Clôture définissable.

Définition. Soit $(R, \ldots)$ une structure, et $A \subseteq R$.
On définit $\text{dcl}(A)$, l'ensemble des éléments de $R$ qui sont définissables sur $A$, par la propriété suivante:
$\alpha \in \text{dcl}(A)$ s'il existe une formule $\varphi(x, \overline{y})$ sans paramètre et un uplet $\overline{y}$ dans $A$ tels que $\alpha$ est l'unique élément de $R$ satisfaisant $\varphi(x, \overline{y})$.

Donc $\text{dcl}(A)$ n'est pas définissable, mais c'est une union d'ensembles définissables, et il contient $A$. On montre facilement que $\text{dcl}(A) = \text{dcl}(\text{dcl}(A))$.

Proposition 64. Soit $(R, +, -, 0, 1, \cdot, <, \ldots)$ $0$-minimale et $A \subseteq R$.
Alors $\text{dcl}(A) \leq R$.

Démo. Rappel du critère de Tarski.

Soient $M \subseteq N$ des structures d'un langage $L$.
$M \prec N$ veut dire : si $\varphi(x)$ est une formule de $L$, et à un uplet de $M$ alors $\varphi(\overline{a})$ est vraie dans $M$ ($M \models \varphi(\overline{a})$) si et seulement si elle est vraie dans $N$.

Le critère dit : pour avoir $M \prec N$ il suffit de montrer, pour toute formule $\varphi(x, \overline{y})$ (sans paramètres) et uplet $\overline{a}$ dans $M$
$s'il existe $b \in N$ tel que $N \models \varphi(b, \overline{a})$ alors il existe $b \in M$ tel que $N \models \varphi(b, \overline{a})$.

Dém de la proposition : On utilise les fonctions de choix définissable et le critère de Tarski.
Setting:
\[ R = (\mathbb{R}, +, \cdot, 0, 1, <, \ldots ) \]

Study of differentiability.

**Def:** Let \( I \subseteq \mathbb{R} \) be open. A function \( f : I \rightarrow \mathbb{R}^n \) is differentiable at \( x \in I \) with derivative \( a \in \mathbb{R}^n \) iff
\[
\lim_{t \to 0} \frac{f(x+t) - f(x)}{t} = a.
\]

Note: this implies if continuous at \( x \), \( a \) is unique.

Write \( a = f'(x) \).

**Properties:** The following are easy to show: let \( f, g : I \rightarrow \mathbb{R}^n \) be differentiable at \( x \).

Then \( (f+g)'(x) = f'(x) + g'(x) \)
\[
(f \cdot g)'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)
\]
(\( \cdot \) dot product in \( \mathbb{R}^n \): \( (y_1, -y_n) \cdot (x_1, -x_n) = y_1x_1 \))

If \( n = 1 \), \( g(y) \neq 0 \) \( \forall y \in I \),
\[
(f/g)'(x) = (f'(x)g(x) - g'(x)f(x))/g(x)^2\]
constant maps have derivative 0.

The identity map has derivative 1.

If \( I, J \subseteq \mathbb{R} \) open, \( f : I \rightarrow \mathbb{R} \) continuous differentiable at \( x \),
\( g : J \rightarrow \mathbb{R} \) continuous differentiable at \( f(x) \in J \).

Then \( g \circ f \), defined on \( I \cap f'(J) \), is continuous differentiable at \( x \), with
\[
(g \circ f)'(x) = g'(f(x)) \cdot f'(x).
\]
Directional derivative:

Let \( f: U \to \mathbb{R}^n \), \( U \subseteq \mathbb{R}^m \) open, \( x \in U, \nu \in \mathbb{R}^m \)

\( f \) is differentiable at \( x \) in the \( \nu \)-direction with derivative \( a \in \mathbb{R}^n \).

If \( g(t) = f(x + tv) \) is differentiable at \( 0 \in \mathbb{R} \) with derivative \( a \).

We write \( d_x f(\nu) = a \).

Usual \( \frac{\partial f}{\partial x_i}(x) \) as \( \nu = (0, \ldots, 1, 0, \ldots) \)

\( i \)-th place.

Differential of a map:

Let \( f = (f_1, \ldots, f_n): U \to \mathbb{R}^n \), \( U \subseteq \mathbb{R}^m \) open.

