6.6.4, we have:

$$\varepsilon_\psi(\sigma)=\varepsilon(\mathrm{Sym}^3\pi)\cdot\varepsilon(\pi)=\varepsilon(\mathbf{I}_{3w}+\mathbf{I}_w)\cdot\varepsilon(\mathbf{I}_w)=(-1)^{(3w+1)/2}.$$

On the other side, $\mu_1=\frac{3w+1}{2}$ comes from $\mathrm{Sym}^3\pi[2]$ and $\mu_4=w$ comes from $\mathrm{Sym}^2\pi[3]$. Since σ acts on $\mathrm{Sym}^3\pi[2]$ by -1 and on $\mathrm{Sym}^2\pi[3]$ by 1, we have $\rho_\psi^\vee(\sigma)=-1$ by Proposition 7.2.1. By Arthur's multiplicity formula, $\mathrm{m}(\pi_\psi)=1$ if and only if $w\equiv 3\bmod 4$.

Case (ii): $\psi=(\mathrm{Sym}^3\pi_1\otimes\pi_2)\oplus(\mathrm{Sym}^2\pi_1\otimes\mathrm{Sym}^2\pi_2)\oplus(\pi_1\otimes\mathrm{Sym}^3\pi_2)\oplus[1]$, where $\pi_1,\pi_2\in\Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL_2})$ have motivic weights $w_1>w_2$ respectively. In this case, ε_ψ is trivial. On the other side, $\mu_1=\frac{3w_1+w_2}{2}$ and $\mu_4=w_1$ or $\frac{w_1+3w_2}{2}$ or $\frac{3w_1-w_2}{2}$. By Proposition 7.2.1, $\rho_\psi^\vee$ is trivial if and only if $\mu_4=\frac{w_1+3w_2}{2}$ or $\frac{3w_1-w_2}{2}$.

(1) $\mu_4=\frac{w_1+3w_2}{2}$ if and only if $\frac{3w_1-w_2}{2}>\frac{w_1+3w_2}{2}>w_1$, which is equivalent to $2w_2<w_1<3w_2$.
(2) $\mu_4=\frac{3w_1-w_2}{2}$ if and only if $\frac{w_1+3w_2}{2}>\frac{3w_1-w_2}{2}$, which is equivalent to $w_1<2w_2$.

By Arthur's multiplicity formula, $\mathrm{m}(\pi_\psi)=1$ if and only if $w_2<w_1<3w_2$ and $w_1\neq 2w_2$. Notice that $w_1\neq 2w_2$ holds automatically since w_1 is odd. □

7.3.5 $H = \mathrm{A}_1^{[7^3,1^5]} \times \mathrm{A}_1^{[5,3^7]}$

By Section 5.6.7, the restriction of the 26-dimensional irreducible representation J_0 of F_4 to H is isomorphic to

$$\mathrm{Sym}^6 \mathrm{St} \otimes \mathrm{Sym}^2 \mathrm{St} + \mathbf{1} \otimes \mathrm{Sym}^4 \mathrm{St},$$

and the centralizer of H in F_4 is trivial.

For $\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ satisfying $\mathrm{H}(\psi) = H$ and $\mathrm{m}(\pi_\psi) = 1$, there are three possible endoscopic types:

(i) $(2,(7,3),(1,5))$. A global Arthur parameter of this type is of the form:

$$\mathrm{Sym}^6 \pi[3] \oplus [5],\ \pi \in \Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_2).$$

(ii) $(2,(5,1),(3,7))$. A global Arthur parameter of this type is of the form:

$$\mathrm{Sym}^4 \pi \oplus \mathrm{Sym}^2 \pi[7],\ \pi \in \Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_2).$$

(iii) $(2,(21,1),(5,1))$. A global Arthur parameter of this type is of the form:

$$\left(\mathrm{Sym}^6 \pi_1 \otimes \mathrm{Sym}^2 \pi_2\right) \oplus \mathrm{Sym}^4 \pi_2,\ \pi_1, \pi_2 \in \Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_2).$$

Proposition* 7.3.9. *A discrete global Arthur parameter $\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ satisfying $\mathrm{H}(\psi) = H$ and $\mathrm{m}(\pi_\psi) = 1$ must be of one of the following parameters:*

- $\mathrm{Sym}^6 \pi[3] \oplus [5]$, *where* $\pi \in \Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_2)$.
- $\mathrm{Sym}^4 \pi \oplus \mathrm{Sym}^2 \pi[7]$, *where* $\pi \in \Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_2)$.
- $\left(\mathrm{Sym}^6 \pi_1 \otimes \mathrm{Sym}^2 \pi_2\right) \oplus \mathrm{Sym}^4 \pi_2$, *where* $\pi_1, \pi_2 \in \Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_2)$ *have motivic weights* w_1, w_2 *respectively such that* $w_2 \neq w_1$ *and* $w_2 \neq 3w_1$.

Proof. This follows from the fact that C_ψ is trivial. The conditions $w_2 \neq w_1$ and $w_2 \neq 3w_1$ in the third case are equivalent to that $\psi = \left(\mathrm{Sym}^6 \pi_1 \otimes \mathrm{Sym}^2 \pi_2\right) \oplus \mathrm{Sym}^4 \pi_2 \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4)$. ☐

7.3.6 $H = \mathrm{A}_1^{[5,3^7]} \times \left(\mathrm{A}_1^{[3^3,2^6,1^5]} \times \mathrm{A}_1^{[2^6,1^{14}]}\right)/\mu_2^{\Delta}$

By Section 5.6.8, the restriction of the 26-dimensional irreducible representation J_0 of F_4 to H is isomorphic to

$$\mathrm{Sym}^4 \mathrm{St} \otimes \mathbf{1} \otimes \mathbf{1} + \mathrm{Sym}^2 \mathrm{St} \otimes \left(\mathrm{St} \otimes \mathrm{St} + \mathrm{Sym}^2 \mathrm{St} \otimes \mathbf{1}\right),$$

and the centralizer of H in F_4 is $\mathrm{Z}(H) \simeq \mathbb{Z}/2\mathbb{Z}$.

For $\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ satisfying $\mathrm{H}(\psi) = H$ and $\mathrm{m}(\pi_\psi) = 1$, there are four possible endoscopic types:

(i) $(3,(6,2),(5,1),(3,3))$. A global Arthur parameter of this type is of the form:

$$\mathrm{Sym}^4 \pi_1 \oplus (\mathrm{Sym}^2 \pi_1 \otimes \pi_2[2]) \oplus \mathrm{Sym}^2 \pi_1[3],\ \pi_1, \pi_2 \in \Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_2).$$

(ii) $(3,(9,1),(6,2),(5,1))$. A global Arthur parameter of this type is of the form:

$$\mathrm{Sym}^4 \pi_1 \oplus (\mathrm{Sym}^2 \pi_1 \otimes \pi_2[2]) \oplus (\mathrm{Sym}^2 \pi_1 \otimes \mathrm{Sym}^2 \pi_2),\ \pi_1, \pi_2 \in \Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_2).$$

(iii) $(3,(4,3),(3,3),(1,5))$. A global Arthur parameter of this type is of the form:

$$\mathrm{Sym}^2 \pi_1[3] \oplus (\pi_1 \otimes \pi_2[3]) \oplus [5],\ \pi_1, \pi_2 \in \Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_2).$$

(iv) $(3,(12,1),(9,1),(5,1))$. A global Arthur parameter of this type is of the form:

$$\mathrm{Sym}^4 \pi_1 \oplus (\mathrm{Sym}^2 \pi_1 \otimes \pi_2 \otimes \pi_3) \oplus (\mathrm{Sym}^2 \pi_1 \otimes \mathrm{Sym}^2 \pi_3),\ \pi_1, \pi_2, \pi_3 \in \Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_2).$$

For this subgroup H of F_4, the restriction of the adjoint representation $\mathfrak{f}_4$ of F_4 to H is isomorphic to

$$\begin{aligned}\mathrm{Sym}^4 \mathrm{St} \otimes \left(\mathrm{St} \otimes \mathrm{St} + \mathrm{Sym}^2 \mathrm{St} \otimes \mathbf{1}\right) + \mathrm{Sym}^2 \mathrm{St} \otimes \mathbf{1} \otimes \mathbf{1} \\ + \mathbf{1} \otimes \left(\mathrm{Sym}^2 \mathrm{St} \otimes \mathbf{1} + \mathbf{1} \otimes \mathrm{Sym}^2 \mathrm{St} + \mathrm{Sym}^3 \mathrm{St} \otimes \mathrm{St}\right).\end{aligned}$$

Proposition* 7.3.10. *For a discrete global Arthur parameter $\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ satisfying $\mathrm{H}(\psi) = H$, the multiplicity $\mathrm{m}(\pi_\psi) = 1$ if and only if ψ is one of the following parameters:*

- $\mathrm{Sym}^4 \pi_1 \oplus (\mathrm{Sym}^2 \pi_1 \otimes \pi_2[2]) \oplus \mathrm{Sym}^2 \pi_1[3]$, *where $\pi_1, \pi_2 \in \Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_2)$ have motivic weights w_1, w_2 respectively such that $w_2 < 2w_1 - 1$ or $w_2 > 4w_1 + 1$.*
- $\mathrm{Sym}^4 \pi_1 \oplus (\mathrm{Sym}^2 \pi_1 \otimes \pi_2[2]) \oplus (\mathrm{Sym}^2 \pi_1 \otimes \mathrm{Sym}^2 \pi_2)$, *where $\pi_1, \pi_2 \in \Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_2)$ have motivic weights w_1, w_2 respectively and satisfy one of the following conditions:*
  - $2w_1 + 1 < w_2 < 4w_1 - 1,\ w_2 \equiv 1 \bmod 4$;
  - $w_2 < 2w_1 - 1$ *or* $w_2 > 4w_1 + 1$, *and* $w_2 \equiv 3 \bmod 4,\ w_1 \neq w_2$.
- $\mathrm{Sym}^2 \pi_1[3] \oplus (\pi_1 \otimes \pi_2[3]) \oplus [5]$, *where $\pi_1, \pi_2 \in \Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_2)$ have motivic weights w_1, w_2 respectively such that $w_2 > 3w_1$.*
- $\mathrm{Sym}^4 \pi_1 \oplus (\mathrm{Sym}^2 \pi_1 \otimes \pi_2 \otimes \pi_3) \oplus (\mathrm{Sym}^2 \pi_1 \otimes \mathrm{Sym}^2 \pi_3)$, *where $\pi_1, \pi_2, \pi_3 \in \Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_2)$ have motivic weights w_1, w_2, w_3 respectively such that one of the following conditions holds:*
  - $w_2 > \max(3w_3, 4w_1 + w_3)$;
  - $2w_1 + w_3 < w_2 < 4w_1 - w_3$;
  - $3w_3 < w_2 < 2w_1 - w_3$;
  - $2w_1 + w_3 < w_2 < \min(4w_1 + w_3, 3w_3)$;
  - $|4w_1 - w_3| < w_2 < w_3 - 2w_1$;
  - $|2w_1 - w_3| < w_2 < \min(4w_1 - w_3, 3w_3)$ *and* $w_3 \neq w_1,\ w_3 \neq w_2$.

Proof. We denote the generator of C_ψ by $\gamma = (1, -1, 1) \in \mathrm{Z}(H)$.

Case (i): $\psi = \mathrm{Sym}^4 \pi_1 \oplus \mathrm{Sym}^2 \pi_1 \otimes \pi_2[2] \oplus \mathrm{Sym}^2 \pi_1[3]$, where $\pi_1, \pi_2 \in \Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_2)$ have motivic weights w_1, w_2 respectively. In this case the restriction of $\mathfrak{f}_4$ along ψ is isomorphic to

$$\left(\mathrm{Sym}^4 \pi_1 \otimes \pi_2[2]\right) \oplus \mathrm{Sym}^4 \pi_1[3] \oplus \mathrm{Sym}^2 \pi_1 \oplus \mathrm{Sym}^2 \pi_2 \oplus \pi_2[4] \oplus [3].$$

By Proposition 6.6.4 we have $\varepsilon_\psi(\gamma)=\varepsilon(\mathrm{Sym}^4\pi_1\otimes\pi_2)\cdot\varepsilon(\pi_2)$. Notice that

$$\varepsilon(\mathbf{I}_w\otimes\mathbf{I}_{w'})=\varepsilon(\mathbf{I}_{w+w'}+\mathbf{I}_{|w-w'|})=i^{w+w'+|w-w'|+2}=(-1)^{\max(w,w')+1},$$

thus

$$\varepsilon_\psi(\gamma)=\varepsilon\left((\mathbf{I}_{4w_1}+\mathbf{I}_{3w_1}+\mathbf{I}_{2w_1}+\mathbf{I}_{w_1})\otimes\mathbf{I}_{w_2}\right)=(-1)^{\max(4w_1,w_2)+\max(2w_1,w_2)}.$$

Hence $\varepsilon_\psi(\gamma)=1$ if and only if $w_2<2w_1$ or $w_2>4w_1$. On the other side, $\mu_1=2w_1$ or $w_1+\frac{w_2+1}{2}$. The generator γ of C_ψ acts on $\mathrm{Sym}^4\pi_1$ and $\mathrm{Sym}^2\pi_1[3]$ by 1 and on $\mathrm{Sym}^2\pi_1\otimes\pi_2[2]$ by -1. We also notice that $\psi\in\Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ implies that $w_2\notin\{2w_1\pm1,4w_1\pm1\}$.

(1) If $w_2<2w_1-1$, then $\mu_1=2w_1$. Now we have $2w_1>w_1+\frac{w_2+1}{2}>w_1+\frac{w_2-1}{2}>w_1+1$ and they are larger than other Hodge weights, thus $\mu_4=w_1+1$. Hence $\rho_\psi^\vee(\gamma)=1$.

(2) If $w_2>2w_1+1$, then $\mu_1=w_1+\frac{w_2+1}{2}$. Now

$$w_1+\frac{w_2+1}{2}>w_1+\frac{w_2-1}{2}>\max(2w_1,\frac{w_2+1}{2})>\min(2w_1,\frac{w_2-1}{2})\geq w_1+1$$

and they are larger than other weights. So $\mu_4=2w_1$ or $\frac{w_2+1}{2},\frac{w_2-1}{2}$. However, if $\mu_4=2w_1$, then we must have $\frac{w_2-1}{2}<2w_1<\frac{w_2+1}{2}$, which is absurd because there is no integer between $\frac{w_2-1}{2}$ and $\frac{w_2+1}{2}$. Hence $\mu_4=\frac{w_2\pm1}{2}$ and $\rho_\psi^\vee(\gamma)=1$.

In conclusion, $\rho_\psi^\vee(\gamma)=1$ for any π_1,π_2. By Arthur's multiplicity formula, $\mathrm{m}(\pi_\psi)=1$ if and only if $w_2<2w_1-1$ or $w_2>4w_1+1$.

Case (ii): $\psi=\mathrm{Sym}^4\pi_1\oplus(\mathrm{Sym}^2\pi_1\otimes\pi_2[2])\oplus(\mathrm{Sym}^2\pi_1\otimes\mathrm{Sym}^2\pi_2)$, where $\pi_1,\pi_2\in\Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_2)$ have motivic weights w_1,w_2 respectively. In this case the restriction of $\mathfrak{f}_4$ along ψ is isomorphic to

$$\left(\mathrm{Sym}^4\pi_1\otimes\mathrm{Sym}^2\pi_2\right)\oplus\left(\mathrm{Sym}^4\pi_1\otimes\pi_2[2]\right)\oplus\mathrm{Sym}^3\pi_2[2]\oplus\mathrm{Sym}^2\pi_1\oplus\mathrm{Sym}^2\pi_2\oplus[3].$$

By Proposition 6.6.4 we have:

$$\varepsilon_\psi(\gamma)=\varepsilon(\mathrm{Sym}^4\pi_1\otimes\pi_2)\cdot\varepsilon(\mathrm{Sym}^3\pi_2)=(-1)^{\max(4w_1,w_2)+\max(2w_1,w_2)+(w_2-1)/2}.$$

On the other side, γ acts on $\mathrm{Sym}^4\pi_1,\mathrm{Sym}^2\pi_1\otimes\mathrm{Sym}^2\pi_2$ by 1 and on $\mathrm{Sym}^2\pi_1\otimes\pi_2[2]$ by -1.

(1) If $w_1>w_2$,then $\mu_1=2w_1$. Now μ_4 must be $w_1+\frac{w_2-1}{2}$ and we have $\rho_\psi^\vee(\gamma)=-1$.

(2) If $w_1<w_2$, then $\mu_1=w_1+w_2$. Now $\rho_\psi^\vee(\gamma)=1$ if and only if μ_4 comes from $\mathrm{Sym}^4\pi_1$ or $\mathrm{Sym}^2\pi_1\otimes\mathrm{Sym}^2\pi_2$. We can easily verify that none of the weights of these two irreducible summands is possible to be μ_4.

In conclusion, $\rho_\psi^\vee(\gamma)=-1$. By Arthur's multiplicity formula, for $\psi\in\Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ the multiplicity $\mathrm{m}(\pi_\psi)=1$ if and only if one of the following conditions holds:

- $2w_1+1<w_2<4w_1-1$, $w_2\equiv1\bmod4$;
- $w_2<2w_1-1$ or $w_2>4w_1+1$, and $w_2\equiv3\bmod4$, $w_1\neq w_2$.

Case (iii): $\psi = \mathrm{Sym}^2\pi_1[3] \oplus (\pi_1 \otimes \pi_2[3]) \oplus [5]$, where $\pi_1, \pi_2 \in \Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_2)$ have motivic weights w_1, w_2 respectively. In this case, the representations of $\mathbf{SL}_2(\mathbb{C})$ in the restriction of $\mathfrak{f}_4$ along ψ are all odd dimensional, thus $\varepsilon_\psi(\gamma) = 1$ by Proposition 6.6.4. On the other side, γ acts on $\mathrm{Sym}^2\pi_1[3]$ by 1 and on $\pi_1 \otimes \pi_2[3]$ by -1. We have $\mu_1 = w_1 + 1$ or $\frac{w_1+w_2}{2} + 1$.

(1) If $w_1 > w_2$, then $\mu_1 = w_1 + 1$. The condition that $\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ implies that $w_1 > w_2 + 4$, thus $w_1 + 1 > w_1 > w_1 - 1 > \frac{w_1+w_2}{2} + 1$, which are larger than other weights. So $\mu_4 = \frac{w_1+w_2}{2} + 1$ and $\rho_\psi^\vee(\gamma) = -1$.
(2) If $w_1 < w_2$, then $\mu_1 = \frac{w_1+w_2}{2} + 1$. Similarly, we have $w_1 < w_2 - 4$. Now μ_4 must be $w_1 + 1$ or $\frac{w_2-w_1}{2} + 1$, so $\rho_\psi^\vee(\gamma) = 1$ if and only if $\mu_4 = \frac{w_2-w_1}{2} + 1$. This is equivalent to $w_2 > 3w_1$.

By Arthur's multiplicity formula, $\mathrm{m}(\pi_\psi) = 1$ if and only if $w_2 > 3w_1$.

Case (iv): $\psi = \mathrm{Sym}^4\pi_1 \oplus (\mathrm{Sym}^2\pi_1 \otimes \pi_2 \otimes \pi_3) \oplus (\mathrm{Sym}^2\pi_1 \otimes \mathrm{Sym}^2\pi_3)$, where $\pi_1, \pi_2, \pi_3 \in \Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_2)$ have motivic weights w_1, w_2, w_3 respectively. In this case, $\varepsilon_\psi(\gamma) = 1$ since the parameter is tempered. On the other side, γ acts on $\mathrm{Sym}^4\pi_1$ and $\mathrm{Sym}^2\pi_1 \otimes \mathrm{Sym}^2\pi_3$ by 1 and on $\mathrm{Sym}^2\pi_1 \otimes \pi_2 \otimes \pi_3$ by -1. We denote the ratios $w_1/w_3, w_2/w_3$ by r_1, r_2 respectively, and denote the multiset of elements μ/w_3, μ running over the eigenvalues of $\mathrm{c}_\infty(\psi)$, by $\widetilde{\mathcal{W}}$. We still order the elements of $\widetilde{\mathcal{W}}$ by $\mu_1 > \mu_2 > \cdots > \mu_{26}$. The largest one μ_1 must be $r_1 + 1$ or $2r_1$ or $r_1 + \frac{r_2+1}{2}$.

(1) If $r_1 < 1, r_2 < 1$, then $\mu_1 = r_1 + 1$. Now $\mu_2 = 2r_1$ or 1 or $r_1 + \frac{r_2+1}{2}$.
   (a) If $r_1 > 1/2$ and $r_2 < 2r_1 - 1$, then $\mu_2 = 2r_1$. Now $r_1 + 1 > 2r_1 > r_1 + \frac{r_2+1}{2} > r_1 + \frac{1-r_2}{2}$, which are larger than other 22 elements, thus $\mu_4 = r_1 + \frac{1-r_2}{2}$ and $\rho_\psi^\vee(\gamma) = -1$.
   (b) If $r_1 < 1/2$ and $r_2 < 1 - 2r_1$, then $\mu_2 = 1$. Now $\rho_\psi^\vee(\gamma) = 1$ if and only if $\mu_4 = 1 - r_1$, which is equivalent to $|4r_1 - 1| < r_2$.
   (c) If $r_2 > |2r_1 - 1|$, then $\mu_2 = r_1 + \frac{r_2+1}{2}$. Now $\rho_\psi^\vee(\gamma) = 1$ if and only if $\mu_4 = 2r_1$ or 1, which is equivalent to $r_2 < 4r_1 - 1$.
(2) If $r_1 > 1, r_2 < 2r_1 - 1$, then $\mu_1 = 2r_1$. Now $\rho_\psi^\vee(\gamma) = 1$ if and only if $\mu_4 = r_1 + 1$, which is equivalent to $r_2 > 3$.
(3) If $r_2 > 1, r_2 > 2r_1 - 1$, then $\mu_1 = r_1 + \frac{r_2+1}{2}$. Now μ_2 belongs to the (multi)set $\{r_1 + 1, 2r_1, r_1 + \frac{r_2-1}{2}, \frac{r_2+1}{2}\}$.
   (a) If $r_1 < 1$ and $r_2 < 2r_1 + 1$, then $\mu_2 = r_1 + 1$. Now $\rho_\psi^\vee(\gamma) = 1$ if and only if $\mu_4 = \frac{r_2+1}{2}$, which is equivalent to $r_2 < 4r_1 - 1$.
   (b) If $r_1 > 1$ and $r_2 < 2r_1 + 1$, then $\mu_2 = 2r_1$. Now $\mu_4 = \min(r_1 + 1, r_1 + \frac{r_2-1}{2})$, thus $\rho_\psi^\vee(\gamma) = 1$ if and only if $r_2 < 3$.
   (c) If $r_1 > 1$ and $r_2 > 2r_1 + 1$, then $\mu_2 = r_1 + \frac{r_2-1}{2}$. Now $\rho_\psi^\vee(\gamma) = 1$ if and only if $\mu_4 = \frac{r_2 \pm 1}{2}$, which is equivalent to $r_2 < 4r_1 - 1$ or $r_2 > 4r_1 + 1$.
   (d) If $r_1 < 1$ and $r_2 > 2r_1 + 1$, then $\mu_2 = \frac{r_2+1}{2}$. Now $\rho_\psi^\vee(\gamma) = 1$ if and only if $\mu_4 = r_1 + \frac{r_2-1}{2}$ or $\frac{r_2+1}{2} - r_1$, which is equivalent to that $r_2 < \min(3, 4r_1 + 1)$ or $r_2 > \max(3, 4r_1 + 1)$.

In conclusion, by Arthur's multiplicity formula, $\mathrm{m}(\pi_\psi) = 1$ if and only if w_1, w_2, w_3 satisfy one of the conditions listed in the proposition. $\square$

7.3.7 $H = \left(\mathrm{A}_1^{[5,4^4,1^5]} \times \mathrm{A}_1^{[2^6,1^{14}]} \times \mathrm{A}_1^{[2^6,1^{14}]}\right)/\mu_2^{\Delta}$

By Section 5.6.9, the restriction of the 26-dimensional irreducible representation J_0 of F_4 to H is isomorphic to

$$\mathbf{1} + \mathbf{1} \otimes \mathrm{St} \otimes \mathrm{St} + \mathrm{Sym}^3 \mathrm{St} \otimes (\mathrm{St} \otimes \mathbf{1} + \mathbf{1} \otimes \mathrm{St}) + \mathrm{Sym}^4 \mathrm{St} \otimes \mathbf{1} \otimes \mathbf{1},$$

and the centralizer of H in F_4 is $\mathrm{Z}(H) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

For $\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ satisfying $\mathrm{H}(\psi) = H$ and $\mathrm{m}(\pi_\psi) = 1$, there are three possible endoscopic types:

(i) $(5,(8,1),(5,1),(4,2),(2,2),(1,1))$. A global Arthur parameter of this type is of the form:

$$\mathrm{Sym}^4 \pi_1 \oplus (\mathrm{Sym}^3 \pi_1 \otimes \pi_2) \oplus \mathrm{Sym}^3 \pi_1[2] \oplus \pi_2[2] \oplus [1],\ \pi_1, \pi_2 \in \Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_2).$$

(ii) $(5,(4,1),(2,4),(2,4),(1,5),(1,1))$. A global Arthur parameter of this type is of the form:

$$(\pi_1 \otimes \pi_2) \oplus \pi_1[4] \oplus \pi_2[4] \oplus [5] \oplus [1],\ \pi_1, \pi_2 \in \Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_2).$$

(iii) $(5,(8,1),(8,1),(5,1),(4,1),(1,1))$. A global Arthur parameter of this type is of the form:

$$\mathrm{Sym}^4 \pi_1 \oplus (\mathrm{Sym}^3 \pi_1 \otimes \pi_2) \oplus (\mathrm{Sym}^3 \pi_1 \otimes \pi_3) \oplus (\pi_2 \otimes \pi_3) \oplus [1],\ \pi_1, \pi_2, \pi_3 \in \Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_2).$$

For this subgroup H of F_4, the restriction of the adjoint representation $\mathfrak{f}_4$ of F_4 to H is isomorphic to

$$\mathbf{1} \otimes \left(\mathrm{Sym}^2 \mathrm{St} \otimes \mathbf{1} + \mathbf{1} \otimes \mathrm{Sym}^2 \mathrm{St}\right) + \mathrm{Sym}^2 \mathrm{St} \otimes \mathbf{1} \otimes \mathbf{1} + \mathrm{Sym}^3 \mathrm{St} \otimes (\mathrm{St} \otimes \mathbf{1} + \mathbf{1} \otimes \mathrm{St})$$
$$+ \mathrm{Sym}^4 \mathrm{St} \otimes \mathrm{St} \otimes \mathrm{St} + \mathrm{Sym}^6 \mathrm{St} \otimes \mathbf{1} \otimes \mathbf{1}$$

Proposition* 7.3.11. *For a discrete global Arthur parameter $\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ satisfying $\mathrm{H}(\psi) = H$, the multiplicity $\mathrm{m}(\pi_\psi) = 1$ if and only if ψ is one of the following parameters:*

- $\mathrm{Sym}^4 \pi_1 \oplus (\mathrm{Sym}^3 \pi_1 \otimes \pi_2) \oplus \mathrm{Sym}^3 \pi_1[2] \oplus \pi_2[2] \oplus [1]$, *where $\pi_1, \pi_2 \in \Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_2)$ have motivic weights w_1, w_2 respectively and satisfy one of the following conditions*
  - $w_2 < w_1$ *or* $w_2 > 4w_1 + 1$, *and* $w_2 \equiv 3 \bmod 4$;
  - $3w_1 < w_2 < 4w_1 - 1$ *and* $w_2 \equiv 1 \bmod 4$.
- $(\pi_1 \otimes \pi_2) \oplus \pi_1[4] \oplus \pi_2[4] \oplus [5] \oplus [1]$, *where $\pi_1, \pi_2 \in \Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_2)$ have motivic weights $w_1 > w_2$ respectively and $w_1 \equiv 3 \bmod 4,\ w_2 \equiv 1 \bmod 4,\ w_2 < w_1 - 4$.*
- $\mathrm{Sym}^4 \pi_1 \oplus (\mathrm{Sym}^3 \pi_1 \otimes \pi_2) \oplus (\mathrm{Sym}^3 \pi_1 \otimes \pi_3) \oplus (\pi_2 \otimes \pi_3) \oplus [1]$, *where $\pi_1, \pi_2, \pi_3 \in \Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_2)$ have motivic weights w_1 and $w_2 > w_3$ respectively satisfying one of the following conditions:*
  - $w_1 > w_3$ *and* $2w_1 - w_3 < w_2 < 2w_1 + w_3$;
  - $w_3 < 3w_1 < w_2 < 2w_1 + w_3$;
  - $w_1 < w_3 < 3w_1,\ w_2 > 4w_1 + w_3$.

Proof. We take a set of generators $\{\sigma = (-1,1,1), \sigma_1 = (1,1,-1)\}$ of $\mathrm{C}_\psi = \mathrm{Z}(H) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Let χ_1, χ_2 be two generators of the character group of C_ψ such that $\chi_1(\sigma) = \chi_2(\sigma_1) = -1, \chi_1(\sigma_1) = \chi_2(\sigma) = 1$.

Case (i): $\psi = \mathrm{Sym}^4\pi_1 \oplus (\mathrm{Sym}^3\pi_1 \otimes \pi_2) \oplus \mathrm{Sym}^3\pi_1[2] \oplus \pi_2[2] \oplus [1]$, where $\pi_1, \pi_2 \in \Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_2)$ have motivic weights w_1, w_2 respectively. In this case, the restriction of $\mathfrak{f}_4$ along ψ is isomorphic to:

$$\mathrm{Sym}^6\pi_1 \oplus \Big(\mathrm{Sym}^4\pi_1 \otimes \pi_2[2]\Big) \oplus \Big(\mathrm{Sym}^3\pi_1 \otimes \pi_2\Big) \oplus \mathrm{Sym}^3\pi_1[2] \oplus \mathrm{Sym}^2\pi_1 \oplus \mathrm{Sym}^2\pi_2 \oplus [3].$$

By Proposition 6.6.4 we have:

$$\varepsilon_\psi(\sigma) = \varepsilon(\mathrm{Sym}^3\pi_1) = \varepsilon(\mathbf{I}_{3w_1} + \mathbf{I}_{w_1}) = (-1)^{(3w_1+1)/2+(w_1+1)/2} = -1,$$
$$\varepsilon_\psi(\sigma_1) = \varepsilon(\mathrm{Sym}^4\pi_1 \otimes \pi_2) \cdot \varepsilon(\mathrm{Sym}^3\pi_1) = (-1)^{\max(4w_1,w_2)+\max(2w_1,w_2)+(w_2-1)/2}.$$

So $\varepsilon_\psi = \chi_1$ or $\chi_1\chi_2$. On the other side, the largest weight μ_1 is $2w_1$ or $\frac{3w_1+w_2}{2}$.

(1) If $w_1 > w_2$, then $\mu_1 = 2w_1$. Now $2w_1 > \frac{3w_1+w_2}{2} > \frac{3w_1+1}{2} > \frac{3w_1-1}{2}$ and they are larger than other weights, thus $\mu_4 = \frac{3w_1-1}{2}$ and $\rho_\psi^\vee = \chi_1\chi_2$.

(2) If $w_1 < w_2$, then $\mu_1 = \frac{3w_1+w_2}{2}$. Now $\mu_2 = 2w_1$ or $\frac{w_1+w_2}{2}$.

   (a) If $w_2 < 3w_1$, then $\mu_2 = 2w_1$. Now $\mu_4 = \frac{w_1+w_2}{2}$ or $\frac{3w_1\pm 1}{2}$, thus $\rho_\psi^\vee = 1$ or χ_2.

   (b) If $w_2 > 3w_1$, then $\mu_2 = \frac{w_1+w_2}{2}$. Now $\mu_4 = 2w_1$ or $\frac{w_2\pm 1}{2}$, thus $\rho^\vee = \chi_1$ or $\chi_1\chi_2$. Notice that $\mu_4 = 2w_1$ if and only if $2w_1$ lies between $\frac{w_2+1}{2}$ and $\frac{w_2-1}{2}$, which can not happen. So $\rho_\psi^\vee = \chi_1\chi_2$ for any $w_2 > 3w_1$ and $w_2 \neq 4w_1 \pm 1$.

Hence by Arthur's multiplicity formula, $\mathrm{m}(\pi_\psi) = 1$ if and only if one of the following conditions holds:

- $w_2 < w_1$ or $w_2 > 4w_1 + 1$, and $w_2 \equiv 3 \bmod 4$;
- $3w_1 < w_2 < 4w_1 - 1$, and $w_2 \equiv 1 \bmod 4$.

Case (ii): $\psi = (\pi_1 \otimes \pi_2) \oplus \pi_1[4] \oplus \pi_2[4] \oplus [5] \oplus [1]$, where $\pi_1, \pi_2 \in \Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_2)$ have motivic weights $w_1 > w_2$ respectively. In this case, the restriction of $\mathfrak{f}_4$ along ψ is isomorphic to

$$\mathrm{Sym}^2\pi_1 \oplus \mathrm{Sym}^2\pi_2 \oplus (\pi_1 \otimes \pi_2[5]) \oplus \pi_1[4] \oplus \pi_2[4] \oplus [7] \oplus [3].$$

By Proposition 6.6.4 we have:

$$\varepsilon_\psi(\sigma) = \varepsilon(\pi_1) \cdot \varepsilon(\pi_2) = \varepsilon(\mathbf{I}_{w_1}) \cdot \varepsilon(\mathbf{I}_{w_2}) = (-1)^{(w_1+w_2)/2+1}$$
$$\varepsilon_\psi(\sigma_1) = \varepsilon(\pi_2) = \varepsilon(\mathbf{I}_{w_2}) = (-1)^{(w_2+1)/2}.$$

On the other side, the condition $\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ implies that $w_2 < w_1 - 4$. Since

$$\frac{w_1+w_2}{2} > \frac{w_1+3}{2} > \frac{w_1+1}{2} > \frac{w_1-1}{2}$$

and they are larger than other weights, we have $\mu_1 = \frac{w_1+w_2}{2}$ and $\mu_4 = \frac{w_1-1}{2}$. The global component group C_ψ acts on $\pi_1 \otimes \pi_2$ and $\pi_1[4]$ by χ_2 and χ_1 respectively, thus by Proposition 7.2.1 the character $\rho_\psi^\vee = \chi_1\chi_2$. By Arthur's multiplicity formula, $\mathrm{m}(\pi_\psi) = 1$ if and only if $w_1 \equiv 3 \bmod 4, w_2 \equiv 1 \bmod 4$ and $w_2 < w_1 - 4$.

Case (iii): $\psi = \mathrm{Sym}^4 \pi_1 \oplus (\mathrm{Sym}^3 \pi_1 \otimes \pi_2) \oplus (\mathrm{Sym}^3 \pi_1 \otimes \pi_3) \oplus (\pi_2 \otimes \pi_3) \oplus [1]$, where $\pi_1, \pi_2, \pi_3 \in \Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_2)$ have motivic weights w_1, w_2, w_3 respectively and we assume that $w_2 > w_3$. In this case ε_ψ is trivial since ψ is tempered. On the other side, C_ψ acts on the four summands $\mathrm{Sym}^4 \pi_1, \mathrm{Sym}^3 \pi_1 \otimes \pi_2, \mathrm{Sym}^3 \pi_1 \otimes \pi_3$ and $\pi_2 \otimes \pi_3$ by $1, \chi_1, \chi_1\chi_2$ and χ_2 respectively. Denote the ratios $w_1/w_3, w_2/w_3$ by r_1, r_2 respectively and the corresponding multiset by $\widetilde{\mathcal{W}}$ as in the proof of Proposition 7.3.10. We still order the elements of $\widetilde{\mathcal{W}}$ by $\mu_1 > \mu_2 > \cdots > \mu_{26}$, then by Proposition 7.2.1 the character $\rho_\psi^\vee = 1$ if and only if μ_1 and μ_4 come from the same irreducible summand of ψ. The largest element μ_1 is $2r_1$ or $\frac{3r_1+r_2}{2}$ or $\frac{r_2+1}{2}$.

(1) If $r_2 < r_1$, then $\mu_1 = 2r_1$. Now $2r_1 > \frac{3r_1+r_2}{2} > \frac{3r_1+1}{2} > \frac{3r_1-r_2}{2} > r_1$, thus $\rho_\psi^\vee$ is not trivial.

(2) If $r_2 > r_1$ and $r_1 > 1/3$, then $\mu_1 = \frac{3r_1+r_2}{2}$.

   (a) If $r_1 > 1$, then $\rho_\psi^\vee = 1$ if and only if $\mu_4 = \frac{r_1+r_2}{2}$, which is equivalent to $2r_1 - 1 < r_2 < 2r_1 + 1$.

   (b) If $r_1 < 1$, then $\rho_\psi^\vee = 1$ if and only if $\mu_4 = \frac{r_2 \pm r_1}{2}$.

      (I) $\mu_4 = \frac{r_2+r_1}{2}$ if and only if $2r_1 < \frac{r_2+r_1}{2} < \frac{3r_1+1}{2} \Leftrightarrow 3r_1 < r_2 < 2r_1 + 1$.

      (II) $\mu_4 = \frac{r_2-r_1}{2}$ if and only if $\frac{r_2-r_1}{2} > \frac{3r_1+1}{2} \Leftrightarrow r_2 > 4r_1 + 1$.

(3) If $r_1 < 1/3$, then $\mu_1 = \frac{r_2+1}{2}$. Now $\frac{r_2+1}{2}, \frac{r_2 \pm r_1}{2}, \frac{3r_1+r_2}{2}$ are larger than $\frac{r_2-1}{2}$, so $\frac{r_2-1}{2}$ can not be μ_4 and thus $\rho_\psi^\vee \neq 1$.

In conclusion, by Arthur's multiplicity formula, $\mathrm{m}(\pi_\psi) = 1$ if and only if w_1, w_2, w_3 satisfy one of the three conditions in Proposition 7.3.11. □

7.3.8 $H = \prod_{i=1}^{4} \mathrm{A}_1^{[2^6,1^{14}]}/\mu_2^{\Delta}$

By Section 5.6.13, the restriction of the 26-dimensional irreducible representation J_0 of F_4 to H is isomorphic to

$$\mathbf{1}^{\oplus 2} + \sum_{\mathrm{Sym}} \mathrm{St} \otimes \mathrm{St} \otimes \mathbf{1} \otimes \mathbf{1},$$

and the centralizer of H in F_4 is $\mathrm{Z}(H) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

For $\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ satisfying $\mathrm{H}(\psi) = H$ and $\mathrm{m}(\pi_\psi) = 1$, there are two possible endoscopic types:

(i) $(8, (4,1), (4,1), (4,1), (2,2), (2,2), (2,2), (1,1), (1,1))$. A global Arthur parameter of this type is of the form:

$$\left(\bigoplus_{1 \leq i < j \leq 3} \pi_i \otimes \pi_j \right) \oplus \left(\bigoplus_{1 \leq i \leq 3} \pi_i[2] \right) \oplus [1] \oplus [1],\ \pi_1, \pi_2, \pi_3 \in \Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_2).$$

(ii) $(8, (4,1), (4,1), (4,1), (4,1), (4,1), (4,1), (1,1), (1,1))$. A global Arthur parameter of this type is of the form:

$$\left(\bigoplus_{1\le i<j\le 4} \pi_i\otimes\pi_j\right)\oplus[1]\oplus[1],\ \pi_1,\pi_2,\pi_3,\pi_4\in\Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_2).$$

For this subgroup H of F_4, the restriction of the adjoint representation $\mathfrak{f}_4$ of F_4 to H is isomorphic to

$$\sum_{\mathrm{Sym}}\mathrm{Sym}^2\,\mathrm{St}\otimes\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1}+\sum_{\mathrm{Sym}}\mathrm{St}\otimes\mathrm{St}\otimes\mathbf{1}\otimes\mathbf{1}+\mathrm{St}\otimes\mathrm{St}\otimes\mathrm{St}\otimes\mathrm{St}.$$

Proposition* 7.3.12. *For a discrete global Arthur parameter $\psi\in\Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ satisfying $\mathrm{H}(\psi)=H$, the multiplicity $\mathrm{m}(\pi_\psi)=1$ if and only if ψ has the form:*

$$\psi=\left(\bigoplus_{1\le i<j\le 3}\pi_i\otimes\pi_j\right)\oplus\left(\bigoplus_{1\le i\le 3}\pi_i[2]\right)\oplus[1]\oplus[1],$$

where $\pi_1,\pi_2,\pi_3\in\Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_2)$ have motivic weights $w_1>w_2>w_3$ respectively such that one of the following conditions holds:

- $w_1>w_2+w_3+1$, *and* $w_1\equiv w_3\equiv 3 \bmod 4$, $w_2\equiv 1\bmod 4$*;*
- $w_1<w_2+w_3-1$, *and* $w_1\equiv w_3\equiv 1\bmod 4$, $w_2\equiv 3\bmod 4$.

Proof. We take a set of generators $\{\gamma=(-1,1,1,1),\gamma_1=(1,-1,1,1),\gamma_2=(1,1,-1,1)\}$ of $\mathrm{C}_\psi=\mathrm{Z}(H)\simeq\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$.

Case (i): $\psi=(\bigoplus_{1\le i<j\le 3}\pi_i\otimes\pi_j)\oplus(\bigoplus_{1\le i\le 3}\pi_i[2])\oplus[1]\oplus[1]$, where $\pi_1,\pi_2,\pi_3\in\Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_2)$ have motivic weights $w_1>w_2>w_3$ respectively. In this case, the restriction of $\mathfrak{f}_4$ along ψ is isomorphic to

$$(\pi_1\otimes\pi_2\otimes\pi_3[2])\oplus\left(\bigoplus_{1\le i<j\le 3}\pi_i\otimes\pi_j\right)\oplus\left(\bigoplus_{1\le i\le 3}\mathrm{Sym}^2\,\pi_i\right)\oplus\left(\bigoplus_{1\le i\le 3}\pi_i[2]\right)\oplus[3].$$

By Proposition 6.6.4 we have:

$$\varepsilon_\psi(\gamma)=\varepsilon(\pi_1)\cdot\varepsilon(\pi_1\otimes\pi_2\otimes\pi_3)=(-1)^{\max(w_1,w_2+w_3)+(w_1-1)/2},$$
$$\varepsilon_\psi(\gamma_1)=\varepsilon(\pi_2)\cdot\varepsilon(\pi_1\otimes\pi_2\otimes\pi_3)=(-1)^{\max(w_1,w_2+w_3)+(w_2-1)/2},$$
$$\varepsilon_\psi(\gamma_2)=\varepsilon(\pi_3)\cdot\varepsilon(\pi_1\otimes\pi_2\otimes\pi_3)=(-1)^{\max(w_1,w_2+w_3)+(w_3-1)/2}.$$

On the other side, the largest element μ_1 must be $\frac{w_1+w_2}{2}$ and μ_4 is the middle one of

$$\left\{\frac{w_1+1}{2},\frac{w_1-1}{2},\frac{w_2+w_3}{2}\right\}.$$

Since there is no integer between $\frac{w_1+1}{2}$ and $\frac{w_1-1}{2}$, we have $\mu_4\neq\frac{w_2+w_3}{2}$. So $\rho_\psi^\vee$ is the product of two characters of C_ψ coming from $\pi_1\otimes\pi_2$ and $\pi_1[2]$ respectively, thus $\rho_\psi^\vee(\gamma)=\rho_\psi^\vee(\gamma_2)=1$ and $\rho_\psi^\vee(\gamma_1)=-1$.

By Arthur's multiplicity formula, $\mathrm{m}(\pi_\psi) = 1$ if and only if one of the following conditions holds:

- $w_1 > w_2 + w_3 + 1$, and $w_1 \equiv w_3 \equiv 3 \bmod 4, w_2 \equiv 1 \bmod 4$;
- $w_1 < w_2 + w_3 - 1$, and $w_1 \equiv w_3 \equiv 1 \bmod 4, w_2 \equiv 3 \bmod 4$.

Case (ii): $\psi = (\bigoplus_{1\leq i<j\leq 4} \pi_i \otimes \pi_j) \oplus [1] \oplus [1]$, where $\pi_1, \pi_2, \pi_3, \pi_4 \in \Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_2)$ have motivic weights $w_1 > w_2 > w_3 > w_4$ respectively. In this case, ε_ψ is trivial. On the other side, μ_1 must be $\frac{w_1+w_2}{2}$. Notice that C_ψ acts on 6 components $\pi_i \otimes \pi_j$ via 6 different characters, so $\rho_\psi^\vee$ is trivial if and only if $\mu_4 = \frac{w_1-w_2}{2}$. However,

$$\frac{w_1-w_2}{2} < \frac{w_1-w_3}{2} < \frac{w_1-w_4}{2} < \frac{w_1+w_4}{2} < \frac{w_1+w_3}{2} < \frac{w_1+w_2}{2},$$

thus $\rho_\psi^\vee \neq 1$ and $\mathrm{m}(\pi_\psi) = 0$. □

7.3.9 $H = \mathrm{A}_1^{[5,3^7]} \times \mathrm{G}_2$

In this case, we need to consider cuspidal representations $\pi \in \Pi_{\mathrm{alg,reg}}^{\mathrm{o}}(\mathbf{PGL}_7)$ such that the image of the corresponding irreducible representation $\mathcal{L}_\mathbb{Z} \to \mathbf{SL}_7(\mathbb{C})$ is a compact Lie group of type G_2. This kind of representations correspond to discrete automorphic representations of the unique semisimple anisotropic $\mathbb{Z}$-group of type G_2 with stable tempered type, which have been studied in [ChenevierRenard, 2015, §8], conditional to the existence of $\mathcal{L}_\mathbb{Z}$ and Arthur's multiplicity formula. We denote by $\Pi_{\mathrm{alg}}^{\mathbf{G}_2}(\mathbf{PGL}_7) \subset \Pi_{\mathrm{alg,reg}}^{\mathrm{o}}(\mathbf{PGL}_7)$ the subset of these representations. The Hodge weights of a representation $\pi \in \Pi_{\mathrm{alg}}^{\mathbf{G}_2}(\mathbf{PGL}_7)$ have the form $w + v > w > v$, where w, v are even integers.