Let \( T: \mathbb{R}^m \to \mathbb{R}^n \) be a linear map, \( x \in U \).

We say \( f \) is differentiable at \( x \) with differential \( T \) if for each \( \varepsilon > 0 \) we have some \( \delta > 0 \) s.t.

\[ |v| < \delta \implies |f(x + v) - f(x) - T(v)| < \varepsilon |v|. \]

Then \( f \) is continuous at \( x \), \( T \) is unique. Write \( T = d_x f \).

\[ d_x f(\nu) = T(\nu). \]

\[ m = 1 \quad d_x f(1) = f'(x). \]

\( f = (f_1, \ldots, f_n) \) is differentiable at \( x \) if each \( f_i \) is, and then the matrix \( \left( \frac{\partial f_i}{\partial x_j}(x) \right) \) is the matrix of \( T \) relative to the standard basis in \( \mathbb{R}^m \) to \( \mathbb{R}^n \).
Properties: usual ones:
\[ f, g : U \to \mathbb{R}^n \text{ differentiable at } x \in U, \ U \text{ open in } \mathbb{R}^m \]
\[ d_x (f + g) = d_x f + d_x g \]
\[ d_x cf = c \cdot d_x f \quad \text{for } c \in \mathbb{R} \]
\[ h : V \to \mathbb{R}^n \text{ differentiable at } f(x) \in V, \ f \text{ continuous} \]
then \( h \circ f \), defined on \( U \cap f^{-1}(V) \), is differentiable at \( x \),
\[ d_x (h \circ f) = d_{f(x)} (h) \cdot d_x f \]

Now assume
\( (\mathbb{R}, +, \cdot, -, 0, 1, <, \ldots) \) is a minimal.
So real closed.

**Lemme (Rolle)** Let \( a < b \), and suppose the function
\[ f : [a, b] \to \mathbb{R} \] is definable, continuous, \( f(a) = f(b) \) and
\( f \) is differentiable on \((a, b)\). Then there is \( c \in (a, b) \)
such that \( f'(c) = 0 \).

If \( c \in (a, b) \) such that \( f(c) \) is minimum/maximum,
Show \( f''(c) = 0 \).
(Exercise)

**Mean Value Theorem** \( a < b \), \( f : [a, b] \to \mathbb{R} \) definable,
continuous, differentiable on \((a, b)\). Then for some \( c \in (a, b) \),
\[ f(b) - f(a) = (b-a) f'(c) \]

Let \( g(t) : [0, 1] \to \mathbb{R} \) defined by
\[ g(t) = f(a + t(b-a)) - t f(b) - f(a) \]
\[ g(0) = f(a) = 0 \]
\[ g(1) = f(b) - (f(b) - f(a)) = 0 \]
\[ g'(t) = f'(a + t(b-a))(b-a) + f(a) - f(b) \]
Lemma \( f : [a,b] \to \mathbb{R} \) continuous, defined on \((a,b)\).

If \( f'(x) = 0 \) for all \( x \in (a,b) \) then \( f(x) \) is constant.

Goal let \( f : I \to \mathbb{R} \) be definable, \( I \subseteq \mathbb{R} \) an interval. Then \( f \) is differentiable at all but finitely many points of \( I \).

Need several lemmas

\[ x \in I \quad \text{Define} \quad f(x^+) = \lim_{t \to 0^+} \frac{f(x+t) - f(x)}{t} \quad \epsilon \mathbb{R} \quad f^+ = \alpha^+ \]

\[ f(x^-) = \lim_{t \to 0^-} \frac{f(x+t) - f(x)}{t} \]

\( f \) differentiable at \( x \) : \( f(x^+) = f(x^-) \in \mathbb{R} \).

Lemma Assume \( f \) is continuous, \( f'(x^+) > 0 \) for all \( x \in I \).
Then \( f \) is strictly increasing, \( f' : f(I) \to \mathbb{R} \) satisfies \( (f^{-1})'(y^+) = 1/f'(x^+) \) for \( x \in I \), \( f(x) = y \). \((1/\epsilon \to 0)\).

If \( f \) were not strictly increasing, then there would be a subinterval \( J \) on which \( f \) is constant \( (f' = 0)\), or strictly decreasing, which contradicts \( f'(x^+) > 0 \).

For \( \epsilon > 0 \) sufficiently small, we have

\[ \lim_{t \to 0^+} \frac{f(x+t) - f(x)}{t} = \lim_{t \to 0^+} \left( f^{-1}(y+u) - f^{-1}(x) \right)^{-1} \]

\[ = \lim_{u \to 0^+} \left( f^{-1}(y+u) - f^{-1}(u) \right)^{-1} \]
Lemma: \( f : I \to \mathbb{R} \) is differentiable, \( x \to f'(x) \), and \( x' \to f'(x^-) \) are \( \mathbb{R} \)-valued continuous on \( I \). Then \( f' \) is continuous on \( I \), and \( f' \) is continuous on \( I \).

If \( f'(a^+) = f'(a^-) \) for all \( a \in I \).

Otherwise, say \( f'(a^+) > f'(a^-) \), let \( c \in \mathbb{R} \) be between \( f'(a^+) \) and \( f'(a^-) \), and let \( J \subseteq I \) be such that \( f'(x^+) > c > f'(x^-) \) on \( J \). Then \( g : J \to \mathbb{R} \), \( g(x) = f(x) - cx \), satisfies \( g'(x^+) > 0 \), \( g'(x^-) < 0 \) for all \( x \in J \). So \( g \) is both strictly increasing and strictly decreasing on \( J \).

(Lemme précédent appliqué à \( -g \))

Lemma: let \( f : I \to \mathbb{R} \) be differentiable. There are only finitely many \( x \in I \) at which \( f(x) = \pm \infty \).

Suppose \( A = \{ x \in I \} : f(x) = \pm \infty \) is infinite.

Then \( A \) contains an interval, and wlog \( A = I \), \( f \) is continuous on \( I \). Then \( f' \) is strictly increasing, and therefore \( f'(x^-) \geq 0 \) for all \( x \in I \).

(We want to reach a contradiction, so we are allowed to shrink \( I \) to non-empty subintervals.) So after shrinking \( I \), we may assume we are in one of the following two cases:

(i) \( f'(x^-) = +\infty \) for all \( x \in I \).
(ii) \( f'(x^-) \in \mathbb{R} \) for all \( x \in I \), and \( x \to f'(x^-) \) is continuous on \( I \).

In subcase (i), we have \((f^{-1})'(y^-) = 0 = (f^{-1})'(y^+)\).

i.e., \( f^{-1} \) is constant. This contradicts \( f'(x^+) > 0 \).

In subcase (ii), let \( x \in I \), and \( c > f'(a^-) \). Then there is a subinterval \( J \subseteq I \) on which \( f'(x^-) < c \).
So looking at \( g(x) = f(x) - cx \) we have:
\[
\begin{align*}
g'(x^+) &= f'(x^+) - c = f'(x^+) = +\infty \\
g'(x^-) &= f'(x^-) - c < 0
\end{align*}
\]

Contradiction, so \( |A| < +\infty \).

Replacing \( f(x) \) by \( f(-x) \), we get that the set of
\( x \in I, \ f(x) = \pm \infty \) is finite.

Proof of the Proposition: if \( f: I \to \mathbb{R} \) is differentiable,
then there are only finitely many points at which
\( f \) is not differentiable.

If: We saw in the previous lemma that the set
of points \( A \) such that one of \( f'(x^+) \), \( f'(x^-) \) is \( \pm \infty \),
is finite. Furthermore, throwing away finitely many
points, we may assume that on each subinterval of
\( I \setminus A \), the maps \( f'(x^+) \) and \( f'(x^-) \) are continuous.
Hence, at all points of \( I \setminus A \), \( f'(x^+) = f'(x^-) \)
and \( f \) is differentiable.

Aim: Inverse function theorem, and implicit function theorem:

If the Jacobian is invertible at a point \( a \in \mathbb{R}^n \),
then \( f \) is locally \( a \) homeo around \( a \).
and IFT.
So looking at \( g(x) = f(x) - cx \) we have:
\[
\begin{align*}
    g'(x^+) &= f'(x^+) - c = f'(x^+) = +\infty \\
    g'(x^-) &= f'(x^-) - c < 0.
\end{align*}
\]

Contradiction. So \(|A| < +\infty\).