By Section 5.6.4, the restriction of the 26-dimensional irreducible representation J_0 of F_4 to H is isomorphic to

$$\mathrm{Sym}^2 \mathrm{St} \otimes \mathrm{V}_7 + \mathrm{Sym}^4 \mathrm{St} \otimes \mathbf{1},$$

where V_7 is the 7-dimensional irreducible representation of G_2, and the centralizer of H in F_4 is trivial.

For $\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ satisfying $\mathrm{H}(\psi) = H$ and $\mathrm{m}(\pi_\psi) = 1$, there are two possible endoscopic types:

(i) $(2, (7, 3), (1, 5))$. A global Arthur parameter of this type is of the form:

$$\pi[3] \oplus [5], \ \pi \in \Pi_{\mathrm{alg}}^{\mathbf{G}_2}(\mathbf{PGL}_7).$$

(ii) $(2, (21, 1), (5, 1))$. A global Arthur parameter of this type is of the form:

$$(\pi \otimes \mathrm{Sym}^2 \tau) \oplus \mathrm{Sym}^4 \tau, \ \pi \in \Pi_{\mathrm{alg}}^{\mathbf{G}_2}(\mathbf{PGL}_7), \tau \in \Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_2).$$

Proposition* 7.3.13. *For a discrete global Arthur parameter $\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ satisfying $\mathrm{H}(\psi) = H$, the multiplicity $\mathrm{m}(\pi_\psi) = 1$ if and only if ψ is one of the following parameters:*

- $\pi[3] \oplus [5]$*, where $\pi \in \Pi_{\mathrm{alg}}^{\mathbf{G}_2}(\mathbf{PGL}_7)$ has Hodge weights $w + v > w > v$ such that $v > 4$;*

- $(\pi \otimes \mathrm{Sym}^2 \tau) \oplus \mathrm{Sym}^4 \tau$, *where* $\pi \in \Pi_{\mathrm{alg}}^{\mathbf{G}_2}(\mathbf{PGL}_7)$ *has Hodge weights* $w+v > w > v$ *and* $\tau \in \Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_2)$ *satisfies* $\mathrm{w}(\tau) \notin \{\frac{w+v}{2}, \frac{w}{2}, \frac{v}{2}\}$.

Proof. This follows from the condition $\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ and the fact that C_ψ is trivial. □

7.3.10 $H = \left(\mathrm{A}_1^{[2^6,1^{14}]} \times \mathrm{A}_1^{[2^6,1^{14}]} \times \mathrm{Sp}(2)\right)/\mu_2^{\Delta}$

By Section 5.6.6, the restriction of the 26-dimensional irreducible representation J_0 of F_4 to H is isomorphic to

$$\mathbf{1} + \mathrm{St} \otimes \mathrm{St} \otimes \mathbf{1} + \mathrm{St} \otimes \mathbf{1} \otimes \mathrm{V}_4 + \mathbf{1} \otimes \mathrm{St} \otimes \mathrm{V}_4 + \mathbf{1} \otimes \mathbf{1} \otimes \wedge^* \mathrm{V}_4,$$

where V_4 is the standard representation of $\mathrm{Sp}(2)$ and $\wedge^*\mathrm{V}_4$ is the 5-dimensional irreducible representation of $\mathrm{Sp}(2)$. The centralizer of H in F_4 is $\mathrm{Z}(H) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

For any $\pi \in \Pi_{\mathrm{alg}}^{\mathbf{Sp}_4}(\mathbf{PGL}_4)$, we denote by $\wedge^*\pi$ the representation in $\Pi_{\mathrm{alg,reg}}^{\mathrm{o}}(\mathbf{PGL}_5)$ corresponding to the following irreducible representation of $\mathcal{L}_{\mathbb{Z}}$:

$$\mathcal{L}_{\mathbb{Z}} \xrightarrow{\psi_\pi} \mathrm{Sp}(2) \xrightarrow{\wedge^*} \mathbf{SL}_5(\mathbb{C}).$$

For $\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ satisfying $\mathrm{H}(\psi) = H$ and $\mathrm{m}(\pi_\psi) = 1$, there are two possible endoscopic types:

(i) $(5,(8,1),(5,1),(4,2),(2,2),(1,1))$. A global Arthur parameter of this type is of the form:

$$\wedge^*\pi \oplus (\pi \otimes \tau) \oplus \pi[2] \oplus \tau[2] \oplus [1],\ \pi \in \Pi_{\mathrm{alg}}^{\mathbf{Sp}_4}(\mathbf{PGL}_4), \tau \in \Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_2).$$

(ii) $(5,(8,1),(8,1),(5,1),(4,1),(1,1))$. A global Arthur parameter of this type is of the form:

$$\wedge^*\pi \oplus (\pi \otimes \tau_1) \oplus (\pi \otimes \tau_2) \oplus (\tau_1 \otimes \tau_2) \oplus [1],\ \pi \in \Pi_{\mathrm{alg}}^{\mathbf{Sp}_4}(\mathbf{PGL}_4), \tau_1, \tau_2 \in \Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_2).$$

For this subgroup H of F_4, the restriction of the adjoint representation $\mathfrak{f}_4$ of F_4 to H is isomorphic to

$$\left(\mathrm{Sym}^2 \mathrm{St} \otimes \mathbf{1} + \mathbf{1} \otimes \mathrm{Sym}^2 \mathrm{St}\right) \otimes \mathbf{1} + (\mathrm{St} \otimes \mathbf{1} + \mathbf{1} \otimes \mathrm{St}) \otimes \mathrm{V}_4$$
$$+ \mathrm{St} \otimes \mathrm{St} \otimes \wedge^*\mathrm{V}_4 + \mathbf{1} \otimes \mathbf{1} \otimes \mathrm{Sym}^2 \mathrm{V}_4.$$

Proposition* 7.3.14. *For a discrete global Arthur parameter* $\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ *satisfying* $\mathrm{H}(\psi) = H$*, the multiplicity* $\mathrm{m}(\pi_\psi) = 1$ *if and only if* ψ *is one of the following parameters:*

- $\wedge^*\pi \oplus (\pi \otimes \tau) \oplus \pi[2] \oplus \tau[2] \oplus [1]$, *where* $\pi \in \Pi_{\mathrm{alg}}^{\mathbf{Sp}_4}(\mathbf{PGL}_4)$ *has Hodge weights* $w_1 > w_2 > 1$ *and* $\tau \in \Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_2)$ *has motivic weight* v *satisfying one of the following conditions:*
  - $w_1 < v < w_1 + w_2 - 1, w_1 + w_2 \equiv 0 \bmod 4, v \equiv 1 \bmod 4$;
  - $w_1 - w_2 + 1 < v < w_2, w_1 + w_2 \equiv 0 \bmod 4, v \equiv 1 \bmod 4$;
  - $w_2 < v < w_1 - w_2 - 1, w_1 + w_2 \equiv 2 \bmod 4, v \equiv 1 \bmod 4$;
  - $v > w_1 + w_2 + 1, w_1 + w_2 \equiv 0 \bmod 4, v \equiv 3 \bmod 4$;

  – $v < \min(w_1 - w_2 - 1, w_2), w_1 + w_2 \equiv 0 \bmod 4, v \equiv 3 \bmod 4$;
  – $\max(w_1 - w_2 + 1, w_2) < v < w_1, w_1 + w_2 \equiv 2 \bmod 4, v \equiv 3 \bmod 4$.

- $\wedge^*\pi \oplus (\pi \otimes \tau_1) \oplus (\pi \otimes \tau_2) \oplus (\tau_1 \otimes \tau_2) \oplus [1]$, *where* $\pi \in \Pi_{\mathrm{alg}}^{\mathbf{Sp}_4}(\mathbf{PGL}_4)$ *has Hodge weights* $w_1 > w_2$ *and* $\tau_1, \tau_2 \in \Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_2)$ *have motivic weights* $v_1 > v_2$ *respectively satisfying one of the following conditions:*
  – $v_2 < w_2 < v_1$ *and* $w_1 - w_2 - v_2 < v_1 < w_1 - w_2 + v_2$;
  – $w_2 < v_2 < w_1$ *and* $v_1 > w_1 + w_2 + v_2$;
  – $v_2 < w_1 < v_1 < w_1 - w_2 + v_2$.

Proof. We take a set of generators $\{\sigma = (1, 1, -1), \sigma_1 = (-1, 1, 1)\}$ of $\mathrm{C}_\psi = \mathrm{Z}(H) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Let χ_1, χ_2 be two generators of the character group of C_ψ such that $\chi_1(\sigma) = \chi_2(\sigma_1) = -1$ and $\chi_1(\sigma_1) = \chi_2(\sigma) = 1$.

Case (i): $\psi = \wedge^*\pi \oplus (\pi \otimes \tau) \oplus \pi[2] \oplus \tau[2] \oplus [1]$, where $\pi \in \Pi_{\mathrm{alg}}^{\mathbf{Sp}_4}(\mathbf{PGL}_4)$ has Hodge weights $w_1 > w_2 > 1$ and $\tau \in \Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_2)$ has motivic weight v. Here we assume that Arthur's $\mathbf{SL}_2(\mathbb{C})$ is sent to the first A_1-factor of $H_{\mathbb{C}}$. In this case, the restriction of $\mathfrak{f}_4$ along ψ is isomorphic to

$$\mathrm{Sym}^2 \pi \oplus (\wedge^*\pi \otimes \tau[2]) \oplus (\pi \otimes \tau) \oplus \pi[2] \oplus \mathrm{Sym}^2 \tau \oplus [3].$$

By Proposition 6.6.4 we have:

$$\varepsilon_\psi(\sigma) = \varepsilon(\pi) = \varepsilon(\mathbf{I}_{w_1} + \mathbf{I}_{w_2}) = (-1)^{(w_1+w_2)/2+1},$$
$$\varepsilon_\psi(\sigma_1) = \varepsilon(\wedge^*\pi \otimes \tau) = (-1)^{\max(w_1+w_2,v)+\max(w_1-w_2,v)+(v+1)/2}.$$

On the other side, the group C_ψ acts on $\wedge^*\pi, \pi \otimes \tau, \pi[2], \tau[2]$ by $1, \chi_1\chi_2, \chi_1, \chi_2$ respectively. The largest element μ_1 must be $\frac{w_1+w_2}{2}$ or $\frac{w_1+v}{2}$.

(1) If $w_2 > v$, then $\mu_1 = \frac{w_1+w_2}{2}$. Now $\mu_4 = \frac{w_1 \pm 1}{2}$ and $\rho_\psi^\vee = \chi_1$.
(2) If $w_2 < v$, then $\mu_1 = \frac{w_1+v}{2}$. Now μ_2 is $\frac{w_1+w_2}{2}$ or $\frac{w_2+v}{2}$.
   (a) If $w_1 > v$, then $\mu_2 = \frac{w_1+w_2}{2}$. Now $\mu_4 = \frac{w_1 \pm 1}{2}$ and $\rho_\psi^\vee = \chi_2$.
   (b) If $w_1 < v$, then $\mu_2 = \frac{w_2+v}{2}$. Now $\mu_4 = \frac{v \pm 1}{2}$ and $\rho_\psi^\vee = \chi_1$.

By Arthur's multiplicity formula, $\mathrm{m}(\pi_\psi) = 1$ if and only if π and τ satisfy one of the conditions listed in the proposition.

Case (ii): $\psi = \wedge^*\pi \oplus (\pi \otimes \tau_1) \oplus (\pi \otimes \tau_2) \oplus (\tau_1 \otimes \tau_2) \oplus [1]$, where $\pi \in \Pi_{\mathrm{alg}}^{\mathbf{Sp}_4}(\mathbf{PGL}_4)$ has Hodge weights $w_1 > w_2$ and $\tau_1, \tau_2 \in \Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_2)$ have motivic weights $v_1 > v_2$ respectively. In this case ε_ψ is a trivial character. On the other side, since C_ψ acts on four non-trivial irreducible summands of ψ by four different characters, $\rho_\psi^\vee = 1$ if and only if μ_1 and μ_4 come from the same irreducible summand. Now μ_1 must be $\frac{w_1+w_2}{2}$ or $\frac{w_1+v_1}{2}$ or $\frac{v_1+v_2}{2}$.

(1) If $w_2 > v_1$, then $\mu_1 = \frac{w_1+w_2}{2}$ and μ_4 can not be $\frac{w_1-w_2}{2}$, thus $\rho_\psi^\vee$ is not trivial.
(2) If $v_1 > w_2$ and $w_1 > v_2$, then $\mu_1 = \frac{w_1+v_1}{2}$. Now $\rho_\psi^\vee$ is trivial if and only if $\mu_4 = \frac{w_2+v_1}{2}$ or $\frac{v_1-w_2}{2}$.

(a) $\mu_4 = \frac{v_1 - w_2}{2}$ is equivalent to that $v_1 - w_2 > \max(v_1 - v_2, w_1 + w_2, w_1 + v_2)$. This holds if and only if $v_2 > w_2$ and $v_1 > w_1 + w_2 + v_2$.

(b) $\mu_4 = \frac{w_2 + v_1}{2}$ is equivalent to that $w_2 + v_1 > \max(w_1 - w_2, w_1 - v_2)$ and $w_2 + v_1$ is smaller than exactly two of $\{w_1 + w_2, v_1 + v_2, w_1 + v_2\}$. This holds in two cases: $w_1 < v_1 < w_1 - w_2 + v_2$ or

$$w_2 > v_2, w_1 > v_1, w_1 - w_2 - v_2 < v_1 < w_1 - w_2 + v_2.$$

(3) If $v_2 > w_1$, $\mu_1 = \frac{v_1 + v_2}{2}$. We have

$$\frac{v_1 - v_2}{2} < \frac{v_1 - w_1}{2} < \frac{v_1 - w_2}{2} < \frac{v_1 + w_2}{2} < \frac{v_1 + w_1}{2} < \frac{v_1 + v_2}{2},$$

thus μ_4 can not be $\frac{v_1 - v_2}{2}$ and $\rho_\psi^\vee$ is not trivial.

In conclusion, by Arthur's multiplicity formula $\mathrm{m}(\pi_\psi) = 1$ if and only if one of the following conditions holds:

- $v_2 < w_2 < v_1$ and $w_1 - w_2 - v_2 < v_1 < w_1 - w_2 + v_2$;
- $w_2 < v_2 < w_1$ and $v_1 > w_1 + w_2 + v_2$;
- $v_2 < w_1 < v_1 < w_1 - w_2 + v_2$. □

7.3.11 $H = \left(\mathrm{A}_1^{[2^6, 1^{14}]} \times \mathrm{Sp}(3)\right) / \mu_2^\Delta$

By Section 5.6.3, the restriction of the 26-dimensional irreducible representation J_0 of F_4 to H is isomorphic to

$$\mathrm{St} \otimes \mathrm{V}_6 + \mathbf{1} \otimes \mathrm{V}_{14},$$

where V_6 is the standard 6-dimensional representation of $\mathrm{Sp}(3)$, and $\mathrm{V}_{14} = \wedge^* \mathrm{V}_6$ is the 14-dimensional irreducible representation of $\mathrm{Sp}(3)$ that is a sub-representation of $\wedge^2 \mathrm{V}_6$. The centralizer of H in F_4 is $\mathrm{Z}(H) \simeq \mathbb{Z}/2\mathbb{Z}$.

For any $\pi \in \Pi_{\mathrm{alg}}^{\mathbf{Sp}_6}(\mathbf{PGL}_6)$, we denote by $\wedge^* \pi$ the representation in $\Pi_{\mathrm{alg,reg}}^{\mathrm{o}}(\mathbf{PGL}_{14})$ corresponding to the following irreducible representation of $\mathcal{L}_\mathbb{Z}$:

$$\mathcal{L}_\mathbb{Z} \xrightarrow{\psi_\pi} \mathrm{Sp}(3) \xrightarrow{\wedge^*} \mathbf{SL}_{14}(\mathbb{C}).$$

For $\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ satisfying $\mathrm{H}(\psi) = H$ and $\mathrm{m}(\pi_\psi) = 1$, there are two possible endoscopic types:

(i) $(2, (14, 1), (6, 2))$. A global Arthur parameter of this type is of the form:

$$\wedge^* \pi \oplus \pi[2],\ \pi \in \Pi_{\mathrm{alg}}^{\mathbf{Sp}_6}(\mathbf{PGL}_6).$$

(ii) $(2, (14, 1), (12, 1))$. A global Arthur parameter of this type is of the form:

$$\wedge^* \pi \oplus (\pi \otimes \tau),\ \pi \in \Pi_{\mathrm{alg}}^{\mathbf{Sp}_6}(\mathbf{PGL}_6), \tau \in \Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_2).$$

For this subgroup H of F_4, the restriction of the adjoint representation $\mathfrak{f}_4$ of F_4 to H is isomorphic to

$$\mathrm{Sym}^2\,\mathrm{St}\otimes\mathbf{1}+\mathrm{St}\otimes\mathrm{V}_{14}'+\mathbf{1}\otimes\mathrm{Sym}^2\,\mathrm{V}_6,$$

where V_{14}' is another 14-dimensional irreducible representation of $\mathrm{Sp}(3)$ that is not equivalent to $\mathrm{V}_{14}=\wedge^*\mathrm{V}_6$.

Proposition* 7.3.15. *For a discrete global Arthur parameter $\psi\in\Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ satisfying $\mathrm{H}(\psi)=H$, the multiplicity $\mathrm{m}(\pi_\psi)=1$ if and only if ψ is one of the following parameters:*

- $\wedge^*\pi\oplus\pi[2]$, *where $\pi\in\Pi_{\mathrm{alg}}^{\mathbf{Sp}_6}(\mathbf{PGL}_6)$ has Hodge weights $w_1>w_2>w_3>1$ and one of the following conditions holds:*
  - $w_1>w_2+w_3+1$ *and* $w_1+w_2+w_3\equiv 3 \bmod 4$;
  - $w_1<w_2+w_3-1$ *and* $w_1+w_2+w_3\equiv 1 \bmod 4$.
- $\wedge^*\pi\oplus(\pi\otimes\tau)$, *where $\pi\in\Pi_{\mathrm{alg}}^{\mathbf{Sp}_6}(\mathbf{PGL}_6)$ has Hodge weights $w_1>w_2>w_3$ and $\tau\in\Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_2)$ has motivic weight v satisfying one of the following conditions:*
  - $|w_1-w_2-w_3|<v<w_3$;
  - $w_1-w_2+w_3<v<w_2$;
  - $w_3<v<\min(w_2,w_1-w_2-w_3)$;
  - $\max(w_2,w_1-w_2-w_3)<v<w_1-w_2+w_3$;
  - $w_1<v<w_1+w_2-w_3$;
  - $v>w_1+w_2+w_3$.

Proof. We denote the generator $(-1,1)\in\mathrm{Z}(H)=\mathrm{C}_\psi$ by γ.

Case (i): $\psi=\wedge^*\pi\oplus\pi[2]$, where $\pi\in\Pi_{\mathrm{alg}}^{\mathbf{Sp}_6}(\mathbf{PGL}_6)$ has Hodge weights $w_1>w_2>w_3>1$. In this case, the restriction of $\mathfrak{f}_4$ along ψ is isomorphic to

$$\mathrm{Sym}^2\,\pi\oplus\pi'[2]\oplus[3],$$

where $\pi'\in\Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_{14})$ corresponds to

$$\mathcal{L}_{\mathbb{Z}}\xrightarrow{\psi_\pi}\mathrm{Sp}(3)\xrightarrow{\mathrm{V}_{14}'}\mathbf{SL}_{14}(\mathbb{C}).$$

Notice that $\wedge^3\mathrm{V}_6\simeq\mathrm{V}_{14}'\oplus\mathrm{V}_6$, thus the Hodge weights of π' are

$$\pm w_1,\pm w_2,\pm w_3,\pm w_1\pm w_2\pm w_3.$$

By Proposition 6.6.4 we have:

$$\begin{aligned}\varepsilon_\psi(\gamma)&=\varepsilon\left(\mathbf{I}_{w_1}+\mathbf{I}_{w_2}+\mathbf{I}_{w_3}+\mathbf{I}_{w_1+w_2+w_3}+\mathbf{I}_{w_1+w_2-w_3}+\mathbf{I}_{w_1-w_2+w_3}+\mathbf{I}_{|w_1-w_2-w_3|}\right)\\&=(-1)^{(w_1+w_2+w_3+1)/2+\max(w_1,w_2+w_3)}.\end{aligned}$$

On the other side, γ acts on $\wedge^*\pi$ by 1 and on $\pi[2]$ by -1. The largest element μ_1 must be $\frac{w_1+w_2}{2}$. Now $\mu_4 = \frac{w_1\pm 1}{2}$, thus $\rho_\psi^\vee(\gamma) = -1$. By Arthur's multiplicity formula, $\mathrm{m}(\pi_\psi) = 1$ if and only if one of the following conditions holds:

- $w_1 > w_2 + w_3 + 1$ and $w_1 + w_2 + w_3 \equiv 3 \bmod 4$;
- $w_1 < w_2 + w_3 - 1$ and $w_1 + w_2 + w_3 \equiv 1 \bmod 4$.

Case (ii): $\psi = \wedge^*\pi \oplus (\pi \otimes \tau)$, where $\pi \in \Pi_{\mathrm{alg}}^{\mathbf{Sp}_6}(\mathbf{PGL}_6)$ has Hodge weights $w_1 > w_2 > w_3$ and $\tau \in \Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_2)$ has motivic weight v. In this case ε_ψ is trivial. On the other side, the largest μ_1 must be $\frac{w_1+w_2}{2}$ or $\frac{w_1+v}{2}$.

(1) If $v < w_2$, then $\mu_1 = \frac{w_1+w_2}{2}$.

  (a) If $v < w_3$, then μ_4 is the middle one in $\{\frac{w_2+w_3}{2}, \frac{w_1+v}{2}, \frac{w_1-v}{2}\}$. Hence $\rho_\psi^\vee = 1$ if and only if $\mu_4 = \frac{w_2+w_3}{2}$, which is equivalent to $v > |w_1 - w_2 - w_3|$.

  (b) If $v > w_3$, then μ_4 is the middle one in $\{\frac{w_2+v}{2}, \frac{w_1+w_3}{2}, \frac{w_1-w_3}{2}\}$. Hence $\rho_\psi^\vee = 1$ if and only if $\mu_4 = \frac{w_1\pm w_3}{2}$, which is equivalent to $v > w_1 - w_2 + w_3$ or $v < w_1 - w_2 - w_3$.

(2) If $v > w_2$, then $\mu_1 = \frac{w_1+v}{2}$.

  (a) If $v < w_1$, then μ_4 is the middle one in $\{\frac{w_2+v}{2}, \frac{w_1+w_3}{2}, \frac{w_1-w_3}{2}\}$. Hence $\rho_\psi^\vee = 1$ if and only if $\mu_4 = \frac{w_2+v}{2}$, which is equivalent to $w_1 - w_2 - w_3 < v < w_1 - w_2 + w_3$.

  (b) If $v > w_1$, then μ_4 is the middle one in $\{\frac{w_1+w_2}{2}, \frac{v+w_3}{2}, \frac{v-w_3}{2}\}$. Hence $\rho_\psi^\vee = 1$ if and only if $\mu_4 = \frac{v\pm w_3}{2}$, which is equivalent to $v > w_1 + w_2 + w_3$ or $v < w_1 + w_2 - w_3$.

In conclusion, $\mathrm{m}(\pi_\psi) = 1$ if and only if one of the conditions on π, τ listed in the proposition is satisfied. ☐

7.3.12 $H = \mathrm{Spin}(8)$

By Section 5.6.5, the restriction of the 26-dimensional irreducible representation J_0 of F_4 to H is isomorphic to

$$\mathbf{1}^{\oplus 2} + \mathrm{V}_8 + \mathrm{V}_{\mathrm{Spin}}^+ + \mathrm{V}_{\mathrm{Spin}}^-,$$

where V_8 is the 8-dimensional vector representation of $\mathrm{Spin}(8)$, *i.e.* the composition of the projection $\mathrm{Spin}(8) \to \mathrm{SO}(8)$ with the standard 8-dimensional representation of $\mathrm{SO}(8)$, and $\mathrm{V}_{\mathrm{Spin}}^\pm$ are two 8-dimensional spinor representations. The centralizer of H in F_4 is $\mathrm{Z}(H) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

For $\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ satisfying $\mathrm{H}(\psi) = H$ and $\mathrm{m}(\pi_\psi) = 1$, there is only one possible endoscopic type: $(5, (8,1), (8,1), (8,1), (1,1), (1,1))$. A global Arthur parameter of this type is of the form:

$$\psi = \pi \oplus \mathrm{Spin}^+ \pi \oplus \mathrm{Spin}^- \pi \oplus [1] \oplus [1],\ \pi \in \Pi_{\mathrm{alg}}^{\mathbf{SO}_8}(\mathbf{PGL}_8),$$

where we lift $\psi_\pi : \mathcal{L}_\mathbb{Z} \twoheadrightarrow \mathrm{SO}(8) \to \mathbf{SO}_8(\mathbb{C})$ to $\widetilde{\psi_\pi} : \mathcal{L}_\mathbb{Z} \to \mathbf{Spin}_8(\mathbb{C})$, and $\mathrm{Spin}^*\pi, * = \pm$ is the representation corresponding to

$$\mathcal{L}_\mathbb{Z} \xrightarrow{\widetilde{\psi_\pi}} \mathbf{Spin}_8(\mathbb{C}) \xrightarrow{V_{\mathrm{Spin}}^*} \mathbf{SL}_8(\mathbb{C}).$$

Proposition* 7.3.16. *For any discrete global Arthur parameter* $\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ *satisfying* $\mathrm{H}(\psi) = H$, *we have* $\mathrm{m}(\pi_\psi) = 0$.

Proof. Let $\psi = \pi \oplus \mathrm{Spin}^+ \pi \oplus \mathrm{Spin}^- \pi \oplus [1] \oplus [1]$, where $\pi \in \Pi_{\mathrm{alg}}^{\mathbf{SO}_8}(\mathbf{PGL}_8)$ has Hodge weights $2w_1 > 2w_2 > 2w_3 > 2w_4$. The global component group $\mathrm{C}_\psi \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and it acts on $\pi, \mathrm{Spin}^+ \pi, \mathrm{Spin}^- \pi$ by three different characters.

Since ε_ψ is trivial, by Arthur's trace formula $\mathrm{m}(\pi_\psi) = 1$ if and only if $\rho_\psi^\vee = 1$, which is equivalent to that μ_1 and μ_4 come from the same irreducible summand of ψ by Proposition 7.2.1.

In this case, the largest element μ_1 must be w_1 or $\frac{w_1+w_2+w_3+w_4}{2}$.

(1) If $w_1 > w_2 + w_3 + w_4$, then $\mu_1 = w_1$. Now we have

$$w_2 < \frac{w_1 + w_2 - w_3 + w_4}{2} < \frac{w_1 + w_3 + w_3 - w_4}{2} < \frac{w_1 + w_2 + w_3 + w_4}{2} < \mu_1,$$

thus μ_4 does not come from π. Hence $\rho_\psi^\vee$ is not trivial.

(2) If $w_1 < w_2 + w_3 + w_4$, then $\mu_1 = \frac{w_1+w_2+w_3+w_4}{2}$. Now we have

$$\frac{w_1 - w_2 + w_3 - w_4}{2} < \frac{w_1 + w_2 - w_3 - w_4}{2} < \min\left(w_2, \frac{w_1 + w_2 \pm (w_3 - w_4)}{2}\right) < \mu_1$$

and

$$\frac{|w_1 - w_2 - w_3 + w_4|}{2} \leq \max\left(w_4, \frac{-w_1 + w_2 + w_3 + w_4}{2}\right)$$

is also smaller than at least 4 weights, hence

$$\mu_4 \notin \left\{\frac{w_1 - w_2 + w_3 - w_4}{2}, \frac{w_1 + w_2 - w_3 - w_4}{2}, \frac{|w_1 - w_2 - w_3 + w_4|}{2}\right\}.$$

So μ_4 does not come from $\mathrm{Spin}^+ \pi$ and $\rho_\psi^\vee$ is not trivial.

In conclusion, by Arthur's multiplicity formula the multiplicity $\mathrm{m}(\pi_\psi)$ is always 0. □

7.3.13 $H = \mathrm{Spin}(9)$

By Section 5.6.2, the restriction of the 26-dimensional irreducible representation J_0 of F_4 to H is isomorphic to

$$\mathbf{1} + \mathrm{V}_9 + \mathrm{V}_{\mathrm{Spin}},$$

where V_9 is the standard representation of $\mathrm{Spin}(9)$, $\mathrm{V}_{\mathrm{Spin}}$ is the 16-dimensional spinor representations. The centralizer of H in F_4 is $\mathrm{Z}(H) \simeq \mathbb{Z}/2\mathbb{Z}$.

For $\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ satisfying $\mathrm{H}(\psi) = H$ and $\mathrm{m}(\pi_\psi) = 1$, there is only one possible endoscopic type: $(3, (16, 1), (9, 1), (1, 1))$. A global Arthur parameter of this type is of the form:

$$\psi = \pi \oplus \mathrm{Spin}\, \pi \oplus [1],\ \pi \in \Pi_{\mathrm{alg}}^{\mathbf{SO}_9}(\mathbf{PGL}_9),$$

where we lift $\psi_\pi : \mathcal{L}_\mathbb{Z} \twoheadrightarrow \mathrm{SO}(9) \to \mathbf{SO}_9(\mathbb{C})$ to $\widetilde{\psi_\pi} : \mathcal{L}_\mathbb{Z} \to \mathbf{Spin}_9(\mathbb{C})$, and $\mathrm{Spin}\, \pi$ is the representation corresponding to

$$\mathcal{L}_\mathbb{Z} \xrightarrow{\widetilde{\psi_\pi}} \mathbf{Spin}_9(\mathbb{C}) \xrightarrow{\mathrm{V}_{\mathrm{Spin}}} \mathbf{SL}_{16}(\mathbb{C}).$$

Proposition* 7.3.17. *A discrete global Arthur parameter* $\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ *satisfies* $\mathrm{H}(\psi) = H$ *and* $\mathrm{m}(\pi_\psi) = 1$ *if and only if* $\psi = \pi \oplus \mathrm{Spin}\,\pi \oplus [1]$*, where* $\pi \in \Pi_{\mathrm{alg}}^{\mathbf{SO}_9}(\mathbf{PGL}_9)$ *has Hodge weights* $w_1 > w_2 > w_3 > w_4$ *satisfying* $w_2 + w_3 - w_4 < w_1 < w_2 + w_3 + w_4$*.*

Proof. Let $\psi = \pi \oplus \mathrm{Spin}\,\pi \oplus [1]$, where $\pi \in \Pi_{\mathrm{alg}}^{\mathbf{SO}_9}(\mathbf{PGL}_9)$ has Hodge weights $w_1 > w_2 > w_3 > w_4$. The global component group C_ψ is a cyclic 2-group, and it acts on π trivially and on $\mathrm{Spin}\,\pi$ by its non-trivial character.

Since the parameter is tempered, ε_ψ is trivial. By Arthur's multiplicity formula, $\mathrm{m}(\pi_\psi) = 1$ if and only if $\rho_\psi^\vee = 1$, which is equivalent to that μ_1 and μ_4 come from the same irreducible summand of ψ by Proposition 7.2.1. In this case, the largest element $\mu_1 = \frac{w_1}{2}$ or $\frac{w_1+w_2+w_3+w_4}{4}$.

(1) If $w_1 > w_2 + w_3 + w_4$, then $\mu_1 = \frac{w_1}{2}$. By our discussion in the proof of Proposition 7.3.16, μ_4 does not come from π, thus $\rho_\psi^\vee$ is not trivial.
(2) If $w_1 < w_2 + w_3 + w_4$, then $\mu_1 = \frac{w_1+w_2+w_3+w_4}{4}$. Now $\mu_4 = \max\left(\frac{w_2}{2}, \frac{w_1+w_2-w_3+w_4}{4}\right)$. Hence $\rho_\psi^\vee$ is trivial if and only if $w_1 + w_4 > w_2 + w_3$.

In conclusion, $\mathrm{m}(\pi_\psi) = 1$ if and only if $w_2 + w_3 - w_4 < w_1 < w_2 + w_3 + w_4$. □

7.3.14 $H = \mathrm{F}_4$

For stable tempered parameters, the component group is trivial and as a direct consequence we have:

Proposition* 7.3.18. *For any discrete global Arthur parameter* $\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ *satisfying* $\mathrm{H}(\psi) = \mathrm{F}_4$*, we have* $\mathrm{m}(\pi_\psi) = 1$.

7.4 Classification of representations contributing to $\mathcal{A}_{\mathrm{V}_\lambda}(\mathbf{F}_4)$

Recall that in Section 6.1, for each irreducible representation V_λ with highest weight λ of $\mathrm{F}_4 = \mathbf{F}_4(\mathbb{R})$, we have defined its multiplicity space in $\mathcal{L}_{\mathrm{disc}}(\mathbf{F}_4)$:

$$\mathcal{A}_{\mathrm{V}_\lambda}(\mathbf{F}_4) = \mathrm{Hom}_{\mathbf{F}_4(\mathbb{R})}(\mathrm{V}_\lambda, \mathcal{L}_{\mathrm{disc}}(\mathbf{F}_4)^{\mathcal{F}_{4,\mathrm{I}}(\widehat{\mathbb{Z}})}),$$

which parametrizes level one discrete automorphic representation π of $\mathbf{F}_4$ such that $\pi_\infty \simeq \mathrm{V}_\lambda$. We have a dimension formula Corollary 6.1.8 for this space. Now with results in Section 7.3, we can study the discrete global Arthur parameters $\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ whose corresponding representation $\pi_\psi \in \Pi(\mathbf{F}_4)$ has multiplicity 1 in $\mathcal{L}_{\mathrm{disc}}(\mathbf{F}_4)$ and contributes to $\mathcal{A}_{\mathrm{V}_\lambda}(\mathbf{F}_4)$.

According to Lemma 6.1.5, we have:

$$\dim \mathcal{A}_{\mathrm{V}_\lambda}(\mathbf{F}_4) = \sum_{\pi \in \Pi(\mathbf{F}_4),\, \pi_\infty \simeq \mathrm{V}_\lambda} \mathrm{m}(\pi).$$

Using discrete global Arthur parameters, we rewrite this formula as

$$\dim \mathcal{A}_{\mathrm{V}_\lambda}(\mathbf{F}_4) = \sum_{\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4),\, \mathrm{c}_\infty(\psi) = \mathrm{c}_\infty(\mathrm{V}_\lambda)} \mathrm{m}(\pi_\psi) = \sum_{\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4),\, \mathrm{c}_\infty(\psi) = \lambda + \rho} \mathrm{m}(\pi_\psi),$$

where ρ is the half sum of positive roots of $\mathbf{F}_4$.

If the endoscopic type of $\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ is not stable, *i.e.* $\mathrm{H}(\psi)$ is the conjugacy class of a proper subgroup of $\mathrm{F}_4 = \mathbf{F}_4(\mathbb{R})$, then it must have one of the types listed in Section 7.3. For each subgroup H of F_4 listed in Theorem 5.6.7, we can determine the discrete global Arthur parameters $\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ satisfying $\mathrm{H}(\psi) = H$ and $\mathrm{m}(\pi_\psi) = 1$. The difference

$$\dim \mathcal{A}_{\mathrm{V}_\lambda}(\mathbf{F}_4) - \#\{\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4) \,|\, \mathrm{H}(\psi) \neq \mathrm{F}_4, \mathrm{c}_\infty(\psi) = \rho + \lambda, \mathrm{m}(\pi_\psi) = 1\} \tag{7.2}$$

is the number of discrete automorphic representations π of $\mathbf{F}_4$ with archimedean component $\pi_\infty \simeq \mathrm{V}_\lambda$ whose global Arthur parameter is tempered and stable. In other words:

Proposition* 7.4.1. *Let λ be a dominant weight of F_4, we define the number*

$$\mathrm{F}_4(\lambda) := \#\{\pi \in \Pi_{\mathrm{cusp}}(\mathbf{PGL}_{26}) \,|\, \mathrm{c}_\infty(\pi) = \mathrm{r}_0(\lambda + \rho) \in \mathfrak{sl}_{26,\mathrm{ss}}, \mathrm{H}(\pi) \simeq \mathrm{F}_4\},$$

where $\mathrm{r}_0 : \mathfrak{f}_4 \to \mathfrak{sl}_{26}$ is the 26-dimensional irreducible representation of $\mathfrak{f}_4$, and define $\mathrm{w}(\lambda)$ to be twice the maximal eigenvalue of $\lambda + \rho$. Then we have a formula for the number $\mathrm{F}_4(\lambda)$, and we list nonzero $\mathrm{F}_4(\lambda)$ for all the dominant weights λ such that $\mathrm{w}(\lambda) \leq 44$ in Table A.8.

Proof. The formula for $\mathrm{F}_4(\lambda)$ follows from Eq. (7.2) and our classifications in Section 7.3. This formula involves the numbers of elements in one of the following sets with certain Hodge weights:

$$\Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_2), \Pi_{\mathrm{alg}}^{\mathbf{Sp}_4}(\mathbf{PGL}_4), \Pi_{\mathrm{alg}}^{\mathbf{Sp}_6}(\mathbf{PGL}_6), \Pi_{\mathrm{alg}}^{\mathbf{G}_2}(\mathbf{PGL}_7), \Pi_{\mathrm{alg}}^{\mathbf{SO}_9}(\mathbf{PGL}_9).$$

For $\Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_2)$, this number is related to the dimension of cusp forms for $\mathbf{SL}_2(\mathbb{Z})$, as explained in Example 6.4.6. For other four sets, we can find some tables in [Algebraic cusp forms] and [Discrete series multiplicities]. A [PARI/GP] program to compute $\mathrm{F}_4(\lambda)$ for dominant weights λ satisfying $\mathrm{w}(\lambda) \leq 60$ is provided in [Codes and tables]. □

Remark 7.4.2. The formula for $\mathrm{F}_4(\lambda)$ has too many terms, thus it is not reasonable to write it down here. However, under some hypothesis on λ, many terms vanish and this formula becomes much simpler. For example, if

- $\lambda_i > 0$ for $i = 1, 2, 3, 4$,
- $\lambda_1 > \lambda_2 + \lambda_3 + \lambda_4 + 3$,
- and λ_3, λ_4 are both odd,

then we have the following formula:

$$\mathrm{F}_4(\lambda) = \dim \mathcal{A}_{\mathrm{V}_\lambda}(\mathbf{F}_4) - \mathrm{O}^*(\lambda_1', \lambda_2', \lambda_3', \lambda_4'),$$

where $\mathrm{O}^*(w_1, w_2, w_3, w_4)$ is the number of equivalence classes of level one cuspidal orthogonal representations of $\mathbf{PGL}_9$ with Hodge weights $w_1 > w_2 > w_3 > w_4 > 0$, and

$$\lambda_1' = 2\lambda_1 + 6\lambda_2 + 4\lambda_3 + 2\lambda_4 + 14,\ \lambda_2' = 2\lambda_1 + 2\lambda_2 + 2\lambda_3 + 2\lambda_4 + 8,$$
$$\lambda_3' = 2\lambda_1 + 2\lambda_2 + 2\lambda_3 + 6,\ \lambda_4' = 2\lambda_1 + 2\lambda_2 + 4.$$

In Table A.8, we find that the smallest $\mathrm{w}(\lambda)$ for λ such that $\mathrm{F}_4(\lambda) \neq 0$ is 36 and the corresponding dominant weight is $\lambda = \varpi_1 + 2\varpi_2 + 2\varpi_4$. We are now going to prove this fact without using Theorem 7.3.1, in order to give readers who skip the proof of Theorem 7.3.1 an example of how we apply Arthur's conjectures.

Proposition* 7.4.3. *There is a level one cuspidal automorphic representation π of $\mathbf{PGL}_{26}$ with motivic weight* 36*, such that the Sato-Tate group $\mathrm{H}(\pi)$ of π is isomorphic to the compact Lie group F_4.*

Proof. We fix $\lambda = \varpi_1 + 2\varpi_2 + 2\varpi_4$. In Table A.3, we find that $\dim \mathcal{A}_{\mathrm{V}_\lambda}(\mathbf{F}_4) = 1$. We denote the unique automorphic representation contributing to $\mathcal{A}_{\mathrm{V}_\lambda}(\mathbf{F}_4)$ by π_0 and its corresponding discrete global Arthur parameter by ψ_0. The eigenvalues of $\mathrm{c}_\infty(\pi_0) = \lambda + \rho$ are:

$$-18, -16, -13, -12, -9, -9, -7, -6, -5, -4, -3, -2, 0, 0, 2, 3, 4, 5, 6, 7, 9, 9, 12, 13, 16, 18.$$

Now it suffices to show that $\mathrm{H}(\psi_0) = \mathrm{F}_4$.

We can exclude some possibilities of $\mathrm{H}(\psi_0)$ and endoscopic types by an argument of motivic weights. For example, if $\mathrm{H}(\psi_0) = \mathrm{A}_1^{[17,9]}$ and $\psi_0 = \mathrm{Sym}^{16}\,\pi \oplus \mathrm{Sym}^8\,\pi$ for some $\pi \in \Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_2)$, then $\mathrm{w}(\pi_0) = 16\mathrm{w}(\pi) \geq 16 \times 11 = 176$, which contradicts with $\mathrm{w}(\pi_0) = 36$. We also notice that 1 is not an eigenvalue of $\mathrm{c}_\infty(\pi_0)$, thus ψ_0 does not have irreducible summands of the form

$$\pi[d], \text{ where } \pi \in \Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_n), n \equiv 1 \bmod 2 \text{ and } d \geq 3.$$

Now we list all possible types for ψ_0:

(1) ψ_0 is a stable and tempered parameter;
(2) $\psi_0 = (\bigoplus_{1\leq i<j\leq 3} \pi_i \otimes \pi_j) \oplus (\bigoplus_{1\leq i\leq 3} \pi_i[2]) \oplus [1] \oplus [1], \pi_1, \pi_2, \pi_3 \in \Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_2)$;
(3) $\psi_0 = (\bigoplus_{1\leq i<j\leq 4} \pi_i \otimes \pi_j) \oplus [1] \oplus [1], \pi_1, \pi_2, \pi_3, \pi_4 \in \Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_2)$;
(4) $\psi_0 = \wedge^*\pi \oplus (\pi \otimes \tau) \oplus \pi[2] \oplus \tau[2] \oplus [1], \pi \in \Pi_{\mathrm{alg}}^{\mathbf{Sp}_4}(\mathbf{PGL}_4), \tau \in \Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_2)$;
(5) $\psi_0 = \wedge^*\pi \oplus (\pi \otimes \tau_1) \oplus (\pi \otimes \tau_2) \oplus (\tau_1 \otimes \tau_2) \oplus [1], \pi \in \Pi_{\mathrm{alg}}^{\mathbf{Sp}_4}(\mathbf{PGL}_4), \tau_1, \tau_2 \in \Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_2)$;
(6) $\psi_0 = \wedge^*\pi \oplus \pi[2], \pi \in \Pi_{\mathrm{alg}}^{\mathbf{Sp}_6}(\mathbf{PGL}_6)$;
(7) $\psi_0 = \wedge^*\pi \oplus (\pi \otimes \tau), \pi \in \Pi_{\mathrm{alg}}^{\mathbf{Sp}_6}(\mathbf{PGL}_6), \tau \in \Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_2)$;
(8) $\psi_0 = \pi \oplus \mathrm{Spin}^+\,\pi \oplus \mathrm{Spin}^-\,\pi \oplus [1] \oplus [1], \pi \in \Pi_{\mathrm{alg}}^{\mathbf{SO}_8}(\mathbf{PGL}_8)$;
(9) $\psi_0 = \pi \oplus \mathrm{Spin}\,\pi \oplus [1], \pi \in \Pi_{\mathrm{alg}}^{\mathbf{SO}_9}(\mathbf{PGL}_9)$.

The definitions of some notations like $\wedge^*, \mathrm{Spin}^\pm$ can be found in Section 7.3. Now we are going to show that ψ_0 can not be of the types listed above except (1).