Replacing \( f(x) \) by \( f(-x) \), we get that the set of \( x \in I \) where \( f(x) = \pm \infty \) is finite.

Proof of the Proposition: if \( f : I \to \mathbb{R} \) is definable, then there are only finitely many points at which \( f \) is not differentiable.

**If:** We saw in the previous lemma that the set of points \( A \) such that one of \( f'(x^+) \), \( f'(x^-) \) is \( \pm \infty \), is finite. Furthermore, throwing away finitely many points, we may assume that on each subinterval of \( I \setminus A \), the maps \( f'(x^+) \) and \( f'(x^-) \) are continuous. Hence, at all points of \( I \setminus A \), \( f'(x^+) = f'(x^-) \) and \( f \) is differentiable.

**Aim:** Inverse function theorem, and implicit function theorem:

If the Jacobian is invertible at a point \( a \in \mathbb{R}^m \), then \( f \) is locally \( a \) homeo around \( a \) and IFT.
For that we need more lemmas.

Setting \( (f_1, \ldots, f_n) = f : U \to \mathbb{R}^n, \quad U \subseteq \mathbb{R}^m \) open.

Qf. We call \( f \) a \( C^1 \)-map if the partial derivatives \( \frac{df_i}{dx_j} \) are defined on \( U \) and continuous.

One shows easily that:

If \( f \) is \( C^1 \), then \( f \) is differentiable at each point of \( U \), and the map \( x \to \frac{d_x f}{dx} \in \mathbb{R}^{m \times m} = \text{Lin} \left( \mathbb{R}^m, \mathbb{R}^n \right) \) is continuous. And conversely.

(Usual proof for \( \to \)).

If \( T : \mathbb{R}^m \to \mathbb{R}^n \) is \( R \)-linear (i.e. \( \in \mathbb{R}^{m \times m} \)), define \( |T| = \max \{ |T(x)| \mid 1 |x| \leq 1, x \in \mathbb{R}^m \} \).

Then \( |T(x)| \leq |T| |x| \).

Lemma. Let \( f : U \to \mathbb{R}^n \) be \( C^1 \), \( [a, b] = \{ (1-t) a + tb \mid 0 \leq t \leq 1 \} \) be a line segment contained in \( U \).

Then \( |f(b) - f(a)| \leq |b - a| \max_{y \in [a, b]} |dyf| \).

Let \( g(t) : [0, 1] \to U \) with \( g(t) = f((1-t) a + tb) \).

Then \( g'(t) = \text{directional derivative of } f \text{ at } (1-t)a + tb \), in direction \( (b-a) \).

\[ = dyf(b-a), \quad \text{when } y = (1-t)a + tb. \]

So \( |g'(t)| \leq M, \quad M = |b-a| \max_{y \in [a, b]} |dyf|. \)

By MVT, we have \( |f(b) - f(a)| = |g(1) - g(0)| \leq M. \)

Lemma. Same assumptions, \( x \in U \).

\[ |f(b) - f(a) - d_x f(b-a)| \leq |b - a| \max_{y \in [a, b]} |dyf - d_x f|, \]
Consider \( h(y) = f(y) - df(y) \).

Then \( dh = dy \cdot \frac{df}{dx} \).

**Lemma** Same assumptions, \( m = n \), \( a \in U \), and assume that \( df \) is invertible. Then there are \( \varepsilon > 0 \), \( C > 0 \) in \( \mathbb{R} \) such that

\[
|f(x) - f(y)| > C |x - y| \quad \text{for all } x, y \in U \text{ with } |x - a|, |y - a| < \varepsilon.
\]

In particular, \( f \) is invertible on a neighborhood of \( a \).