Type (2): The Hodge weights of the irreducible summand $\pi_i[2], i = 1, 2, 3$ are $\mathrm{w}(\pi_i) \pm 1$, thus there are two consecutive integers $\frac{\mathrm{w}(\pi_i)\pm 1}{2}$ in the eigenvalues of $\mathrm{c}_\infty(\pi_0)$. The possible $\mathrm{w}(\pi_i)$'s are $5, 7, 9, 11, 13, 25$. However, $\Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_2)$ contains no representations with motivic weights $5, 7, 9, 13$, thus we are unable to find three different $\mathrm{w}(\pi_i)$. If $\pi_i \simeq \pi_j$ for some i, j, then $\pi_i \otimes \pi_j$ has two zero weights, which is a contradiction!

Type (3): By the same argument for type (2), ψ_0 can not be of this type.

Type (4): Denote the Hodge weights of $\pi \in \Pi_{\mathrm{alg}}^{\mathbf{Sp}_4}(\mathbf{PGL}_4)$ by $w_1 > w_2$. By a similar argument for type (2), we can see that $w_1, w_2 \in \{5, 7, 9, 11, 13, 25\}$. Via the help of [ChenevierRenard, 2015, Table 5], we have $w_1 = 25$ and $w_2 \in \{5, 7, 9\}$, thus $\mathrm{w}(\tau)$ must be 11. Since $(w_1 + w_2)/2$ has to be an eigenvalue of $\mathrm{c}_\infty(\pi_\infty)$, the smaller Hodge weight w_2 can only be 7.

Now we use Arthur's multiplicity formula. In this case

$$\mathrm{H}(\psi_0) = \left(\mathrm{A}_1^{[2^6,1^{14}]} \times \mathrm{A}_1^{[2^6,1^{14}]} \times \mathrm{Sp}(2)\right)/\mu_2^\Delta,$$

and by Section 5.6.6 the global component group $\mathrm{C}_{\psi_0} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. We take a set of generators $\{\sigma = (1, 1, -1), \sigma_1 = (-1, 1, 1)\}$ of C_{ψ_0}. The restriction of the adjoint representation $\mathfrak{f}_4$ of F_4 along ψ_0 is isomorphic to

$$\mathrm{Sym}^2 \pi \oplus (\wedge^* \pi \otimes \tau[2]) \oplus (\pi \otimes \tau) \oplus \pi[2] \oplus \mathrm{Sym}^2 \tau \oplus [3].$$

By Proposition 6.6.4 we have:

$$\varepsilon_{\psi_0}(\sigma) = \varepsilon(\pi) = \varepsilon(\mathbf{I}_7) \cdot \varepsilon(\mathbf{I}_{25}) = -1.$$

On the other side $\mu_1 = 36$ comes from $\pi \otimes \tau$ and $\mu_4 = 24$ comes from $\pi[2]$. The element σ acts on $\pi \otimes \tau$ and $\pi[2]$ both by -1, thus $\rho_{\psi_0}^\vee(\sigma) = 1$ by Proposition 7.2.1. By Arthur's multiplicity formula, the corresponding representation has multiplicity 0 in $\mathcal{L}_{\mathrm{disc}}(\mathbf{F}_4)$.

Type (5): Denote the Hodge weights of $\pi \in \Pi_{\mathrm{alg}}^{\mathbf{Sp}_4}(\mathbf{PGL}_4)$ by $w_1 > w_2$, and assume that $\mathrm{w}(\tau_1) > \mathrm{w}(\tau_2)$. Since $36 \geq w_1 + \mathrm{w}(\tau_1) \geq w_1 + 15$, we have $w_1 \leq 21$, thus $(w_1, w_2) = (19, 7)$ or $(21, 5), (21, 9), (21, 13)$ by [ChenevierRenard, 2015, Table 5]. We also need $(w_1 \pm w_2)/2$ to be eigenvalues of $\mathrm{c}_\infty(\pi_0)$, so $(w_1, w_2) = (19, 7)$. However, the equalities $36 = w_1 + \mathrm{w}(\tau_1)$ and $32 = w_1 + \mathrm{w}(\tau_2)$ imply that $\mathrm{w}(\tau_1) = 17, \mathrm{w}(\tau_2) = 13$, which contradicts with the non-existence of representations in $\Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_2)$ with Hodge weight 13.

Type (6): Denote the Hodge weights of $\pi \in \Pi_{\mathrm{alg}}^{\mathbf{Sp}_6}(\mathbf{PGL}_6)$ by $w_1 > w_2 > w_3$. We have three pairs of consecutive integers $\frac{w_i \pm 1}{2}$ in the eigenvalues of $\mathrm{c}_\infty(\pi_0)$, thus for $i = 1, 2, 3$ we have $w_i \in \{5, 7, 9, 11, 13, 25\}$. By [ChenevierRenard, 2015, Table 6], (w_1, w_2, w_3) must be $(25, 13, 7)$. However, $\wedge^* \pi$ has 38 as its weight, which is a contradiction.

Type (7): Denote the Hodge weights of $\pi \in \Pi_{\mathrm{alg}}^{\mathbf{Sp}_6}(\mathbf{PGL}_6)$ by $w_1 > w_2 > w_3$. Since $36 \geq w_1 + \mathrm{w}(\tau) \geq w_1 + 11$, we have $23 \leq w_1 \leq 25$. Combining $36 \geq w_1 + w_2$ with [ChenevierRenard, 2015, Table 6], we get $(w_1, w_2, w_3) = (23, 13, 5)$. However, $\mathrm{w}(\tau) = 32 - w_1 = 9 < 11$, which is a contradiction.

Type (8): Denote the Hodge weights of $\pi \in \Pi_{\mathrm{alg}}^{\mathbf{SO}_8}(\mathbf{PGL}_8)$ by $w_1 > w_2 > w_3 > w_4$. The multiset

$$\{\pm w_1/2, \pm w_2/2, \pm w_3/2, \pm w_4/2, \frac{\pm w_1 \pm w_2 \pm w_3 \pm w_4}{4}, 0, 0\}$$

coincides with the multiset of eigenvalues of $\mathrm{c}_\infty(\pi_0)$. The solutions to this system of equations are

$$(w_1, w_2, w_3, w_4) = (26, 24, 18, 4), (32, 18, 12, 10), (36, 14, 8, 6).$$

By the method of Chenevier-Taïbi in [ChenevierTaïbi, 2020], there are no representations in $\Pi_{\mathrm{alg}}^{\mathbf{SO}_8}(\mathbf{PGL}_8)$ with these Hodge weights.

Type (9): By the same argument for type (9), we get the Hodge weights of $\pi \in \Pi_{\text{alg}}^{\mathbf{SO}_9}(\mathbf{PGL}_9)$:

$$(w_1, w_2, w_3, w_4) = (26, 24, 18, 4), (32, 18, 12, 10), (36, 14, 8, 6).$$

Again by the method in [ChenevierTaïbi, 2020], there are no representations in $\Pi_{\text{alg}}^{\mathbf{SO}_9}(\mathbf{PGL}_9)$ with these Hodge weights.

In conclusion, the discrete global Arthur parameter ψ_0 is a stable and tempered parameter, *i.e.* $\mathrm{H}(\psi_0) = \mathrm{F}_4$. Composing this ψ_0 with the 26-dimensional irreducible representation $r_0 : \widehat{\mathbf{F}_4}(\mathbb{C}) \to \mathbf{SL}_{26}(\mathbb{C})$, we get an irreducible 26-dimensional representation of $\mathcal{L}_{\mathbb{Z}}$, and its corresponding cuspidal representation of $\mathbf{PGL}_{26}$ is the desired one. □

For each dominant weight λ of $\mathbf{F}_4$, we define $\Psi_\lambda(\mathbf{F}_4)$ to be the set

$$\{\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4) \,|\, \pi_\psi \in \Pi_{\mathrm{disc}}(\mathbf{F}_4) \text{ and } (\pi_\psi)_\infty \simeq \mathrm{V}_\lambda\}.$$

In Table A.6 and Table A.7, we list the elements of $\Psi_\lambda(\mathbf{F}_4)$ for weights λ such that $\mathrm{w}(\lambda) \leq 36$ and $\Psi_\lambda(\mathbf{F}_4) \neq \emptyset$, where we use the following notations:

Notation 7.4.4. For a representation π in $\Pi_{\text{alg}}^{\mathbf{Sp}_{2n}}(\mathbf{PGL}_{2n}), n = 1, 2, 3$ with Hodge weights $w_1 > w_2 > \cdots > w_n$, we denote it by $\Delta_{w_1,\ldots,w_n}$. If there are $k \geq 1$ equivalence classes of cuspidal representations with these Hodge weights, we give them a superscript $\Delta_{w_1,\ldots,w_n}^{(k)}$, meaning that in this case we have k different choices of cuspidal representations. Similarly, for k different representations π in $\Pi_{\text{alg}}^{\mathbf{SO}_9}(\mathbf{PGL}_9)$ or $\Pi_{\text{alg}}^{\mathbf{G}_2}(\mathbf{PGL}_7)$ with Hodge weights $w_1 > \cdots > w_n$, where $n = 3$ or 4, we denote them by $\Delta_{w_1,\ldots,w_n,0}^{(k)}$ and omit the superscript when $k = 1$, *i.e.* the cuspidal representation with these Hodge weights is unique up to equivalence.

7.5 Some related problems

In this section we explain some representation-theoretic problems motivated by our conjectural classification of discrete global Arthur parameters for $\mathbf{F}_4$.

7.5.1 Theta correspondence between $\mathbf{PGL}_2$ and $\mathbf{F}_4$

Inside an exceptional group $\mathbf{E}_{7,3}$ of Lie type E_7 and $\mathbb{Q}$-rank 3, which is split over every finite prime p, there is a reductive dual pair $\mathbf{PGL}_2 \times \mathbf{F}_4$, so we have an exceptional theta correspondence between representations of $\mathbf{PGL}_2$ and $\mathbf{F}_4$.

For a level one cuspidal automorphic representation $\pi \in \Pi_{\text{alg}}^{\perp}(\mathbf{PGL}_2)$, by Savin's work on this exceptional theta correspondence [Savin, 1994], if the theta lift $\Theta(\pi)$ of π to $\mathbf{F}_4$ is nonzero, then its corresponding discrete global Arthur parameter is $\psi = \pi[6] \oplus [9] \oplus [5]$. By Proposition 7.3.6, we see that $\mathrm{m}(\pi_\psi)$ is always 1, admitting Arthur's conjectures. This predicts that the global theta lift $\Theta(\pi)$ is nonzero for any $\pi \in \Pi_{\text{alg}}^{\perp}(\mathbf{PGL}_2)$, and we will prove this result in Chapter 8.

Remark 7.5.1. For $\pi \in \Pi_{\text{alg}}^{\perp}(\mathbf{PGL}_2)$, the archimedean theta lift of π_∞ is isomorphic to the irreducible representation $\mathrm{V}_{n\varpi_4}$ of F_4 for some n. For readers interested in this exceptional theta correspondence, we list in Table A.4 the dimensions of $\mathrm{V}_{n\varpi_4}^{\mathcal{F}_{4,\mathrm{I}}(\mathbb{Z})}$ and $\mathrm{V}_{n\varpi_4}^{\mathcal{F}_{4,\mathrm{E}}(\mathbb{Z})}$ for $n \leq 40$.

7.5.2 Theta correspondence between $\mathbf{G}_2^{\mathrm{s}}$ and $\mathbf{F}_4$

Inside an exceptional group $\mathbf{E}_{8,4}$ of Lie type E_8 and $\mathbb{Q}$-rank 4, there is a reductive dual pair $\mathbf{G}_2^{\mathrm{s}} \times \mathbf{F}_4$, where $\mathbf{G}_2^{\mathrm{s}}$ is the generic fiber of the split Chevalley group of Lie type G_2.

In [Dalal, 2024], Dalal classifies level one quaternionic discrete automorphic representations of $\mathbf{G}_2^{\mathrm{s}}$. The exceptional theta correspondence from $\mathbf{G}_2^{\mathrm{s}}$ to $\mathbf{F}_4$ is functorial, so for a level one quaternionic discrete automorphic representation of $\mathbf{G}_2^{\mathrm{s}}$, if its global theta lift to $\mathbf{F}_4$ is nonzero, then we can describe the corresponding discrete global Arthur parameters in $\Psi_{\mathrm{AJ}}(\mathbf{F}_4)$. The discrete global Arthur parameters of $\mathbf{F}_4$ involving in this correspondence are:

- $\mathrm{Sym}^2\,\pi[3] \oplus \pi[4] \oplus \pi[2] \oplus [5]$, $\pi \in \Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_2)$,
- $\mathrm{Sym}^2\,\pi_1[3] \oplus (\pi_1 \otimes \pi_2[3]) \oplus [5]$, where $\pi_1, \pi_2 \in \Pi_{\mathrm{alg}}^{\perp}(\mathbf{PGL}_2)$ satisfy $\mathrm{w}(\pi_2) = 3\mathrm{w}(\pi_1) + 2$,
- and $\pi[3] \oplus [5]$, where $\pi \in \Pi_{\mathrm{alg}}^{\mathbf{G}_2}(\mathbf{PGL}_7)$.

According to Proposition 7.3.7, Proposition 7.3.10 and Proposition 7.3.13, for every ψ among these discrete global Arthur parameters, we have $\mathrm{m}(\pi_\psi) = 1$. This predicts that the global theta lift of any level one quaternionic discrete automorphic representation of $\mathbf{G}_2^{\mathrm{s}}$ to $\mathbf{F}_4$ is nonzero, which is proved by Pollack in [Pollack, 2023, §8].

Remark 7.5.2. For any quaternionic discrete series π of $\mathbf{G}_2^{\mathrm{s}}(\mathbb{R})$, the archimedean theta lift of π is isomorphic to the irreducible representation $\mathrm{V}_{n\varpi_3}$ of F_4 for some n. For readers interested in this exceptional theta correspondence, we list in Table A.5 the dimensions of $\mathrm{V}_{n\varpi_3}^{\mathcal{F}_{4,\mathrm{I}}(\mathbb{Z})}$ and $\mathrm{V}_{n\varpi_3}^{\mathcal{F}_{4,\mathrm{E}}(\mathbb{Z})}$ for $n \leq 30$.

Chapter 8

Exceptional theta correspondence $\mathbf{F}_4 \times \mathbf{PGL}_2$ for level one automorphic representations

This chapter corresponds to the preprint [Shan, 2025].

Abstract

Let $\mathbf{F}_4$ be the unique (up to isomorphism) connected semisimple algebraic group over $\mathbb{Q}$ of type F_4, with compact real points and split over $\mathbb{Q}_p$ for all primes p. A conjectural computation [Shan, 2024, Proposition 6.3.6] predicts the existence of a family of level one automorphic representations of $\mathbf{F}_4$, which are expected to be functorial lifts of cuspidal representations of $\mathbf{PGL}_2$ associated with Hecke eigenforms. In this paper, we study the exceptional theta correspondence for $\mathbf{F}_4 \times \mathbf{PGL}_2$, and establish the existence of such a family of automorphic representations for $\mathbf{F}_4$. Motivated by [Pollack, 2023], our main tool is a notion of "exceptional theta series" on $\mathbf{PGL}_2$, arising from certain automorphic representations of $\mathbf{F}_4$. These theta series are holomorphic modular forms on $\mathbf{SL}_2(\mathbb{Z})$, with explicit Fourier expansions, and span the entire space of level one cusp forms.

Contents

8.1 Introduction

Since the last century, automorphic representations of general linear groups and classical groups have been widely studied. For those of *exceptional groups*, *i.e.* algebraic groups with Lie type $\mathrm{G}_2, \mathrm{F}_4, \mathrm{E}_6, \mathrm{E}_7$ or E_8, most of the known results are about the smallest exceptional group $\mathbf{G}_2$, either split or anisotropic. In this paper, we will study a family of automorphic representations for $\mathbf{F}_4$, the unique (up to isomorphism) connected semisimple algebraic group over $\mathbb{Q}$ of type F_4, with compact real points and split over $\mathbb{Q}_p$ for every prime p.

8.1.1 Motivation from [Shan, 2024]

In [Shan, 2024], we compute the number of *level one* automorphic representations for $\mathbf{F}_4$, *i.e.* unramified at every finite place, with any given arbitrary archimedean component. Furthermore, the *discrete global Arthur parameters* of these automorphic representations are classified *conjecturally*, admitting the existence of the (level one) Langlands group and Arthur's multiplicity formula [Arthur, 1989]. In particular, we conjecture the existence of a specific family of automorphic representations for $\mathbf{F}_4$, which are related to classical modular forms for $\mathbf{SL}_2(\mathbb{Z})$. Before recalling this statement, we introduce some notations:

- Let ϖ_4 be the highest weight of the 26-dimensional irreducible representation of $\mathbf{F}_4(\mathbb{R})$.
- There is a morphism $\mathbf{Sp}_6(\mathbb{C}) \times \mathbf{SL}_2(\mathbb{C}) \to \widehat{\mathbf{F}_4}(\mathbb{C}) = \mathbf{F}_4(\mathbb{C})$ whose kernel is a cyclic group of order 2, the image of this morphism is a maximal proper regular closed subgroup of $\mathbf{F}_4(\mathbb{C})$ (see [Shan, 2024, §4.3.2]). Denote by ι the morphism:

$$\mathbf{SL}_2(\mathbb{C}) \times \mathbf{SL}_2(\mathbb{C}) \xhookrightarrow{\text{(principal embedding, id)}} \mathbf{Sp}_6(\mathbb{C}) \times \mathbf{SL}_2(\mathbb{C}) \to \mathbf{F}_4(\mathbb{C}).$$

- Denote by e_p the conjugacy class of $\begin{pmatrix} p^{1/2} & \\ & p^{-1/2} \end{pmatrix}$ in $\mathbf{SL}_2(\mathbb{C})$.

Conjecture 8.1.1. *[Shan, 2024, Proposition 6.3.6] Let π be the level one algebraic automorphic representation of $\mathbf{PGL}_2$ associated to a cuspidal Hecke eigenform of weight $2n+12$ for $\mathbf{SL}_2(\mathbb{Z})$, and c_p the Satake parameter of π_p, viewed as a semisimple conjugacy class in $\widehat{\mathbf{PGL}_2}(\mathbb{C}) = \mathbf{SL}_2(\mathbb{C})$. There exists a level one automorphic representation Π of $\mathbf{F}_4$ such that:*

- *$\Pi_\infty \simeq \mathrm{V}_{n\varpi_4}$, the irreducible representation of $\mathbf{F}_4(\mathbb{R})$ with highest weight $n\varpi_4$;*
- *for every prime p, the Satake parameter of Π_p is the conjugacy class of $\iota(e_p, c_p)$.*

Motivated by the Langlands functoriality principle, the automorphic representation Π in Conjecture 8.1.1 is expected to be a functorial lift of π with respect to the embedding

$$i : \widehat{\mathbf{PGL}_2} = \mathbf{SL}_2 \xhookrightarrow{(1,\mathrm{id})} \mathbf{Sp}_6 \times \mathbf{SL}_2 \hookrightarrow \widehat{\mathbf{F}_4}. \tag{8.1}$$

One useful tool for constructing functorial lifts is the *theta correspondence*, which studies the restriction of a *minimal representation* to reductive dual pairs. There exists a reductive dual pair $\mathbf{PGL}_2 \times \mathbf{F}_4$ inside certain algebraic group $\mathbf{E}_7$ of Lie type E_7 (see Section 8.2 for more details). For the theta correspondence associated with this dual pair over a characteristic 0 local field, one already has the following results (see also Section 8.3):

- Over $\mathbb{R}$, Gross and Savin describe the restriction of the minimal representation of $\mathbf{E}_7(\mathbb{R})$ to $\mathbf{PGL}_2(\mathbb{R}) \times \mathbf{F}_4(\mathbb{R})$ [GrossSavin, 1998, Proposition 3.2], which shows that the theta lift $\Theta(\pi_\infty)$ of π_∞ is isomorphic to $\mathrm{V}_{n\varpi_4}$;
- Over a p-adic field, this theta correspondence is studied by Karasiewicz and Savin in [Savin, 1994; KarasiewiczSavin, 2023]. In particular, they demonstrate that the theta lift $\Theta(\pi_p)$ of the unramified tempered principal series representation π_p is irreducible and has the desired Satake parameter $\iota(e_p, c_p)$.

Based on these local results, it is natural to expect that the functorial lift Π is exactly the global theta lift $\Theta(\pi)$ of π to $\mathbf{F}_4$. The main result in this paper confirms this expectation:

Theorem 8.1.2. *(Theorem 8.6.12) The global theta lift $\Theta(\pi)$ is a non-zero irreducible automorphic representation of $\mathbf{F}_4$, and satisfies the local-global compatibility of theta correspondence $\Theta(\pi) \simeq \otimes'_v \Theta(\pi_v)$. In particular, Conjecture 8.1.1 holds.*

8.1.2 Exceptional theta series

Our main tool is to develop a notion of "*exceptional theta series*", motivated by Pollack's construction of Siegel modular forms for $\mathbf{Sp}_6(\mathbb{Z})$. This is a variant of the classical weighted theta series developed by Jacobi and Hecke, and gives an explicit theta lift from certain automorphic forms of $\mathbf{F}_4$ to $\mathbf{PGL}_2$.

8.1.2.1 Classical theta series

We first recall the classical construction of theta series. Let L be an even unimodular lattice in the Euclidean space $\mathbb{R}^n$, *i.e.* a self-dual lattice for any element v of which the scalar product $v.v$ is even. A well-known result states that the series

$$\vartheta_L(z) = \sum_{v\in L} q^{\frac{v.v}{2}}, \text{ where } q = e^{2\pi i z},\ z \in \mathcal{H} = \{x+iy \mid y>0\},$$

is a modular form of level $\mathbf{SL}_2(\mathbb{Z})$ and weight $n/2$. One can weight this theta series by a homogeneous harmonic polynomial P of degree d over $\mathbb{R}^n$ [Hecke, 1940]:

$$\vartheta_{L,P}(z) = \sum_{v\in L} P(v) q^{\frac{v.v}{2}}, \tag{8.2}$$

and the resulting weighted theta series is a modular form for $\mathbf{SL}_2(\mathbb{Z})$ of weight $\frac{n}{2}+d$. It is a cusp form when $d>0$, and Waldspurger shows in [Waldspurger, 1979] that for a fixed pair of integers (n,d), the space $\mathrm{S}_{\frac{n}{2}+d}(\mathbf{SL}_2(\mathbb{Z}))$ of weight $\frac{n}{2}+d$ cusp forms is spanned by:

$$\{\vartheta_{L,P} \mid L \subseteq \mathbb{R}^n \text{ is an even unimodular lattice, and } P \in \mathscr{H}_d(\mathbb{R}^n)\},$$

where $\mathscr{H}_d(\mathbb{R}^n)$ is the space of homogeneous harmonic polynomials of degree d over $\mathbb{R}^n$.

8.1.2.2 Corresponding structures in the exceptional case

We want to produce a family of modular forms analogous to (8.2), starting from automorphic representations for $\mathbf{F}_4$ with archimedean component $\mathrm{V}_{n\varpi_4}$. The table below highlights the corresponding structures in the classical and exceptional settings:

	classical case	exceptional case
underlying space	Euclidean space $\mathbb{R}^n$	Euclidean Albert $\mathbb{R}$-algebra $\mathrm{J}_{\mathbb{R}}$
group of automorphisms	$\mathbf{O}_n(\mathbb{R})$	$\mathbf{F}_4(\mathbb{R})$
integral structure	even unimodular lattice	Albert lattice
homogeneous polynomials	harmonic polynomials	a polynomial model of $\mathrm{V}_{n\varpi_4}$

Table 8.1: Comparison between classical and exceptional cases

We briefly explain the objects appearing in Table 8.1, and the details will be provided in Section 8.2.2 and Section 8.2.3:

- The 27-dimensional *Euclidean Albert $\mathbb{R}$-algbera (or exceptional Jordan $\mathbb{R}$-algebra)* $\mathrm{J}_{\mathbb{R}} = \mathrm{Her}_3(\mathbb{O}_{\mathbb{R}})$ is the space of "Hermitian" 3-by-3 matrices over the *real octonion division algebra* $\mathbb{O}_{\mathbb{R}}$, equipped with the distinguished element $\mathrm{I} = \mathrm{diag}(1,1,1)$, the adjoint map $\# : \mathrm{J}_{\mathbb{R}} \to \mathrm{J}_{\mathbb{R}}$, and the determinant $\det : \mathrm{J}_{\mathbb{R}} \to \mathbb{R}$. Precisely, together with these structures, $\mathrm{J}_{\mathbb{R}}$ is a *cubic Jordan $\mathbb{R}$-algebra* and furthermore it is an *Albert $\mathbb{R}$-algebra*. We call it Euclidean because its underlying vector space admits a symmetric inner product $(A,B) = \frac{1}{2}\operatorname{Tr}(AB+BA)$ that is positive definite.
- The group of Albert $\mathbb{R}$-algebra automorphisms of $\mathrm{J}_{\mathbb{R}}$ is the real points $\mathbf{F}_4(\mathbb{R})$ of $\mathbf{F}_4$, *i.e.* $\mathbf{F}_4(\mathbb{R}) = \{g \in \mathrm{GL}(\mathrm{J}_{\mathbb{R}}) \,|\, g\mathrm{I} = \mathrm{I},\ \det(gA) = \det(A),\ \text{for any } A \in \mathrm{J}_{\mathbb{R}}\}$.
- By an *Albert lattice*, we mean a lattice $J \subseteq \mathrm{J}_{\mathbb{R}}$ satisfying that $\mathrm{I} \in J$, J is stable under $\#$, $\det(J) \subseteq \mathbb{Z}$, and $(J, \mathrm{I}, \#, \det)$ is an *Albert $\mathbb{Z}$-algebra*.
- In Section 8.4.1.2, we describe a polynomial model $\mathrm{V}_n(\mathrm{J}_{\mathbb{C}})$ of $\mathrm{V}_{n\varpi_4}$: the space spanned by degree n homogeneous polynomials over $\mathrm{J}_{\mathbb{R}}$ of the form:

$$X \mapsto (X,A)^n\,,\ \text{where } 0 \neq A \in \mathrm{J}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C},\ A^2 = 0,\ \operatorname{Tr}(A) = 0.$$

8.1.2.3 Weighting the theta series constructed by Elkies-Gross

The starting point of the exceptional theta series associated with $\mathrm{J}_{\mathbb{R}}$ is the work of Elkies and Gross [ElkiesGross, 1996].

Let $\mathcal{J}$ be the set of Albert lattices, and equip it with the natural $\mathbf{F}_4(\mathbb{R})$-action. This set is the disjoint union of two $\mathbf{F}_4(\mathbb{R})$-orbits [Gross, 1996, Proposition 5.3]. We take a set of representatives $\{J_1, J_2\}$ for these two orbits, where $J_1 = \mathrm{J}_{\mathbb{Z}}$ (see Example 8.2.13) is taken as the base point of $\mathcal{J}$. For $J = J_1$ or J_2, Elkies and Gross construct the following theta series:

$$\vartheta_J(z) = 1 + 240 \sum_{\substack{J \ni T \geq 0,\\ \operatorname{rank}(T)=1}} \sigma_3(\mathrm{c}_J(T)) q^{\operatorname{Tr}(T)},\ q = e^{2\pi i z},\ z \in \mathcal{H},$$

where $c_J(T)$ is the largest integer c such that $T/c \in J$, and $\sigma_3(n) = \sum_{d|n} d^3$. This theta series is a modular form of weight 12 for $\mathbf{SL}_2(\mathbb{Z})$. Moreover,

$$\vartheta_{J_1} = E_{12} - \frac{65520}{691}\Delta,\ \vartheta_{J_2} = E_{12} + \frac{432000}{691}\Delta,$$

where E_{12} is the normalized Eisenstein series of weight 12, and Δ is the discriminant modular form.

Remark 8.1.3. The coefficient $240\sigma_3(\mathrm{c}(T))$ appearing in the Fourier expansion of ϑ_J comes from *Kim's modular form* F_{Kim}, an Eisenstein series on the exceptional tube domain $\mathcal{H}_{\mathrm{J}}$ (see Section 8.4.2.1), which is constructed in [Kim, 1993] and serves as our source for producing theta series.

We extend the construction of Elkies-Gross to *weighted exceptional theta series* as follows:

Theorem 8.1.4. *(Theorem 8.5.2, Corollary 8.5.5) For any Albert lattice $J \in \mathscr{J}$ and a polynomial $P \in \mathrm{V}_n(\mathrm{J}_{\mathbb{C}})$, the theta series*

$$\vartheta_{J,P}(z) := \sum_{\substack{J \ni T \geq 0, \\ \mathrm{rank}(T)=1}} \sigma_3(\mathrm{c}_J(T))P(T)q^{\mathrm{Tr}(T)} \tag{8.3}$$

is a modular form of weight $2n+12$ for $\mathbf{SL}_2(\mathbb{Z})$. When $n = \deg(P) > 0$, $\vartheta_{J,P}$ is a cusp form.

Our proof of Theorem 8.1.4 follows Pollack's method for the proof of [Pollack, 2023, Theorem 1.1.1]. For the automorphic form (or precisely, *algebraic modular form*) of $\mathbf{F}_4$ associated with J and P, we construct its global theta lift to $\mathbf{PGL}_2$, taking certain (iterated) differential of Kim's modular form F_{Kim} as the kernel function. Then we show that this global theta lift arises from a holomorphic modular form, whose Fourier expansion is exactly (8.3).

Remark 8.1.5. Here we explain briefly how we describe the global theta lift from $\mathbf{F}_4$ to $\mathbf{PGL}_2$ in terms of exceptional theta series, and more details can be found in Section 8.4.1.1. The space $\mathcal{A}_{\mathrm{V}_{n\varpi_4}}(\mathbf{F}_4)$ of *level one "vector-valued" automorphic form of* $\mathbf{F}_4$ *with weight* $\mathrm{V}_{n\varpi_4}$ can be identified with the space of functions $f : \mathscr{J} \to \mathrm{V}_n(\mathrm{J}_{\mathbb{C}})$ satisfying $f(gJ) = g.f(J)$ for any $g \in \mathbf{F}_4(\mathbb{R})$ and $J \in \mathscr{J}$. The global theta lift of f to $\mathbf{PGL}_2$ is the modular form

$$\frac{1}{|\Gamma_1|}\vartheta_{J_1,f(J_1)} + \frac{1}{|\Gamma_2|}\vartheta_{J_2,f(J_2)} \in \mathrm{M}_{2n+12}(\mathbf{SL}_2(\mathbb{Z})),$$

where Γ_i is the automorphism group of the Albert $\mathbb{Z}$-algebra J_i, $i = 1, 2$.

8.1.3 Strategy towards Theorem 8.1.2

Now we illustrate our strategy for the proof of Theorem 8.1.2.

Let $\varphi \simeq \otimes\varphi_v \in \pi \simeq \otimes'_v\pi_v$ be the automorphic form of $\mathbf{PGL}_2$ associated to a Hecke eigenform $f \in \mathrm{S}_{2n+12}(\mathbf{SL}_2(\mathbb{Z}))$. We want to show that its global theta lift $\Theta_\phi(\varphi)$, with respect to some vector ϕ in the *minimal representation* of $\mathbf{E}_7(\mathbb{A})$, is non-zero. For this goal, we compute the $\mathbf{Spin}_9$*-period integral* of $\Theta_\phi(\varphi)$, where $\mathbf{Spin}_9$ is a maximal proper regular closed subgroup of

$\mathbf{F}_4$. The $\mathbf{Spin}_9$-period of an automorphic form f on $[\mathbf{F}_4] = \mathbf{F}_4(\mathbb{Q})\backslash\mathbf{F}_4(\mathbb{A})$ is defined as follows, where dg is taken to be the Tamagawa measure:

$$\mathcal{P}_{\mathbf{Spin}_9}(f) := \int_{\mathbf{Spin}_9(\mathbb{Q})\backslash\mathbf{Spin}_9(\mathbb{A})} f(g)dg.$$

Remark 8.1.6. One motivation for considering this $\mathbf{Spin}_9$-period is the global conjecture of Sakellaridis-Venkatesh [SakellaridisVenkatesh, 2017]. The homogeneous space $\mathbf{X} = \mathbf{Spin}_9\backslash\mathbf{F}_4$ is a spherical variety whose *dual group* is $\mathbf{G}_{\mathbf{X}}^{\vee} = \mathbf{SL}_2$, equipped with the embedding $i : \mathbf{G}_{\mathbf{X}}^{\vee} \to \widehat{\mathbf{F}_4}$ as described in (8.1). Roughly speaking, the conjecture of Sakellaridis-Venkatesh predicts that the cuspidal automorphic representations of $\mathbf{F}_4$ with non-zero $\mathbf{Spin}_9$-periods arise from functorial lifts with respect to the embedding $i : \widehat{\mathbf{PGL_2}} \to \widehat{\mathbf{F}_4}$. Therefore, we expect the global theta lift $\Theta_\phi(\varphi)$ to have a non-zero $\mathbf{Spin}_9$-period (for some suitable choice of ϕ).

Using a see-saw duality argument, an *exceptional Siegel-Weil formula* that we prove in Section 8.6.1 and a standard calculation of Rankin-Selberg integral (Section 8.6.2), we rewrite the $\mathbf{Spin}_9$-period of $\Theta_\phi(\varphi)$ as an Eulerian integral over $\mathbf{PGL}_2(\mathbb{A})$. Moreover, we prove the following result, which verifies the prediction of Sakellaridis-Venkatesh [SakellaridisVenkatesh, 2017, §17; Sakellaridis, 2021, Table 1] for the global period associated with spherical variety $\mathbf{Spin}_9\backslash\mathbf{F}_4$:

Theorem 8.1.7. *(Corollary 8.6.9) For any smooth, holomorphic and spherical vector $\phi \simeq \otimes_v \phi_v$ in the minimal representation $\Pi_{\min} \simeq \otimes_v' \Pi_{\min,v}$ of $\mathbf{E}_7(\mathbb{A})$, the $\mathbf{Spin}_9$-period integral of $\Theta_\phi(\varphi)$ is equal to:*

$$\mathcal{P}_{\mathbf{Spin}_9}(\Theta_\phi(\varphi)) = \frac{\mathrm{L}(\pi, \frac{5}{2})\mathrm{L}(\pi, \frac{11}{2})}{\zeta(4)\zeta(8)} \cdot I_\infty(\phi_\infty, \varphi_\infty),$$

where $\mathrm{L}(\pi, s) = \mathrm{L}(f, \frac{2n+11}{2} + s)$ is the standard automorphic L-function of π (as an Euler product over all the finite places), and $I_\infty(\phi_\infty, \varphi_\infty)$ is an integral over $\mathbf{PGL}_2(\mathbb{R})$.

The L-factor in Theorem 8.1.7 is non-zero, thus the non-vanishing of $\mathcal{P}_{\mathbf{Spin}_9}(\Theta_\phi(\varphi))$ is equivalent to that of $I_\infty(\phi_\infty, \varphi_\infty)$.

For any Hecke eigenform f in $\mathrm{S}_{2n+12}(\mathbf{SL}_2(\mathbb{Z}))$, the associated automorphic form $\varphi \simeq \otimes_v \varphi_v$ in $\pi \simeq \otimes_v' \pi_v$ satisfies that φ_∞ is the unique (up to some scalar) lowest weight holomorphic vector of the discrete series $\mathcal{D}(2n+12) \simeq \pi_\infty$. Therefore, fixing a vector $\phi \in \Pi_{\min}$ as in Theorem 8.1.7, $\mathcal{P}_{\mathbf{Spin}_9}(\Theta_\phi(\varphi)) \neq 0$ for *any* such φ, if and only if it holds for *one* such φ. Hence to prove Theorem 8.1.2 it suffices to find a vector $\phi \in \Pi_{\min}$ satisfying the conditions in Theorem 8.1.7 and that $\mathcal{P}_{\mathbf{Spin}_9}(\Theta_\phi(\varphi)) \neq 0$, where φ is the automorphic form associated to certain Hecke eigenform $f \in \mathrm{S}_{2n+12}(\mathbf{SL}_2(\mathbb{Z}))$.

Our proof of the existence of $\phi \in \Pi_{\min}$ relies on an automorphic form of $\mathbf{F}_4$ that is invariant under $\mathbf{Spin}_9(\mathbb{R})$ and has a non-zero global theta lift to $\mathbf{PGL}_2$. As mentioned in Section 8.1.2, in this paper the global theta lifting from $\mathbf{F}_4$ to $\mathbf{PGL}_2$ is realized via exceptional theta series. If we take $J = J_1 = \mathrm{J}_\mathbb{Z}$ and P_n the unique non-zero $\mathbf{Spin}_9(\mathbb{R})$-invariant polynomial in $\mathrm{V}_n(\mathrm{J}_\mathbb{C})$, $n \geq 2$, then Theorem 8.1.4 gives us a weight $2n+12$ cusp form, which can be verified to be non-zero by analyzing the Fourier coefficient of q (Theorem 8.5.6). This implies that the automorphic form for $\mathbf{F}_4$ associated to $\mathrm{J}_\mathbb{Z}$ and P_n is the desired one!

As a corollary of Theorem 8.1.2, we have the following analogue of Waldspurger's result for classical theta series:

Theorem 8.1.8. *(Corollary 8.6.13) For any $n > 0$, the space $\mathrm{S}_{2n+12}(\mathbf{SL}_2(\mathbb{Z}))$ is spanned by the set of weighted exceptional theta series $\{\vartheta_{J,P} \mid J = J_1$ or $J_2, P \in \mathrm{V}_n(\mathrm{J}_\mathbb{C})\}$.*

We end the introduction with a short summary of the contents of this paper. We recall the necessary preliminaries on exceptional groups in Section 8.2, and the results on local theta correspondences in Section 8.3. We establish the global theta correspondence in Section 8.4, then study the Fourier expansions of exceptional theta series and prove Theorem 8.1.4 in Section 8.5. The last section Section 8.6 is for the proof of Theorem 8.1.2, Theorem 8.1.7 and Theorem 8.1.8.

Acknowledgements

The author thanks Gaëtan Chenevier for proposing this interesting project, and also thanks Edmund Karasiewicz, Nhat Hoang Le, Chuhao Huang, Aaron Pollack, Gordan Savin and Jialiang Zou for stimulating discussions. Special thanks go to Wee Teck Gan for pointing out the strategy towards Conjecture 8.1.1 to the author, the invitation for a visit to the National University of Singapore, and a lot of helpful discussions.

8.2 Preliminaries on exceptional groups

In this section we recall the definitions of two reductive algebraic groups $\mathbf{F}_4$ and $\mathbf{E}_7$ over $\mathbb{Q}$ and construct the following two reductive dual pairs[1] inside $\mathbf{E}_7$:

$$\mathbf{F}_4 \times \mathbf{PGL}_2 \text{ and } \mathbf{Spin}_9 \times \mathbf{SO}_{2,2}.$$

8.2.1 Octonions

We first recall the notion of octonions, which are crucial for defining exceptional groups.

Definition 8.2.1. An *octonion algebra* over a commutative ring k is a locally free k-module C of rank 8, equipped with a non-degenerate quadratic form $N : C \to k$ and a (possibly non-associative) k-algebra structure admitting a 2-sided identity element e, such that $N(xy) = N(x)N(y)$, $x, y \in C$. The quadratic form N is referred as the *norm* on C.

Now we recall some basic properties of octonion algebras, for which we refer to [SpringerVeldkamp, 2000]. There is a unique anti-involution of algebra $x \mapsto \overline{x}$ called the *conjugation* on C, with the property that $N(x) = x\overline{x} = \overline{x}x$. The *trace* is defined as the linear map $\mathrm{Tr} : C \to k$, $x \mapsto x + \overline{x}$. The symmetric bilinear form associated with N is $\langle x, y\rangle := N(x + y) - N(x) - N(y) = \mathrm{Tr}(x\overline{y})$.

Although the multiplication law of C is not associative, it is still *trace-associative* in the sense that $\mathrm{Tr}((xy)z) = \mathrm{Tr}(x(yz))$ for all $x, y, z \in C$, and we can define a trilinear form: $\mathrm{Tr}(xyz) := \mathrm{Tr}((xy)z) = \mathrm{Tr}(x(yz))$.

When considering octonion algebras over $\mathbb{R}$, we have the following classification result:

[1] Actually we do not prove in this paper that $\mathbf{Spin}_9 \times \mathbf{SO}_{2,2}$ is indeed a reductive dual pair, instead we only give a homomorphism $\mathbf{Spin}_9 \times \mathbf{SO}_{2,2} \to \mathbf{E}_7$, whose kernel is a central cyclic group of order 2.

Proposition 8.2.2. *[Adams, 1996, Theorem 15.1] Up to $\mathbb{R}$-algebra isomorphism, there is a unique octonion algebra $\mathbb{O}_{\mathbb{R}}$ over $\mathbb{R}$ whose norm* N *is positive definite, which is named as the* real octonion division algebra.

We choose a basis $\{e_0, e_1, \ldots, e_7\}$ as in [Gross, 1996, §4], where e_0 is the 2-sided identity element. Identify the real numbers $\mathbb{R}$ with the subalgebra $\mathbb{R}e_0$ of $\mathbb{O}_{\mathbb{R}}$, and denote the identity element e_0 by 1. On $\mathbb{O}_{\mathbb{R}}$, the conjugation is defined by $\overline{1} = 1$ and $\overline{e_i} = -e_i$ for each i. For any element $x = \sum_{i=0}^{7} x_i e_i \in \mathbb{O}_{\mathbb{R}}$, one has $\mathrm{N}(x) = \sum_{i=0}^{7} x_i^2$ and $\mathrm{Tr}(x) = 2x_0$.

Definition 8.2.3. *Cayley's definite octonion algebra* $\mathbb{O}_{\mathbb{Q}}$ is the sub-$\mathbb{Q}$-algebra of $\mathbb{O}_{\mathbb{R}}$, generated by $\{e_1, \ldots, e_7\}$, which is an octonion $\mathbb{Q}$-algebra with the norm $\mathrm{N}|_{\mathbb{O}_{\mathbb{Q}}}$.

The following definition gives an integral structure of Cayley's definite octonion algebra:

Definition 8.2.4. *Coxeter's integral order* $\mathbb{O}_{\mathbb{Z}}$ in $\mathbb{O}_{\mathbb{Q}}$ is the lattice spanned by $\mathbb{Z} \oplus \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_7$ and

$$h_1 = (1 + e_1 + e_2 + e_4)/2,\ h_2 = (1 + e_1 + e_3 + e_7)/2,$$
$$h_3 = (1 + e_1 + e_5 + e_6)/2,\ h_4 = (e_1 + e_2 + e_3 + e_5)/2,$$

which is an octonion $\mathbb{Z}$-algebra with the norm $\mathrm{N}|_{\mathbb{O}_{\mathbb{Z}}}$.

8.2.2 Albert algebras

In this section, we will not generally define either an Albert algebra or a (cubic) Jordan algebra, where precise definitions and details can be found in [GaribaldiPeterssonRacine, 2023]. Instead, we recall some examples and properties of Albert algebras that are important for us.

8.2.2.1 Hermitian 3-by-3 matrices over octonion algebras

Given an octonion algebra C over a commutative ring k, we consider the space $\mathrm{Her}_3(C)$ consisting of "Hermitian matrices" in $\mathrm{M}_3(C)$, *i.e.* matrices of the form

$$[a, b, c\,; x, y, z] := \begin{pmatrix} a & z & \overline{y} \\ \overline{z} & b & x \\ y & \overline{x} & c \end{pmatrix},\ a, b, c \in k,\ x, y, z \in C,$$

equipped with the following structures, where the maps are all polynomial laws in the sense of [Roby, 1963]:

- the identity matrix $\mathrm{I} = \mathrm{diag}(1, 1, 1)$,
- the adjoint map $\# : \mathrm{Her}_3(C) \to \mathrm{Her}_3(C)$, which is a quadratic map over k:

$$\begin{pmatrix} a & z & \overline{y} \\ \overline{z} & b & x \\ y & \overline{x} & c \end{pmatrix} \mapsto \begin{pmatrix} bc - N(x) & \overline{xy} - cz & zx - b\overline{y} \\ xy - c\overline{z} & ca - N(y) & \overline{yz} - ax \\ \overline{zx} - by & yz - a\overline{x} & ab - N(z) \end{pmatrix}, \tag{8.4}$$

- and the determinant, which is a cubic form over k:

$$\det([a,b,c\,;x,y,z]) := abc + \mathrm{Tr}(xyz) - aN(x) - bN(y) - cN(z). \tag{8.5}$$

One can construct more polynomial laws from these structures:

- There exists a symmetric bilinear form on $\mathrm{Her}_3(C)$:

$$(A,B) := (\nabla_A \det)(\mathrm{I}) \cdot (\nabla_B \det)(\mathrm{I}) - (\nabla_A \nabla_B \det)(\mathrm{I}).$$

  If $A = [a,b,c\,;x,y,z]$ and $B = [a',b',c'\,;x',y',z']$, then

$$(A,B) = aa' + bb' + cc' + \langle x,x'\rangle + \langle y,y'\rangle + \langle z,z'\rangle.$$

- The *trace* $\mathrm{Tr} : \mathrm{Her}_3(M) \to k$ is defined as $\mathrm{Tr}(A) = (A,\mathrm{I})$.
- The linearization of # gives a symmetric cross product $A \times B := (A+B)^\# - A^\# - B^\#$.

With these structures, we can define the rank of a matrix in $\mathrm{Her}_3(C)$:

Definition 8.2.5. The *rank* of $A \in \mathrm{Her}_3(C)$ is defined as follows:

- If $A = 0$, then $\mathrm{rank}(A) = 0$;
- If $A \neq 0$ and $A^\# = 0$, then $\mathrm{rank}(A) = 1$;
- If $A \neq 0, A^\# \neq 0$ and $\det(A) = 0$, then $\mathrm{rank}(A) = 2$;
- Otherwise, $\mathrm{rank}(A) = 3$.

8.2.2.2 Euclidean exceptional Jordan $\mathbb{R}$-algebra and its $\mathbb{Q}$-structure

One of the most important Albert algebras appearing in this article is the following:

Definition 8.2.6. The *Euclidean exceptional Jordan $\mathbb{R}$-algebra* (or *Euclidean Albert $\mathbb{R}$-algebra*) is defined to be $\mathrm{J}_\mathbb{R} := \mathrm{Her}_3(\mathbb{O}_\mathbb{R})$, where $\mathbb{O}_\mathbb{R}$ is the real octonion division algebra.

The space $\mathrm{J}_\mathbb{R}$ is a commutative but not associative $\mathbb{R}$-algebra with respect to the $\mathbb{R}$-bilinear multiplication law $A \circ B := \frac{1}{2}(AB + BA)$, where AB and BA denote the matrix multiplication, and I is its 2-sided identity element. One can easily check that the symmetric bilinear form $(\ ,\ )$ satisfies $(A,B) = \mathrm{Tr}(A \circ B)$ for any $A, B \in \mathrm{J}_\mathbb{R}$, and it is positive definite.

Definition 8.2.7. A matrix $A = [a,b,c\,;x,y,z] \in \mathrm{J}_\mathbb{R}$ is *positive semi-definite* if its seven minor determinants

$$a, b, c, bc - \mathrm{N}(x), ca - \mathrm{N}(y), ab - \mathrm{N}(z), \det(A)$$

are all non-negative, and we write $A \geq 0$. When these minor determinants are all positive, we call A *positive definite* and write $A > 0$.

Similarly to Definition 8.2.3, this algebra $\mathrm{J}_\mathbb{R}$ admits a rational structure:

Definition 8.2.8. The *Euclidean exceptional Jordan $\mathbb{Q}$-algebra* $\mathrm{J}_\mathbb{Q}$ is the sub-$\mathbb{Q}$-algebra of $\mathrm{J}_\mathbb{R}$ consisting of $[a,b,c\,;x,y,z], a,b,c \in \mathbb{Q}, x,y,z \in \mathbb{O}_\mathbb{Q}$ equipped with the multiplication $\circ$.

Notation 8.2.9. Here we fix some elements in $\mathrm{J}_{\mathbb{Q}}$ that will be used frequently in this paper:

$$\mathrm{E}_1 := [1,0,0\,;0,0,0],\ \mathrm{E}_2 := [0,1,0\,;0,0,0],\ \mathrm{E}_3 := [0,0,1\,;0,0,0].$$

8.2.2.3 Albert algebras over $\mathbb{Z}$

Let k be a commutative ring. *Albert k-algebras* are defined in [GaribaldiPeterssonRacine, 2023, Definition 7.1] Roughly speaking, an Albert k-algebra is a projective k-module J together with a distinguished point 1_J, a quadratic map $\# : J \to J$ and a cubic form $d : J \to k$ (as polynomial laws in the sense of [Roby, 1963]) satisfying certain equations, such that for some faithfully flat k-algebra K, $J \otimes_k K$ is isomorphic to $\mathrm{Her}_3(C_K)$ as *Jordan K-algebras*, where C_K is an octonion K-algebra. For any ring homomorphism $k \to k'$, the scalar extension $J \otimes_k k'$ of an Albert k-algebra J is an Albert k'-algebra.

Definition 8.2.10. [GaribaldiPeterssonRacine, 2023, Lemma 10.3] An isomorphism of Albert k-algebras $\phi : J \to J'$ is a k-module isomorphism such that $\phi(1_J) = 1_{J'}$ and $d_{J'} \circ \phi = d_J$[2] as polynomial laws.

Example 8.2.11. The space of 3-by-3 Hermitian matrices $\mathrm{Her}_3(C)$ defined in Section 8.2.2.1 is an Albert k-algebra. In particular, $\mathrm{J}_{\mathbb{R}}$ and $\mathrm{J}_{\mathbb{Q}}$ defined in and Section 8.2.2.2 are Albert algebras over $\mathbb{R}$ and $\mathbb{Q}$ respectively.

Here are several classification results in [SpringerVeldkamp, 2000, §5.8; GaribaldiPetersson-Racine, 2023, §11, §14] about Albert algebras that will be useful for us:

(1) There is a unique isomorphism class of Albert $\mathbb{R}$-algebras that are *Euclidean*, *i.e.* the associated symmetric bilinear form is positive definite, and this class is represented by $(\mathrm{J}_{\mathbb{R}}, \mathrm{I}, \#, \det)$ defined in Section 8.2.2.2.
(2) Euclidean Albert $\mathbb{Q}$-algebras are also unique up to isomorphism.
(3) Albert $\mathbb{Z}_p$-algebras are unique up to isomorphism.
(4) There are exactly two isomorphism classes of Euclidean Albert $\mathbb{Z}$-algebras.

In this article, we concentrate on the following family of Euclidean Albert $\mathbb{Z}$-algebras:

Definition 8.2.12. An *Albert lattice* of $\mathrm{J}_{\mathbb{R}}$ is a lattice $J \subseteq \mathrm{J}_{\mathbb{R}}$ satisfying:

- The identity matrix $\mathrm{I} = \mathrm{diag}(1,1,1) \in \mathrm{J}_{\mathbb{R}}$ is contained in J;
- It is stable under the adjoint map $\#$ defined in (8.4);
- The cubic form det defined in (8.5) takes integral values on J;
- Together with $\mathrm{I}, \#$ and det, J is an Albert $\mathbb{Z}$-algebra.

Denote the set of Albert lattices inside $\mathrm{J}_{\mathbb{R}}$ by $\mathscr{J}$.

Example 8.2.13. Let $\mathrm{J}_{\mathbb{Z}} := \mathrm{Her}_3(\mathbb{O}_{\mathbb{Z}})$, *i.e.* the rank 27 lattice

$$\{[a,b,c\,;x,y,z] \in \mathrm{J}_{\mathbb{Q}} \mid a,b,c \in \mathbb{Z}, x,y,z \in \mathbb{O}_{\mathbb{Z}}\}$$

inside $\mathrm{J}_{\mathbb{Q}}$. It satisfies the conditions in Definition 8.2.12, thus it is an Albert lattice.

[2] Here $\circ$ means the composition, not the multiplication defined in Section 8.2.2.2.

Example 8.2.14. An Albert $\mathbb{Z}$-algebra not isomorphic to $(\mathrm{J}_{\mathbb{Z}}, \mathrm{I}, \#, \det)$ defined in Example 8.2.13 is constructed as follows, following [Gross, 1996, §4; GaribaldiPeterssonRacine, 2023, Definition 14.1]. Take

$$\mathrm{E} = [2, 2, 2; \beta, \beta, \beta],\ \beta = \frac{1}{2}(-1 + \mathrm{e}_1 + \mathrm{e}_2 + \cdots + \mathrm{e}_7) \in \mathbb{O}_{\mathbb{Z}}.$$

This element $\mathrm{E} \in \mathrm{J}_{\mathbb{Z}}$ is positive definite and has determinant 1. Under the adjoint map $\#$ on $\mathrm{J}_{\mathbb{R}}$ defined as (8.4), we have $\mathrm{E}^{\#} = [2, 2, 2\,; \overline{\beta}, \overline{\beta}, \overline{\beta}] \in \mathrm{J}_{\mathbb{Z}}$. Using this element, we define another quadratic map $\#^{\mathrm{E}}$ on $\mathrm{J}_{\mathbb{Z}}$ by $X^{\#^{\mathrm{E}}} := (\mathrm{E}^{\#}, X^{\#})\mathrm{E}^{\#} - \mathrm{E} \times X^{\#}$. Set $\mathrm{J}_{\mathbb{Z}}^{(\mathrm{E})} := (\mathrm{J}_{\mathbb{Z}}, \mathrm{E}^{\#}, \#^{\mathrm{E}}, \det)$, where det is still the restriction of $\det : \mathrm{J}_{\mathbb{R}} \to \mathbb{R}$ to $\mathrm{J}_{\mathbb{Z}}$. This "*isotopy*" $\mathrm{J}_{\mathbb{Z}}^{(\mathrm{E})}$ is an Albert $\mathbb{Z}$-algebra [GaribaldiPeterssonRacine, 2023, Corollary 13.11], and it is not isomorphic to $(\mathrm{J}_{\mathbb{Z}}, \mathrm{I}, \#, \det)$ as Albert $\mathbb{Z}$-algebras [ElkiesGross, 1996, Proposition 5.5].

The associated symmetric bilinear form $(\ ,\ )$ on $\mathrm{J}_{\mathbb{Z}}^{(\mathrm{E})}$ is positive definite [ElkiesGross, 1996, Proposition 2.10], thus $\mathrm{J}_{\mathbb{Z}}^{(\mathrm{E})}$ is Euclidean. By the classification result about Euclidean Albert $\mathbb{R}$-algebras listed above, we have an isomorphism $\varphi : \mathrm{J}_{\mathbb{Z}}^{(\mathrm{E})} \otimes_{\mathbb{Z}} \mathbb{R} \to \mathrm{J}_{\mathbb{R}}$ of Albert $\mathbb{R}$-algebras. Its image $\varphi(\mathrm{J}_{\mathbb{Z}}^{(\mathrm{E})})$ is an Albert lattice of $\mathrm{J}_{\mathbb{R}}$ in the sense of Definition 8.2.12.

Question. *Can we find a simpler description of Albert lattices of* $\mathrm{J}_{\mathbb{R}}$*? For example, is it true that a unimodular lattice* $J \subset \mathrm{J}_{\mathbb{R}}$ *such that* J *contains* I *as a characteristic vector and* J *is stable under* $\#$ *(or equivalently, under* $A \mapsto A^2$*) is an Albert lattice in* $\mathrm{J}_{\mathbb{R}}$*?*

8.2.3 $\mathbf{F}_4$

We start to define exceptional algebraic groups.

Definition 8.2.15. Define $\mathbf{F}_4$ to be the closed subgroup of the algebraic $\mathbb{Q}$-group $\mathbf{GL}_{\mathrm{J}_{\mathbb{Q}}}$, that (as a functor) sends a commutative $\mathbb{Q}$-algebra R to the group

$$\mathbf{F}_4(R) := \{g \in \mathrm{GL}(\mathrm{J}_{\mathbb{Q}} \otimes_{\mathbb{Q}} R) \,|\, g(A \circ B) = g(A) \circ g(B),\ \text{for any } A, B \in \mathrm{J}_{\mathbb{Q}} \otimes_{\mathbb{Q}} R\}.$$

By [SpringerVeldkamp, 2000, Theorem 7.2.1], $\mathbf{F}_4$ is a semisimple and simply-connected $\mathbb{Q}$-group of Lie type F_4. The real points $\mathrm{F}_4 := \mathbf{F}_4(\mathbb{R})$ of $\mathbf{F}_4$ is contained in the isometry group $\mathrm{O}(\mathrm{J}_{\mathbb{R}}, \mathrm{q})$ of the positive definite quadratic form q, thus it is compact. For every prime p, $\mathbf{F}_4$ is split over $\mathbb{Q}_p$. By [SpringerVeldkamp, 2000, Proposition 5.9.4], the $\mathbb{Q}$-group $\mathbf{F}_4$ coincides with the algebraic group consisting of the Albert algebra automorphisms of $\mathrm{J}_{\mathbb{Q}}$, *i.e.* sending any commutative $\mathbb{Q}$-algebra R to

$$\{g \in \mathrm{GL}(\mathrm{J}_{\mathbb{Q}} \otimes_{\mathbb{Q}} R) \,|\, g(\mathrm{I}) = g, \det(gA) = \det(A),\ \text{for any } A \in \mathrm{J}_{\mathbb{Q}} \otimes_{\mathbb{Q}} R\}.$$

With this coincidence, we construct *reductive* $\mathbb{Z}$*-models* of $\mathbf{F}_4$ in the sense of [Gross, 1996] as group of Albert algebra automorphisms of elements $J \in \mathcal{J}$.

Definition 8.2.16. Given an Albert lattice $J \in \mathcal{J}$, define $\mathbf{Aut}_{J/\mathbb{Z}}$ to be the $\mathbb{Z}$-group scheme sending a commutative $\mathbb{Z}$-algebra R to the group

$$\mathbf{Aut}_{J/\mathbb{Z}}(R) := \{g \in \mathrm{GL}(J \otimes_{\mathbb{Z}} R) \,|\, g(\mathrm{I}) = \mathrm{I}, \det(gA) = \det(A), \text{ for any } A \in J \otimes_{\mathbb{Z}} R\}\,.$$

If we take J to be $\mathrm{J}_{\mathbb{Z}}$ defined in Example 8.2.13, we denote the group scheme $\mathbf{Aut}_{\mathrm{J}_{\mathbb{Z}}/\mathbb{Z}}$ by $\mathcal{F}_{4,\mathrm{I}}$.

The following result shows that $\mathbf{Aut}_{J/\mathbb{Z}}$ is a reductive $\mathbb{Z}$-model of $\mathbf{F}_4$:

Proposition 8.2.17. *[GaribaldiPeterssonRacine, 2023, Lemma 9.1] For any choice of Albert lattice $J \in \mathcal{J}$, the group scheme $\mathbf{Aut}_{J/\mathbb{Z}}$ is smooth and its fiber $\mathbf{Aut}_{J/\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z}$ is semisimple for every prime p. Moreover, the generic fiber of $\mathbf{Aut}_{J/\mathbb{Z}}$ is $\mathbf{F}_4$.*

In [Gross, 1996, Proposition 5.3], Gross proves that there are exactly two $\mathbf{F}_4(\mathbb{Q})$-orbits on the equivalence classes of reductive $\mathbb{Z}$-models of $\mathbf{F}_4$ in the *genus* of $\mathcal{F}_{4,\mathrm{I}}$. From now on we fix a reductive $\mathbb{Z}$-model $\mathcal{F}_{4,\mathrm{I}}$ of $\mathbf{F}_4$, and we have the following formulation of the double cosets space $\mathbf{F}_4(\mathbb{Q})\backslash\mathbf{F}_4(\mathbb{A})/\mathcal{F}_{4,\mathrm{I}}(\widehat{\mathbb{Z}})$.

Proposition 8.2.18. *There is a bijection $\mathcal{J} \xrightarrow{\simeq} \mathbf{F}_4(\mathbb{Q})\backslash\mathbf{F}_4(\mathbb{A})/\mathcal{F}_{4,\mathrm{I}}(\widehat{\mathbb{Z}})$ sending the base point $\mathrm{J}_{\mathbb{Z}}$ to the double coset of the identity of $\mathbf{F}_4(\mathbb{A})$.*

Proof. For any $J \in \mathcal{J}$, the Albert $\mathbb{Q}$-algebras $J \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\mathrm{J}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$ are isomorphic, so there exists an element $g_\infty \in \mathbf{F}_4(\mathbb{R})$ inducing $J \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\simeq} \mathrm{J}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$. Set $J' = g_\infty(J)$, which is an Albert $\mathbb{Z}$-algebra inside $\mathrm{J}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q} = \mathrm{J}_{\mathbb{Q}}$. Since $J' \otimes_{\mathbb{Z}} \mathbb{Z}_p$ and $\mathrm{J}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}_p$ are isomorphic as Albert $\mathbb{Z}_p$-algebras, we can choose an element $g_p \in \mathbf{F}_4(\mathbb{Q}_p)$ that induces $\mathrm{J}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}_p \xrightarrow{\simeq} J' \otimes_{\mathbb{Z}} \mathbb{Z}_p$. For almost all prime numbers p, we have $J' \otimes_{\mathbb{Z}} \mathbb{Z}_p = \mathrm{J}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}_p$, so the element g_p lies in $\mathcal{F}_{4,\mathrm{I}}(\mathbb{Z}_p)$ for almost all p.

In this way, we associate with $J \in \mathcal{J}$ an element $(g_\infty, g_2, g_3, \ldots) \in \mathbf{F}_4(\mathbb{A})$, and it can be easily verified that its image in $\mathbf{F}_4(\mathbb{Q})\backslash\mathbf{F}_4(\mathbb{A})/\mathcal{F}_{4,\mathrm{I}}(\widehat{\mathbb{Z}})$ does not depend on the choice of g_∞ and g_p. So we have a well-defined map $\mathcal{J} \to \mathbf{F}_4(\mathbb{Q})\backslash\mathbf{F}_4(\mathbb{A})/\mathcal{F}_{4,\mathrm{I}}(\widehat{\mathbb{Z}})$, and its inverse is:

$$(g_v) \mapsto g_\infty^{-1}\left(\bigcap_p \left(g_p\left(\mathrm{J}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}_p\right) \cap \mathrm{J}_{\mathbb{Q}}\right)\right) \in \mathcal{J}. \qquad \square$$

Notation 8.2.19. We choose a set of representatives $\{1, \gamma_{\mathrm{E}}\}$ of $\mathbf{F}_4(\mathbb{Q})\backslash\mathbf{F}_4(\mathbb{A}_f)/\mathcal{F}_{4,\mathrm{I}}(\widehat{\mathbb{Z}})$, and denote by $\mathrm{J}_{\mathrm{E}} \subseteq \mathrm{J}_{\mathbb{Q}}$ the Albert lattice corresponding to γ_{E}. Equipped with the natural $\mathbf{F}_4(\mathbb{R})$-action, $\mathcal{J}$ is the disjoint union of the $\mathbf{F}_4(\mathbb{R})$-orbits of $\mathrm{J}_{\mathbb{Z}}$ and J_{E}.

8.2.3.1 An algebraic group of type E_6

If we remove the condition of fixing the identity element I in the definition of $\mathcal{F}_{4,\mathrm{I}}$, we get the following group of type E_6:

Definition 8.2.20. Define $\mathbf{M}_{\mathrm{J}}$ to be the $\mathbb{Z}$-group scheme sending any commutative ring R to

$$\left\{(\lambda(g), g) \in R^\times \times \mathrm{GL}(\mathrm{J}_{\mathbb{Z}} \otimes_{\mathbb{Z}} R) \,\middle|\, \det(gA) = \lambda(g)\det(A), \text{ for any } A \in \mathrm{J}_{\mathbb{Z}} \otimes_{\mathbb{Z}} R\right\},$$

and $\mathbf{M}_{\mathrm{J}}^1$ to be $\ker\lambda$.

By [Conrad, 2015, Proposition 6.5], $\mathbf{M}_{\mathrm{J}}^1$ is a simply-connected semisimple group scheme of type E_6, and its generic fiber has $\mathbb{Q}$-rank 2.

Remark 8.2.21. Notice that the bilinear form $(\ ,\ )$ on $\mathrm{J}_{\mathbb{Z}} \otimes_{\mathbb{Z}} R$ is not $\mathbf{M}_{\mathrm{J}}(R)$-invariant. For any $m \in \mathbf{M}_{\mathrm{J}}(R)$, we denote by m^* the unique element in $\mathbf{M}_{\mathrm{J}}(R)$ such that $(m(X), m^*(Y)) = (m^*(X), m(Y)) = (X, Y)$ for any $X, Y \in \mathrm{J}_{\mathbb{Z}} \otimes R$.

Observe that we have already seen two Albert $\mathbb{Z}$-algebras $\mathrm{J}_{\mathbb{Z}}^{(\mathrm{E})}$ and J_{E} that are both not isomorphic to $\mathrm{J}_{\mathbb{Z}}$ and their extensions to $\mathbb{Q}$ are isomorphic to $\mathrm{J}_{\mathbb{Q}}$, by the classification result listed in Section 8.2.2.3 they are isomorphic, although they have different distinguished points. This fact gives us an element that will be used in the proof of Theorem 8.5.2:

Lemma 8.2.22. *There exists an element $\delta \in \mathbf{M}_{\mathrm{J}}^1(\mathbb{Q})$ that induces an isomorphism of Albert $\mathbb{Z}$-algebras $\mathrm{J}_{\mathbb{Z}}^{(\mathrm{E})} \xrightarrow{\simeq} \mathrm{J}_{\mathrm{E}}$. Moreover, if we denote the image of δ under the diagonal embedding $\mathbf{M}_{\mathrm{J}}^1(\mathbb{Q}) \hookrightarrow \mathbf{M}_{\mathrm{J}}^1(\mathbb{A}) = \mathbf{M}_{\mathrm{J}}^1(\mathbb{R}) \times \mathbf{M}_{\mathrm{J}}^1(\mathbb{A}_f)$ by $(\delta_\infty, \delta_f)$, then $\delta_\infty(\mathrm{J}_{\mathbb{Z}}) = \mathrm{J}_{\mathrm{E}}$, $\delta_\infty(\mathrm{E}) = \mathrm{I}$ and $\delta_f^{-1}\gamma_{\mathrm{E}} \in \mathbf{M}_{\mathrm{J}}^1(\widehat{\mathbb{Z}})$.*

Proof. Since the Albert $\mathbb{Z}$-algebras $\mathrm{J}_{\mathbb{Z}}^{(\mathrm{E})}, \mathrm{J}_{\mathrm{E}}$ contained in $\mathrm{J}_{\mathbb{Q}}$ are isomorphic, there is a $\mathbb{Q}$-linear isomorphism δ of $\mathrm{J}_{\mathbb{Q}}$ such that $\delta(\mathrm{J}_{\mathbb{Z}}^{(\mathrm{E})}) = \mathrm{J}_{\mathrm{E}}$, $\delta(\mathrm{E}) = \mathrm{I}$ and $\det(\delta A) = \delta(A)$ for any $A \in \mathrm{J}_{\mathbb{Q}}$. In other words, δ is our desired element in $\mathbf{M}_{\mathrm{J}}^1(\mathbb{Q})$. The properties of δ_∞ follows immediately. Forgetting the Albert algebra structures, $\delta_f^{-1}\gamma_{\mathrm{E}} : \mathrm{J}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}} \to \mathrm{J}_{\mathbb{Z}}^{(\mathrm{E})} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$ is a linear automorphism of $\mathrm{J}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$ preserving the determinant, thus $\delta_f^{-1}\gamma_{\mathrm{E}} \in \mathbf{M}_{\mathrm{J}}^1(\widehat{\mathbb{Z}})$. □

8.2.4 E_7

Now we recall the definition of $\mathbf{E}_7$, a larger algebraic group over $\mathbb{Q}$ containing $\mathbf{F}_4$, and our main references are [Pollack, 2020, §2.2; KimYamauchi, 2016, §3][3].

Consider the 56-dimensional vector space $\mathrm{W}_{\mathrm{J}} = \mathrm{J}_{\mathbb{Q}} \oplus \mathbb{Q} \oplus \mathrm{J}_{\mathbb{Q}} \oplus \mathbb{Q}$[4], equipped with the following structures:

- A symplectic form: for $w_i = (X_i, \xi_i, X_i', \xi_i') \in \mathrm{W}_{\mathrm{J}}$, $i = 1, 2$,

$$\langle w_1, w_2 \rangle_{\mathrm{J}} := \xi_1 \xi_2' - \xi_2 \xi_1' + (X_1, X_2') - (X_2, X_1');$$

- A quartic form: for $w = (X, \xi, X', \xi') \in \mathrm{W}_{\mathrm{J}}$,

$$\mathrm{Q}(w) = (\xi\xi' - (X, X'))^2 + 4\xi \det(X) + 4\xi' \det(X') - 4(X^\#, X'^\#).$$

Definition 8.2.23. Define $\mathbf{H}_{\mathrm{J}}$ to be the algebraic subgroup of $\mathbf{GL}_{\mathrm{W}_{\mathrm{J}}}$ consisting of elements that preserve the forms $\langle\ ,\ \rangle_{\mathrm{J}}$ and Q up to some similitude $\nu : \mathbf{H}_{\mathrm{J}} \to \mathbf{G}_{\mathrm{m}}$, *i.e.*

$$\mathbf{H}_{\mathrm{J}} = \left\{ (\nu(g), g) \in \mathbf{G}_{\mathrm{m}} \times \mathbf{GL}_{\mathrm{W}_{\mathrm{J}}} \,\middle|\, \langle gv, gw \rangle_{\mathrm{J}} = \nu(g) \langle v, w \rangle_{\mathrm{J}}, \mathrm{Q}(gv) = \nu(g)^2 \mathrm{Q}(v), \forall v, w \in \mathrm{W}_{\mathrm{J}} \right\}.$$

[3] Notice that there are some slight mistakes in [KimYamauchi, 2016, §3] and the correction is in [KimYamauchi, 2023, §2].

[4] In [Pollack, 2020], Pollack considers the space $\mathbb{Q} \oplus \mathrm{J}_{\mathbb{Q}} \oplus \mathrm{J}_{\mathbb{Q}}^\vee \oplus \mathbb{Q}$. An element $(X, \xi, X', \xi') \in \mathrm{W}_{\mathrm{J}}$ corresponds to $(a, b, c, d) = (\xi', X, (-, X'), \xi)$ under the notations of Pollack.

Define $\mathbf{H}_{\mathrm{J}}^1$ to be the kernel of ν, which is simply-connected and has $\mathbb{Q}$-rank 3 and Lie type E_7 [Freudenthal, 1954], and $\mathbf{E}_7$ to be the adjoint group of $\mathbf{H}_{\mathrm{J}}$.

Remark 8.2.24. The center of $\mathbf{H}_{\mathrm{J}}$ consists of scalars, and it contains a specific element $\iota^2 = -\mathrm{Id}_{\mathrm{W}_{\mathrm{J}}}$, where $\iota \in \mathbf{H}_{\mathrm{J}}$ is defined as

$$\iota(X, \xi, X', \xi') = (-X', -\xi', X, \xi). \tag{8.6}$$

In [Gross, 1996], we know that $\mathbf{E}_7$ has a unique (up to equivalence) reductive $\mathbb{Z}$-models, and we will also denote this $\mathbb{Z}$-group scheme by $\mathbf{E}_7$ when there is no confusion. Note that $\mathbf{E}_7(\mathbb{Z})$ is the stabilizer in $\mathbf{E}_7(\mathbb{R})$ of the lattice $\mathrm{J}_{\mathbb{Z}} \oplus \mathbb{Z} \oplus \mathrm{J}_{\mathbb{Z}} \oplus \mathbb{Z} \subseteq \mathrm{W}_{\mathrm{J}}$.

8.2.4.1 Siegel parabolic subgroup of $\mathbf{E}_7$

Definition 8.2.25. The *Siegel parabolic subgroup* $\mathbf{P}_{\mathrm{J,sc}}$ of $\mathbf{H}_{\mathrm{J}}^1$ is defined as the stabilizer of the line $\mathbb{Q}(0,1,0,0) \subseteq \mathrm{W}_{\mathrm{J}}$. A Levi subgroup of $\mathbf{P}_{\mathrm{J,sc}}$ can be defined as the subgroup that also stabilizes $\mathbb{Q}(0,0,0,1)$. Denote by $\mathbf{P}_{\mathrm{J}}$ the image of $\mathbf{P}_{\mathrm{J,sc}}$ in $\mathbf{E}_7$.

This Levi subgroup is isomorphic to $\mathbf{M}_{\mathrm{J}}$, and the action of $(\lambda(m), m) \in \mathbf{M}_{\mathrm{J}}$ on W_{J} is

$$m(X, \xi, X', \xi') = (m^* X, \lambda(m)\xi, mX', \lambda(m)^{-1}\xi').$$

The unipotent radical $\mathbf{N}_{\mathrm{J}}$ of $\mathbf{P}_{\mathrm{J,sc}}$ is abelian and satisfies $\mathbf{N}_{\mathrm{J}}(\mathbb{Q}) \simeq \mathrm{J}_{\mathbb{Q}}$, and any element of $\mathbf{N}_{\mathrm{J}}(\mathbb{Q})$ has the following form:

$$\mathrm{n}(A)(X, \xi, X', \xi') = \left(X + \xi' A, \xi + (A, X') + (A^{\#}, X) + \xi' \det(A), X' + A \times X + \xi' A^{\#}, \xi'\right), \ A \in \mathrm{J}_{\mathbb{Q}}.$$

We have the Levi decomposition $\mathbf{P}_{\mathrm{J,sc}} = \mathbf{M}_{\mathrm{J}}\mathbf{N}_{\mathrm{J}}$, and the action of $\mathbf{M}_{\mathrm{J}}$ on $\mathbf{N}_{\mathrm{J}}$ is given by the following lemma:

Lemma 8.2.26. *For any $m \in \mathbf{M}_{\mathrm{J}}(\mathbb{Q}) \subseteq \mathbf{P}_{\mathrm{J,sc}}$ and $A \in \mathrm{J}_{\mathbb{Q}}$, we have the following identity:*

$$m\mathrm{n}(A)m^{-1} = \mathrm{n}\left(\lambda(m) m^* A\right).$$

Proof. This follows from a direct calculation using the property: for any $m \in \mathbf{M}_{\mathrm{J}}(\mathbb{Q})$ and $X, Y \in \mathrm{J}_{\mathbb{Q}}$, we have $m(X \times Y) = \lambda(m)(m^* X) \times (m^* Y)$. □

The Levi subgroup of $\mathbf{P}_{\mathrm{J}} \subseteq \mathbf{E}_7$ induced by $\mathbf{M}_{\mathrm{J}}$ is the quotient of $\mathbf{M}_{\mathrm{J}}$ by μ_2, where μ_2 is generated by the element $X \mapsto -X$ in $\mathbf{M}_{\mathrm{J}}$. We identify this Levi subgroup with $\mathbf{M}_{\mathrm{J}}$ via the short exact sequence:

$$1 \to \mu_2 \to \mathbf{M}_{\mathrm{J}} \xrightarrow{m \mapsto \lambda(m) m^*} \mathbf{M}_{\mathrm{J}} \to 1. \tag{8.7}$$

Hence we still have the Levi decomposition $\mathbf{P}_{\mathrm{J}} \simeq \mathbf{M}_{\mathrm{J}}\mathbf{N}_{\mathrm{J}}$, but with a different action:

$$m\mathrm{n}(A)m^{-1} = \mathrm{n}(mA), \text{ for any } m \in \mathbf{M}_{\mathrm{J}}(\mathbb{Q}), A \in \mathrm{J}_{\mathbb{Q}}.$$

Remark 8.2.27. For any $A \in \mathrm{J}_{\mathbb{Q}}$, we define $\mathrm{n}^{\vee}(A) = \iota \mathrm{n}(-A)\iota^{-1}$. Set $\overline{\mathbf{N}_{\mathrm{J}}} = \iota \mathbf{N}_{\mathrm{J}} \iota^{-1}$, then $\overline{\mathbf{P}_{\mathrm{J,sc}}} = \mathbf{M}_{\mathrm{J}}\overline{\mathbf{N}_{\mathrm{J}}}$ is the parabolic subgroup opposite to $\mathbf{P}_{\mathrm{J,sc}}$. The action of $\mathbf{M}_{\mathrm{J}}$ on $\overline{\mathbf{N}_{\mathrm{J}}}$ is:

$$m\mathrm{n}^{\vee}(A)m^{-1} = \mathrm{n}^{\vee}\left(\lambda(m)^{-1}mA\right), \text{ for any } m \in \mathbf{M}_{\mathrm{J}}(\mathbb{Q}), A \in \mathrm{J}_{\mathbb{Q}}.$$

8.2.4.2 The Lie algebra $\mathfrak{e}_7$

Denote the Lie algebra of $\mathbf{H}^1_{\mathrm{J}}(\mathbb{C})$ by $\mathfrak{e}_7$, which admits a decomposition

$$\mathfrak{e}_7 = \mathrm{n}^{\vee}_{\mathrm{L}}(\mathrm{J}_{\mathbb{C}}) \oplus \mathfrak{m}_{\mathrm{J}} \oplus \mathrm{n}_{\mathrm{L}}(\mathrm{J}_{\mathbb{C}}), \tag{8.8}$$

where

- $\mathfrak{m}_{\mathrm{J}} = \mathrm{Lie}(\mathbf{M}_{\mathrm{J}}(\mathbb{C}))$;
- for any $A \in \mathrm{J}_{\mathbb{C}}$, define $\mathrm{n}_{\mathrm{L}}(A)$ to be the element in $\mathrm{Lie}(\mathbf{N}_{\mathrm{J}}(\mathbb{C}))$ such that $\exp(\mathrm{n}_{\mathrm{L}}(A)) = \mathrm{n}(A)$, and denote $\mathrm{Lie}(\mathbf{N}_{\mathrm{J}}(\mathbb{C}))$ by $\mathrm{n}_{\mathrm{L}}(\mathrm{J}_{\mathbb{C}})$;
- for any $A \in \mathrm{J}_{\mathbb{C}}$, define $\mathrm{n}_{\mathrm{L}}(A)$ to be the element in $\mathrm{Lie}(\overline{\mathbf{N}_{\mathrm{J}}}(\mathbb{C}))$ such that $\exp(\mathrm{n}_{\mathrm{L}}(A)) = \mathrm{n}^{\vee}(A)$, and denote $\mathrm{Lie}(\overline{\mathbf{N}_{\mathrm{J}}}(\mathbb{C}))$ by $\mathrm{n}^{\vee}_{\mathrm{L}}(\mathrm{J}_{\mathbb{C}})$.

Besides this decomposition, we also have the Cartan decomposition of $\mathfrak{e}_7$. Let $\mathrm{K}_{\mathrm{E}_7}$ be the subgroup of $\mathbf{H}^1_{\mathrm{J}}(\mathbb{R})$ that fixes the line in $\mathrm{W}_{\mathrm{J}} \otimes \mathbb{C}$ spanned by $(i\mathrm{I}, -i, -\mathrm{I}, 1)$, which is a maximal compact subgroup of $\mathbf{H}^1_{\mathrm{J}}(\mathbb{R})$. Take $\mathfrak{k}_{\mathrm{E}_7}$ to be the complexified Lie algebra of $\mathrm{K}_{\mathrm{E}_7}$, then we have the following Cartan decomposition of $\mathfrak{e}_7$:

$$\mathfrak{e}_7 = \mathfrak{p}^-_{\mathrm{J}} \oplus \mathfrak{k}_{\mathrm{E}_7} \oplus \mathfrak{p}^+_{\mathrm{J}}, \tag{8.9}$$

where $\mathfrak{p}^+_{\mathrm{J}} \oplus \mathfrak{p}^-_{\mathrm{J}}$ is the natural decomposition of the -1 eigenspace for the Cartan involution. We have the following relation between these two decompositions (8.8) and (8.9) of $\mathfrak{e}_7$:

Proposition 8.2.28. *[Pollack, 2023, Proposition 6.1.1] There exists an element* $\mathrm{C}_h \in \mathbf{H}^1_{\mathrm{J}}(\mathbb{C})$*, called the* Cayley transform*, satisfying:*

(1) $\mathrm{C}_h^{-1}\mathrm{n}_{\mathrm{L}}(\mathrm{J}_{\mathbb{C}})\mathrm{C}_h = \mathfrak{p}^+_{\mathrm{J}}$;
(2) $\mathrm{C}_h^{-1}\mathrm{n}^{\vee}_{\mathrm{L}}(\mathrm{J}_{\mathbb{C}})\mathrm{C}_h = \mathfrak{p}^-_{\mathrm{J}}$;
(3) $\mathrm{C}_h^{-1}\mathfrak{m}_{\mathrm{J}}\mathrm{C}_h = \mathfrak{k}_{\mathrm{E}_7}$.

By Proposition 8.2.28, we make the following identifications:

- Identify the factor $\mathfrak{p}^+_{\mathrm{J}}$ as $\mathrm{J}^{\vee}_{\mathbb{C}}$, via the map

$$\mathfrak{p}^+_{\mathrm{J}} \ni \mathrm{X}^+_A := i\mathrm{C}_h^{-1}\mathrm{n}_{\mathrm{L}}(A)\mathrm{C}_h \mapsto (-, A) \in \mathrm{J}_{\mathbb{C}},$$

  and equip it with the following $\mathbf{M}_{\mathrm{J}}(\mathbb{C})$-action:

$$(m.\ell)(X) = \ell\left(\lambda(m)m^{-1}(X)\right), \text{ for any } m \in \mathbf{M}_{\mathrm{J}}(\mathbb{C}), \ell \in \mathrm{J}^{\vee}_{\mathbb{C}}, X \in \mathrm{J}_{\mathbb{C}}.$$

- Identify $\mathfrak{p}_{\mathrm{J}}^-$ as $\mathrm{J}_{\mathbb{C}}$, via the map

$$\mathfrak{p}_{\mathrm{J}}^- \ni \mathrm{X}_A^- := i\mathrm{C}_h^{-1}\mathrm{n}_{\mathrm{L}}^{\vee}(A)\mathrm{C}_h \mapsto A \in \mathrm{J}_{\mathbb{C}},$$

  and equip it with the following $\mathbf{M}_{\mathrm{J}}(\mathbb{C})$-action:

$$m.X = \lambda(m)^{-1}m(X) \text{ for any } m \in \mathbf{M}_{\mathrm{J}}(\mathbb{C}), X \in \mathrm{J}_{\mathbb{C}}.$$

The natural $\mathbf{M}_{\mathrm{J}}(\mathbb{C})$-invariant pairing $\{-,-\} : \mathrm{J}_{\mathbb{C}} \times \mathrm{J}_{\mathbb{C}}^{\vee} \to \mathbb{C}$ can be extended to

$$\{-,-\} : \mathrm{J}_{\mathbb{C}}^{\otimes n} \times (\mathrm{J}_{\mathbb{C}}^{\vee})^{\otimes n} \to \mathbb{C}, (X_1 \otimes \cdots \otimes X_n, \ell_1 \otimes \cdots \otimes \ell_n) \mapsto \frac{\sum_{\sigma \in S_n} \prod_{i=1}^n \{X_i, \ell_{\sigma(i)}\}}{n!}, \tag{8.10}$$

which factors through $\mathrm{Sym}^n \mathrm{J}_{\mathbb{C}} \times \mathrm{Sym}^n (\mathrm{J}_{\mathbb{C}}^{\vee})$.

Example 8.2.29. Identifying $\mathrm{Sym}^n (\mathrm{J}_{\mathbb{C}}^{\vee})$ with the space $\mathrm{P}_n(\mathrm{J}_{\mathbb{C}})$ of degree n homogeneous polynomials over $\mathrm{J}_{\mathbb{C}}$, the $\mathbf{M}_{\mathrm{J}}(\mathbb{C})$-action on it is $(m.P)(X) = P(\lambda(m)m^{-1}(X))$ for any $m \in \mathbf{M}_{\mathrm{J}}(\mathbb{C})$, $P \in \mathrm{P}_n(\mathrm{J}_{\mathbb{C}})$ and $T \in \mathrm{J}_{\mathbb{C}}$, and the pairing $\{T^{\otimes n}, P\}$ is equal to $P(T)$.

8.2.5 Dual pairs

Now we explain the two reductive dual pairs $\mathbf{F}_4 \times \mathbf{PGL}_2$ and $\mathbf{Spin}_9 \times \mathbf{SO}_{2,2}$ in $\mathbf{E}_7$.

8.2.5.1 $\mathbf{F}_4 \times \mathbf{PGL}_2$

We study first the centralizer of $\mathbf{F}_4$ in $\mathbf{M}_{\mathrm{J}}$. For any element g in the centralizer $\mathbf{C}_{\mathbf{M}_{\mathrm{J}}}(\mathbf{F}_4)$, it stabilizes the subspace $\mathrm{J}_{\mathbb{Q}}^{\mathbf{F}_4(\mathbb{Q})}$, which is a line spanned by I, thus $g(\mathrm{I})$ is a non-zero multiple of I. So we obtain a morphism $\mathbf{C}_{\mathbf{M}_{\mathrm{J}}}(\mathbf{F}_4) \to \mathbf{G}_{\mathrm{m}}$ by restricting to the line spanned by I.

- This morphism is injective, since the center of $\mathbf{F}_4$ is trivial;
- For any scalar $\lambda \in \mathbb{Q}^{\times}$, the map $X \mapsto \lambda X$ is an element of $\mathbf{C}_{\mathbf{M}_{\mathrm{J}}}(\mathbf{F}_4)(\mathbb{Q})$, thus morphism is also surjective.

Hence the centralizer of $\mathbf{F}_4$ in the Levi subgroup $\mathbf{M}_{\mathrm{J}}$ of $\mathbf{H}_{\mathrm{J}}^1$ is a rank 1 torus.

The centralizer of $\mathbf{F}_4$ in $\mathbf{P}_{\mathrm{J,sc}}$ is generated by $\mathbf{C}_{\mathbf{M}_{\mathrm{J}}}(\mathbf{F}_4)$ and the subgroup $\{\mathrm{n}(x\mathrm{I}), x \in \mathbf{G}_{\mathrm{a}}\}$ of $\mathbf{N}_{\mathrm{J}}$, and it is isomorphic to the standard Borel subgroup of $\mathbf{SL}_2$ via:

$$(X \mapsto uX) \mapsto \begin{pmatrix} u & \\ & u^{-1} \end{pmatrix}, \mathrm{n}(x\mathrm{I}) \mapsto \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}.$$

Similarly, the centralizer of $\mathbf{F}_4$ in $\overline{\mathbf{P}_{\mathrm{J,sc}}}$ is isomorphic to the opposite Borel subgroup of $\mathbf{SL}_2$. As a consequence, we get a subgroup $\mathbf{F}_4 \times \mathbf{SL}_2$ inside $\mathbf{H}_{\mathrm{J}}^1$, which is a maximal proper subgroup of $\mathbf{H}_{\mathrm{J}}^1$ [KarasiewiczSavin, 2023, Lemma 2.4], so it gives a reductive dual pair in $\mathbf{H}_{\mathrm{J}}^1$, and induces a dual pair $\mathbf{F}_4 \times \mathbf{GL}_2$ (*resp.* $\mathbf{F}_4 \times \mathbf{PGL}_2$) inside $\mathbf{H}_{\mathrm{J}}$ (*resp.* $\mathbf{E}_7$).

8.2.5.2 $\mathbf{Spin}_9 \times \mathbf{SO}_{2,2}$

By [Yokota, 2009, Theorem 2.7.4], the stabilizer of $\mathrm{E}_1 = [1,0,0\,;0,0,0]$ in $\mathbf{F}_4$ is isomorphic to $\mathbf{Spin}_9$, the spin group of a positive definite 9-dimensional quadratic space. In the sequel we refer to this group as $\mathbf{Spin}_9$. The 9-dimensional quadratic space can be found inside $\mathrm{J}_{\mathbb{Q}}$:

Lemma 8.2.30. *The group* $\mathbf{Spin}_9$ *preserves respectively the following subspaces of* $\mathrm{J}_{\mathbb{Q}}$*:*

$$\mathrm{J}_1 := \{[0,\xi,-\xi\,;x,0,0] \,|\, \xi \in \mathbb{Q}, x \in \mathbb{O}_{\mathbb{Q}}\}$$

and

$$\mathrm{J}_2 := \{[0,0,0\,;0,y,z] \,|\, y,z \in \mathbb{O}_{\mathbb{Q}}\}\,.$$

Proof. Since

$$\mathrm{J}_1 = \{X \in \mathrm{J}_{\mathbb{Q}} \,|\, \mathrm{E}_1 \circ X = 0, \mathrm{Tr}(X) = 0\}$$

and

$$\mathrm{J}_2 = \{X \in \mathrm{J}_{\mathbb{Q}} \,|\, 2\mathrm{E}_1 \circ X = X\}\,,$$

the lemma follows from the definition that $\mathbf{Spin}_9$ is the stabilizer of E_1 in $\mathbf{F}_4$. □

Notation 8.2.31. In this article, $\mathbf{SO}_{2,2}$ is defined to be the special orthogonal group of a split 4-dimensional quadratic space over $\mathbb{Q}$, and we define $\mathbf{Spin}_{2,2}, \mathbf{GSpin}_{2,2}$ similarly. Notice that $\mathbf{GSpin}_{2,2} \simeq \{(g_1,g_2) \in \mathbf{GL}_2 \times \mathbf{GL}_2, \det(g_1) = \det(g_2)\}$, $\mathbf{Spin}_{2,2} \simeq \mathbf{SL}_2 \times \mathbf{SL}_2$, and $\mathbf{SO}_{2,2} \simeq \mathbf{GSpin}_{2,2}/\mathbf{G}_{\mathrm{m}}^{\Delta} \simeq \mathbf{Spin}_{2,2}/\mu_2^{\Delta}$.