**Proof** Let \( \varepsilon > 0 \) be small enough so that \( B(a, \varepsilon) \subset U \).

By the previous lemma, we have

\[
|f(x) - f(y) - df(x - y)| < (x - y) \max_{g \in [x, y]} |dg_f - df| \\
\]

\[
|df(x - y)| - |f(x) - f(y)| \\
\]

\[
\Rightarrow |f(x) - f(y)| \geq |df(x - y)| - |x - y| \max_{g \in [x, y]} |dg_f - df| \\
\]

As \( df \) is invertible, there is \( c' \), not depending on \( x, y \), such that

\[
|df(x - y)| \geq c' |x - y|.
\]

Indeed, we have \( |g| = |df^{-1} \circ df^{-1} (g)| \leq |df^{-1}||df^{-1} (g)| \)

\[
\Rightarrow |df (g)| \geq |g||df^{-1}|^{-1}.
\]

Decreasing \( \varepsilon \), we may assume that

\[
|g_{df} - df| < c' \frac{1}{2} \quad \text{for all } g \in B(a, \varepsilon)
\]

hence

\[
|f(x) - f(y)| \geq c' |x - y| - c' \frac{1}{2} |x - y| \geq c' \frac{1}{2} |x - y|.
\]
Inverse function theorem

Let \( f : U \to \mathbb{R}^m \) be a definable \( C^1 \) map on a definable open set \( U \subseteq \mathbb{R}^m \), \( \alpha \in U \) s.t. \( \text{def} : \mathbb{R}^m \to \mathbb{R}^m \) is invertible.

Then there are a definable open \( U' \ni \alpha \), \( U' \subseteq U \), and a definable nbhd \( V' \) of \( f(\alpha) \) such that \( f \) maps \( U' \) homeomorphically onto \( V' \) and \( f^{-1} : V' \to U' \) is also \( C^1 \).

**Proof** Since they define the same topology, one may replace \( \alpha \) on \( \mathbb{R}^m \) by \( \alpha \) on \( \mathbb{R}^m \) by \( \| \alpha \| = \sqrt{x_1^2 + x_2^2 + \cdots + x_m^2} \).

We can find \( c, \varepsilon > 0 \) such that

\[
\| x - \alpha \| < \varepsilon \to x \in U \text{ and } d_x f \text{ is invertible.}
\]

\[
\| x - \alpha \|, \| y - \alpha \| \leq \varepsilon \to \| f(x) - f(y) \| \geq c \| x - y \|.
\]

**Claim** \( \{ y \mid \| y - f(\alpha) \| < \frac{1}{2} c \varepsilon \} \subseteq \{ f(x) \mid \| x - \alpha \| \leq \varepsilon \} \)

Assume \( \| y - f(\alpha) \| < \frac{1}{2} c \varepsilon \)

Consider \( P(x) = \| f(x) - y \|^2 \) on the ball \( \| x - \alpha \| \leq \varepsilon \).

As the ball is closed and bounded, \( P(x) \) assumes its minimum value on it. However, if \( \| x - \alpha \| = \varepsilon \),

then \( P(x) = \| f(x) - f(\alpha) \| ^2 + \| y - f(\alpha) \| ^2 \geq \frac{1}{2} c \varepsilon \)

So the minimum value is attained at \( b, \| b - \alpha \| < \varepsilon \).

So \( 0 = \frac{\partial P}{\partial x j} (b) = \sum_{i=1}^{m} (f_i(b) - y_i) \cdot \frac{\partial f_i}{\partial x j} (b) \)

for \( j \), \( d_b f (f(b) - y) = 0 \).
df invertible implies f(b) = y, which proves the claim.
So the image by f of the open set of \( \|x-a\| < \varepsilon \) contains the open set of \( \|y-f(a)\| < \frac{\varepsilon}{2} \) ce \( \mathbb{R}^n \).

Let \( U' = \{ x \times 1 \mid \|x-a\| < \varepsilon \} \). Then, reasoning in the same way with all points \( a' \in U' \) we get that f is open on U'. It is also injective, hence it is a homeomorphism between U' and \( V' = f(U') \).

It remains to show that \( f^{-1} : V' \rightarrow U' \) is C^1.

By definition

\[
\frac{(f(b) - f(a) - daf(b-a))}{\|b-a\|} \rightarrow 0 \text{ as } b \rightarrow a.
\]

Also \( \|b-a\| < c^{-1} \|f(b) - f(a)\| \). Apply \((daf)^{-1}\)

\[
(daf)^{-1} \frac{(f(b) - f(a)) - (b-a)}{\|f(b) - f(a)\|} \rightarrow 0
\]
as \( f(b) \rightarrow f(a) \).

i.e.: \( f^{-1} \) is differentiable at \( f(a) \), with differential \( d_{f(a)} f^{-1} = (daf)^{-1} \).