We study first the centralizer of $\mathbf{Spin}_9$ in the Levi subgroup $\mathbf{M}_{\mathrm{J}} \subseteq \mathbf{H}_{\mathrm{J}}^1$:

Lemma 8.2.32. *The centralizer* $\mathbf{C}_{\mathbf{M}_{\mathrm{J}}}(\mathbf{Spin}_9)$ *is an extension of* $\mathbf{G}_{\mathrm{m}} \times \mathbf{G}_{\mathrm{m}}$ *by* μ_2*.*

Proof. For any element $g \in \mathbf{C}_{\mathbf{M}_{\mathrm{J}}}(\mathbf{Spin}_9)$, it stabilizes the subspace $\mathrm{J}_{\mathbb{Q}}^{\mathbf{Spin}_9(\mathbb{Q})}$, which is spanned by E_1 and $\mathrm{I} - \mathrm{E}_1 = \mathrm{E}_2 + \mathrm{E}_3$. The rank 1 elements in this subspace are non-zero multiples of E_1, and the rank 2 elements are non-zero multiples of $\mathrm{E}_2 + \mathrm{E}_3$. As elements of $\mathbf{M}_{\mathrm{J}}$ preserve the rank, g acts on E_1 (*resp.* $\mathrm{E}_2 + \mathrm{E}_3$) by a scalar. So we obtain a morphism from $\mathbf{C}_{\mathbf{M}_{\mathrm{J}}}(\mathbf{Spin}_9)$ to $\mathbf{G}_{\mathrm{m}} \times \mathbf{G}_{\mathrm{m}}$, whose kernel is the center of $\mathbf{Spin}_9$, a cyclic group generated by the involution $[a,b,c\,;x,y,z] \mapsto [a,b,c\,;x,-y,-z]$ [Shan, 2024, §4.3.1]. This morphism of algebraic groups is also surjective, since for any non-zero scalars λ,μ, we have the following element in $\mathbf{C}_{\mathbf{M}_{\mathrm{J}}}(\mathbf{Spin}_9)$:

$$m_{\lambda,\mu} : [a,b,c\,;x,y,z] \mapsto [\lambda^{-1}\mu^2 a, \lambda b, \lambda c\,; \lambda x, \mu y, \mu z].$$ □

Let $\mathbf{C}'$ be the subgroup of $\mathbf{C}_{\mathbf{M}_{\mathrm{J}}}(\mathbf{Spin}_9)$ consisting of $m_{\lambda,\mu}$, then we have the following commutative diagram:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mu_2 & \longrightarrow & \mathbf{C}_{\mathbf{M}_{\mathrm{J}}}(\mathbf{Spin}_9) & \longrightarrow & \mathbf{G}_{\mathrm{m}} \times \mathbf{G}_{\mathrm{m}} & \longrightarrow & 1 \\
 & & \| & & \uparrow & & \| & & \\
1 & \longrightarrow & \mu_2 & \xrightarrow{\mu \mapsto m_{1,\mu}} & \mathbf{C}' & \xrightarrow{m_{\lambda,\mu} \mapsto (\lambda^{-1}\mu^2,\lambda)} & \mathbf{G}_{\mathrm{m}} \times \mathbf{G}_{\mathrm{m}} & \longrightarrow & 1
\end{array},$$

which shows that $\mathbf{C}_{\mathbf{M}_\mathrm{J}}(\mathbf{Spin}_9) = \mathbf{C}'$ is a split torus of rank 2. The centralizer of $\mathbf{Spin}_9$ in $\mathbf{P}_{\mathrm{J,sc}}$ is generated by $\mathbf{C}_{\mathbf{M}_\mathrm{J}}(\mathbf{Spin}_9)$ and $\{\mathrm{n}(x\mathrm{E}_1 + y(\mathrm{E}_2+\mathrm{E}_3)), x, y \in \mathbb{Q}\} \subseteq \mathbf{N}_\mathrm{J}$, and it is isomorphic to the standard Borel subgroup of $\mathbf{Spin}_{2,2} = \mathbf{SL}_2 \times \mathbf{SL}_2$ via:

$$m_{\lambda,\mu} \mapsto \left(\begin{pmatrix}\lambda & \\ & \lambda^{-1}\end{pmatrix}, \begin{pmatrix}\mu & \\ & \mu^{-1}\end{pmatrix}\right), \ \mathrm{n}(x\mathrm{E}_1 + y(\mathrm{E}_2+\mathrm{E}_3)) \mapsto \left(\begin{pmatrix}1 & x\\ & 1\end{pmatrix}, \begin{pmatrix}1 & y\\ & 1\end{pmatrix}\right). \tag{8.11}$$

Similarly, the centralizer of $\mathbf{Spin}_9$ in $\overline{\mathbf{P}_{\mathrm{J,sc}}}$ is isomorphic to the opposite Borel subgroup of $\mathbf{Spin}_{2,2}$, thus we get a morphism $\mathbf{Spin}_9 \times \mathbf{Spin}_{2,2} \to \mathbf{H}_\mathrm{J}^1$. The kernel of this morphism is $\{(\mathrm{id},\mathrm{id}), (m_{1,-1}, m_{1,-1})\}$, and we denote by $\mathbf{Spin}_9 \times_{\mu_2} \mathbf{Spin}_{2,2}$ the quotient of $\mathbf{Spin}_9 \times \mathbf{Spin}_{2,2}$ by this kernel. The morphism $\mathbf{Spin}_9 \times_{\mu_2} \mathbf{Spin}_{2,2} \hookrightarrow \mathbf{H}_\mathrm{J}^1$ induces an embedding of $\mathbf{Spin}_9 \times_{\mu_2} \mathbf{GSpin}_{2,2}$ (*resp.* $\mathbf{Spin}_9 \times_{\mu_2} \mathbf{SO}_{2,2}$) into $\mathbf{H}_\mathrm{J}$ (*resp.* $\mathbf{E}_7$).

The centralizer $\mathbf{C}_{\mathbf{E}_7}(\mathbf{F}_4) \simeq \mathbf{PGL}_2$ is embedded into $\mathbf{SO}_{2,2} \subseteq \mathbf{C}_{\mathbf{E}_7}(\mathbf{Spin}_9)$ via the map induced from the diagonal embedding $\mathbf{GL}_2 \to \mathbf{GSpin}_{2,2}$.

8.3 Local theta correspondence

In this section we recall some results on the minimal representation of $\mathbf{E}_7$ and the local theta correspondences for the exceptional dual pairs constructed in Section 8.2.5.

8.3.1 Minimal representation of $\mathbf{E}_7$

The theory of theta correspondences studies the restrictions of minimal representations to reductive dual pairs, so we first recall the definition of the minimal representation of $\mathbf{E}_7(F)$ for $F = \mathbb{Q}_p$ or $\mathbb{R}$, and also some properties that will be used.

Definition 8.3.1. (i) The *minimal representation* $\Pi_{\mathrm{min},p}$ *of* $\mathbf{E}_7(\mathbb{Q}_p)$ is the unramified representation whose Satake parameter is the $\widehat{\mathbf{E}_7}(\mathbb{C})$-conjugacy class of $\varphi\begin{pmatrix}p^{1/2} & \\ & p^{-1/2}\end{pmatrix}$. Here the morphism $\varphi : \mathbf{SL}_2(\mathbb{C}) \to \widehat{\mathbf{E}_7}(\mathbb{C})$ corresponds to the subregular unipotent orbit of $\widehat{\mathbf{E}_7}(\mathbb{C}) = \mathbf{H}_\mathrm{J}^1(\mathbb{C})$.

(ii) Let Π^+ be the holomorphic representation of $\mathbf{H}_\mathrm{J}^1(\mathbb{R})$ with the smallest Gelfand-Kirillov dimension among non-trivial representations, and Π^- be the anti-holomorphic representation contragradient to Π^+. The *minimal representation* $\Pi_{\mathrm{min},\infty}$ *of* $\mathbf{E}_7(\mathbb{R})$ is the unique representation whose restriction to $\mathbf{H}_\mathrm{J}^1(\mathbb{R})$ is $\Pi^+ \oplus \Pi^-$.

The first property that we need is the following relation between the minimal representation and a principal series:

Proposition 8.3.2. *[Savin, 1994, Proposition 6.1][Sahi, 1993] For $v = p$ or ∞, the minimal representation $\Pi_{\mathrm{min},v}$ of $\mathbf{E}_7(\mathbb{Q}_v)$ is the unique irreducible submodule of the normalized degenerate principal series*

$$\mathrm{Ind}_{\mathbf{P}_\mathrm{J}(\mathbb{Q}_v)}^{\mathbf{E}_7(\mathbb{Q}_v)} \delta_{\mathbf{P}_\mathrm{J}}^{-1/2} |\lambda|^2,$$

where $\delta_{\mathbf{P}_\mathrm{J}}$ is the modulus character of $\mathbf{P}_\mathrm{J}(\mathbb{Q}_v)$, and $\lambda : \mathbf{M}_\mathrm{J}(\mathbb{Q}_v) \to \mathbb{Q}_v^\times$ is the similitude character of $\mathbf{M}_\mathrm{J}(\mathbb{Q}_v)$.

The sections of $\mathrm{Ind}_{\mathbf{P}_{\mathrm{J}}(\mathbb{Q}_p)}^{\mathbf{E}_7(\mathbb{Q}_v)}\delta_{\mathbf{P}_{\mathrm{J}}}^{-1/2}|\lambda|^2$ are smooth functions $f:\mathbf{P}_{\mathrm{J}}(\mathbb{Q}_p)\to\mathbb{C}$ such that

$$f(pg)=|\lambda(p)|_p^2 f(g),\ \text{ for all } p\in\mathbf{P}_{\mathrm{J}}(\mathbb{Q}_p), g\in\mathbf{E}_7(\mathbb{Q}_p). \tag{8.12}$$

From now on, we identify $\Pi_{\min,v}$ as the unique irreducible submodule of $\mathrm{Ind}_{\mathbf{P}_{\mathrm{J}}(\mathbb{Q}_v)}^{\mathbf{E}_7(\mathbb{Q}_v)}\delta_{\mathbf{P}_{\mathrm{J}}}^{-1/2}|\lambda|^2$, and normalize the spherical vector Φ_p in $\Pi_{\min,v}$ by the condition that $\Phi_p(1)=1$.

The second property is the $\mathrm{K}_{\mathrm{E}_7}$-types of the holomorphic part Π^+ of $\Pi_{\min}$. The maximal compact subgroup $\mathrm{K}_{\mathrm{E}_7}$ of $\mathbf{H}_{\mathrm{J}}^1(\mathbb{R})$ is isomorphic to $\mathrm{E}_6\times\mathrm{U}(1)$, where E_6 is the simply-connected compact Lie group of type E_6.

Definition 8.3.3. (1) Define $\mathrm{E}(n)$ to be the irreducible representation of the compact Lie group E_6 with highest weight $n\lambda$, where λ is the highest weight of $\mathfrak{p}_{\mathrm{J}}^+$ as a E_6-representation.
(2) For $n,k\in\mathbb{N}$, define $\mathrm{E}(n,k)$ to be the irreducible representation of $\mathrm{K}_{\mathrm{E}_7}$ such that its restriction to E_6 is isomorphic to $\mathrm{E}(n)$ and its restriction to $\mathrm{U}(1)$ is the character $z\mapsto z^k$.

The restriction of Π^+ to $\mathrm{K}_{\mathrm{E}_7}$ is given in [Wallach, 1979]:

$$\Pi^+|_{\mathrm{K}_{\mathrm{E}_7}}\simeq\bigoplus_{n=0}^{\infty}\mathrm{E}(n,2n+12). \tag{8.13}$$

8.3.2 p-adic correspondence for $\mathbf{F}_4\times\mathbf{PGL}_2$

Over $\mathbb{Q}_p$, the exceptional theta correspondence for $\mathbf{F}_4\times\mathbf{PGL}_2$ has been studied in [Savin, 1994; KarasiewiczSavin, 2023]. Now we recall some results that we need.

Definition 8.3.4. Let π be a smooth irreducible representation of $\mathbf{PGL}_2(\mathbb{Q}_p)$, then the maximal π-isotypic quotient of $\Pi_{\min,p}$ admits an action of $\mathbf{F}_4(\mathbb{Q}_p)$ and factors as $\pi\boxtimes\Theta(\pi)$. We call $\Theta(\pi)$ the *big theta lift* of π, and its maximal semisimple quotient $\theta(\pi)$ the *small theta lift* of π.

Let $\mathbf{B}_0=\mathbf{T}_0\mathbf{N}_0$ be the Borel subgroup of $\mathbf{PGL}_2$ consisting of upper triangular matrices, and $\overline{\mathbf{B}_0}$ be the opposite Borel subgroup. Let χ be a character of $\mathbf{T}_0(\mathbb{Q}_p)=\{\left(\begin{smallmatrix} t & \\ & 1\end{smallmatrix}\right), t\in\mathbb{Q}_p^\times\}$ satisfying $\chi=|-|^s\cdot\chi_0$, where $s\geq 0$ and χ_0 is a unitary character of $\mathbf{T}_0(\mathbb{Q}_p)$. When $s\neq\frac{1}{2}$ or $\chi_0^2\neq 1$, the principal series $\mathrm{Ind}_{\overline{\mathbf{B}_0}(\mathbb{Q}_p)}^{\mathbf{PGL}_2(\mathbb{Q}_p)}(\chi)$ is irreducible. It turns out the theta lift of this principal series to $\mathbf{F}_4(\mathbb{Q}_p)$ is also a principal series. Before stating the result of Karasiewicz-Savin, we introduce a maximal parabolic subgroup of $\mathbf{F}_4$.

Definition 8.3.5. Using Bourbaki's labeling for simple roots of F_4, we define $\mathbf{Q}$ to be the maximal parabolic subgroup of $\mathbf{F}_4$ obtained by removing α_4 from the Dynkin diagram.

The Levi subgroup of $\mathbf{Q}$ is isomorphic to $\mathbf{GSpin}_7$, whose similitude map $\mathbf{GSpin}_7\to\mathbf{GL}_1$ is given by the fundamental weight ϖ_4. Notice that $\widehat{\mathbf{Q}}\simeq\mathbf{GSp}_6\simeq\mathbf{Sp}_6\times\mathbf{G}_m$.

Proposition 8.3.6. *[KarasiewiczSavin, 2023, Proposition 6.4] Let $\chi=|-|^s\cdot\chi_0$ be a character of $\mathbf{T}_0(\mathbb{Q}_p)$ such that χ_0 is unitary and $0\leq s<1/2$, then the big theta lift of $\mathrm{Ind}_{\overline{\mathbf{B}_0}(\mathbb{Q}_p)}^{\mathbf{PGL}_2(\mathbb{Q}_p)}(\chi)$ to $\mathbf{F}_4(\mathbb{Q}_p)$ is irreducible, and*

$$\Theta(\mathrm{Ind}_{\overline{\mathbf{B}_0}(\mathbb{Q}_p)}^{\mathbf{PGL}_2(\mathbb{Q}_p)}(\chi))=\theta(\mathrm{Ind}_{\overline{\mathbf{B}_0}(\mathbb{Q}_p)}^{\mathbf{PGL}_2(\mathbb{Q}_p)}(\chi))\simeq\mathrm{Ind}_{\mathbf{Q}(\mathbb{Q}_p)}^{\mathbf{F}_4(\mathbb{Q}_p)}(\chi\circ\varpi_4).$$

Remark 8.3.7. If χ is unramified, then Proposition 8.3.6 tells that the Satake parameter of $\theta(\mathrm{Ind}_{\mathbf{B}_0(\mathbb{Q}_p)}^{\mathbf{PGL}_2(\mathbb{Q}_p)}(\chi))$ is the $\widehat{\mathbf{F}_4}(\mathbb{C})$-conjugacy class of the image of (e_p, c_p) under the embedding $\mathbf{SL}_2 \times \mathbf{SL}_2 \to \mathbf{Sp}_6 \times \mathbf{SL}_2 \to \widehat{\mathbf{F}_4}$, where $c_p = \mathrm{diag}(\chi(p), \chi(p)^{-1})$ and $e_p = \mathrm{diag}(p^{1/2}, p^{-1/2})$.

8.3.3 Archimedean theta correspondence

For the dual pair $\mathbf{F}_4(\mathbb{R}) \times \mathbf{PGL}_2(\mathbb{R})$ inside $\mathbf{E}_7(\mathbb{R})$, we have the following result:

Proposition 8.3.8. *[GrossSavin, 1998, Proposition 3.2] The restriction of* $\Pi_{\min,\infty}$ *to* $\mathbf{F}_4(\mathbb{R}) \times \mathbf{PGL}_2(\mathbb{R})$ *is isomorphic to*

$$\bigoplus_{n\geq 0} \mathrm{V}_{n\varpi_4} \boxtimes \mathcal{D}(2n+12),$$

where $\mathrm{V}_{n\varpi_4}$ *is the irreducible representation of* $\mathbf{F}_4(\mathbb{R})$ *with highest weight* $n\varpi_4$*, and* $\mathcal{D}(m)$ *is the unitary completion of* $d_{hol}(m) \oplus d_{anti\text{-}holo}(m)$*,* $d_{hol}(m)$ *being the holomorphic discrete series representation of* $\mathbf{SL}_2(\mathbb{R})$ *with minimal* $\mathbf{SO}_2(\mathbb{R})$ *type* m *and* $d_{anti\text{-}holo}(m)$ *being its contragradient.*

Before stating the result for $\mathbf{Spin}_9 \times \mathbf{SO}_{2,2}$, we define some notations for $\mathbf{Spin}_9(\mathbb{R})$.

Notation 8.3.9. Let λ_1 be the highest weight of the standard 9-dimensional representation of $\mathbf{Spin}_9(\mathbb{R})$, and λ_2 that of the 16-dimensional spinor representation. Denote by $\mathrm{U}_{m,n}$ the irreducible representation of $\mathbf{Spin}_9(\mathbb{R})$ with highest weight $m\lambda_1 + n\lambda_2$.

Proposition 8.3.10. *The restriction of* $\Pi_{\min,\infty}$ *to* $\mathbf{Spin}_9(\mathbb{R}) \times \mathbf{SO}_{2,2}(\mathbb{R})$ *is isomorphic to*

$$\bigoplus_{m,n\geq 0} \mathrm{U}_{m,n} \boxtimes \mathcal{D}(n+4) \boxtimes \mathcal{D}(2m+n+8),$$

where we view $\mathcal{D}(n+4) \boxtimes \mathcal{D}(2m+n+8)$ *as a representation of* $\mathbf{SO}_{2,2}(\mathbb{R})$*.*

Proof. The proof is parallel to the argument in [GrossSavin, 1998, §3] for $\mathbf{G}_2 \times \mathbf{PGSp}_6$, using the branching laws in [Lepowsky, 1970]. □

8.4 Global theta correspondence

In this section, we recall an automorphic realization of the minimal representation of $\mathbf{E}_7(\mathbb{A})$, and then use it to define global theta lifts.

8.4.1 Automorphic forms

Let $\mathbf{G}$ be a connected reductive group over $\mathbb{Q}$ which admits a (reductive) $\mathbb{Z}$-model $\mathscr{G}$, in the sense of [Gross, 1996]. Let $\widehat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$, $\mathbb{A}_f = \widehat{\mathbb{Z}} \otimes \mathbb{Q}$, and $\mathbb{A} = \mathbb{R} \times \mathbb{A}_f$. We fix a maximal compact subgroup K_∞ of $\mathbf{G}(\mathbb{R})$ and let $\mathfrak{g} = \mathbb{C} \otimes_{\mathbb{R}} \mathrm{Lie}(\mathbf{G}(\mathbb{R}))$.

For the simplicity we assume that the center of $\mathbf{G}$ is anisotropic, and denote the quotient space $\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})$ by $[\mathbf{G}]$. This topological space $[\mathbf{G}]$ admits a right invariant finite Haar measure μ, with respect to which we can define the space $\mathrm{L}^2([\mathbf{G}])$ of square-integrable functions on $[\mathbf{G}]$. The topological group $\mathbf{G}(\mathbb{A})$ acts on $\mathrm{L}^2([\mathbf{G}])$ by right translation, and the Petersson inner product makes it a unitary $\mathbf{G}(\mathbb{A})$-representation.

Definition 8.4.1. (1) An irreducible unitary representation π of $\mathbf{G}(\mathbb{A})$ is *(square-integrable) discrete automorphic* in the sense of [BorelJacquet, 1979, §4.6], if π is isomorphic to a $\mathbf{G}(\mathbb{A})$-invariant closed subspace of $\mathrm{L}^2([\mathbf{G}])$. We denote by $\Pi_{\mathrm{disc}}(\mathbf{G})$ the set of equivalence classes of discrete automorphic representations of $\mathbf{G}$, and by $\mathrm{L}^2_{\mathrm{disc}}([\mathbf{G}])$ the discrete part of $\mathrm{L}^2([\mathbf{G}])$.
(2) An irreducible unitary representation π of $\mathbf{G}(\mathbb{A})$ has *level one* if π can be decomposed as $\pi = \pi_\infty \otimes \pi_f$, where π_∞ is an irreducible unitary representation of $\mathbf{G}(\mathbb{R})$ and π_f is a smooth irreducible representation of $\mathbf{G}(\mathbb{A}_f)$ such that $\pi_f^{\mathscr{G}(\widehat{\mathbb{Z}})} \neq 0$. We denote the subset of $\Pi_{\mathrm{disc}}(\mathbf{G})$ consisting of those with level one by $\Pi^{\mathrm{unr}}_{\mathrm{disc}}(\mathbf{G})$.
(3) The space of *(square-integrable) automorphic forms* $\mathcal{A}(\mathbf{G})$ is defined to be the space of $K_\infty \times \mathscr{G}(\widehat{\mathbb{Z}})$-finite and $\mathrm{Z}(\mathrm{U}(\mathfrak{g}))$-finite functions in the discrete spectrum $\mathrm{L}^2_{\mathrm{disc}}([\mathbf{G}])$.

Definition 8.4.2. (1) A square-integrable Borel function $f : [\mathbf{G}] \to \mathbb{C}$ is *cuspidal* if for the unipotent radical $\mathbf{U}$ of every proper parabolic subgroup of $\mathbf{G}$, we have

$$\int_{[\mathbf{U}]} f(ug)du = 0$$

for almost all $g \in \mathbf{G}(\mathbb{A})$. We denote the subspace of $\mathrm{L}^2([\mathbf{G}])$ consisting of the classes of cuspidal functions by $\mathrm{L}^2_{\mathrm{cusp}}([\mathbf{G}])$, and the subspace of $\mathcal{A}(\mathbf{G})$ consisting of cuspidal automorphic forms by $\mathcal{A}_{\mathrm{cusp}}(\mathbf{G})$.
(2) A discrete automorphic representation of $\mathbf{G}$ is *cuspidal* if it is a subrepresentation of $\mathrm{L}^2_{\mathrm{cusp}}([\mathbf{G}])$. Denote by $\Pi_{\mathrm{cusp}}(\mathbf{G})$ (*resp.* $\Pi^{\mathrm{unr}}_{\mathrm{cusp}}(\mathbf{G})$) the subset of $\Pi_{\mathrm{disc}}(\mathbf{G})$ (*resp.* $\Pi^{\mathrm{unr}}_{\mathrm{disc}}(\mathbf{G})$) consisting of cuspidal representations.

8.4.1.1 Automorphic forms of $\mathbf{F}_4$

Now we concentrate on the level one automorphic forms of $\mathbf{F}_4$, and describe them in a manner similar to the case for orthogonal groups [ChenevierLannes, 2019, §4.4]. The adelic quotient $[\mathbf{F}_4]$ us compact, so $\mathrm{L}^2([\mathbf{F}_4]) = \mathrm{L}^2_{\mathrm{disc}}([\mathbf{F}_4]) = \mathrm{L}^2_{\mathrm{cusp}}([\mathbf{F}_4])$, and every automorphic representation of $\mathbf{F}_4$ is discrete and cuspidal.

A level one automorphic representation of $\mathbf{F}_4$ is generated by some automorphic form $\varphi \in \mathcal{A}(\mathbf{F}_4)^{\mathcal{F}_{4,\mathrm{I}}(\widehat{\mathbb{Z}})} \subseteq \mathrm{L}^2([\mathbf{F}_4])^{\mathcal{F}_{4,\mathrm{I}}(\widehat{\mathbb{Z}})}$. The latter space can be viewed as the space of square-integrable functions on $\mathbf{F}_4(\mathbb{Q})\backslash\mathbf{F}_4(\mathbb{A})/\mathcal{F}_{4,\mathrm{I}}(\widehat{\mathbb{Z}})$, endowed with the Radon measure that is the image of μ by the canonical map $[\mathbf{F}_4] \to \mathbf{F}_4(\mathbb{Q})\backslash\mathbf{F}_4(\mathbb{A})/\mathcal{F}_{4,\mathrm{I}}(\widehat{\mathbb{Z}})$. By the Peter-Weyl theorem, $\mathrm{L}^2([\mathbf{F}_4])^{\mathcal{F}_{4,\mathrm{I}}(\widehat{\mathbb{Z}})}$ can be decomposed into a direct sum of irreducible representations:

Lemma 8.4.3. *Denote by* $\mathrm{Irr}(\mathbf{F}_4(\mathbb{R}))$ *the set of equivalence classes of irreducible representations of* $\mathbf{F}_4(\mathbb{R})$*, then we have:*

$$\mathrm{L}^2([\mathbf{F}_4])^{\mathcal{F}_{4,\mathrm{I}}(\widehat{\mathbb{Z}})} \simeq \overline{\bigoplus_{V\in\mathrm{Irr}(\mathbf{F}_4(\mathbb{R}))} V \otimes \mathcal{A}_V(\mathbf{F}_4)},$$

where $\mathcal{A}_V(\mathbf{F}_4)$ *is defined as*

$$\left\{ f : \mathbf{F}_4(\mathbb{Q})\backslash\mathbf{F}_4(\mathbb{A})/\mathcal{F}_{4,\mathrm{I}}(\widehat{\mathbb{Z}}) \to V \,\middle|\, f(gh) = h^{-1}.f(g),\ \textit{for any } g \in \mathbf{F}_4(\mathbb{A}), h \in \mathbf{F}_4(\mathbb{R}) \right\}. \quad (8.14)$$

Under this isomorphism, an automorphic form $\varphi \in \mathcal{A}(\mathbf{F}_4)^{\mathcal{F}_{4,\mathrm{I}}(\widehat{\mathbb{Z}})}$ is identified with an element of $\bigoplus_{V \in \mathrm{Irr}(\mathbf{F}_4(\mathbb{R}))} V \otimes \mathcal{A}_V(\mathbf{F}_4)$. The number of $\pi \in \Pi_{\mathrm{disc}}^{\mathrm{unr}}(\mathbf{F}_4)$ such that $\pi_\infty \simeq V$, counted with multiplicities, is exactly $\dim \mathcal{A}_V(\mathbf{F}_4)$, which is computed explicitly in [Shan, 2024].

Using Proposition 8.2.18, we identify $\mathbf{F}_4(\mathbb{Q})\backslash\mathbf{F}_4(\mathbb{A})/\mathcal{F}_{4,\mathrm{I}}(\widehat{\mathbb{Z}})$ with the set $\mathcal{J}$ of Albert lattices, and equip $\mathcal{J}$ with the corresponding right $\mathbf{F}_4(\mathbb{R})$-invariant Radon measure. We can thus identify $\mathrm{L}^2([\mathbf{F}_4])^{\mathcal{F}_{4,\mathrm{I}}(\widehat{\mathbb{Z}})}$ with $\mathrm{L}^2(\mathcal{J})$, equipped with the induced $\mathbf{F}_4(\mathbb{R})$ action:

$$(g.f)(J) = f(g^{-1}J), \text{ for any } g \in \mathbf{F}_4(\mathbb{R}), J \in \mathcal{J},$$

and identify $\mathcal{A}_V(\mathbf{F}_4)$ with the space

$$\{f : \mathcal{J} \to V \mid f(gJ) = g.f(J), \text{ for any } g \in \mathbf{F}_4(\mathbb{R}), J \in \mathcal{J}\}.$$

We will use either of these two formulations of $\mathcal{A}_V(\mathbf{F}_4)$, depending on convenience.

A function $f \in \mathcal{A}_V(\mathbf{F}_4)$ is determined by its values on the set of representatives $\{1, \gamma_{\mathrm{E}}\}$ for $\mathbf{F}_4(\mathbb{Q})\backslash\mathbf{F}_4(\mathbb{A}_f)/\mathcal{F}_{4,\mathrm{I}}(\widehat{\mathbb{Z}})$ chosen in Notation 8.2.19. Furthermore, we have:

Lemma 8.4.4. *The evaluation map $f \mapsto (f(1), f(\gamma_{\mathrm{E}}))$ (or equivalently $f \mapsto (f(\mathrm{J}_{\mathbb{Z}}), f(\mathrm{J}_{\mathrm{E}}))$) induces an isomorphism of vector spaces:*

$$\mathrm{M}_V(\mathbf{F}_4) \simeq V^{\Gamma_{\mathrm{I}}} \oplus V^{\Gamma_{\mathrm{E}}},$$

where $\Gamma_{\mathrm{I}} = \mathcal{F}_{4,\mathrm{I}}(\mathbb{Z})$ is the automorphism group of the Albert algebra $\mathrm{J}_{\mathbb{Z}}$, and Γ_{E} is that of J_{E}.

8.4.1.2 A polynomial model of $\mathrm{V}_{n\varpi_4}$

In this paper, we focus on automorphic representations of $\mathbf{F}_4$ with archimedean component $V = \mathrm{V}_{n\varpi_4}$. Now we give a polynomial model of this family of irreducible representations.

When $n = 1$, a natural model for the 26-dimensional representation V_{ϖ_4} is the trace 0 part of $\mathrm{J}_{\mathbb{C}} \simeq \mathfrak{p}_{\mathrm{J}}^-$. We choose the realization dual to this one, *i.e.* the subspace of $\mathrm{P}_1(\mathrm{J}_{\mathbb{C}}) \simeq \mathfrak{p}_{\mathrm{J}}^+$ consisting of linear functions ℓ on $\mathrm{J}_{\mathbb{C}}$ such that $\ell(\mathrm{I}) = 0$.

For $n \geq 1$, $\mathrm{V}_{n\varpi_4}$ is a subrepresentation of $\mathrm{Sym}^n \mathrm{V}_{\varpi_4} \subseteq \mathrm{Sym}^n \mathfrak{p}_{\mathrm{J}}^+ = \mathrm{P}_n(\mathrm{J}_{\mathbb{C}})$, where the action of $\mathbf{F}_4(\mathbb{R})$ on $\mathrm{P}_n(\mathrm{J}_{\mathbb{C}})$ is given as:

$$(g.P)(X) = P(g^{-1}x), \text{ for any } g \in \mathbf{F}_4(\mathbb{R}), P \in \mathrm{P}_n(\mathrm{J}_{\mathbb{C}}) \text{ and } X \in \mathrm{J}_{\mathbb{C}}.$$

Definition 8.4.5. Define $\mathbb{X}$ to be the following $\mathbf{F}_4(\mathbb{C})$-orbit in $\mathrm{J}_{\mathbb{C}}$:

$$\mathbb{X} := \{A \in \mathrm{J}_{\mathbb{C}} \mid \mathrm{Tr}(A) = 0, \mathrm{rank}(A) = 1\} = \{A \in \mathrm{J}_{\mathbb{C}} \mid A \neq 0, \mathrm{Tr}(A) = 0, \mathrm{rank}(A) = 1\}.$$

For any $n \geq 1$, we define $\mathrm{V}_n(\mathrm{J}_{\mathbb{C}})$ to be the subspace of $\mathrm{P}_n(\mathrm{J}_{\mathbb{C}})$ spanned by polynomials of the form $X \mapsto (\mathrm{Tr}\,(X \circ A))^n$, $A \in \mathbb{X}$.

Lemma 8.4.6. *For any $n \geq 1$, $\mathrm{V}_n(\mathrm{J}_{\mathbb{C}})$ is an irreducible representation of $\mathbf{F}_4(\mathbb{R})$, and its highest weight is $n\varpi_4$.*

Proof. This lemma follows from the fact that $\mathbb{X}$ is the set of highest vectors in the irreducible $\mathbf{F}_4(\mathbb{R})$-representation $\{A \in \mathrm{J}_{\mathbb{C}}, \mathrm{Tr}(A) = 0\} \simeq \mathrm{V}_{\varpi_4}$, and $\mathbf{F}_4(\mathbb{R})$ acts on it transitively. □

8.4.2 Automorphic realization of minimal representation

Let $\Pi_{\min} = \otimes'_v \Pi_{\min,v}$ be the (adelic) minimal representation of $\mathbf{E}_7(\mathbb{A})$. To establish the global theta correspondence for dual pairs inside $\mathbf{E}_7$, we need to choose an automorphic realization of $\Pi_{\min}$, *i.e.* an $\mathbf{E}_7(\mathbb{A})$-equivariant embedding $\theta : \Pi_{\min} \hookrightarrow \mathrm{L}^2([\mathbf{E}_7])$. In this section, we follow [KimYamauchi, 2016, §6] to give θ via an explicit modular form constructed by Kim in [Kim, 1993].

8.4.2.1 Exceptional modular forms

Definition 8.4.7. The *exceptional tube domain* $\mathcal{H}_{\mathrm{J}}$ of complex dimension 27 is the open subset of $\mathrm{J}_{\mathbb{C}} = \mathrm{J}_{\mathbb{R}} + i\mathrm{J}_{\mathbb{R}}$ consisting of $Z = X + iY$ with Y positive definite.

For any element $Z \in \mathrm{J}_{\mathbb{C}}$, set $\mathrm{r}_1(Z) := \left(Z, \det(Z), Z^{\#}, 1\right) \in \mathrm{W}_{\mathrm{J}} \otimes \mathbb{C}$. By [Pollack, 2020, Proposition 2.3.1], for any $g \in \mathbf{H}^1_{\mathrm{J}}(\mathbb{R})$ and $Z \in \mathcal{H}_{\mathrm{J}}$, there exist a unique scalar $J(g, Z) \in \mathbb{C}^{\times}$, which is called the *automorphy factor* for $\mathbf{H}^1_{\mathrm{J}}(\mathbb{R})$, and a unique $Z' \in \mathcal{H}_{\mathrm{J}}$ such that

$$g.\mathrm{r}_1(Z) = J(g, Z)\mathrm{r}_1(Z').$$

Definition 8.4.8. The action of $\mathbf{H}^1_{\mathrm{J}}(\mathbb{R})$-action on $\mathcal{H}_{\mathrm{J}}$ is defined as follows: for $g \in \mathbf{H}^1_{\mathrm{J}}(\mathbb{R})$ and $Z \in \mathcal{H}_{\mathrm{J}}$, $g.Z$ is the unique $Z' \in \mathcal{H}_{\mathrm{J}}$ satisfying $g.\mathrm{r}_1(Z) \in \mathbb{C}^{\times}\mathrm{r}_1(Z')$.

Example 8.4.9. We list the actions of some elements in $\mathbf{H}^1_{\mathrm{J}}(\mathbb{R})$:

- For $\mathrm{n}(A) \in \mathbf{N}_{\mathrm{J}}(\mathbb{R})$, $\mathrm{n}(A).Z = Z + A$ and $J(\mathrm{n}(A), Z) = 1$;
- For $m \in \mathbf{M}_{\mathrm{J}}(\mathbb{R})$, $m.(X + iY) = \lambda(m)(\lambda(X) + i\lambda(Y))$ and $J(m, Z) = \lambda(m)^{-1}$;
- For ι defined by (8.6), $\iota.Z = -Z^{-1}$ and $J(\iota, Z) = \det(Z)$.

The center $\pm 1 \simeq \langle \iota^2 \rangle$ of $\mathbf{H}^1_{\mathrm{J}}(\mathbb{R})$ acts trivially on $\mathcal{H}_{\mathrm{J}}$, and the group of holomorphic transformations of $\mathcal{H}_{\mathrm{J}}$ is $\mathbf{H}^1_{\mathrm{J}}(\mathbb{R})/\pm 1$, the connected component of $\mathbf{E}_7(\mathbb{R})$.

Definition 8.4.10. A holomorphic function $F : \mathcal{H}_{\mathrm{J}} \to \mathbb{C}$ is a *modular form of level* 1 *and weight* k if for any $Z \in \mathcal{H}_{\mathrm{J}}$ and $\gamma \in \mathbf{H}^1_{\mathrm{J}}(\mathbb{Z})$ we have

$$F(\gamma.Z) = J(\gamma, Z)^k \cdot F(Z).$$

Kim's modular form F_{Kim} is defined by the following Fourier expansion:

$$\mathrm{F}_{Kim}(Z) := 1 + 240 \sum_{\substack{\mathrm{J}_{\mathbb{Z}} \ni T \geq 0, \\ \mathrm{rank}(T)=1}} \sigma_3\left(\mathrm{c}_{\mathrm{J}_{\mathbb{Z}}}(T)\right) e^{2\pi i (T,Z)}, \text{ for any } Z \in \mathcal{H}_{\mathrm{J}}, \tag{8.15}$$

where $\mathrm{c}_{\mathrm{J}_{\mathbb{Z}}}(T)$ is the content of T, *i.e.* the largest integer c such that $T/c \in \mathrm{J}_{\mathbb{Z}}$, and $\sigma_3(n) = \sum_{d|n} d^3$. The function F_{Kim} defined by (8.15) is a modular form of level 1 and weight 4.

8.4.2.2 Kim's automorphic form

Kim's modular form F_{Kim} gives rise to a level one automorphic form of $\mathbf{E}_7$. Using the strong approximation property of $\mathbf{E}_7$, we have the following natural homemorphisms:

$$\mathbf{E}_7(\mathbb{Q})\backslash\mathbf{E}_7(\mathbb{A})/\mathbf{E}_7(\widehat{\mathbb{Z}}) \simeq \mathbf{E}_7(\mathbb{Z})\backslash\mathbf{E}_7(\mathbb{R}) \simeq \mathbf{H}^1_{\mathrm{J}}(\mathbb{Z})\backslash\mathbf{H}^1_{\mathrm{J}}(\mathbb{R}),$$

thus we write any element $g \in \mathbf{E}_7(\mathbb{A})$ as $g = g_{\mathbb{Q}}g_\infty g_{\widehat{\mathbb{Z}}}$, where $g_{\mathbb{Q}} \in \mathbf{E}_7(\mathbb{Q})$, $g_{\widehat{\mathbb{Z}}} \in \mathbf{E}_7(\widehat{\mathbb{Z}})$ and $g_\infty \in \mathbf{E}_7(\mathbb{R})$ is the image of an element in $\mathbf{H}^1_{\mathrm{J}}(\mathbb{R})$ under the projection $\mathbf{H}_{\mathrm{J}}(\mathbb{R}) \to \mathbf{E}_7(\mathbb{R})$. In other words, g_∞ is an element of $\mathbf{H}^1_{\mathrm{J}}(\mathbb{R})/\pm 1$, the group of holomorphic automorphisms of $\mathcal{H}_{\mathrm{J}}$. Now for $g = g_{\mathbb{Q}}g_\infty g_{\widehat{\mathbb{Z}}} \in \mathbf{E}_7(\mathbb{A})$, we define

$$\Theta_{Kim}(g) := J(g_\infty, i\mathrm{I})^{-4} \cdot \mathrm{F}_{Kim}(g_\infty.i\mathrm{I}),$$

which is a well-defined[5] automorphic form of $\mathbf{E}_7$. Using the explicit action on $\mathcal{H}_{\mathrm{J}}$ given in Example 8.4.9, one gets the following:

Lemma 8.4.11. *The automorphic form $\Theta_{Kim} \in \mathcal{A}(\mathbf{E}_7)$ is invariant under $\mathbf{F}_4(\mathbb{R}) \times \mathbf{E}_7(\widehat{\mathbb{Z}})$.*

Now we use Θ_{Kim} to embed $\Pi_{\min}$ into $\mathrm{L}^2([\mathbf{E}_7])$:

Definition 8.4.12. Let $\Phi_p \in \Pi_{\min,p}$ be the normalized spherical vector, $\Phi_\infty \in \Pi^+ \subseteq \Pi_{\min,\infty}$ the unique (up to scalar) holomorphic vector with the minimal $\mathrm{K}_{\mathrm{E}_7}$-type, and $\Phi_0 := \Phi_\infty \otimes \Phi_f = \otimes_v \Phi_v \in \Pi_{\min}$. The automorphic realization $\theta : \Pi_{\min} \hookrightarrow \mathrm{L}^2([\mathbf{E}_7])$ is defined to be the unique $\mathbf{E}_7(\mathbb{A})$-equivariant map sending Φ_0 to Θ_{Kim}.

8.4.2.3 Constructing automorphic forms with non-minimal K_{E_7}-types

The holomorphic vector Φ_∞ lies in the minimal $\mathrm{K}_{\mathrm{E}_7}$-type of $\Pi^+ \subseteq \Pi_{\min,\infty}$, and we follow the method in [Pollack, 2020] to produce (holomorphic) automorphic forms with higher $\mathrm{K}_{\mathrm{E}_7}$-types.

For the two summands $\mathfrak{p}_{\mathrm{J}}^{\pm}$ in the Cartan decomposition (8.9) of $\mathfrak{e}_7$, choose a basis $\{\mathrm{X}_\alpha\}_\alpha$ of $\mathfrak{p}_{\mathrm{J}}^{+}$ and its dual basis $\{\mathrm{X}_\alpha^\vee\}_\alpha$ of $\mathfrak{p}_{\mathrm{J}}^{-}$ with respect to $\mathfrak{p}_{\mathrm{J}}^{+} \times \mathfrak{p}_{\mathrm{J}}^{-} \simeq \mathrm{J}_{\mathbb{C}}^\vee \times \mathrm{J}_{\mathbb{C}} \xrightarrow{\{-,-\}} \mathbb{C}$.

Definition 8.4.13. We define a linear differential operator $\mathrm{D} : \mathcal{A}(\mathbf{E}_7) \to \mathcal{A}(\mathbf{E}_7) \otimes \mathfrak{p}_{\mathrm{J}}^{-}$ by

$$\mathrm{D}\varphi(g) := \sum_\alpha (\mathrm{X}_\alpha\varphi)(g) \otimes \mathrm{X}_\alpha^\vee, \text{ for every } \varphi \in \mathcal{A}(\mathbf{E}_7),$$

which is independent of the choice of $\{\mathrm{X}_\alpha\}_\alpha$. For any integer $n \geq 0$, set D^n to be the n-times composition of D.

Applying the differential operator D^n defined in Definition 8.4.13 to Θ_{Kim}, we obtain

$$\Theta_n := \mathrm{D}^n\Theta_{Kim} \in \mathcal{A}(\mathbf{E}_7) \otimes (\mathfrak{p}_{\mathrm{J}}^{-})^{\otimes n},$$

whose coordinates belong to the $\mathrm{K}_{\mathrm{E}_7}$-type $\mathrm{E}(n, 2n+12)$ in (8.13).

[5] Here we use the fact that $J(\gamma, Z) = \pm 1$ for any $\gamma \in \mathbf{H}^1_{\mathrm{J}}(\mathbb{Z})$ and $Z \in \mathcal{H}_{\mathrm{J}}$.

Notation 8.4.14. (1) For any Albert lattice $J \in \mathcal{J}$, denote by J^+ the set of rank 1 and positive semi-definite elements in J, and set $\mathrm{a}_J(T) := \sigma_3(\mathrm{c}_J(T))$ for any $T \in J$, where $\mathrm{c}_J(T)$ is the content of T in J.

(2) For any element $T \in \mathrm{J}_{\mathbb{R}}$, denote by h_T the function:

$$\mathbf{H}^1_{\mathrm{J}}(\mathbb{R}) \to \mathbb{C},\ g_\infty \mapsto J(g_\infty, i\mathrm{I})^{-4} \cdot e^{2\pi i(T, g_\infty.i\mathrm{I})}.$$

With these notations, for any $n \geq 1$, we rewrite Θ_n as:

$$\Theta_n(g) = 240 \sum_{T \in \mathrm{J}^+_{\mathbb{Z}}} \mathrm{a}_{\mathrm{J}_{\mathbb{Z}}}(T) \cdot \mathrm{D}^n h_T(g) = 240 \sum_{T \in \mathrm{J}^+_{\mathbb{Z}}} \mathrm{a}_{\mathrm{J}_{\mathbb{Z}}}(T) \cdot \mathrm{D}^n h_T(g_\infty), \tag{8.16}$$

where $g = g_{\mathbb{Q}} g_\infty g_{\widehat{\mathbb{Z}}}$ as in Section 8.4.2.2. We end this section by the following property of Θ_n:

Lemma 8.4.15. *For any $g_\infty \in \mathbf{H}^1_{\mathrm{J}}(\mathbb{R})$ and $h_\infty \in \mathbf{F}_4(\mathbb{R})$, we have $\Theta_n(g_\infty h_\infty) = h_\infty^{-1}.\Theta_n(g_\infty)$, where the action of h_∞^{-1} is applied on $(\mathfrak{p}_{\mathrm{J}}^-)^{\otimes n}$.*

Proof. By the definition of $\Theta_n = \mathrm{D}^n \Theta_{Kim}$, we have:

$$\begin{aligned}
\Theta_n(g_\infty h_\infty) &= \sum_{\alpha_1,\dots,\alpha_n} (\mathrm{X}_{\alpha_n} \cdots \mathrm{X}_{\alpha_1} \Theta_{Kim})(g_\infty h_\infty) \otimes \mathrm{X}^\vee_{\alpha_1} \otimes \cdots \otimes \mathrm{X}^\vee_{\alpha_n} \\
&= \sum_{\alpha_1,\dots,\alpha_n} \frac{d}{dt_n}\Big|_{t_n=0} \cdots \frac{d}{dt_1}\Big|_{t_1=0} \Theta_{Kim}(g_\infty h_\infty e^{t_n \mathrm{X}_{\alpha_n}} \cdots e^{t_1 \mathrm{X}_{\alpha_1}}) \otimes \mathrm{X}^\vee_{\alpha_1} \otimes \cdots \otimes \mathrm{X}^\vee_{\alpha_n} \\
&= \sum_{\alpha_1,\dots,\alpha_n} \frac{d}{dt_n}\Big|_{t_n=0} \cdots \frac{d}{dt_1}\Big|_{t_1=0} \Theta_{Kim}(g_\infty e^{t_n \mathrm{Ad}(h_\infty)\mathrm{X}_{\alpha_n}} \cdots e^{t_1 \mathrm{Ad}(h_\infty)\mathrm{X}_{\alpha_1}} h_\infty) \otimes \mathrm{X}^\vee_{\alpha_1} \otimes \cdots \otimes \mathrm{X}^\vee_{\alpha_n} \\
&= \sum_{\alpha_1,\dots,\alpha_n} \frac{d}{dt_n}\Big|_{t_n=0} \cdots \frac{d}{dt_1}\Big|_{t_1=0} \Theta_{Kim}(g_\infty e^{t_n h_\infty.\mathrm{X}_{\alpha_n}} \cdots e^{t_1 h_\infty.\mathrm{X}_{\alpha_1}}) \otimes \mathrm{X}^\vee_{\alpha_1} \otimes \cdots \otimes \mathrm{X}^\vee_{\alpha_n},
\end{aligned}$$

where $h_\infty.\mathrm{X}_\alpha = \mathrm{Ad}(h_\infty)\mathrm{X}_\alpha$ and the last equality follows from Lemma 8.4.11. Since $\mathbf{F}_4(\mathbb{R})$ is a subgroup of the maximal compact subgroup $\mathrm{K}_{\mathrm{E}_7}$ of $\mathbf{E}_7(\mathbb{R})$, $\{h_\infty.\mathrm{X}_\alpha\}_\alpha$ also gives a basis of $\mathfrak{p}_{\mathrm{J}}^+$, and its dual basis of $\mathfrak{p}_{\mathrm{J}}^-$ is $\{h_\infty.\mathrm{X}^\vee_\alpha\}_\alpha$. As the differential operator D is independent of the choice of $\{\mathrm{X}_\alpha\}_\alpha$, we have:

$$\Theta_n(g_\infty h_\infty) = \sum_{\alpha_1,\dots,\alpha_n} (\mathrm{X}_{\alpha_n} \cdots \mathrm{X}_{\alpha_1} \Theta_{Kim})(g_\infty) \otimes h_\infty^{-1}.\mathrm{X}^\vee_{\alpha_1} \otimes \cdots \otimes h_\infty^{-1}.\mathrm{X}^\vee_{\alpha_n} = h_\infty^{-1}.\Theta_n(g_\infty). \qquad \square$$

8.4.3 Global theta lifts

Let $\mathbf{G} \times \mathbf{H}$ be one of the two reductive dual pairs given in Section 8.2.5, *i.e.* $\mathbf{G} \times \mathbf{H} = \mathbf{F}_4 \times \mathbf{PGL}_2$ or $\mathbf{Spin}_9 \times \mathbf{SO}_{2,2}$.

Definition 8.4.16. For $\varphi \in \mathcal{A}(\mathbf{H})$ and $\phi \in \Pi_{\min}$, the *global theta lift of φ with respect to ϕ* is the automorphic form of $\mathbf{G}$ defined by the following absolutely convergent integral:

$$\Theta_\phi(\varphi)(g) := \int_{[\mathbf{H}]} \theta(\phi)(gh)\overline{\varphi(h)}dh, \text{ for any } g \in \mathbf{G}(\mathbb{A}).$$

For a cuspidal automorphic representation $\pi \in \Pi_{\mathrm{cusp}}(\mathbf{H})$, its *global theta lift* $\Theta(\pi)$ is the $\mathbf{G}(\mathbb{A})$-subspace of $\mathrm{L}^2([\mathbf{G}])$ generated by $\{\Theta_\phi(\varphi) \mid \varphi \in \pi,\ \phi \in \Pi_{\min}\}$.

Remark 8.4.17. In this paper, we are always in the situation that either $[\mathbf{H}]$ is compact or $\varphi \in \mathcal{A}(\mathbf{H})$ is cuspidal. For the second case, the absolute convergence comes from the rapid decay of φ.

We also define the global theta lift of a "vector-valued automorphic form" $\alpha \in \mathcal{A}_{\mathrm{V}_{n\varpi}}(\mathbf{F}_4)$ defined as (8.14), which is compatible with Definition 8.4.16:

Definition 8.4.18. For a function $\alpha : \mathbf{F}_4(\mathbb{Q})\backslash\mathbf{F}_4(\mathbb{A})/\mathcal{F}_{4,\mathrm{I}}(\widehat{\mathbb{Z}}) \to \mathrm{V}_{n\varpi_4}$ in $\mathcal{A}_{\mathrm{V}_{n\varpi_4}}(\mathbf{F}_4)$, its *global theta lift* $\Theta(\alpha)$ is defined as:

$$\Theta(\alpha)(g) = \int_{[\mathbf{F}_4]} \{\Theta_n(gh), \alpha(h)\} dh, \text{ for any } g \in \mathbf{PGL}_2(\mathbb{A}), \tag{8.17}$$

where $\{-,-\} : \mathrm{J}_{\mathbb{C}}^{\otimes n} \times (\mathrm{J}_{\mathbb{C}}^{\vee})^{\otimes n} \to \mathbb{C}$ is the pairing defined in (8.10), and we view $\alpha(h) \in \mathrm{V}_{n\varpi_4}$ as a homogeneous polynomial over $\mathrm{J}_{\mathbb{C}}$.

8.5 Exceptional theta series

In this section, we compute the Fourier expansion of the theta lift $\Theta(\alpha)$ of $\alpha \in \mathcal{A}_{\mathrm{V}_{n\varpi_4}}(\mathbf{F}_4)$, and prove Theorem 8.1.4 in the introduction. From now on, we will identify α with its values $\alpha_{\mathrm{I}} \in \mathrm{V}_n(\mathrm{J}_{\mathbb{C}})^{\Gamma_{\mathrm{I}}}, \alpha_{\mathrm{E}} \in \mathrm{V}_n(\mathrm{J}_{\mathbb{C}})^{\Gamma_{\mathrm{E}}}$ at $1, \gamma_{\mathrm{E}}$ as in Lemma 8.4.4.

8.5.1 Fourier expansions of global theta lifts

Normalize the Haar measure dh of $\mathbf{F}_4(\mathbb{A})$ in (8.17) so that $\mathbf{F}_4(\mathbb{R})\mathcal{F}_{4,\mathrm{I}}(\widehat{\mathbb{Z}})$ has measure 1. Write $g \in \mathbf{PGL}_2(\mathbb{A})$ as $g = g_{\mathbb{Q}} g_{\infty} g_{\widehat{\mathbb{Z}}}$, where $g_{\mathbb{Q}} \in \mathbf{PGL}_2(\mathbb{Q}), g_{\widehat{\mathbb{Z}}} \in \mathbf{PGL}_2(\widehat{\mathbb{Z}})$ and g_{∞} is the image of an element in $\mathbf{SL}_2(\mathbb{R})$, then using Lemma 8.2.22, Lemma 8.4.15 and the $\mathbf{F}_4(\mathbb{R})$-invariance of $\{-,-\}$, we obtain:

$$\begin{aligned}\Theta(\alpha)(g) =& \frac{1}{|\Gamma_{\mathrm{I}}|}\int_{\mathbf{F}_4(\mathbb{R})\mathcal{F}_{4,\mathrm{I}}(\widehat{\mathbb{Z}})} \{\Theta_n(gh_{\infty}h_{\widehat{\mathbb{Z}}}), \alpha(h_{\infty}h_{\widehat{\mathbb{Z}}})\}dh + \frac{1}{|\Gamma_{\mathrm{E}}|}\int_{\mathbf{F}_4(\mathbb{R})\mathcal{F}_{4,\mathrm{I}}(\widehat{\mathbb{Z}})} \{\Theta_n(gh_{\infty}\gamma_{\mathrm{E}}h_{\widehat{\mathbb{Z}}}), \alpha(h_{\infty}\gamma_{\mathrm{E}}h_{\widehat{\mathbb{Z}}})dh\} \\ =& \frac{1}{|\Gamma_{\mathrm{I}}|}\int_{\mathbf{F}_4(\mathbb{R})\mathcal{F}_{4,\mathrm{I}}(\widehat{\mathbb{Z}})} \{h_{\infty}^{-1}.\Theta_n(g_{\infty}), h_{\infty}^{-1}.\alpha_{\mathrm{I}}\}dh + \frac{1}{|\Gamma_{\mathrm{E}}|}\int_{\mathbf{F}_4(\mathbb{R})\mathcal{F}_{4,\mathrm{I}}(\widehat{\mathbb{Z}})} \{h_{\infty}^{-1}.\Theta_n(\delta_{\infty}^{-1}g_{\infty}), h_{\infty}^{-1}.\alpha_{\mathrm{E}}\} \\ =& \frac{1}{|\Gamma_{\mathrm{I}}|}\{\Theta_n(g_{\infty}), \alpha_{\mathrm{I}}\} + \frac{1}{|\Gamma_{\mathrm{E}}|}\{\Theta_n(\delta_{\infty}^{-1}g_{\infty}), \alpha_{\mathrm{E}}\}.\end{aligned} \tag{8.18}$$

If the global theta lift $\Theta(\alpha) \in \mathcal{A}(\mathbf{PGL}_2)$ is non-zero, then the following result shows that it arises from a weight $2n+12$ classical holomorphic modular form on $\mathbf{SL}_2(\mathbb{Z})$:

Proposition 8.5.1. *Let $\mathcal{H} \subseteq \mathbb{C}$ be the Poincaré half plane, and $j : \mathbf{SL}_2(\mathbb{R}) \times \mathcal{H} \to \mathbb{C}^{\times}$ the automorphy factor given by $j\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), z\right) = cz + d$. For any $\alpha \in \mathcal{A}_{\mathrm{V}_{n\varpi_4}}(\mathbf{F}_4)$, the function*

$$f_{\Theta(\alpha)}(z) := j(g,i)^{2n+12}\Theta(\alpha)(g),\ z = g.i \in \mathcal{H},\ g \in \mathbf{SL}_2(\mathbb{R}),$$

is well-defined and is a level one holomorphic modular form of weight $2n+12$. Furthermore, it is a cusp form when $n > 0$.

We postpone the proof of Proposition 8.5.1 to Section 8.5.3, and prove the following main theorem on the Fourier expansion of $f_{\Theta(\alpha)}$:

Theorem 8.5.2. *(Theorem 8.1.4 in Section 8.1) Let $\alpha \in \mathcal{A}_{\mathrm{V}_{n\varpi_4}}(\mathbf{F}_4)$, $n > 0$ and $f_{\Theta(\alpha)}$ the cusp form associated to its global theta lift $\Theta(\alpha)$. Up to a non-zero constant, $f_{\Theta(\alpha)}$ has the following Fourier expansion:*

$$f_{\Theta(\alpha)}(z) = \frac{1}{|\Gamma_{\mathrm{I}}|}\sum_{T\in \mathrm{J}_{\mathbb{Z}}^{+}} \mathrm{a}_{\mathrm{J}_{\mathbb{Z}}}(T)\alpha_{\mathrm{I}}(T)q^{\mathrm{Tr}(T)} + \frac{1}{|\Gamma_{\mathrm{E}}|}\sum_{T\in \mathrm{J}_{\mathrm{E}}^{+}} \mathrm{a}_{\mathrm{J}_{\mathrm{E}}}(T)\alpha_{\mathrm{E}}(T)q^{\mathrm{Tr}(T)},\ q = e^{2\pi i z}.$$

Remark 8.5.3. The case when $n = 0$ is studied by Elkies and Gross in [ElkiesGross, 1996]. In this case $\alpha \in \mathcal{A}_{\mathbf{1}}(\mathbf{F}_4)$ can be identified as a pair of complex numbers. For α corresponding to $(|\Gamma_{\mathrm{I}}|, 0)$, $f_{\Theta(\alpha)} = E_{12} + \frac{432000}{691}\Delta$; for α corresponding to $(0, |\Gamma_{\mathrm{E}}|)$, $f_{\Theta(\alpha)} = E_{12} - \frac{65520}{691}\Delta$, where $E_{12}(z) = 1 + \frac{2}{\zeta(-11)}\sum_{n\geq 1}\sigma_{11}(n)q^n$ is the normalized weight 12 Eisenstein series, and $\Delta(z) = q\prod_{n\geq 1}(1-q^n)^{24}$ is the discriminant modular form.

Before proving Theorem 8.5.2, we state a result that will be used in the proof, whose proof is also postponed to Section 8.5.3.

Theorem 8.5.4. *Let $P \in \mathrm{V}_n(\mathrm{J}_{\mathbb{C}}) \simeq \mathrm{V}_{n\varpi_4}$ for any $n > 0$, T an element of $\mathrm{J}_{\mathbb{R}}$, and $h_T(g) = J(g_\infty, i\mathrm{I})^{-4}\cdot e^{2\pi i(T, g_\infty . i\mathrm{I})}$ the function given in Notation 8.4.14, then we have:*

$$\{(\mathrm{D}^n h_T)(g), P\} = (-4\pi)^n \cdot j(g,i)^{-2n-12}P\,(T)\,e^{2\pi i(T,\,g.i\mathrm{I})},\ \ \textit{for any } g \in \mathbf{SL}_2(\mathbb{R}).$$

Proof of Theorem 8.5.2. By (8.18), we have

$$f_{\Theta(\alpha)}(z) = j(g,i)^{2n+12}\left(\frac{1}{|\Gamma_{\mathrm{I}}|}\{\Theta_n(g),\alpha_{\mathrm{I}}\} + \frac{1}{|\Gamma_{\mathrm{E}}|}\{\Theta_n(\delta_\infty^{-1}g),\alpha_{\mathrm{E}}\}\right),\ z = g.i \in \mathcal{H}. \tag{8.19}$$

Using the Fourier expansion (8.16) of Θ_n and Theorem 8.5.4, the first term in (8.19) equals

$$\begin{aligned}\frac{1}{|\Gamma_{\mathrm{I}}|}j(g,i)^{2n+12}\{\Theta_n(g),\alpha_{\mathrm{I}}\} &= \frac{240}{|\Gamma_{\mathrm{I}}|}j(g,i)^{2n+12}\sum_{T\in\mathrm{J}_{\mathbb{Z}}^{+}}\mathrm{a}_{\mathrm{J}_{\mathbb{Z}}}(T)\{\mathrm{D}^n h_T(g),\alpha_{\mathrm{I}}\}\\ &= \frac{240(-4\pi)^n}{|\Gamma_{\mathrm{I}}|}\sum_{T\in\mathrm{J}_{\mathbb{Z}}^{+}}\mathrm{a}_{\mathrm{J}_{\mathbb{Z}}}(T)\alpha_{\mathrm{I}}(T)q^{(T,\mathrm{I})_{\mathrm{I}}},\end{aligned}$$

and the second term in (8.19) equals

$$\begin{aligned}\frac{1}{|\Gamma_{\mathrm{E}}|}j(g,i)^{2n+12}\{\Theta_n(\delta_\infty^{-1}g),\alpha_{\mathrm{E}}\} &= \frac{240}{|\Gamma_{\mathrm{E}}|}j(g,i)^{2n+12}\sum_{T\in\mathrm{J}_{\mathbb{Z}}^{+}}\mathrm{a}_{\mathrm{J}_{\mathbb{Z}}}(T)\{\mathrm{D}^n h_T(\delta_\infty^{-1}g),\alpha_{\mathrm{E}}\}\\ &= \frac{240}{|\Gamma_{\mathrm{E}}|}j(g,i)^{2n+12}\sum_{T\in\mathrm{J}_{\mathbb{Z}}^{+}}\mathrm{a}_{\mathrm{J}_{\mathbb{Z}}}(T)\left\{\mathrm{D}^n h_{\delta_\infty^* T}(g),\alpha_{\mathrm{E}}\right\}\\ &= \frac{240(-4\pi)^n}{|\Gamma_{\mathrm{E}}|}\sum_{T\in\mathrm{J}_{\mathbb{Z}}^{+}}\mathrm{a}_{\mathrm{J}_{\mathbb{Z}}}(T)\alpha_{\mathrm{E}}(\delta_\infty^* T)e^{2\pi i(\delta_\infty^* T, g.i\mathrm{I})}.\end{aligned}$$

Since $\mathbf{M}^1_{\mathrm{J}}(\mathbb{R})$ preserves the rank and stabilizes the set of positive semi-definite elements [Elkies-Gross, 1996, Proposition 2.4], we have $\mathrm{J}_{\mathrm{E}}^+ = \delta_\infty(\mathrm{J}_{\mathbb{Z}}^+)$, thus

$$\sum_{T\in \mathrm{J}_{\mathbb{Z}}^+} \mathrm{a}_{\mathrm{J}_{\mathbb{Z}}}(T)\alpha_{\mathrm{E}}(\delta_\infty^* T)e^{2\pi i(\delta_\infty^* T, g.i\mathrm{I})} = \sum_{T\in \mathrm{J}_{\mathrm{E}}^+} \mathrm{a}_{\mathrm{J}_{\mathrm{E}}}(T)\alpha_{\mathrm{E}}(\delta_\infty^*\delta_\infty^{-1} T)e^{2\pi i(\delta_\infty^*\delta_\infty^{-1} T, g.i\mathrm{I})}.$$

The element $\delta_\infty^*\delta_\infty^{-1}$ is the archimedean part of $\delta^*\delta^{-1} \in \mathbf{M}^1_{\mathrm{J}}(\mathbb{Q})$. By Lemma 8.2.22, $\delta_f^{-1}\gamma_{\mathrm{E}} \in \mathbf{M}^1_{\mathrm{J}}(\widehat{\mathbb{Z}})$, so $\delta_f^*\delta_f^{-1} \in \gamma_{\mathrm{E}}^*\mathbf{M}^1_{\mathrm{J}}(\widehat{\mathbb{Z}})\gamma_{\mathrm{E}}^{-1} = \gamma_{\mathrm{E}}\mathbf{M}^1_{\mathrm{J}}(\widehat{\mathbb{Z}})\gamma_{\mathrm{E}}^{-1} = \mathrm{Aut}(\mathrm{J}_{\mathrm{E}} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}, \det)$. As a direct consequence, $\delta^*\delta^{-1}$ induces an automorphism of the lattice J_{E}, thus we have:

$$\sum_{T\in \mathrm{J}_{\mathrm{E}}^+} \mathrm{a}_{\mathrm{J}_{\mathrm{E}}}(T)\alpha_{\mathrm{E}}(\delta_\infty^*\delta_\infty^{-1} T)e^{2\pi i(\delta_\infty^*\delta_\infty^{-1} T, g.i\mathrm{I})} = \sum_{T\in \mathrm{J}_{\mathrm{E}}^+} \mathrm{a}_{\mathrm{J}_{\mathrm{E}}}(T)\alpha_{\mathrm{E}}(T)q^{\mathrm{Tr}(T)}. \qquad \square$$

A direct corollary of Theorem 8.5.2 is the following:

Corollary 8.5.5. *For any Albert lattice $J \in \mathcal{J}$ and any polynomial $P \in \mathrm{V}_n(\mathrm{J}_{\mathbb{C}})$, the (weighted) theta series*

$$\vartheta_{J,P}(z) := \sum_{T\in J^+} \mathrm{a}_J(T)P(T)q^{\mathrm{Tr}(T)},\ z \in \mathcal{H}, q = e^{2\pi i}, \tag{8.20}$$

is a modular form on $\mathbf{SL}_2(\mathbb{Z})$ of weight $2n + 12$, and it is cuspidal if P is not constant.

Proof. Since the theta series (8.20) is invariant under the $\mathbf{F}_4(\mathbb{R})$-action on the pair (J, P) in the sense that $\vartheta_{gJ,gP} = \vartheta_{J,P}$, it suffices to prove the modularity for $J \in \{\mathrm{J}_{\mathbb{Z}}, \mathrm{J}_{\mathrm{E}}\}$. Here we give the proof for $J = \mathrm{J}_{\mathbb{Z}}$, and that for J_{E} is almost the same.

Let $\alpha : \mathcal{J} \to \mathrm{V}_n(\mathrm{J}_{\mathbb{C}})$ be the element in $\mathcal{A}_{\mathrm{V}_n(\mathrm{J}_{\mathbb{C}})}(\mathbf{F}_4)$ that is supported on the $\mathbf{F}_4(\mathbb{R})$-orbit of $\mathrm{J}_{\mathbb{Z}}$ and takes the value $\sum_{\gamma\in\Gamma_{\mathrm{I}}} \gamma.P$ at $\mathrm{J}_{\mathbb{Z}} \in \mathcal{J}$. By Theorem 8.5.2 and Remark 8.5.3, $f_{\Theta(\alpha)}$ is a modular form on $\mathbf{SL}_2(\mathbb{Z})$ of weight $2n + 12$. On the other hand, $\mathrm{J}_{\mathbb{Z}}^+$ is stable under the action of Γ_{I}, thus one has:

$$\begin{aligned} f_{\Theta(\alpha)}(z) &= \frac{1}{|\Gamma_{\mathrm{I}}|}\sum_{T\in \mathrm{J}_{\mathbb{Z}}^+} \mathrm{a}_{\mathrm{J}_{\mathbb{Z}}}(T)\left(\sum_{\gamma\in\Gamma_{\mathrm{I}}} P(\gamma^{-1}T)\right) q^{\mathrm{Tr}(T)} \\ &= \frac{1}{|\Gamma_{\mathrm{I}}|}\sum_{\gamma\in\Gamma_{\mathrm{I}}}\left(\sum_{T\in \mathrm{J}_{\mathbb{Z}}} \mathrm{a}_{\mathrm{J}_{\mathbb{Z}}}(\gamma T)P(T)q^{\mathrm{Tr}(\gamma T)}\right) \\ &= \vartheta_{\mathrm{J}_{\mathbb{Z}},P}(z) \end{aligned} \qquad \square$$

If we view $\alpha \in \mathcal{A}_{\mathrm{V}_{n\varpi_4}}(\mathbf{F}_4)$ as a function $\alpha : \mathcal{J} \to \mathrm{V}_n(\mathrm{J}_{\mathbb{C}})$, the modular form $f_{\Theta(\alpha)}$ can be written in the following forms:

$$f_{\Theta(\alpha)} = \frac{1}{|\Gamma_{\mathrm{I}}|}\vartheta_{\mathrm{J}_{\mathbb{Z}},\alpha(\mathrm{J}_{\mathbb{Z}})} + \frac{1}{|\Gamma_{\mathrm{E}}|}\vartheta_{\mathrm{J}_{\mathrm{E}},\alpha(\mathrm{J}_{\mathrm{E}})}.$$

8.5.2 Theta series attached to $\mathbf{Spin}_9(\mathbb{R})$-invariant polynomials

As an application of Theorem 8.5.2, we are going to show that for every weight k with $\mathrm{S}_k(\mathbf{SL}_2(\mathbb{Z})) \neq 0$, there exists a polynomial $P \in \mathrm{V}_{\frac{k-12}{2}}(\mathrm{J}_\mathbb{C})$ such that the weighted theta series $\vartheta_{\mathrm{J}_\mathbb{Z},P}$ defined as (8.20) is non-zero. This result will be used later in Section 8.6.4.

The $\mathrm{F}_4 \downarrow \mathrm{B}_4$ branching law [Lepowsky, 1970, §2, Theorem 7] says that $\dim \mathrm{V}_{n\varpi_4}^{\mathbf{Spin}_9(\mathbb{R})} = 1$ for any $n > 0$, where $\mathbf{Spin}_9$ is defined as the stabilizer of $\mathrm{E}_1 = [1,0,0\,;0,0,0]$ in $\mathbf{F}_4$, thus the $\mathbf{Spin}_9(\mathbb{R})$-invariant polynomial in $\mathrm{V}_n(\mathrm{J}_\mathbb{C})$ is unique up to a non-zero scalar.

Theorem 8.5.6. *For $n \geq 2$ and any non-zero polynomial $P \in \mathrm{V}_n(\mathrm{J}_\mathbb{C})^{\mathbf{Spin}_9(\mathbb{R})}$, the weighted theta series $\vartheta_{\mathrm{J}_\mathbb{Z},P}$ is non-zero.*

Proof. We first construct an explicit polynomial $P_n \in \mathrm{V}_n(\mathrm{J}_\mathbb{C})^{\mathbf{Spin}_9(\mathbb{R})}$. In the real definite octonion algebra $\mathbb{O}_\mathbb{R}$, we pick three purely imaginary elements x_0, y_0, z_0 such that $\mathbb{R} \oplus \mathbb{R}x_0 \oplus \mathbb{R}y_0 \oplus \mathbb{R}z_0$ is isomorphic to Hamilton's quaternion algebra, *i.e.*

$$x_0^2 = y_0^2 = z_0^2 = -1 \text{ and } x_0y_0 = -y_0x_0 = z_0.$$

Take $x_1 = x_0$, $y_1 = \sqrt{-2}y_0$ and $z_1 = \sqrt{-2}z_0$, and choose $B = [2,-1,-1\,;x_1,y_1,z_1] \in \mathrm{J}_\mathbb{C}$. It can be easily verified that $B \in \mathbb{X}$, thus the polynomial $Q_n(X) := (\mathrm{Tr}(X \circ B))^n = (X,B)^n$ lies in $\mathrm{V}_n(\mathrm{J}_\mathbb{C})$, and take $P_n(X) := \int_{\mathbf{Spin}_9(\mathbb{R})} k.Q_n(X)dk = \int_{\mathbf{Spin}_9(\mathbb{R})}(X,kB)^n dk$ to be the average of Q_n over $\mathbf{Spin}_9(\mathbb{R})$. Now it suffices to show that the associated theta series $\vartheta_{\mathrm{J}_\mathbb{Z},P_n} \neq 0$.

Consider the first Fourier coefficient a_1 of $\vartheta_{\mathrm{J}_\mathbb{Z},P_n}$. The elements in $\mathrm{J}_\mathbb{Z}^+$ having contributions to the coefficient of q are E_1, E_2 and E_3, thus:

$$a_1 = \sum_{i=1}^{3} P_n(\mathrm{E}_i) = \int_{\mathbf{Spin}_9(\mathbb{R})} \left(\sum_{i=1}^{3} (\mathrm{E}_i, kB)^n \right) dk. \tag{8.21}$$

By Lemma 8.2.30, $\mathbf{Spin}_9(\mathbb{R})$ preserves the subspaces $\mathrm{J}_1 = \{[0,\xi,-\xi\,;x,0,0] \mid \xi \in \mathbb{R}, x \in \mathbb{O}_\mathbb{R}\}$ and $\mathrm{J}_2 = \{[0,0,0\,;0,y,z] \mid y,z \in \mathbb{O}_\mathbb{R}\}$ respectively. So for any $k \in \mathbf{Spin}_9(\mathbb{R})$ we set:

$$k[0,0,0\,;x_1,0,0] = [0,\xi(k),-\xi(k)\,;x(k),0,0] \in \mathrm{J}_1,$$
$$k[0,0,0\,;0,y_1,z_1] = [0,0,0\,;0,y(k),z(k)] \in \mathrm{J}_2 \otimes \mathbb{C}.$$

We have the equality $2\xi(k)^2 + \langle x(k), x(k)\rangle = \langle x_1, x_1\rangle = 2$, as k preserves the inner product on $\mathrm{J}_\mathbb{R}$, which implies that $|\xi(k)| \leq 1$. The three diagonal entries of kB are $2, -1+\xi(k)$ and $-1-\xi(k)$, thus $\sum_{i=1}^{3}(\mathrm{E}_i, kB)^n = 2^n + (-1+\xi(k))^n + (-1-\xi(k))^n \in \mathbb{R}_{\geq 0}$. When we take $k = 1$, $\sum_{i=1}^{3}(\mathrm{E}_i, B)^n = 2^n + (-1)^n + (-1)^n$ is positive for any $n \geq 2$. Hence the integral in (8.21) is strictly positive, and as a consequence the weighted theta series $\vartheta_{\mathrm{J}_\mathbb{Z},P_n}$ is non-zero. □

8.5.3 Proof of Theorem 8.5.4

In this section, we will prove Proposition 8.5.1 and Theorem 8.5.4, following a similar strategy to that of Pollack in [Pollack, 2023, §6].

We first define a basis $\{\mathrm{X}_\alpha\}_\alpha$ of $\mathfrak{p}_{\mathrm{J}}^+$ as follows: for any $A \in \mathrm{J}_{\mathbb{C}}$, write $\mathrm{X}_A := \mathrm{X}_A^+ = i\mathrm{C}_h^{-1}\mathrm{n}_{\mathrm{L}}(A)\mathrm{C}_h$ as in Section 8.2.4.2, which is an element of $\mathfrak{p}_{\mathrm{J}}^+$ by Proposition 8.2.28. Choose a $\mathbb{C}$-basis $\{e_1, \ldots, e_{27}\}$ of $\mathrm{J}_{\mathbb{C}}$, then we have a basis $\{\mathrm{X}_{e_i}\}_{1\leq i\leq 27}$ of $\mathfrak{p}_{\mathrm{J}}^+$, and we denote its dual basis by $\{\mathrm{X}_{e_i}^\vee\}_{1\leq i\leq 27}$. In [Pollack, 2023, §6.2], Pollack calculates the action of $\mathrm{X}_{A_n}\cdots\mathrm{X}_{A_1}$ on $h_T|_{\mathbf{M}_{\mathrm{J}}(\mathbb{R})}$. Before recalling his result, we explain some notations that will appear in the statement.

Let $\mathrm{T}(\mathrm{J}_{\mathbb{C}}) = \bigoplus_{k=0}^{\infty} \mathrm{J}_{\mathbb{C}}^{\otimes k}$ be the tensor algebra of $\mathrm{J}_{\mathbb{C}}$. Define a family of $\mathbf{F}_4(\mathbb{R})$-equivariant maps $\mathscr{P}_k : \mathrm{J}_{\mathbb{C}}^{\otimes k} \to \mathrm{T}(\mathrm{J}_{\mathbb{C}})$ inductively:

- let $\mathscr{P}_0 = 1$ be the constant map;
- for $k \geq 0$, define[6]

$$\begin{aligned}\mathscr{P}_{k+1}(A_1 \otimes \cdots \otimes A_k \otimes A_{k+1}) =& \mathscr{P}_k(A_1 \otimes \cdots \otimes A_k) \otimes A_{k+1} + 4\,\mathrm{Tr}(A_{k+1})\mathscr{P}_k(A_1 \otimes \cdots \otimes A_k) \\ &+ A_{k+1} \circ \mathscr{P}_k(A_1 \otimes \cdots \otimes A_k) + \mathscr{P}_k(A_{k+1} \circ (A_1 \otimes \cdots \otimes A_k)),\end{aligned}$$

  where $A \circ (A_1 \otimes \cdots \otimes A_r) := \sum_{j=1}^r A_1 \otimes \cdots \otimes (A \circ A_j) \otimes \cdots \otimes A_r$.

For any $T \in \mathrm{J}_{\mathbb{R}}$ and $m \in \mathbf{M}_{\mathrm{J}}(\mathbb{R})$, we define a linear form $w_{T,m}$ on $\mathrm{T}(\mathrm{J}_{\mathbb{C}})$ by:

$$w_{T,m}(A_1 \otimes \cdots \otimes A_r) = (-4\pi)^r \prod_{j=1}^r (T, m(A_j))\,, \text{ for any } r \geq 0.$$

Proposition 8.5.7. *[Pollack, 2023, Proposition 6.2.2] Let the notations be as above, then for any $m \in \mathbf{M}_{\mathrm{J}}(\mathbb{R})$ and $A_1, \ldots, A_n \in \mathrm{J}_{\mathbb{C}}$, we have*

$$\mathrm{X}_{A_n}\cdots\mathrm{X}_{A_1} h_T(m) = w_{T,\lambda(m)m^*}(\mathscr{P}_n(A_1 \otimes \cdots \otimes A_n))h_T(m).$$

Remark 8.5.8. There is a slight mistake in [Pollack, 2023, Proposition 6.2.2], whose correct formula should be

$$\mathrm{X}_{A_n}\cdots\mathrm{X}_{A_1} h_T(M(\delta, m)) = w_{T,m}(\mathscr{P}_n(A_1 \otimes \cdots \otimes A_n))h_T(M(\delta, m)),$$

where $M(\delta, m)$ is the element of $\mathbf{M}_{\mathrm{J}}(\mathbb{R})$ such that $M(\delta, m)\mathrm{n}(A)M(\delta, m)^{-1} = \mathrm{n}(m(A))$.

Observe that $\mathscr{P}_n(A_1 \otimes \cdots \otimes A_n)$ is the sum of $A_1 \otimes \cdots \otimes A_n$ with tensors of smaller degrees. The following lemma enables us to consider only the leading term of $\mathscr{P}_n$.

Lemma 8.5.9. *Let P be an element in $\mathrm{V}_n(\mathrm{J}_{\mathbb{C}}) \simeq \mathrm{V}_{n\varpi_4}$, then:*

$$\sum_{i_1,\ldots,i_n} \mathscr{P}_n(e_{i_1} \otimes \cdots \otimes e_{i_n})\{\mathrm{X}_{e_{i_1}}^\vee \otimes \cdots \otimes \mathrm{X}_{e_{i_n}}^\vee, P\} = \sum_{i_1,\ldots,i_n} e_{i_1} \otimes \cdots \otimes e_{i_n}\{\mathrm{X}_{e_{i_1}}^\vee \otimes \cdots \otimes \mathrm{X}_{e_{i_n}}^\vee, P\}. \tag{8.22}$$

Proof. Since the pairing $\{-,-\}$ is $\mathbf{F}_4(\mathbb{R})$-invariant and $\mathscr{P}_n$ is $\mathbf{F}_4(\mathbb{R})$-equivariant, for any $g \in \mathbf{F}_4(\mathbb{R})$, we have:

$$\sum_{i_1,\ldots,i_n} \mathscr{P}_n(e_{i_1} \otimes \cdots \otimes e_{i_n})\{\mathrm{X}_{e_{i_1}}^\vee \otimes \cdots \otimes \mathrm{X}_{e_{i_n}}^\vee, g.P\}$$

[6] In [Pollack, 2023, §6.2], the Jordan product $A \circ B$ is denoted by $\frac{1}{2}\{A, B\}$, where $\{A, B\} = AB + BA$ is defined in [Pollack, 2020, §3.3.1].

$$
\begin{aligned}
&= \sum_{i_1,\dots,i_n} \mathscr{P}_n(e_{i_1} \otimes \cdots \otimes e_{i_n}) \{ \mathrm{X}^\vee_{g^{-1}.e_{i_1}} \otimes \cdots \otimes \mathrm{X}^\vee_{g^{-1}.e_{i_n}}, P \} \\
&= \sum_{i_1,\dots,i_n} \mathscr{P}_n(g.e_{i_1} \otimes \cdots \otimes g.e_{i_n}) \{ \mathrm{X}^\vee_{e_{i_1}} \otimes \cdots \otimes \mathrm{X}^\vee_{e_{i_n}}, P \} \\
&= \sum_{i_1,\dots,i_n} g.\mathscr{P}_n(e_{i_1} \otimes \cdots \otimes e_{i_n}) \{ \mathrm{X}^\vee_{e_{i_1}} \otimes \cdots \otimes \mathrm{X}^\vee_{e_{i_n}}, P \}.
\end{aligned}
$$

Comparing this with the right-hand side of (8.22), it suffices to prove (8.22) for one non-zero vector in $\mathrm{V}_{n\varpi_4}$, so we take P to be $(\mathrm{Tr}(X \circ A))^n \in \mathrm{V}_n(\mathrm{J}_\mathbb{C})$ for an arbitrary $A \in \mathbb{X}$, as explained in Section 8.4.1.2.

Both sides of (8.22) are independent of the choice of the basis $\{e_i\}_{1\leq i\leq 27}$ of $\mathrm{J}_\mathbb{C}$, thus we choose a specific basis $\{e_i\}_{1\leq i\leq 27}$ such that $e_1 = A$. With this choice, it suffices to prove $\mathscr{P}_n(e_1^{\otimes n}) = e_1^{\otimes n}$, which follows from the inductive definition of $\mathscr{P}_n$ and the fact that $\mathrm{Tr}(e_1) = 0$, $e_1 \circ e_1 = 0$. □

Proposition 8.5.10. *For $m \in \mathbf{M}_\mathrm{J}(\mathbb{R})$ and $P \in \mathrm{V}_n(\mathrm{J}_\mathbb{C}) \simeq \mathrm{V}_{n\varpi_4}$, we have*

$$
\{\mathrm{D}^n h_T(m), P\} = (-4\pi)^n P\left(\lambda(m) m^{-1} T\right) h_T(m).
$$

Proof. Combining Proposition 8.5.7 and Lemma 8.5.9 together, we have:

$$
\begin{aligned}
\{\mathrm{D}^n h_T(m), P\} &= \sum_{i_1,\dots,i_n} \mathrm{X}_{e_{i_n}} \cdots \mathrm{X}_{e_{i_1}} h_T(m) \left\{ \mathrm{X}^\vee_{e_{i_1}} \otimes \cdots \otimes \mathrm{X}^\vee_{e_{i_n}}, P \right\} \\
&= \sum_{i_1,\dots,i_n} w_{T,\lambda(m)m^*}(\mathscr{P}_n(e_{i_1} \otimes \cdots \otimes e_{i_n})) h_T(m) \left\{ \mathrm{X}^\vee_{e_{i_1}} \otimes \cdots \otimes \mathrm{X}^\vee_{e_{i_n}}, P \right\} \\
&= h_T(m) \sum_{i_1,\dots,i_n} w_{T,\lambda(m)m^*}(e_{i_1} \otimes \cdots \otimes e_{i_n}) \left\{ \mathrm{X}^\vee_{e_{i_1}} \otimes \cdots \otimes \mathrm{X}^\vee_{e_{i_n}}, P \right\} \\
&= (-4\pi)^n h_T(m) \sum_{i_1,\dots,i_n} \left(\prod_{j=1}^n (T, \lambda(m) m^*(e_{i_j})) \right) \left\{ \mathrm{X}^\vee_{e_{i_1}} \otimes \cdots \otimes \mathrm{X}^\vee_{e_{i_n}}, P \right\} \\
&= (-4\pi)^n h_T(m) \sum_{i_1,\dots,i_n} \left(\prod_{j=1}^n \left(\lambda(m) m^{-1} T, e_{i_j}\right) \right) \left\{ \mathrm{X}^\vee_{e_{i_1}} \otimes \cdots \otimes \mathrm{X}^\vee_{e_{i_n}}, P \right\} \\
&= (-4\pi)^n h_T(m) \left\{ \left(\lambda(m) m^{-1} T\right)^{\otimes n}, P \right\} \\
&= (-4\pi)^n P\left(\lambda(m) m^{-1} T\right) h_T(m).
\end{aligned}
$$

□

To prove Theorem 8.5.4, we use the Iwasawa decomposition to write $g \in \mathbf{SL}_2(\mathbb{R})$ as:

$$
g = tnk, \text{ where } t = \begin{pmatrix} u & \\ & u^{-1} \end{pmatrix}, n = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}, k = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.
$$

By a direct calculation, we have the following:

Lemma 8.5.11. *For $A_1, \dots, A_n \in \mathrm{J}_\mathbb{C}$, we have the following identities:*

(1) $\mathrm{X}_{A_n} \cdots \mathrm{X}_{A_1} h_T(mn(A)) = e^{2\pi i (T, \lambda(m) m^* A)} \mathrm{X}_{A_n} \cdots \mathrm{X}_{A_1} h_T(m)$, $\forall A \in \mathrm{J}_\mathbb{C}, m \in \mathbf{M}_\mathrm{J}(\mathbb{R})$;

(2) $\mathrm{X}_{A_n} \cdots \mathrm{X}_{A_1} h_T(gk) = J(k, i\mathrm{I})^{-4} (k.\mathrm{X}_{A_n}) \cdots (k.\mathrm{X}_{A_1}) h_T(g)$, $\forall k \in \mathrm{K}_{\mathrm{E}_7}, g \in \mathbf{H}^1_\mathrm{J}(\mathbb{R})$.

Proof of Theorem 8.5.4. Let the notations be as above. By Lemma 8.5.11, we have:

$$\begin{aligned}
\mathrm{D}^n h_T(g) &= \mathrm{D}^n h_T(tnk) \\
&= J(k, i\mathrm{I})^{-4} \sum_{i_1,\dots,i_n} (k.\mathrm{X}_{e_{i_1}} \cdots k.\mathrm{X}_{e_{i_n}}) h_T(tn) \otimes \mathrm{X}^\vee_{e_{i_1}} \otimes \cdots \otimes \mathrm{X}^\vee_{e_{i_n}} \\
&= J(k, i\mathrm{I})^{-4} e^{2\pi i (T, u^2 x\mathrm{I})} \sum_{i_1,\dots,i_n} (k.\mathrm{X}_{e_{i_1}} \cdots k.\mathrm{X}_{e_{i_n}}) h_T(t) \otimes \mathrm{X}^\vee_{e_{i_1}} \otimes \cdots \otimes \mathrm{X}^\vee_{e_{i_n}}. \\
&= j(k, i)^{-2n-12} e^{2\pi i (T, u^2 x\mathrm{I})} \cdot \mathrm{D}^n h_T(t),
\end{aligned}$$

where the last equality follows from $k.\mathrm{X}_A = (\cos\theta + i \sin\theta)^2 \mathrm{X}_A = j(k, i)^{-2}\mathrm{X}_A$. Now we take the pairing of $\mathrm{D}^n h_T(g)$ with P, and use Proposition 8.5.10 to obtain the desired identity:

$$\begin{aligned}
\{\mathrm{D}^n h_T(g), P\} &= j(k, i)^{-2n-12} e^{2\pi i (T, u^2 x\mathrm{I})} (-4\pi)^n P\left(u^2 T\right) J(t, i\mathrm{I})^{-4} e^{2\pi i (T, t.i\mathrm{I})} \\
&= (-4\pi)^n j(k, i)^{-2n-12} j(t, i)^{-12} u^{2n} P(T) e^{2\pi i (T, t.(i\mathrm{I}+x\mathrm{I}))} \\
&= (-4\pi)^n j(g, i)^{-2n-12} P(T)\, e^{2\pi i (T, g.i\mathrm{I})}.
\end{aligned}$$

□

Proof of Proposition 8.5.1. To show that $f_{\Theta(\alpha)}(z) := j(g, i)^{2n+12}\Theta(\alpha)(g)$ is well-defined, it suffices to verify that for k in the maximal compact subgroup of $\mathbf{SL}_2(\mathbb{R})$, we have:

$$\Theta(\alpha)(gk) = j(k, i)^{-2n-12}\Theta(\alpha)(g), \text{ for any } g \in \mathbf{SL}_2(\mathbb{R}).$$

This follows from Lemma 8.5.11 and the identity $k.\mathrm{X}_A = j(k, i)^{-2} \cdot \mathrm{X}_A$. By the definition of $\Theta(\alpha)$ and Proposition 8.3.8, $f_{\Theta(\alpha)}$ is a level one holomorphic modular form with weight $2n+12$, and when $n > 0$ it is a cusp form. □

8.6 Global theta lifts from $\mathbf{PGL}_2$ to $\mathbf{F}_4$

We look at the other direction of the global theta correspondence, *i.e.* from $\mathbf{PGL}_2$ to $\mathbf{F}_4$. Let $\pi \simeq \otimes'_v \pi_v$ be a level one algebraic cuspidal automorphic representation of $\mathbf{PGL}_2$ associated to a Hecke eigenform of $\mathbf{SL}_2(\mathbb{Z})$ with weight $2n + 12$, $n > 0$. We take an automorphic form $\varphi \in \pi$ corresponding to $\otimes'\varphi_v$ under the isomorphism $\pi \simeq \otimes'\pi_v$, such that:

- φ_∞ is the unique lowest weight holomorphic vector in the discrete series representation $\mathcal{D}(2n + 12)$ of $\mathbf{PGL}_2(\mathbb{R})$;
- for each prime p, φ_p is chosen to be the normalized spherical vector in the principal series representation π_p of $\mathbf{PGL}_2(\mathbb{Q}_p)$.

Our goal is to prove $\Theta(\pi) \neq 0$. In other words, we need to find a vector $\phi \in \Pi_{\min}$ such that $\Theta_\phi(\varphi) \neq 0$. The strategy is to calculate the $\mathbf{Spin}_9$-*period* of the global theta lift $\Theta_\phi(\varphi)$:

$$\mathcal{P}_{\mathbf{Spin}_9}(\Theta_\phi(\varphi)) := \int_{[\mathbf{Spin}_9]} \Theta_\phi(\varphi)(g)dg.$$

As stated in Remark 8.1.6, one motivation for considering this period integral is the conjecture of Sakellaridis-Venkatesh.