This reasoning works at every point \( a' \in U' \), and therefore \( f^{-1} \) is C^1 (\( x \mapsto (dx f)^{-1} \) is continuous).
Corollary (Implicit function theorem)

Let \( U \subseteq \mathbb{R}^{m+n} \) be definable open, \( f_1, \ldots, f_n : U \to \mathbb{R} \) be definable \( C^1 \). Let \((x_0, y_0) \in \mathbb{R}^{m+n} \) be s.t.

\[
f_1(x_0, y_0) = \ldots = f_n(x_0, y_0) = 0,
\]

and the matrix

\[
\begin{pmatrix}
\frac{\partial f_1}{\partial y_1}(x_0, y_0) & \cdots & \frac{\partial f_1}{\partial y_n}(x_0, y_0)
\end{pmatrix}
\]

is invertible. Then there is an open definable nbhd \( V \) of \( x_0 \) in \( \mathbb{R}^m \), and \( W \) of \( y_0 \) in \( \mathbb{R}^n \), and a definable \( C^1 \) map \( \phi : V \to W \) such that \( V \times W \subseteq U \), and for all \((x, y) \in V \times W \) we have

\[
f_1(x, y) = \ldots = f_n(x, y) = 0 \quad \Rightarrow \quad y = \phi(x).
\]

Proof. Apply the inverse function theorem to the map

\[
y : (x, y) \mapsto (x, f_1(x, y), \ldots, f_n(x, y))
\]

\[
U \to \mathbb{R}^{m+n}.
\]

Consider the map \( \pi g(x, 0) \) near \( x_0 \).

What else:

- \( \frac{d}{dx} \) form: \( I \subseteq \mathbb{R} \) interval, \( f : I \to \mathbb{R} \) definable with \( f'(x) \neq 0 \) for all \( x \in I \) in a nbhd of a \( \lim_{x \to a} g(x) = 0 = \lim_{x \to a} f(x) \). Then \( \lim_{x \to a} \frac{g(x)}{f(x)} = \lim_{x \to a} \frac{f(x)}{g(x)} \).
- L'Hôpital's rule: If \( f : I \to \mathbb{R} \) is \((m+1)\)-times differentiable on \( I \), and \( a, b \) are in \( I \). Then

\[
f(b) = f(a) + f'(a)(b - a) + \frac{f''(a)}{2!}(b-a)^2 + \cdots
\]

\[
+ \frac{f^{(m)}(a)}{m!}(b-a)^m + \frac{f^{(m+1)}(\xi)}{(m+1)!}(b-a)^{m+1}
\]

for some \( \xi \in (a, b) \).
Def: Let $A \subseteq \mathbb{R}^m$, $f: A \rightarrow \mathbb{R}^n$ a definable map. Then $f$ is a $C^1$-map if there are a definable open $U \supseteq A$, and a definable $C^1$-map $F: U \rightarrow \mathbb{R}^n$, such that $F|_{A} = f$.

A $C^1$ cell is a cell in which all defining functions are $C^1$.

Thm (C$^1$ cell decomposition)

(Iim) If $A_{1}, \ldots , A_{k} \subseteq \mathbb{R}^m$ are definable, there is a decomposition of $\mathbb{R}^m$ into $C^1$-cells partitioning $A_{1}, \ldots , A_{k}$.

(IIim) If $A \subseteq \mathbb{R}^m$ and $f: A \rightarrow \mathbb{R}$ is definable, then there is a decomposition of $\mathbb{R}^m$ into $C^1$-cells, partitioning $A$ and such that if $C \subseteq \mathbb{R}, C \subseteq A$ then $f|_{C}$ is $C^1$.

If $f$ and $A$ are as in IIim, $p \in \text{Int}(A)$, write

$$\nabla f(p) = (\frac{\partial f}{\partial x_1}(p), \ldots , \frac{\partial f}{\partial x_m}(p))$$

provided they exist.

$A' = \{ p \in A \mid p \in \text{int}(A) \text{ and } \nabla f \text{ is defined at } p \}$

Then

(IIIim) $A \setminus A'$ has empty interior.

[On utilize IIIim proof: prove IIIim]