Plugging the definition of the global theta lift $\Theta_\phi(\varphi)$ in this period integral and changing the order of integration, we obtain:

$$\begin{aligned}\mathcal{P}_{\mathbf{Spin}_9}(\Theta_\phi(\varphi)) &= \int_{[\mathbf{Spin}_9]}\int_{[\mathbf{PGL}_2]} \theta(\phi)\,(gh)\,\overline{\varphi(h)}dhdg \\ &= \int_{[\mathbf{PGL}_2]} \overline{\varphi(h)}\left(\int_{[\mathbf{Spin}_9]} \theta(\phi)\,(gh)\,dg\right)dh.\end{aligned} \tag{8.23}$$

8.6.1 Exceptional Siegel-Weil formula

The integral $\int_{[\mathbf{Spin}_9]}\theta(\phi)(gh)dg$ appearing in (8.23), as a function of $h \in \mathbf{SO}_{2,2}(\mathbb{A})$, is the global theta lift of the constant function on $[\mathbf{Spin}_9]$ to $\mathbf{SO}_{2,2}$. In this section, we will prove an *exceptional Siegel-Weil formula* for $\mathbf{Spin}_9 \times \mathbf{SO}_{2,2}$, which represents this theta lift as an Eisenstein series on $\mathbf{SO}_{2,2}$.

Definition 8.6.1. Let $\mathbf{B} = \mathbf{TN}$ be the Borel subgroup of

$$\mathbf{SO}_{2,2} = \mathbf{GSpin}_{2,2}/\mathbf{G}_{\mathrm{m}} = \{(g_1, g_2) \in \mathbf{GL}_2 \times \mathbf{GL}_2 \mid \det g_1 = \det g_2\}/\mathbf{G}_{\mathrm{m}}^{\Delta}$$

consisting of the equivalence classes of (g_1, g_2), where $g_1$and g_2 are upper triangular matrices. For $s_1, s_2 \in \mathbb{C}$, we define a character χ_{s_1,s_2} on $\mathbf{T}(\mathbb{A})$ by:

$$\chi_{s_1,s_2}\left(\left(\begin{smallmatrix} a_1 & \\ & b_1\end{smallmatrix}\right), \left(\begin{smallmatrix} a_2 & \\ & b_2\end{smallmatrix}\right)\right) := |a_1/b_1|^{\frac{s_1}{2}} \cdot |a_2/b_2|^{\frac{s_2}{2}},$$

and define $\mathrm{I}(s_1, s_2)$ to be the (normalized) degenerate principal series $\mathrm{Ind}_{\mathbf{B}(\mathbb{A})}^{\mathbf{SO}_{2,2}(\mathbb{A})}\chi_{s_1,s_2}$.

By Proposition 8.3.2, we identify the (adelic) minimal representation $\Pi_{\min}$ of $\mathbf{E}_7(\mathbb{A})$ as a subrepresentation of $\mathrm{Ind}_{\mathbf{P}_{\mathrm{J}}(\mathbb{A})}^{\mathbf{E}_7(\mathbb{A})}\delta_{\mathbf{P}_{\mathrm{J}}}^{-1/2}|\lambda|^2$.

Lemma 8.6.2. *The restriction of sections gives a morphism* $\mathrm{Ind}_{\mathbf{P}_{\mathrm{J}}(\mathbb{A})}^{\mathbf{E}_7(\mathbb{A})}\delta_{\mathbf{P}_{\mathrm{J}}}^{-1/2}|\lambda|^2 \to \mathrm{I}(3, 7)$.

Proof. A section $f \in \mathrm{Ind}_{\mathbf{P}_{\mathrm{J}}(\mathbb{A})}^{\mathbf{E}_7(\mathbb{A})}\delta_{\mathbf{P}_{\mathrm{J}}}^{-1/2}|\lambda|^2$ satisfies the functional equation (8.12). Combining the explicit morphisms (8.7) and (8.11), the image of $\left(\left(\begin{smallmatrix} a_1 & \\ & b_1\end{smallmatrix}\right), \left(\begin{smallmatrix} a_2 & \\ & b_2\end{smallmatrix}\right)\right) \in \mathbf{T}(\mathbb{A})$ in $\mathbf{M}_{\mathrm{J}} \subseteq \mathbf{E}_7$ has similitude $(a_1/b_1)\cdot(a_2/b_2)^2$, thus the restriction of f to $\mathbf{SO}_{2,2}(\mathbb{A})$ satisfies:

$$f(tng) = \chi_{4,8}(t)f(g), \text{ for any } t \in \mathbf{T}(\mathbb{A}), n \in \mathbf{N}(\mathbb{A}), g \in \mathbf{SO}_{2,2}(\mathbb{A}).$$

This shows that $f|_{\mathbf{SO}_{2,2}(\mathbb{A})}$ is a section of $\mathrm{Ind}_{\mathbf{B}(\mathbb{A})}^{\mathbf{SO}_{2,2}(\mathbb{A})}\delta_{\mathbf{B}}^{-1/2}\chi_{4,8} = \mathrm{I}(3, 7)$. □

Lemma 8.6.2 gives us a $\mathbf{SO}_{2,2}(\mathbb{A})$-equivariant map:

$$\mathrm{Res} : \Pi_{\min} \hookrightarrow \mathrm{Ind}_{\mathbf{P}_{\mathrm{J}}(\mathbb{A})}^{\mathbf{E}_7(\mathbb{A})}\delta_{\mathbf{P}_{\mathrm{J}}}^{-1/2}|\lambda|^2 \to \mathrm{I}(3, 7).$$

Given a smooth vector $\phi \in \Pi_{\min}$, we have the following two automorphic forms on $\mathbf{SO}_{2,2}$:

- The theta integral:

$$\Theta_\phi(1)(g) = \int_{[\mathbf{Spin}_9]} \theta(\phi)(gh)dh, \text{ for any } g \in \mathbf{SO}_{2,2}(\mathbb{A}),$$

- The Eisenstein series associated to $\widetilde{\phi} := \operatorname{Res}(\phi) \in \mathrm{I}(3,7)$:

$$E(\widetilde{\phi})(g) := \sum_{\gamma \in \mathbf{B}(\mathbb{Q}) \backslash \mathbf{SO}_{2,2}(\mathbb{Q})} \widetilde{\phi}(\gamma g), \text{ for any } g \in \mathbf{SO}_{2,2}(\mathbb{A}).$$

Theorem 8.6.3. *Let $\Phi_f := \otimes_p \Phi_p$ be the normalized spherical vector in $\Pi_{\min,f}$ chosen in Section 8.4.2, then for any smooth holomorphic vector $\phi_\infty \in \Pi_{\min,\infty}$, up to some scalar we have:*

$$E(\operatorname{Res}(\phi_\infty \otimes \Phi_f)) = \Theta_{\phi_\infty \otimes \Phi_f}(1).$$

Before proving this formula for any smooth vector $\phi_\infty \in \Pi_{\min,\infty}$, we verify it for the specific vector Φ_∞ chosen in Section 8.4.2.

Proposition 8.6.4. *For the vector $\Phi_0 = \Phi_\infty \otimes \Phi_f \in \Pi_{\min}$, up to some scalar we have:*

$$E(\operatorname{Res}(\Phi_0)) = \Theta_{\Phi_0}(1).$$

Proof. By the choice of Φ_0, $\operatorname{Res}(\Phi_0)_p$ is the normalized spherical vector of $\mathrm{I}(3,7)_p$ for each prime p, and $\operatorname{Res}(\Phi_0)_\infty$ is the unique holomorphic vector in $\mathrm{I}(3,7)_\infty$ with minimal $\mathrm{K}_{\mathrm{E}_7} \cap \mathbf{Spin}_{2,2}(\mathbb{R})$-type. As a result, the Eisenstein series $E(\operatorname{Res}(\Phi_0))$ is a non-zero multiple of the automorphic form associated to $E_4 \boxtimes E_8$, where E_k is the normalized holomorphic Eisenstein series in $\mathrm{M}_k(\mathbf{SL}_2(\mathbb{Z}))$.

On the other side, the global theta lift is a non-zero multiple of

$$(g_1, g_2) \in \mathbf{SO}_{2,2}(\mathbb{A}) \mapsto j(g_{1,\infty})^{-4} j(g_{2,\infty})^{-8} \mathrm{F}_{Kim}\left(\operatorname{diag}(g_{1,\infty}.i, g_{2,\infty}.i, g_{2,\infty}.i)\right),$$

where $(g_{1,\infty}, g_{2,\infty}) \in \mathbf{Spin}_{2,2}(\mathbb{R})$ is the archimedean component of (g_1, g_2) (up to some left translation by $\mathbf{SO}_{2,2}(\mathbb{Q})$). It suffices to show that $\mathrm{F}_{Kim}(\operatorname{diag}(z_1, z_2, z_2))$, as a function on $\mathcal{H} \times \mathcal{H}$, is a non-zero multiple of $E_4(z_1)E_8(z_2)$.

Since the space of modular forms $\mathrm{M}_k(\mathbf{SL}_2(\mathbb{Z}))$, $k = 4$ or 8, is 1-dimensional and spanned by E_k, it suffices to show that as a function for the variable z_1 (*resp.* z_2), $\mathrm{F}_{Kim}(\operatorname{diag}(z_1, z_2, z_2))$ is a modular form of weight 4 (*resp.* 8). The only hard part in the proof of the modularity is to show that

$$z_1^{-4} \mathrm{F}_{Kim}(\operatorname{diag}(-1/z_1, z_2, z_2)) = \mathrm{F}_{Kim}(\operatorname{diag}(z_1, z_2, z_2)) = z_2^{-8} \mathrm{F}_{Kim}(\operatorname{diag}(z_1, -1/z_2, -1/z_2)).$$

We only give the proof for the first equality here, and the second one can be proved similarly. From the explicit actions on $\mathcal{H}_{\mathrm{J}}$ given in Example 8.4.9, we have

$$\operatorname{diag}(-1/z_1, z_2, z_2) = \left(\mathrm{n}(\mathrm{E}_1) \cdot \iota \cdot \mathrm{n}(\mathrm{E}_1) \cdot \iota \cdot \mathrm{n}(\mathrm{E}_1)\right).\operatorname{diag}(z_1, z_2, z_2),$$

then the desired functional equation is implied by the modularity of F_{Kim}:

$$\begin{aligned}&\mathrm{F}_{Kim}(\mathrm{diag}(-1/z_1, z_2, z_2))\\ =&J(\iota, \mathrm{diag}(z_1/(z_1+1), -1/z_2, -1/z_2))J(\iota^{-1}, \mathrm{diag}(z_1+1, z_2, z_2))\mathrm{F}_{Kim}(\mathrm{diag}(z_1, z_2, z_2))\\ =&\left(\frac{z_1}{(z_1+1)z_2^2}\right)^4 \cdot \left(-(z_1+1)\, z_2^2\right)^4 \cdot \mathrm{F}_{Kim}(\mathrm{diag}(z_1, z_2, z_2))\\ =&z_1^4 \mathrm{F}_{Kim}(\mathrm{diag}(z_1, z_2, z_3)).\end{aligned}$$

□

Proof of Theorem 8.6.3. For a smooth vector $\phi_\infty \in \Pi^+ \subseteq \Pi_{\min,\infty}$ whose restriction $\mathrm{Res}(\phi_\infty \otimes \Phi_f)$ to $\mathbf{SO}_{2,2}(\mathbb{A})$ vanishes, we know from Proposition 8.3.10 that it is orthogonal to the space $(\Pi^+)^{\mathbf{Spin}_9(\mathbb{R})}$, thus the theta lift $\Theta_{\phi_\infty\otimes\Phi_f}(1) = 0$.

Now we can assume that the smooth vector $\phi_\infty \in (\Pi^+)^{\mathbf{Spin}_9(\mathbb{R})}$ lies in the $\mathbf{Spin}_{2,2}(\mathbb{R})$-orbit of Φ_∞, then the theorem follows from Proposition 8.6.4 and the fact that the maps $E(\mathrm{Res}(-))$ and $\Theta_-(1)$ are both $\mathbf{SO}_{2,2}(\mathbb{A})$-equivariant. □

8.6.2 Unfolding the period integral

Take the smooth vector $\phi \in \Pi_{\min}$ to be $\phi_\infty \otimes \Phi_f$, where Φ_f is the normalized spherical vector and ϕ_∞ is a vector in $\Pi^+ \subseteq \Pi_{\min,\infty}$ such that $\widetilde{\phi} := \mathrm{Res}(\phi) \in \mathrm{I}(3,7)$ is non-zero. Using the Siegel-Weil formula Theorem 8.6.3 for $\mathbf{Spin}_9 \times \mathbf{SO}_{2,2}$, we write the period integral (8.23) as a Rankin-Selberg type integral and unfold it:

$$\begin{aligned}\mathcal{P}_{\mathbf{Spin}_9}(\Theta_\phi(\varphi)) &= \int_{[\mathbf{PGL}_2]} \overline{\varphi(h)} E(\mathrm{Res}(\phi))(h^\Delta) dh\\ &= \int_{[\mathbf{PGL}_2]} \overline{\varphi(h)} \sum_{\mathbf{B}(\mathbb{Q})\backslash \mathbf{SO}_{2,2}(\mathbb{Q})} \widetilde{\phi}\left(\gamma h^\Delta\right) dh\\ &= \sum_{\gamma\in \mathbf{B}(\mathbb{Q})\backslash \mathbf{SO}_{2,2}(\mathbb{Q})/\mathbf{PGL}_2^\Delta(\mathbb{Q})} \int_{{}^\gamma\mathbf{G}(\mathbb{Q})\backslash \mathbf{PGL}_2(\mathbb{A})} \widetilde{\phi}(\gamma h^\Delta)\overline{\varphi(h)} dh,\end{aligned} \tag{8.24}$$

where h^Δ denotes the image of $h \in \mathbf{PGL}_2(\mathbb{A})$ under $\mathbf{PGL}_2(\mathbb{A}) \to \mathbf{SO}_{2,2}(\mathbb{A})$, and the reductive subgroup ${}^\gamma\mathbf{G}$ of $\mathbf{PGL}_2$ is defined to be $\mathbf{PGL}_2^\Delta \cap \gamma^{-1}\mathbf{B}\gamma$.

By an easy calculation of orbits, the double coset in the summation of (8.24) has two orbits, represented by $1 = (\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right), \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right))$ and $\gamma_0 = (w_0, 1) := (\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right), \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right))$ respectively. For the first orbit, ${}^1\mathbf{G} = \mathbf{B}_0 = \mathbf{T}_0\mathbf{N}_0$ is the standard Borel subgroup of $\mathbf{PGL}_2$, and its contribution to the Rankin-Selberg integral (8.24) is zero since φ is cuspidal. For the second orbit, ${}^{\gamma_0}\mathbf{G} = \mathbf{T}_0$ is the maximal torus consisting of diagonal matrices, thus we have:

$$\mathcal{P}_{\mathbf{Spin}_9}(\Theta_\phi(\varphi)) = \int_{\mathbf{T}_0(\mathbb{Q})\backslash \mathbf{PGL}_2(\mathbb{A})} \widetilde{\phi}(\gamma_0 g^\Delta)\overline{\varphi(g)} dg. \tag{8.25}$$

Before calculating this integral, we make some normalization on the measure dg of $\mathbf{PGL}_2(\mathbb{A})$:

Notation 8.6.5. Fix a Haar measure dx on $\mathbb{Q}_p$ such that $dx(\mathbb{Z}_p) = 1$, and let $d^\times t$ be the Haar measure $(1-p^{-1})^{-1} \cdot \frac{dt}{|t|}$ on $\mathbb{Q}_p^\times$ so that $d^\times t(\mathbb{Z}_p^\times) = 1$. We choose the following left-invariant

Haar measure db on $\mathbf{B}_0(\mathbb{Q}_p)$:

$$db := d^\times t dx = \frac{dtdx}{|t|}, \text{ for } b = \begin{pmatrix} t & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \in \mathbf{B}_0(\mathbb{Q}_p).$$

On the hyperspecial subgroup $\mathbf{PGL}_2(\mathbb{Z}_p)$, we choose the invariant Haar measure dk such that the volume of $\mathbf{PGL}_2(\mathbb{Z}_p)$ is 1. Via the Iwasawa decomposition, we give $\mathbf{PGL}_2(\mathbb{Q}_p)$ the product measure $dg_p = dbdk$, which makes $\mathbf{PGL}_2(\mathbb{Z}_p)$ have measure 1. Take a non-trivial invariant Haar measure dg_∞ on $\mathbf{PGL}_2(\mathbb{R})$ and set $dg = \otimes'_v dg_v$.

The first step to calculate (8.25) is to rewrite it as an Euler product, for which we need the following:

Definition 8.6.6. Fix a non-trivial continuous unitary character $\psi = \psi_\infty \otimes \psi_f = \otimes_v \psi_v$ of $\mathbb{Q}\backslash\mathbb{A}$ such that the conductor of ψ_p is $\mathbb{Z}_p$ for each p and $\psi_\infty(x) = e^{2\pi i x}$ for all $x \in \mathbb{R}$. The ψ-*Whittaker coefficient* of $\varphi \in \mathcal{A}_{\mathrm{cusp}}(\mathbf{PGL}_2)$ is defined to be:

$$W_{\varphi,\psi}(g) := \int_{[\mathbf{N}_0]} \varphi(ng)\psi^{-1}(n)dn.$$

The global Whittaker function $W_{\varphi,\psi}$ satisfies $W_{\varphi,\psi}(ng) = \psi(n)W_{\varphi,\psi}(g)$ for any $g \in \mathbf{PGL}_2(\mathbb{A})$ and $n \in \mathbf{N}_0(\mathbb{A})$, and it factors as $W_{\varphi,\psi}(g) = \prod_v W_{\varphi_v,\psi_v}(g_v)$ [Cogdell, 2004, Corollary 4.1.3], where W_{φ_p,ψ_p} is a spherical Whittaker function on $\mathbf{PGL}_2(\mathbb{Q}_p)$. We normalize the spherical vector $\varphi_p \in \pi_p$ so that $W_{\varphi_p,\psi_p}|_{\mathbf{PGL}_2(\mathbb{Z}_p)} = 1$.

Expanding the automorphic form φ along $\mathbf{N}_0$, the right-hand side of (8.25) becomes:

$$\int_{\mathbf{T}_0(\mathbb{Q})\backslash\mathbf{PGL}_2(\mathbb{A})} \widetilde{\phi}(\gamma_0 g^\Delta) \overline{\sum_{a\in\mathbb{Q}^\times} W_{\varphi,\psi}\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right)} dg = \int_{\mathbf{PGL}_2(\mathbb{A})} \widetilde{\phi}(\gamma_0 g^\Delta)\overline{W_{\varphi,\psi}(g)}dg.$$

So far we have proved the following:

Proposition 8.6.7. *Let $\phi = \phi_\infty \otimes \Phi_f \in \Pi_{\min}$ be a smooth vector such that $\widetilde{\phi} = \mathrm{Res}(\phi) \neq 0$, then we have*

$$\mathcal{P}_{\mathbf{Spin}_9}(\Theta_\phi(\varphi)) = \int_{\mathbf{PGL}_2(\mathbb{A})} \widetilde{\phi}(\gamma_0 g^\Delta)\overline{W_{\varphi,\psi}(g)}dg = \prod_v I_v(\widetilde{\phi}_v, \varphi_v, \psi_v),$$

where the local zeta integral $I_v(\widetilde{\phi_v}, \varphi_v, \psi_v)$ is defined by:

$$I_v(\widetilde{\phi}_v, \varphi_v, \psi_v) := \int_{\mathbf{PGL}_2(\mathbb{Q}_v)} \widetilde{\phi}_v(\gamma_{0,v} g_v^\Delta)\overline{W_{\varphi_v,\psi_v}(g_v)}dg_v.$$

8.6.3 Unramified calculations

The goal of this section is to calculate the local zeta integral $I_p(\widetilde{\phi}_p, \varphi_p, \psi_p)$:

Proposition 8.6.8. *Let φ_p be the normalized spherical vector of the unramified principal series π_p of $\mathbf{PGL}_2(\mathbb{Q}_p)$ whose Satake parameter is $\begin{pmatrix} \alpha_p & \\ & \alpha_p^{-1} \end{pmatrix} \in \mathbf{SL}_2(\mathbb{C})_{\mathrm{ss}}$, and $\widetilde{\phi}_p = \mathrm{Res}(\Phi_p)$ the* *normalized spherical section of* $\mathrm{I}(3,7)_p$*, then we have:*

$$I_p(\widetilde{\phi}_p,\varphi_p,\psi_p)=\frac{(1-p^{-4})(1-p^{-8})}{(1-p^{-\frac{5}{2}}\alpha_p)(1-p^{-\frac{5}{2}}\alpha_p^{-1})(1-p^{-\frac{11}{2}}\alpha_p)(1-p^{-\frac{11}{2}}\alpha_p^{-1})}.$$

Proof. With the choice of measures in Notation 8.6.5, we write I_p as a double integral:

$$\begin{aligned} &I_p(\widetilde{\phi}_p,\varphi_p,\psi_p)\\ =&\int_{\mathbf{B_0}(\mathbb{Q}_p)}\int_{\mathbf{PGL}_2(\mathbb{Z}_p)}\widetilde{\phi}_p(\gamma_0 b^\Delta k^\Delta)\overline{W_{\varphi_p,\psi_p}(bk)}dbdk\\ =&\int_{\mathbb{Q}_p^\times}\int_{\mathbb{Q}_p}\widetilde{\phi}_p\left(\gamma_0\begin{pmatrix}t&\\&1\end{pmatrix}^\Delta\begin{pmatrix}1&x\\&1\end{pmatrix}^\Delta\right)\overline{W_{\varphi_p,\psi_p}\left(\begin{pmatrix}t&\\&1\end{pmatrix}\begin{pmatrix}1&x\\&1\end{pmatrix}\right)}d^\times tdx \end{aligned}\tag{8.26}$$

As the normalized spherical section of $\mathrm{I}(3,7)_p$, $\widetilde{\phi}_p$ satisfies that:

$$\widetilde{\phi}_p\left(\gamma_0\begin{pmatrix}t&\\&1\end{pmatrix}^\Delta\begin{pmatrix}1&x\\&1\end{pmatrix}^\Delta\right)=\begin{cases}|t|^2 & ,\ x\in\mathbb{Z}_p\\ |t|^2\cdot|x|^{-4} & ,\ x\notin\mathbb{Z}_p\end{cases}\tag{8.27}$$

On the other hand, the values of the spherical Whittaker function W_{φ_p,ψ_p} comes from a standard result [Cogdell, 2004, Proposition 7.4]:

$$W_{\varphi_p,\psi_p}\left(\begin{pmatrix}t&\\&1\end{pmatrix}\begin{pmatrix}1&x\\&1\end{pmatrix}\right)=\begin{cases}0 & ,\ t\notin\mathbb{Z}_p\\ p^{-n/2}\psi_p(tx)\cdot\frac{\alpha_p^{n+1}-\alpha_p^{-n-1}}{\alpha_p-\alpha_p^{-1}} & ,\ t\in p^n\mathbb{Z}_p^\times \text{ for some } n\geq 0\end{cases}\tag{8.28}$$

Plugging (8.27) and (8.28) into Eq. (8.26), we have:

$$I_p(\widetilde{\phi}_p,\varphi_p,\psi_p)=\sum_{n=0}^{\infty}\int_{p^n\mathbb{Z}_p^\times}p^{-\frac{5}{2}n}\frac{\alpha_p^{n+1}-\alpha_p^{-n-1}}{\alpha_p-\alpha_p^{-1}}I_n(t)d^\times t\tag{8.29}$$

where

$$I_n(t)=\int_{\mathbb{Z}_p}\overline{\psi_p(tx)}dx+\int_{\mathbb{Q}_p\setminus\mathbb{Z}_p}|x|^{-4}\overline{\psi_p(tx)}dx=1+\sum_{m=1}^{\infty}\int_{p^{-m}\mathbb{Z}_p^\times}p^{-4m}\overline{\psi_p(tx)}dx.$$

We set $t=p^nt_0$, $t_0\in\mathbb{Z}_p^\times$ and change the variable of integration by $x=p^{-m}t_0^{-1}y$, which induces that $dx=p^mdy$, then we have:

$$\int_{p^{-m}\mathbb{Z}_p^\times}p^{-4m}\overline{\psi_p(tx)}dx=p^{-3m}\int_{\mathbb{Z}_p^\times}\overline{\psi_p(p^{n-m}y)}dy=\begin{cases}p^{-3m}(1-p^{-1}) & ,\ m\leq n\\ -p^{-3(n+1)}\cdot p^{-1} & ,\ m=n+1\\ 0 & ,\ m>n+1\end{cases}$$

Hence the integral $I_n(t)$ is independent of $t\in p^n\mathbb{Z}_p^\times$ and

$$I_n(t)=1+\sum_{m=1}^{n}p^{-3m}(1-p^{-1})-p^{-3(n+1)-1}=\frac{(1-p^{-4})(1-p^{-3n-3})}{1-p^{-3}}.$$

Putting this value in (8.29), we obtain:

$$\begin{aligned} I_p(\widetilde{\phi}_p, \varphi_p, \psi_p) &= \frac{(1-p^{-4})}{(1-p^{-3})(\alpha_p - \alpha_p^{-1})} \sum_{n=0}^{\infty} p^{-\frac{5}{2}n}(\alpha_p^{n+1} - \alpha_p^{-n-1})(1-p^{-3n-3}) \\ &= \frac{(1-p^{-4})}{(1-p^{-3})(\alpha_p - \alpha_p^{-1})} \left(\frac{\alpha_p}{1-p^{-\frac{5}{2}}\alpha_p} - \frac{\alpha_p^{-1}}{1-p^{-\frac{5}{2}}\alpha_p^{-1}} - \frac{p^{-3}\alpha_p}{1-p^{-\frac{11}{2}}\alpha_p} + \frac{p^{-3}\alpha_p^{-1}}{1-p^{-\frac{11}{2}}\alpha_p^{-1}} \right) \\ &= \frac{(1-p^{-4})(1-p^{-8})}{(1-p^{-\frac{5}{2}}\alpha_p)(1-p^{-\frac{5}{2}}\alpha_p^{-1})(1-p^{-\frac{11}{2}}\alpha_p)(1-p^{-\frac{11}{2}}\alpha_p^{-1})}. \end{aligned}$$

□

As a direct consequence of Proposition 8.6.8, we have the following result, which corresponds to Theorem 8.1.7 in the introduction:

Corollary 8.6.9. *(Theorem 8.1.7 in Section 8.1) Let $\phi = \phi_\infty \otimes \Phi_f$ be a smooth holomorphic vector in $\Pi_{\min}$ such that $\widetilde{\phi} = \mathrm{Res}(\phi) \neq 0$, and $\varphi \simeq \varphi_\infty \otimes \varphi_f \in \pi$ the automorphic form of $\mathbf{PGL}_2$ associated to a (normalized) Hecke eigenform for $\mathbf{SL}_2(\mathbb{Z})$ of weight $2n+12$, $n > 0$. Then we have:*

$$\mathcal{P}_{\mathbf{Spin}_9}(\Theta_\phi(\varphi)) = \frac{\mathrm{L}(\pi, \frac{5}{2})\mathrm{L}(\pi, \frac{11}{2})}{\zeta(4)\zeta(8)} \cdot I_\infty(\mathrm{Res}(\phi_\infty), \varphi_\infty, \psi_\infty). \tag{8.30}$$

The L-function $\mathrm{L}(\pi, s)$ appearing in (8.30) is the standard automorphic L-function of π, defined as the Euler product $\prod_p (1-p^{-s}\alpha_p)(1-p^{-s}\alpha_p^{-1})$, where the $\mathbf{SL}_2(\mathbb{C})$-conjugacy class of $\mathrm{diag}(\alpha_p, \alpha_p^{-1})$ is the Satake parameter of π_p.

Remark 8.6.10. The L-factor $\mathrm{L}(\pi, \frac{5}{2})\mathrm{L}(\pi, \frac{11}{2})$ appearing in (8.30) agrees with the prediction of the global conjecture [SakellaridisVenkatesh, 2017, §17; Sakellaridis, 2021, Table 1] of Sakellaridis-Venkatesh for the spherical variety $\mathbf{Spin}_9\backslash\mathbf{F}_4$.

It is well-known that the standard automorphic L-function $\mathrm{L}(\pi, s)$ has no zero at $s = \frac{5}{2}$ or $\frac{11}{2}$. As a consequence, the non-vanishing of $\mathcal{P}_{\mathbf{Spin}_9}(\Theta_\phi(\varphi))$ is equivalent to that of the archimedean zeta integral $I_\infty(\mathrm{Res}(\phi_\infty), \varphi_\infty, \psi_\infty)$.

8.6.4 Non-vanishing of $\Theta_\phi(\varphi)$

By Corollary 8.6.9, for the non-vanishing of $\Theta(\pi)$, it suffices to find some smooth vector $\phi_\infty \in \Pi^+ \subseteq \Pi_{\min,\infty}$ such that $I_\infty(\mathrm{Res}(\phi_\infty), \varphi_\infty, \psi_\infty) \neq 0$. Notice that for the cuspidal automorphic form φ associated to *any* Hecke eigenform of weight $2n+12$, its archimedean component φ_∞ is the unique (up to some scalar) holomorphic lowest weight vector in $d_{hol}(2n+12) \subseteq \mathcal{D}(2n+12)$, thus we only need to prove the following:

Proposition 8.6.11. *For any $n > 1$, there exist an automorphic form $\varphi_n \in \mathcal{A}_{\mathrm{cusp}}(\mathbf{PGL}_2)$ associated to some Hecke eigenform in $\mathrm{S}_{2n+12}(\mathbf{SL}_2(\mathbb{Z}))$, and a smooth vector $\phi_n \in \Pi^+ \subseteq \Pi_{\min,\infty}$, such that $I_\infty(\mathrm{Res}(\phi_n), \varphi_{n,\infty}, \psi_\infty) \neq 0$, or equivalently, $\mathcal{P}_{\mathbf{Spin}_9}(\Theta_{\phi_n \otimes \Phi_f}(\varphi_n)) \neq 0$.*

Proof. For each $n > 1$, Theorem 8.5.6 shows that there exists a non-zero $\mathbf{Spin}_9(\mathbb{R})$-invariant polynomial P_n in $\mathrm{V}_n(\mathrm{J}_\mathbb{C})$ such that the weighted theta series $\vartheta_{\mathrm{J}_\mathbb{Z}, P_n}$ defined as (8.20) is non-zero. Let $\alpha_n \in \mathcal{A}_{\mathrm{V}_{n\varpi_4}}(\mathbf{F}_4)$ to be the vector-valued automorphic form such that $\alpha_n(1) = \sum_{\gamma \in \Gamma_\mathrm{I}} \gamma.P_n \in$

$V_n(J_{\mathbb{C}})^{\Gamma_I}$ and $\alpha_n(\gamma_E) = 0$, then the global theta lift $\Theta(\alpha_n)$ is a non-zero holomorphic cuspidal automorphic form of $\mathbf{PGL}_2$. Hence there exists an automorphic form $\varphi_n \in \mathcal{A}_{\mathrm{cusp}}(\mathbf{PGL}_2)$ associated to some Hecke eigenform in $S_{2n+12}(\mathbf{SL}_2(\mathbb{Z}))$, such that the Petersson inner product

$$\int_{[\mathbf{PGL}_2]} \Theta(\alpha_n)(g)\overline{\varphi_n(g)}dg \tag{8.31}$$

is non-zero. Putting the definition of $\Theta(\alpha_n)$ into (8.31), we have:

$$0 \neq \frac{1}{|\Gamma_I|}\int_{[\mathbf{PGL}_2]} \left\{\Theta_n(g), \sum_{\gamma\in\Gamma_I} \gamma.P_n\right\}\overline{\varphi_n(g)}dg = \int_{[\mathbf{PGL}_2]} \{\Theta_n(g), P_n\}\,\overline{\varphi_n(g)}dg. \tag{8.32}$$

Take the following smooth vector in $\Pi^+ \subseteq \Pi_{\min,\infty}$:

$$\phi_n := \{D^n\Phi_\infty, P_n\} = \sum_{\alpha_1,\dots,\alpha_n} \{X_{\alpha_1}^\vee \otimes \cdots X_{\alpha_n}^\vee, P_n\} \cdot (X_{\alpha_n}\cdots X_{\alpha_1}.\Phi_\infty),$$

where Φ_∞ is the specific vector chosen in Section 8.4.2 and D is the operator $\Pi^+ \to \Pi^+ \otimes \mathfrak{p}_J^-$ sending ϕ to $\sum_\alpha X_\alpha\phi\otimes X_\alpha^\vee$, with an arbitrary choice of basis $\{X_\alpha\}$ of $\mathfrak{p}_J^+$ and its dual basis $\{X_\alpha^\vee\}$. By Definition 8.4.12, the automorphic realization $\theta : \Pi_{\min} \hookrightarrow L^2([\mathbf{E}_7])$ maps $\phi_n \otimes \Phi_f$ to

$$\theta(\phi_n \otimes \Phi_f) = \{D^n\theta(\Phi_\infty \otimes \Phi_f), P_n\} = \{D^n\Theta_{Kim}, P_n\} = \{\Theta_n, P_n\}.$$

Use $\theta(\phi_n \otimes \Phi_f)$ as the kernel function to define a global theta lift of φ_n, then we calculate the $\mathbf{Spin}_9$-period integral of this global theta lift:

$$\mathcal{P}_{\mathbf{Spin}_9}(\Theta_{\phi_n\otimes\Phi_f}(\varphi_n)) = \int_{[\mathbf{PGL}_2]\times[\mathbf{Spin}_9]} \{\Theta_n(gh), P_n\}\overline{\varphi_n(g)}dgdh.$$

Since we have the strong approximation property $\mathbf{Spin}_9(\mathbb{A}) = \mathbf{Spin}_9(\mathbb{Q})\mathbf{Spin}_9(\mathbb{R})\mathbf{Spin}_9(\widehat{\mathbb{Z}})$, the $\mathbf{Spin}_9$-period integral is a non-zero multiple of

$$\begin{aligned}\int_{[\mathbf{PGL}_2]}\int_{\mathbf{Spin}_9(\mathbb{R})} \{\Theta_n(gh_\infty), P_n\}\overline{\varphi_n(g)}dgdh_\infty &= \int_{[\mathbf{PGL}_2]}\int_{\mathbf{Spin}_9(\mathbb{R})} \{h_\infty^{-1}.\Theta_n(g), P_n\}\overline{\varphi_n(g)}dgdh_\infty \\ &= \int_{\mathbf{Spin}_9(\mathbb{R})} dh_\infty \cdot \int_{[\mathbf{PGL}_2]} \{\Theta_n(g), P_n\}\overline{\varphi_n(g)}dg,\end{aligned}$$

where we use Lemma 8.4.15 and the $\mathbf{Spin}_9(\mathbb{R})$-invariance of P_n. Combining this with (8.32), we obtain the non-vanishing of $\mathcal{P}_{\mathbf{Spin}_9}(\Theta_{\phi_n\otimes\Phi_f}(\varphi_n))$, which is equivalent to the non-vanishing of $I_\infty(\mathrm{Res}(\phi_n), \varphi_{n,\infty}, \psi_\infty)$ by Corollary 8.6.9. □

Our main theorem is a direct consequence of Corollary 8.6.9 and Proposition 8.6.11:

Theorem 8.6.12. *(Theorem 8.1.2 in Section 8.1) Let $\pi \in \Pi_{\mathrm{cusp}}^{\mathrm{unr}}(\mathbf{PGL}_2)$ be the automorphic representation associated to a Hecke eigenform in $S_k(\mathbf{SL}_2(\mathbb{Z}))$, then its global theta lift $\Theta(\pi)$ to $\mathbf{F}_4$ is non-zero. Furthermore, we have the local-global compatibility of theta correspondence,* i.e.

$$\Theta(\pi) \simeq \otimes_v' \theta(\pi_v).$$

Proof. The case when $k \geq 16$ is a corollary of Proposition 8.6.11 and Corollary 8.6.9. When $k = 12$, this is a result in [ElkiesGross, 1996] (see also Remark 8.5.3). The local-global compatibility of theta correspondence follows from Proposition 8.3.6 and Proposition 8.3.8. □

Corollary 8.6.13. *(Theorem 8.1.8 in Section 8.1) For $n \geq 2$, the following map is surjective:*

$$\mathcal{A}_{\mathrm{V}_{n\varpi_4}}(\mathbf{F}_4) \to \mathrm{S}_{2n+12}(\mathbf{SL}_2(\mathbb{Z}))$$
$$(\alpha : \mathcal{J} \to \mathrm{V}_{n\varpi_4}) \mapsto f_{\Theta(\alpha)} = \frac{1}{|\Gamma_{\mathrm{I}}|}\vartheta_{\mathrm{J}_{\mathbb{Z}},\alpha(\mathrm{J}_{\mathbb{Z}})} + \frac{1}{|\Gamma_{\mathrm{E}}|}\vartheta_{\mathrm{J}_{\mathrm{E}},\alpha(\mathrm{J}_{\mathrm{E}})}$$

Proof. Suppose that the map $\alpha \mapsto f_{\Theta(\alpha)}$ is not surjective, then there exists a non-zero Hecke eigenform $f \in \mathrm{S}_{2n+12}(\mathbf{SL}_2(\mathbb{Z}))$, such that its associated automorphic form $\varphi \in \mathcal{A}(\mathbf{PGL}_2)$ is orthogonal to $\Theta(\alpha)$ for all $\alpha \in \mathcal{A}_{\mathrm{V}_{n\varpi_4}}(\mathbf{F}_4)$, with respect to the Petersson inner product. In particular, φ is orthogonal to $\Theta(\alpha_n)$, where α_n is the algebraic modular form chosen in the proof of Proposition 8.6.11. Take $\phi_n \in \Pi_{\min}$ to be the one in Proposition 8.6.11, we have:

$$0 = \int_{[\mathbf{PGL}_2]\times[\mathbf{Spin}_9]} \{\Theta_n(gh), P_n\}\overline{\varphi(g)}dgdh = \mathcal{P}_{\mathbf{Spin}_9}(\Theta_{\phi_n\otimes\Phi_f}(\varphi)),$$

which leads to a contradiction. □

Appendix A

Figures and tables

$$
\begin{pmatrix}
4 & 2 & 2 & -3 & -1 & -1 & -1 & -3 & -3 & -3 & -2 & -2 & -2 & -2 & -2 & -4 & -4 & -4 & -4 & -2 & -2 & -2 & -2 & -4 & -4 & -4 & -4 \\
2 & 4 & 2 & -2 & -2 & -2 & -2 & -4 & -4 & -4 & -4 & -3 & -1 & -1 & -1 & -3 & -3 & -3 & -2 & -2 & -2 & -2 & -2 & -4 & -4 & -4 & -4 \\
2 & 2 & 4 & -2 & -2 & -2 & -2 & -4 & -4 & -4 & -4 & -2 & -2 & -2 & -2 & -4 & -4 & -4 & -4 & -3 & -1 & -1 & -1 & -3 & -3 & -3 & -2 \\
-3 & -2 & -2 & 5 & 1 & 1 & 1 & 4 & 4 & 4 & 2 & 2 & 2 & 2 & 2 & 4 & 4 & 4 & 4 & 2 & 2 & 2 & 2 & 4 & 4 & 4 & 4 \\
-1 & -2 & -2 & 1 & 5 & 1 & 1 & 4 & 4 & 4 & 4 & 2 & 0 & 2 & 2 & 2 & 2 & 2 & 3 & 2 & 0 & 0 & 0 & 2 & 2 & 2 & 0 \\
-1 & -2 & -2 & 1 & 1 & 5 & 1 & 4 & 2 & 2 & 4 & 2 & 0 & 0 & 2 & 2 & 2 & 2 & 1 & 2 & 2 & 0 & 0 & 2 & 3 & 3 & 2 \\
-1 & -2 & -2 & 1 & 1 & 1 & 5 & 2 & 4 & 2 & 4 & 2 & 0 & 0 & 0 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 0 & 3 & 2 & 3 & 2 \\
-3 & -4 & -4 & 4 & 4 & 4 & 2 & 8 & 6 & 6 & 6 & 4 & 2 & 2 & 3 & 5 & 5 & 6 & 5 & 4 & 2 & 2 & 2 & 5 & 6 & 5 & 4 \\
-3 & -4 & -4 & 4 & 4 & 2 & 4 & 6 & 8 & 6 & 6 & 4 & 2 & 3 & 2 & 6 & 5 & 5 & 5 & 4 & 2 & 2 & 2 & 5 & 5 & 6 & 4 \\
-3 & -4 & -4 & 4 & 4 & 2 & 2 & 6 & 6 & 8 & 6 & 4 & 2 & 3 & 3 & 5 & 6 & 5 & 5 & 4 & 2 & 2 & 2 & 6 & 5 & 5 & 4 \\
-2 & -4 & -4 & 2 & 4 & 4 & 4 & 6 & 6 & 6 & 8 & 4 & 0 & 2 & 2 & 4 & 4 & 4 & 3 & 4 & 3 & 1 & 1 & 5 & 5 & 5 & 3 \\
-2 & -3 & -2 & 2 & 2 & 2 & 2 & 4 & 4 & 4 & 4 & 5 & 1 & 1 & 1 & 4 & 4 & 4 & 2 & 2 & 2 & 2 & 2 & 4 & 4 & 4 & 4 \\
-2 & -1 & -2 & 2 & 0 & 0 & 0 & 2 & 2 & 2 & 0 & 1 & 5 & 1 & 1 & 4 & 4 & 4 & 4 & 2 & 0 & 2 & 2 & 2 & 2 & 2 & 3 \\
-2 & -1 & -2 & 2 & 2 & 0 & 0 & 2 & 3 & 3 & 2 & 1 & 1 & 5 & 1 & 4 & 2 & 2 & 4 & 2 & 0 & 0 & 2 & 2 & 2 & 2 & 1 \\
-2 & -1 & -2 & 2 & 2 & 2 & 0 & 3 & 2 & 3 & 2 & 1 & 1 & 1 & 5 & 2 & 4 & 2 & 4 & 2 & 0 & 0 & 0 & 2 & 2 & 2 & 1 \\
-4 & -3 & -4 & 4 & 2 & 2 & 2 & 5 & 6 & 5 & 4 & 4 & 4 & 4 & 2 & 8 & 6 & 6 & 6 & 4 & 2 & 2 & 3 & 5 & 5 & 6 & 5 \\
-4 & -3 & -4 & 4 & 2 & 2 & 2 & 5 & 5 & 6 & 4 & 4 & 4 & 2 & 4 & 6 & 8 & 6 & 6 & 4 & 2 & 3 & 2 & 6 & 5 & 5 & 5 \\
-4 & -3 & -4 & 4 & 2 & 2 & 2 & 6 & 5 & 5 & 4 & 4 & 4 & 2 & 2 & 6 & 6 & 8 & 6 & 4 & 2 & 3 & 3 & 5 & 6 & 5 & 5 \\
-4 & -2 & -4 & 4 & 3 & 1 & 1 & 5 & 5 & 5 & 3 & 2 & 4 & 4 & 4 & 6 & 6 & 6 & 8 & 4 & 0 & 2 & 2 & 4 & 4 & 4 & 3 \\
-2 & -2 & -3 & 2 & 2 & 2 & 2 & 4 & 4 & 4 & 4 & 2 & 2 & 2 & 2 & 4 & 4 & 4 & 4 & 5 & 1 & 1 & 1 & 4 & 4 & 4 & 2 \\
-2 & -2 & -1 & 2 & 0 & 2 & 2 & 2 & 2 & 2 & 3 & 2 & 0 & 0 & 0 & 2 & 2 & 2 & 0 & 1 & 5 & 1 & 1 & 4 & 4 & 4 & 4 \\
-2 & -2 & -1 & 2 & 0 & 0 & 2 & 2 & 2 & 2 & 1 & 2 & 2 & 0 & 0 & 2 & 3 & 3 & 2 & 1 & 1 & 5 & 1 & 4 & 2 & 2 & 4 \\
-2 & -2 & -1 & 2 & 0 & 0 & 0 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 0 & 3 & 2 & 3 & 2 & 1 & 1 & 1 & 5 & 2 & 4 & 2 & 4 \\
-4 & -4 & -3 & 4 & 2 & 2 & 3 & 5 & 5 & 6 & 5 & 4 & 2 & 2 & 2 & 5 & 6 & 5 & 4 & 4 & 4 & 4 & 2 & 8 & 6 & 6 & 6 \\
-4 & -4 & -3 & 4 & 2 & 3 & 2 & 6 & 5 & 5 & 5 & 4 & 2 & 2 & 2 & 5 & 5 & 6 & 4 & 4 & 4 & 2 & 4 & 6 & 8 & 6 & 6 \\
-4 & -4 & -3 & 4 & 2 & 3 & 3 & 5 & 6 & 5 & 5 & 4 & 2 & 2 & 2 & 6 & 5 & 5 & 4 & 4 & 4 & 2 & 2 & 6 & 6 & 8 & 6 \\
-4 & -4 & -2 & 4 & 0 & 2 & 2 & 4 & 4 & 4 & 3 & 4 & 3 & 1 & 1 & 5 & 5 & 5 & 3 & 2 & 4 & 4 & 4 & 6 & 6 & 6 & 8
\end{pmatrix}
$$

Figure A.1: The gram matrix of $(\mathrm{J}_{\mathbb{Z}}, \langle\, ,\, \rangle_{\mathrm{E}})$ in the basis $\mathcal{B}$ given in Eq. (3.6)

$$\sigma_1 = \left(\begin{array}{rrrrrrrrrrrrrrrrrrrrrrrrrrr}
1 & 2 & 1 & 0 & -1 & 0 & -2 & -1 & -2 & -1 & -2 & -1 & 0 & -1 & 0 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -2 & -2 & -1 & -1 \\
1 & 1 & 1 & -1 & 0 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & -1 & -1 & -1 & -1 & 0 & -1 & -1 & -1 & -1 & 0 & -2 & -1 & -1 & -1 \\
1 & 2 & 1 & -1 & 0 & -2 & -1 & -2 & -1 & -1 & -2 & -1 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & -2 & -1 & -1 & -2 & -2 & -1 & -2 \\
-1 & -2 & -1 & 1 & 0 & 2 & 2 & 2 & 1 & 1 & 2 & 1 & 0 & 0 & 1 & 1 & 2 & 1 & 1 & 1 & 2 & 2 & 0 & 3 & 2 & 1 & 2 \\
-1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 2 & 1 & 1 & 1 & 2 & 1 & 1 & 2 & 2 & 1 & 2 \\
0 & 0 & -1 & 0 & -1 & 1 & 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
1 & 1 & 1 & -1 & 1 & -2 & -1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -1 & -1 & -2 & -1 & 0 & -2 & -2 & -2 & -2 \\
0 & 1 & 1 & -1 & 0 & -1 & -1 & -2 & -1 & -1 & -1 & -1 & -1 & 0 & -1 & -1 & -2 & -1 & -1 & -1 & 0 & -1 & -1 & -1 & -1 & 0 & -1 \\
1 & 1 & 0 & 0 & -1 & -1 & 0 & -1 & 0 & -1 & -1 & -1 & 1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & -1 & -1 & 0 & -2 & -1 & 0 & -1 \\
0 & 0 & 1 & -1 & 1 & 0 & -1 & 0 & -1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & 0 & 0 & 0 & -1 & -1 & 1 & -1 & 0 & -1 & -1 & 0 \\
-1 & -3 & -1 & 1 & 0 & 2 & 3 & 2 & 3 & 1 & 3 & 2 & 0 & 1 & -1 & 2 & 1 & 1 & 0 & 1 & 2 & 2 & 2 & 3 & 3 & 2 & 3 \\
-1 & -2 & -1 & 1 & 2 & 2 & 2 & 3 & 3 & 1 & 3 & 2 & 0 & 1 & 0 & 2 & 1 & 2 & 1 & 1 & 2 & 1 & 1 & 2 & 3 & 2 & 2 \\
-1 & -1 & 0 & 2 & 2 & 0 & 0 & 2 & 2 & 2 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 2 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
1 & 3 & 1 & -1 & -2 & -2 & -2 & -3 & -3 & -3 & -4 & -2 & 1 & -1 & 0 & -1 & -1 & -1 & 0 & -1 & -2 & -1 & -1 & -3 & -3 & -2 & -2 \\
0 & 1 & 0 & 0 & 0 & -1 & -1 & 0 & -1 & 1 & -1 & 0 & 0 & -1 & 1 & -1 & 1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & -1 & -1 \\
0 & 1 & 1 & 0 & 1 & -1 & -1 & -1 & -1 & 0 & -1 & -1 & -1 & 0 & 1 & -1 & -1 & -1 & 0 & -1 & 0 & -1 & -1 & -1 & -1 & 0 & -1 \\
1 & 0 & -1 & -1 & -2 & 1 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & -1 & 1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \\
-1 & -2 & 0 & 1 & -1 & 0 & 2 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 2 & 2 & 1 & 3 & 2 & 1 & 2 \\
0 & -1 & 0 & 0 & -1 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & -1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\
1 & 0 & 1 & -1 & -1 & 0 & -1 & -1 & -1 & -2 & -2 & 0 & 0 & -2 & -1 & -1 & -1 & -1 & -2 & -1 & 0 & 0 & -1 & -1 & -1 & 0 & 0 \\
1 & 0 & 0 & -2 & -1 & 1 & 0 & -1 & -1 & -1 & 0 & 0 & -1 & -1 & -1 & -1 & -1 & -1 & -2 & 0 & 0 & -1 & -1 & -1 & -1 & 0 & -1 \\
0 & 1 & 0 & -1 & 1 & 1 & -1 & 0 & -1 & 0 & 1 & 0 & -1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -2 & 0 & -1 & 0 & 0 & -1 \\
0 & 1 & 0 & 1 & 0 & 0 & -1 & 1 & 0 & -1 & -1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\
1 & 0 & 0 & -1 & 1 & 1 & -1 & 1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & -1 & -1 & -1 & -1 & 0 & -1 \\
-1 & 0 & 0 & 2 & 0 & -2 & 0 & 0 & 0 & 2 & 0 & -1 & 1 & 1 & 1 & 0 & 1 & 1 & 2 & 0 & 0 & 1 & 2 & 1 & 1 & -1 & 1
\end{array}\right)$$

$$\sigma_2 = \left(\begin{array}{rrrrrrrrrrrrrrrrrrrrrrrrrrr}
2 & 2 & 1 & -2 & 0 & -1 & -1 & -2 & -2 & -1 & -1 & -1 & -1 & -1 & -1 & -2 & -2 & -2 & -2 & -1 & -1 & -2 & -2 & -2 & -3 & -2 & -3 \\
2 & 2 & 2 & -1 & -1 & 0 & -2 & -2 & -3 & -2 & -2 & -1 & -2 & -2 & 0 & -3 & -2 & -3 & -3 & -2 & 0 & -2 & -2 & -2 & -2 & -2 & -2 \\
1 & 1 & 1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & -1 & -1 \\
-1 & -1 & -1 & 0 & 0 & -1 & 2 & 0 & 2 & -1 & 0 & -1 & 1 & 1 & 0 & 1 & 0 & 1 & 2 & 0 & -1 & 1 & 1 & -1 & 0 & 0 & 0 \\
-1 & 0 & -1 & 0 & 1 & 0 & 2 & 0 & 2 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 2 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & -1 \\
0 & -1 & 0 & 0 & 2 & 1 & 1 & 1 & 2 & 1 & 2 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & 1 & 1 & 2 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & -1 & 0 & -1 & -1 & 0 & 1 & 1 & -2 & 1 & -1 & 1 & 0 \\
1 & 1 & 1 & -1 & -2 & 1 & -1 & -1 & -3 & -1 & -1 & 0 & 0 & -2 & 0 & -1 & 0 & -1 & -2 & 0 & 0 & 0 & -1 & 0 & -1 & -1 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & -1 & 2 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 2 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 & 1 & -1 & 0 & 0 & 1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 1 & 0 & -1 & 1 & 0 & 2 & 1 \\
1 & 1 & 0 & -1 & -1 & -2 & -2 & -1 & -2 & -2 & -3 & -2 & 1 & 0 & 0 & -1 & -1 & -1 & 0 & -2 & -3 & -1 & 1 & -3 & -2 & -4 & -2 \\
-1 & -2 & 0 & 1 & -1 & -1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & -1 & 1 & 0 & 1 & 1 & 0 & 1 & 2 & 3 & 1 & 2 & 1 & 3 \\
0 & 0 & 1 & 0 & 0 & -2 & -1 & -1 & -1 & -1 & -2 & -1 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & -2 & -1 & 1 & 1 & -1 & -1 & -2 & 0 \\
0 & 1 & 1 & -1 & 0 & -1 & 0 & -1 & -2 & -1 & -1 & -1 & -1 & -1 & 0 & -2 & -1 & -1 & -1 & -1 & 0 & 1 & -1 & 0 & -1 & -2 & -1 \\
-1 & 1 & 0 & 1 & 0 & -1 & -1 & 0 & -1 & 0 & -1 & -1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\
0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & -1 & -1 & -1 & 0 & -1 & 1 & -1 \\
1 & 1 & 0 & -2 & 0 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & -2 & 0 & -1 & 0 & 0 & -1 & 1 & -1 & -1 & -2 & -1 & -1 & -1 & -2 \\
0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & -1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & -2 \\
1 & -2 & 0 & 0 & -1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & -2 & 1 & 0 & 0 & -1 & 1 & 1 & 0 & 1 & 1 & 1 & 2 & 2 \\
-3 & -2 & -2 & 2 & -1 & 1 & 3 & 2 & 3 & 1 & 2 & 2 & 2 & 2 & 0 & 4 & 3 & 4 & 3 & 2 & 2 & 3 & 3 & 3 & 4 & 3 & 4 \\
-2 & -1 & -1 & 1 & -1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 0 & 3 & 2 & 2 & 2 & 2 & 1 & 1 & 3 & 2 & 3 & 2 & 3 \\
-1 & 1 & 0 & 0 & -1 & -1 & -2 & -1 & -1 & -1 & -2 & -1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & -1 & -1 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 1 & -1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & -2 & 1 & -1 & 1 & 1 & 0 \\
2 & 0 & 1 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -2 & -1 & 0 & -2 & -2 & -2 & -2 & -1 & 0 & -2 & -1 & -1 & -1 & -1 & -2 \\
1 & 1 & 1 & 0 & 0 & -2 & -1 & -2 & -1 & 0 & -2 & -1 & -1 & -1 & 0 & -2 & -1 & -2 & -1 & -2 & -1 & 0 & -2 & -1 & -3 & -1 & -1 \\
2 & 2 & 2 & -3 & -1 & -1 & -3 & -3 & -4 & -3 & -3 & -2 & -1 & -2 & -1 & -3 & -3 & -3 & -3 & -2 & -2 & -3 & -1 & -4 & -3 & -3 & -3 \\
0 & 0 & -1 & 1 & 0 & 1 & 3 & 1 & 1 & 1 & 2 & 1 & -1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 2 & 2 & -2 & 2 & 0 & 0 & 0
\end{array}\right)$$

Figure A.2: Generators σ_1 and σ_2 of $\mathcal{F}_{4,\mathrm{E}}(\mathbb{Z})$ as 27×27 matrices in the basis $\mathcal{B}$ of $\mathrm{J}_{\mathbb{Z}}$

s	o(c_s)	i(c_s)	s	o(c_s)	i(c_s)	s	o(c_s)	i(c_s)
(1,0,0,0,0)	1	(27,351,2925,52)	(2,1,1,0,1)	9	(3,3,0,1)	(4,4,2,0,1)	20	(4,3,-8,0)
(0,0,0,0,1)	2	(-5,-1,45,20)	(0,1,0,1,2)	10	(-2,1,0,6)	(7,0,1,1,3)	20	(4,9,16,4)
(0,1,0,0,0)	2	(3,-9,-35,-4)	(0,2,0,1,1)	10	(0,-1,0,0)	(2,1,3,1,2)	21	(0,0,2,0)
(0,0,1,0,0)	3	(0,0,9,-2)	(4,2,0,0,1)	10	(10,49,160,10)	(4,2,1,2,1)	21	(2,1,-1,0)
(1,0,0,0,1)	3	(0,0,9,7)	(0,0,0,1,4)	12	(-4,0,21,15)	(0,4,0,1,6)	24	(-2,0,3,7)
(1,1,0,0,0)	3	(9,36,90,7)	(0,1,0,2,1)	12	(-1,2,-2,1)	(0,6,0,1,4)	24	(0,-2,-1,1)
(0,0,0,1,0)	4	(-1,3,-3,0)	(0,2,0,1,2)	12	(-1,0,0,3)	(1,2,3,2,1)	24	(0,0,3,-1)
(0,1,0,0,1)	4	(-1,-1,1,4)	(0,4,0,1,0)	12	(2,-6,-15,-3)	(2,4,2,1,2)	24	(1,-2,-2,-1)
(1,0,1,0,0)	4	(3,3,1,0)	(1,0,3,0,1)	12	(0,0,5,-1)	(3,1,3,1,3)	24	(0,0,1,1)
(2,0,0,0,1)	4	(7,27,77,8)	(1,1,1,1,1)	12	(0,0,1,0)	(3,5,1,1,2)	24	(2,-2,-7,-1)
(2,1,0,0,0)	4	(15,111,545,20)	(1,3,1,0,1)	12	(2,-4,-11,-2)	(4,0,2,1,5)	24	(-1,0,2,5)
(1,1,0,0,1)	5	(2,1,0,2)	(1,4,1,0,0)	12	(3,-6,-26,-3)	(4,2,2,1,3)	24	(1,0,0,1)
(0,0,0,1,1)	6	(-2,2,-3,5)	(2,0,0,1,3)	12	(-2,0,5,8)	(4,2,4,1,0)	24	(2,0,-1,-1)
(0,1,0,0,2)	6	(-3,0,10,11)	(2,0,2,1,0)	12	(1,0,2,-1)	(6,2,0,3,1)	24	(3,4,2,1)
(0,1,0,1,0)	6	(0,0,1,-1)	(2,1,0,1,2)	12	(0,0,1,3)	(6,2,4,0,1)	24	(4,6,3,1)
(0,2,0,0,1)	6	(1,-4,-6,-1)	(2,2,0,1,1)	12	(2,0,-3,0)	(7,2,1,1,3)	24	(4,8,11,3)
(1,0,1,0,1)	6	(0,0,1,2)	(2,4,0,0,1)	12	(4,0,-19,-1)	(2,4,2,1,4)	28	(0,-1,0,1)
(1,1,1,0,0)	6	(3,0,-8,-1)	(3,0,1,1,1)	12	(2,2,1,1)	(3,4,1,3,1)	28	(1,-1,-1,-1)
(2,0,0,1,0)	6	(4,8,9,2)	(3,3,1,0,0)	12	(6,12,5,2)	(2,4,6,0,1)	30	(1,-2,1,-2)
(2,1,0,0,1)	6	(6,18,37,5)	(4,0,2,0,1)	12	(5,12,18,3)	(3,6,1,1,4)	30	(1,-2,-3,0)
(3,0,1,0,0)	6	(12,72,289,14)	(4,1,0,0,3)	12	(3,6,14,5)	(6,1,0,5,1)	30	(1,1,0,0)
(4,0,0,0,1)	6	(16,128,681,23)	(5,0,1,1,0)	12	(8,32,85,7)	(6,4,2,2,1)	30	(3,2,-3,0)
(4,1,0,0,0)	6	(21,216,1450,35)	(6,1,0,0,2)	12	(11,62,238,13)	(8,0,2,1,6)	30	(1,1,4,4)
(1,0,0,1,1)	7	(-1,1,-1,3)	(2,1,1,1,1)	13	(1,0,0,0)	(12,1,0,3,2)	30	(7,25,60,6)
(2,1,1,0,0)	7	(6,15,20,3)	(2,2,2,0,1)	14	(2,-1,-4,-1)	(1,4,3,4,1)	36	(0,0,2,-1)
(0,0,0,1,2)	8	(-3,1,5,10)	(4,1,0,1,2)	14	(3,5,7,3)	(2,8,2,1,4)	36	(1,-3,-4,-1)
(0,1,0,1,1)	8	(-1,1,-1,2)	(1,0,2,1,2)	15	(-1,1,0,2)	(4,6,2,1,7)	40	(0,-1,0,2)
(0,2,0,1,0)	8	(1,-3,-3,-2)	(4,2,1,1,0)	15	(5,10,9,2)	(8,2,6,1,3)	40	(2,1,0,0)
(1,1,1,0,1)	8	(1,-1,-1,0)	(1,1,3,1,1)	18	(0,0,4,-1)	(1,6,5,1,5)	42	(0,-1,1,0)
(1,2,1,0,0)	8	(3,-3,-17,-2)	(2,2,2,1,1)	18	(1,-1,0,-1)	(10,2,4,1,6)	42	(2,2,2,2)
(2,0,0,1,1)	8	(1,1,1,2)	(4,1,0,1,4)	18	(0,0,4,5)	(1,12,7,2,3)	60	(1,-3,-2,-2)
(2,2,0,0,1)	8	(5,9,5,2)	(6,2,2,0,1)	18	(7,23,48,5)	(6,4,6,1,12)	60	(-1,0,1,4)
(3,1,1,0,0)	8	(9,39,111,8)	(2,4,2,1,0)	20	(2,-3,-8,-2)	(10,2,10,1,6)	60	(1,0,1,0)
(1,1,0,1,1)	9	(0,0,0,1)	(3,0,1,3,1)	20	(0,1,0,0)	(11,12,1,3,5)	60	(3,1,-6,0)

Table A.1: Kac coordinates, Orders and invariants i (defined in Section 4.5) of the rational torsion conjugacy classes of F_4

s	$n_1(s)$	$n_2(s)$	s	$n_1(s)$	$n_2(s)$
(1,0,0,0,0)	1	1	(1,1,1,1,1)	435456000	105670656
(0,0,0,0,1)	723	819	(1,3,1,0,1)	101606400	0
(0,1,0,0,0)	459900	68796	(2,0,0,1,3)	1612800	0
(0,0,1,0,0)	6540800	2283008	(2,0,2,1,0)	24192000	13208832
(1,0,0,0,1)	121920	139776	(2,1,0,1,2)	43545600	0
(1,1,0,0,0)	268800	34944	(2,2,0,1,1)	14515200	17611776
(0,0,0,1,0)	249480	137592	(2,4,0,0,1)	4112640	0
(0,1,0,0,1)	2835000	0	(3,0,1,1,1)	7257600	0
(1,0,1,0,0)	14968800	3302208	(3,3,1,0,0)	4838400	0
(2,0,0,0,1)	23400	58968	(4,0,2,0,1)	14515200	4402944
(2,1,0,0,0)	37800	0	(5,0,1,1,0)	3628800	0
(1,1,0,0,1)	1741824	0	(2,1,1,1,1)	0	48771072
(0,0,0,1,1)	497280	0	(2,2,2,0,1)	223948800	11321856
(0,1,0,1,0)	44150400	8805888	(4,2,1,1,0)	34836480	0
(0,2,0,0,1)	10483200	2201472	(1,1,3,1,1)	232243200	0
(1,0,1,0,1)	74995200	17611776	(2,2,2,1,1)	154828800	105670656
(1,1,1,0,0)	67737600	8805888	(6,2,2,0,1)	19353600	0
(2,0,0,1,0)	1881600	2935296	(2,4,2,1,0)	87091200	0
(2,1,0,0,1)	604800	0	(4,4,2,0,1)	52254720	0
(3,0,1,0,0)	806400	0	(2,1,3,1,2)	199065600	30191616
(4,0,0,0,1)	6720	0	(4,2,1,2,1)	0	60383232
(1,0,0,1,1)	0	4313088	(0,4,0,1,6)	7257600	0
(2,1,1,0,0)	24883200	539136	(0,6,0,1,4)	21772800	0
(0,0,0,1,2)	272160	0	(1,2,3,2,1)	174182400	0
(0,1,0,1,1)	10886400	0	(2,4,2,1,2)	174182400	52835328
(0,2,0,1,0)	22680000	6604416	(3,1,3,1,3)	261273600	0
(1,1,1,0,1)	342921600	0	(3,5,1,1,2)	87091200	0
(1,2,1,0,0)	32659200	0	(4,2,2,1,3)	58060800	52835328
(2,0,0,1,1)	5443200	6604416	(4,2,4,1,0)	65318400	0
(2,2,0,0,1)	5715360	0	(6,2,4,0,1)	50803200	0
(3,1,1,0,0)	5443200	0	(2,4,2,1,4)	149299200	22643712
(1,1,0,1,1)	77414400	0	(2,4,6,0,1)	34836480	0
(2,1,1,0,1)	19353600	35223552	(6,4,2,2,1)	139345920	0
(0,2,0,1,1)	38320128	0	(2,8,2,1,4)	116121600	0
(4,2,0,0,1)	1741824	0	(4,6,2,1,7)	104509440	0
(0,2,0,1,2)	29030400	8805888	(8,2,6,1,3)	104509440	0
(0,4,0,1,0)	10886400	0	(6,4,6,1,12)	69672960	0
(1,0,3,0,1)	47174400	0			

Table A.2: Kac coordinates of the conjugacy classes of F_4 whose intersections with $\mathcal{F}_{4,\mathrm{I}}(\mathbb{Z})$ and $\mathcal{F}_{4,\mathrm{E}}(\mathbb{Z})$ are not both empty

λ	$d(\lambda)$	λ	$d(\lambda)$	λ	$d(\lambda)$	λ	$d(\lambda)$	λ	$d(\lambda)$
(0,0,0,2)	1	(0,0,1,9)	7	(0,1,1,7)	7	(0,0,0,13)	8	(2,0,4,1)	13
(0,0,0,3)	1	(0,0,2,7)	6	(0,1,2,5)	9	(0,0,1,11)	15	(2,1,0,6)	16
(0,0,0,4)	1	(0,0,3,5)	6	(0,1,3,3)	14	(0,0,2,9)	20	(2,1,1,4)	17
(0,0,2,0)	1	(0,0,4,3)	4	(0,1,4,1)	4	(0,0,3,7)	27	(2,1,2,2)	25
(0,0,0,5)	1	(0,0,5,1)	1	(0,2,0,6)	11	(0,0,4,5)	34	(2,1,3,0)	8
(0,0,1,3)	1	(0,1,0,8)	2	(0,2,1,4)	9	(0,0,5,3)	30	(2,2,0,3)	4
(0,0,0,6)	3	(0,1,1,6)	3	(0,2,2,2)	15	(0,0,6,1)	14	(2,2,1,1)	9
(0,0,2,2)	1	(0,1,2,4)	4	(0,2,3,0)	2	(0,1,0,10)	11	(2,3,0,0)	6
(0,0,0,7)	1	(0,1,3,2)	3	(0,3,0,3)	3	(0,1,1,8)	23	(3,0,0,7)	1
(0,0,1,5)	1	(0,1,4,0)	1	(0,3,1,1)	3	(0,1,2,6)	39	(3,0,1,5)	9
(0,0,2,3)	1	(0,2,0,5)	1	(0,4,0,0)	6	(0,1,3,4)	44	(3,0,2,3)	7
(0,0,0,8)	4	(0,2,1,3)	3	(1,0,0,10)	3	(0,1,4,2)	37	(3,0,3,1)	8
(0,0,1,6)	1	(0,2,2,1)	1	(1,0,1,8)	7	(0,1,5,0)	13	(3,1,0,4)	12
(0,0,2,4)	1	(0,3,0,2)	2	(1,0,2,6)	10	(0,2,0,7)	11	(3,1,1,2)	7
(0,0,4,0)	2	(1,0,0,9)	1	(1,0,3,4)	11	(0,2,1,5)	32	(3,1,2,0)	8
(0,0,0,9)	4	(1,0,1,7)	3	(1,0,4,2)	8	(0,2,2,3)	36	(4,0,0,5)	2
(0,0,1,7)	2	(1,0,2,5)	2	(1,0,5,0)	4	(0,2,3,1)	26	(4,0,1,3)	3
(0,0,2,5)	1	(1,0,3,3)	3	(1,1,0,7)	2	(0,3,0,4)	21	(4,0,2,1)	2
(0,0,3,3)	2	(1,0,4,1)	1	(1,1,1,5)	9	(0,3,1,2)	21	(4,1,0,2)	4
(0,1,3,0)	1	(1,1,0,6)	3	(1,1,2,3)	8	(0,3,2,0)	14	(4,1,1,0)	1
(0,3,0,0)	1	(1,1,1,4)	2	(1,1,3,1)	9	(0,4,0,1)	5	(5,0,1,1)	1
(1,1,0,4)	1	(1,1,2,2)	4	(1,2,0,4)	8	(1,0,0,11)	3	(5,1,0,0)	3
(3,1,0,0)	1	(1,2,1,1)	2	(1,2,1,2)	5	(1,0,1,9)	13		
(0,0,0,10)	5	(1,3,0,0)	1	(1,2,2,0)	5	(1,0,2,7)	20		
(0,0,1,8)	4	(2,0,0,7)	1	(1,3,0,1)	1	(1,0,3,5)	32		
(0,0,2,6)	6	(2,0,1,5)	2	(2,0,0,8)	5	(1,0,4,3)	26		
(0,0,4,2)	3	(2,0,2,3)	1	(2,0,1,6)	4	(1,0,5,1)	21		
(0,0,5,0)	1	(2,0,3,1)	1	(2,0,2,4)	10	(1,1,0,8)	18		
(0,1,1,5)	1	(2,1,0,4)	2	(2,0,3,2)	4	(1,1,1,6)	27		
(0,1,3,1)	1	(2,1,1,2)	1	(2,0,4,0)	5	(1,1,2,4)	46		
(0,2,0,4)	1	(2,1,2,0)	1	(2,1,1,3)	5	(1,1,3,2)	31		
(0,2,2,0)	1	(3,0,1,3)	1	(2,1,2,1)	2	(1,1,4,0)	20		
(1,0,0,8)	1	(3,1,0,2)	1	(2,2,0,2)	8	(1,2,0,5)	10		
(1,0,1,6)	1	(0,0,0,12)	13	(3,0,0,6)	4	(1,2,1,3)	28		
(1,0,2,4)	1	(0,0,1,10)	6	(3,0,1,4)	3	(1,2,2,1)	16		
(1,0,3,2)	1	(0,0,2,8)	15	(3,0,2,2)	3	(1,3,0,2)	18		
(1,2,0,2)	1	(0,0,3,6)	15	(3,0,3,0)	2	(1,3,1,0)	2		
(2,0,0,6)	2	(0,0,4,4)	15	(3,2,0,0)	2	(2,0,0,9)	4		
(2,0,2,2)	1	(0,0,5,2)	4	(4,0,0,4)	3	(2,0,1,7)	12		
(2,2,0,0)	1	(0,0,6,0)	11	(4,0,2,0)	2	(2,0,2,5)	16		
(0,0,0,11)	5	(0,1,0,9)	2	(6,0,0,0)	3	(2,0,3,3)	21		

Table A.3: The nonzero $d(\lambda)$ for $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ such that $2\lambda_1 + 3\lambda_2 + 2\lambda_3 + \lambda_4 \leq 13$

n	$d_1(n)$	$d_2(n)$	n	$d_1(n)$	$d_2(n)$	n	$d_1(n)$	$d_2(n)$	n	$d_1(n)$	$d_2(n)$
1	0	0	11	4	1	21	83	209	31	4112	24425
2	1	0	12	8	5	22	130	413	32	6294	38234
3	1	0	13	6	2	23	169	590	33	8904	54760
4	1	0	14	12	8	24	280	1138	34	13284	82989
5	1	0	15	13	8	25	368	1629	35	18664	117447
6	2	1	16	20	18	26	601	2915	36	27332	173760
7	1	0	17	22	22	27	835	4253	37	38024	242971
8	3	1	18	37	58	28	1323	7161	38	54627	351485
9	3	1	19	39	63	29	1868	10455	39	75354	486013
10	4	1	20	67	150	30	2919	16962	40	106332	689219

Table A.4: Dimensions $d_1(n) = \dim \mathrm{V}_{n\varpi_4}^{\mathcal{F}_{4,\mathrm{I}}(\mathbb{Z})}$ and $d_2(n) = \dim \mathrm{V}_{n\varpi_4}^{\mathcal{F}_{4,\mathrm{E}}(\mathbb{Z})}$ for $n \leq 40$

n	$d_1(n)$	$d_2(n)$	n	$d_1(n)$	$d_2(n)$
1	0	0	16	699558	4607562
2	1	0	17	1899450	12528178
3	0	0	18	4951537	32636950
4	1	1	19	12298529	81088431
5	0	1	20	29444006	194120684
6	4	7	21	67821302	447181025
7	2	14	22	151304284	997568542
8	32	136	23	326873722	2155210696
9	84	583	24	686811782	4528418428
10	497	2936	25	1404333622	9259307898
11	1765	11764	26	2802604042	18478677233
12	7111	46299	27	5463354204	36021961176
13	24173	159701	28	10425639768	68740584631
14	80166	526081	29	19491910968	128517811865
15	241776	1594526	30	35762551274	235797459916

Table A.5: Dimensions $d_1(n) = \dim \mathrm{V}_{n\varpi_3}^{\mathcal{F}_{4,\mathrm{I}}(\mathbb{Z})}$ and $d_2(n) = \dim \mathrm{V}_{n\varpi_3}^{\mathcal{F}_{4,\mathrm{E}}(\mathbb{Z})}$ for $n \leq 30$

w(λ)	λ	$\dim \mathcal{A}_{\mathrm{V}_\lambda}(\mathbf{F}_4)$	$\Psi_\lambda(\mathbf{F}_4)$
16	(0,0,0,0)	2	$[9] \oplus [17]$
			$\Delta_{11}[6] \oplus [5] \oplus [9]$
20	(0,0,0,2)	1	$\Delta_{15}[6] \oplus [5] \oplus [9]$
22	(0,0,0,3)	1	$\Delta_{17}[6] \oplus [5] \oplus [9]$
24	(0,0,0,4)	1	$\Delta_{19}[6] \oplus [5] \oplus [9]$
	(0,0,2,0)	1	$\mathrm{Sym}^2\,\Delta_{11}[3] \oplus \Delta_{11}[4] \oplus \Delta_{11}[2] \oplus [5]$
26	(0,0,0,5)	1	$\Delta_{21}[6] \oplus [5] \oplus [9]$
	(0,0,1,3)	1	$\Delta_{24,16,8,0}[3] \oplus [5]$
28	(0,0,0,6)	3	$\Delta_{23}^{(2)}[6] \oplus [5] \oplus [9]$
			$\Delta_{26,20,6,0}[3] \oplus [5]$
	(0,0,2,2)	1	$\Delta_{26,16,10,0}[3] \oplus [5]$
30	(0,0,0,7)	1	$\Delta_{25}[6] \oplus [5] \oplus [9]$
	(0,0,1,5)	1	$\Delta_{28,20,8,0}[3] \oplus [5]$
	(0,0,2,3)	1	$\Delta_{28,18,10,0}[3] \oplus [5]$
32	(0,0,0,8)	4	$\Delta_{27}^{(2)}[6] \oplus [5] \oplus [9]$
			$\Delta_{30,24,6,0}^{(2)}[3] \oplus [5]$
	(0,0,1,6)	1	$\Delta_{30,22,8,0}[3] \oplus [5]$
	(0,0,2,4)	1	$\Delta_{30,20,10,0}[3] \oplus [5]$
	(0,0,4,0)	2	$\mathrm{Sym}^2\,\Delta_{15}[3] \oplus \Delta_{15}[4] \oplus \Delta_{15}[2] \oplus [5]$
			$\Delta_{30,16,14,0}[3] \oplus [5]$
34	(0,0,0,9)	4	$\Delta_{29}^{(2)}[6] \oplus [5] \oplus [9]$
			$\Delta_{32,26,6,0}^{(2)}[3] \oplus [5]$
	(0,0,1,7)	2	$\Delta_{32,24,8,0}^{(2)}[3] \oplus [5]$
	(0,0,2,5)	1	$\Delta_{32,22,10,0}[3] \oplus [5]$
	(0,0,3,3)	2	$\Delta_{32,20,12,0}^{(2)}[3] \oplus [5]$
	(0,1,3,0)	1	$\Delta_{32,16,14,6,0} \oplus \mathrm{Spin}\,\Delta_{32,16,14,6,0} \oplus [1]$
	(0,3,0,0)	1	$\mathrm{Sym}^3\,\Delta_{11}[2] \oplus \mathrm{Sym}^2\,\Delta_{11}[3] \oplus \Delta_{11}[4] \oplus [1]$
	(1,1,0,4)	1	$\Delta_{30,20,10,8,0} \oplus \mathrm{Spin}\,\Delta_{30,20,10,8,0} \oplus [1]$
	(3,1,0,0)	1	$\wedge^*\Delta_{19,7} \oplus (\Delta_{19,7} \otimes \Delta_{15}) \oplus \Delta_{19,7}[2] \oplus \Delta_{15}[2] \oplus [1]$

Table A.6: Elements of nonempty $\Psi_\lambda(\mathbf{F}_4)$ for the weights λ such that $\mathrm{w}(\lambda) \leq 34$

λ	$\dim \mathcal{A}_{\mathrm{V}_\lambda}(\mathbf{F}_4)$	$\Psi_\lambda(\mathbf{F}_4)$
(0,0,0,10)	5	$\Delta_{31}^{(2)}[6] \oplus [5] \oplus [9]$
		$\Delta_{34,28,6,0}^{(3)}[3] \oplus [5]$
(0,0,1,8)	4	$\wedge^* \Delta_{21,13} \oplus (\Delta_{21,13} \otimes \Delta_{15}) \oplus (\Delta_{21,13} \otimes \Delta_{11}) \oplus (\Delta_{15} \otimes \Delta_{11}) \oplus [1]$
		$\Delta_{34,26,8,0}^{(3)}[3] \oplus [5]$
(0,0,2,6)	6	$\Delta_{34,24,10,0}^{(5)}[3] \oplus [5]$
		$\Delta_{34,24,10,4,0} \oplus \mathrm{Spin}\, \Delta_{34,24,10,4,0} \oplus [1]$
(0,0,4,2)	3	$\Delta_{34,20,14,0}^{(2)}[3] \oplus [5]$
		$\Delta_{34,20,14,4,0} \oplus \mathrm{Spin}\, \Delta_{32,20,14,4,0} \oplus [1]$
(0,0,5,0)	1	$\mathrm{Sym}^2 \Delta_{17}[3] \oplus \Delta_{17}[4] \oplus \Delta_{17}[2] \oplus [5]$
(0,1,1,5)	1	$\Delta_{34,22,10,6,0} \oplus \mathrm{Spin}\, \Delta_{34,22,10,6,0} \oplus [1]$
(0,1,3,1)	1	$\Delta_{34,18,14,6,0} \oplus \mathrm{Spin}\, \Delta_{34,18,14,6,0} \oplus [1]$
(0,2,0,4)	1	$\Delta_{34,20,10,8,0} \oplus \mathrm{Spin}\, \Delta_{32,16,14,6,0} \oplus [1]$
(0,2,2,0)	1	$\wedge^* \Delta_{21,13} \oplus (\Delta_{21,13} \otimes \Delta_{15}) \oplus \Delta_{21,13}[2] \oplus \Delta_{15}[2] \oplus [1]$
(1,0,0,8)	1	$\Delta_{32,26,8,6,0} \oplus \mathrm{Spin}\, \Delta_{32,26,8,6,0} \oplus [1]$
(1,0,1,6)	1	$\Delta_{32,24,10,6,0} \oplus \mathrm{Spin}\, \Delta_{32,24,10,6,0} \oplus [1]$
(1,0,2,4)	1	$\Delta_{32,22,12,6,0} \oplus \mathrm{Spin}\, \Delta_{32,22,12,6,0} \oplus [1]$
(1,0,3,2)	1	$\Delta_{32,20,14,6,0} \oplus \mathrm{Spin}\, \Delta_{32,20,14,6,0} \oplus [1]$
(1,2,0,2)	1	ψ_0
(2,0,0,6)	2	$\Delta_{30,24,10,8,0}^{(2)} \oplus \mathrm{Spin}\, \Delta_{30,24,10,8,0} \oplus [1]$
(2,0,2,2)	1	$\Delta_{30,20,14,8,0} \oplus \mathrm{Spin}\, \Delta_{30,20,14,8,0} \oplus [1]$
(2,2,0,0)	1	$\wedge^* \Delta_{21,9} \oplus (\Delta_{21,9} \otimes \Delta_{15}) \oplus \Delta_{21,9}[2] \oplus \Delta_{15}[2] \oplus [1]$

Table A.7: Elements of nonempty $\Psi_\lambda(\mathbf{F}_4)$ for the weights λ such that $\mathrm{w}(\lambda) = 36$

λ	$F_4(\lambda)$	λ	$F_4(\lambda)$	λ	$F_4(\lambda)$	λ	$F_4(\lambda)$	λ	$F_4(\lambda)$
(1,2,0,2)	1	(1,2,2,0)	5	(1,1,3,2)	22	(0,1,3,5)	70	(2,0,2,6)	28
(0,1,2,4)	2	(2,0,2,4)	2	(1,1,4,0)	11	(0,1,4,3)	68	(2,0,3,4)	32
(0,1,4,0)	1	(2,0,3,2)	2	(1,2,0,5)	7	(0,1,5,1)	49	(2,0,4,2)	35
(0,2,1,3)	2	(2,1,1,3)	3	(1,2,1,3)	22	(0,2,0,8)	31	(2,0,5,0)	12
(0,3,0,2)	2	(2,1,2,1)	2	(1,2,2,1)	13	(0,2,1,6)	61	(2,1,0,7)	10
(1,0,3,3)	1	(2,2,0,2)	4	(1,3,0,2)	12	(0,2,2,4)	92	(2,1,1,5)	42
(1,1,1,4)	1	(3,0,0,6)	1	(1,3,1,0)	2	(0,2,3,2)	74	(2,1,2,3)	46
(1,1,2,2)	2	(3,0,2,2)	2	(2,0,1,7)	2	(0,2,4,0)	35	(2,1,3,1)	41
(1,2,1,1)	2	(3,2,0,0)	1	(2,0,2,5)	3	(0,3,0,5)	26	(2,2,0,4)	39
(2,1,0,4)	2	(0,0,3,7)	3	(2,0,3,3)	9	(0,3,1,3)	61	(2,2,1,2)	34
(2,1,2,0)	1	(0,0,4,5)	6	(2,0,4,1)	5	(0,3,2,1)	40	(2,2,2,0)	24
(0,0,3,6)	1	(0,0,5,3)	8	(2,1,0,6)	11	(0,4,0,2)	28	(2,3,0,1)	2
(0,0,4,4)	1	(0,0,6,1)	4	(2,1,1,4)	9	(0,4,1,0)	8	(3,0,0,8)	5
(0,0,5,2)	1	(0,1,0,10)	2	(2,1,2,2)	21	(1,0,0,12)	1	(3,0,1,6)	6
(0,0,6,0)	1	(0,1,1,8)	6	(2,1,3,0)	2	(1,0,1,10)	4	(3,0,2,4)	21
(0,1,1,7)	1	(0,1,2,6)	19	(2,2,0,3)	1	(1,0,2,8)	23	(3,0,3,2)	13
(0,1,2,5)	3	(0,1,3,4)	18	(2,2,1,1)	8	(1,0,3,6)	36	(3,0,4,0)	14
(0,1,3,3)	6	(0,1,4,2)	25	(2,3,0,0)	4	(1,0,4,4)	50	(3,1,0,5)	2
(0,1,4,1)	2	(0,1,5,0)	4	(3,0,1,5)	2	(1,0,5,2)	34	(3,1,1,3)	21
(0,2,0,6)	4	(0,2,0,7)	2	(3,0,2,3)	2	(1,0,6,0)	24	(3,1,2,1)	13
(0,2,1,4)	4	(0,2,1,5)	20	(3,0,3,1)	3	(1,1,0,9)	6	(3,2,0,2)	20
(0,2,2,2)	8	(0,2,2,3)	21	(3,1,0,4)	4	(1,1,1,7)	50	(3,2,1,0)	2
(0,2,3,0)	2	(0,2,3,1)	19	(3,1,1,2)	5	(1,1,2,5)	69	(4,0,0,6)	2
(0,3,0,3)	3	(0,3,0,4)	19	(3,1,2,0)	3	(1,1,3,3)	86	(4,0,1,4)	3
(0,3,1,1)	2	(0,3,1,2)	10	(4,1,0,2)	3	(1,1,4,1)	57	(4,0,2,2)	7
(0,4,0,0)	1	(0,3,2,0)	13	(0,0,2,10)	4	(1,2,0,6)	56	(4,0,3,0)	1
(1,0,2,6)	2	(0,4,0,1)	2	(0,0,3,8)	13	(1,2,1,4)	72	(4,1,1,1)	6
(1,0,3,4)	2	(1,0,2,7)	4	(0,0,4,6)	27	(1,2,2,2)	93	(4,2,0,0)	1
(1,0,4,2)	4	(1,0,3,5)	11	(0,0,5,4)	26	(1,2,3,0)	17	(5,0,0,4)	2
(1,1,1,5)	4	(1,0,4,3)	9	(0,0,6,2)	24	(1,3,0,3)	18	(5,0,2,0)	2
(1,1,2,3)	4	(1,0,5,1)	11	(0,0,7,0)	8	(1,3,1,1)	34	(7,0,0,0)	1
(1,1,3,1)	6	(1,1,0,8)	7	(0,1,0,11)	1	(1,4,0,0)	9		
(1,2,0,4)	7	(1,1,1,6)	15	(0,1,1,9)	21	(2,0,0,10)	3		
(1,2,1,2)	3	(1,1,2,4)	27	(0,1,2,7)	44	(2,0,1,8)	9		

Table A.8: The nonzero $F_4(\lambda)$ for the weights λ such that $w(\lambda) \leq 44$

Articles

Yi Shan. "*Level one automorphic representations of anisotropic exceptional group over* $\mathbb{Q}$ *of type* F_4". Submitted (2024).
DOI: 10.48550/arXiv.2407.05859

Yi Shan. "*Exceptional theta correspondence* $\mathbf{F}_4 \times \mathbf{PGL}_2$ *for level one automorphic representations*". Submitted (2025).
DOI: 10.48550/arXiv.2501.19101

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RÉSUMÉ

Dans cette thèse, nous étudions les représentations automorphes de niveau un pour la $\mathbb{Q}$-forme $\mathbf{F}_4$ du groupe exceptionnel compact de type de Lie F_4. Ce travail est divisé en les deux parties suivantes.

Représentations automorphes de niveau un de $\mathbf{F}_4$ avec un poids donné. D'abord, en suivant la méthode de Chenevier et Renard, nous calculons le nombre de représentations automorphes de niveau un pour $\mathbf{F}_4$ avec une composante archimédienne donnée. Plus précisément, nous étudions le groupe d'automorphismes des deux *algèbres d'Albert sur* $\mathbb{Z}$ étudiées par Gross, ainsi que la dimension des invariants de ces groupes dans toute représentation irréductible de $\mathbf{F}_4(\mathbb{R})$.

Ensuite, en admettant les conjectures standards d'Arthur et Langlands sur les représentations automorphes, nous affinons ce comptage en étudiant la contribution des représentations dont le *paramètre global d'Arthur* a n'importe quelle image possible. Cela inclut une description détaillée de toutes ces images, et des énoncés précis pour la formule de multiplicité d'Arthur dans chaque cas. Notre résultat fournit en particulier une formule conjecturale mais explicite pour le nombre de représentations automorphes algébriques, cuspidales, de niveau un de $\mathbf{GL}_{26}$ sur $\mathbb{Q}$ ayant un poids « F_4-*régulier* » donné, et pour groupe de Sato-Tate $\mathbf{F}_4(\mathbb{R})$ tout entier.

Correspondance thêta exceptionnelle pour $\mathbf{F}_4 \times \mathbf{PGL}_2$. Nous étudions la correspondance thêta exceptionnelle globale pour la paire duale réductive $\mathbf{F}_4 \times \mathbf{PGL}_2$. Notre résultat principal affirme que pour toute représentation automorphe de $\mathbf{PGL}_2$ associée à une forme parabolique propre de Hecke pour $\mathbf{SL}_2(\mathbb{Z})$, son Θ-lift global est une représentation automorphe irréductible non nulle de $\mathbf{F}_4$. Cela vérifie un calcul conjectural effectué dans la partie précédente. Motivés par les travaux de Pollack, notre principal outil consiste à construire une famille de *séries thêta exceptionnelles*, qui sont des formes paraboliques holomorphes de $\mathbf{SL}_2(\mathbb{Z})$, et nous montrons que cette famille engendre tout l'espace des formes paraboliques de niveau un.

MOTS CLÉS

Formes automorphes, Groupes exceptionnels, Programme de Langlands, Correspondance thêta

ABSTRACT

In this thesis, we study level one automorphic representations for the $\mathbb{Q}$-form $\mathbf{F}_4$ of the exceptional compact group of Lie type F_4, The work is divided into the following two parts.

Level one automorphic representations of $\mathbf{F}_4$ with a given weight. First, following the method of Chenevier and Renard, we calculate the number of level one automorphic representations for $\mathbf{F}_4$ with any given archimedean component. More explicitly, we study the automorphism group of the two *Albert* $\mathbb{Z}$*-algebras* studied by Gross, as well as the dimension of the invariants of these groups in any irreducible representation of $\mathbf{F}_4(\mathbb{R})$.

Next, assuming standard conjectures by Arthur and Langlands on automorphic representations, we refine this counting by studying the contribution of the representations whose *global Arthur parameter* has any possible image. This includes a detailed description of all those images, and precise statements for the Arthur's multiplicity formula in each case. Our result provides in particular a conjectural but explicit formula for the number of algebraic, cuspidal, level one automorphic representations of $\mathbf{GL}_{26}$ over $\mathbb{Q}$ with any given "F_4-*regular*" weight and of Sato-Tate group $\mathbf{F}_4(\mathbb{R})$.

Exceptional theta correspondence for $\mathbf{F}_4 \times \mathbf{PGL}_2$. We study the global exceptional theta correspondence for the reductive dual pair $\mathbf{F}_4 \times \mathbf{PGL}_2$. Our main result states that for any automorphic representation of $\mathbf{PGL}_2$ associated with a cuspidal Hecke eigenform for $\mathbf{SL}_2(\mathbb{Z})$, its global theta lift to $\mathbf{F}_4$ is a non-zero irreducible automorphic representation. This verifies a conjectural calculation made in the previous part. Motivated by Pollack's work, our main tool is to construct a family of *exceptional theta series*, which are holomorphic cusp forms of $\mathbf{SL}_2(\mathbb{Z})$, and we show that this family spans the entire space of level one cusp forms.

KEYWORDS

Automorphic forms, Exceptional groups, Langlands program, Theta correspondence