Examen du cours spécialisé *Corps réels clos et structures o-minimales* (M2 MathFonda, printemps 2022).

The homework is to be done at home, and hand back on April 28 (2022). You are allowed to look at your course notes and at mine, but not anything else. Do not hesitate to ask me questions, even dumb ones. In all questions, you can suppose known the results of the preceding questions.
All rings and fields are commutative.
In red, you will see some corrections or precisions.

Problem 1.
We are going to describe the pre-positive prime cones of certain polynomial rings. Un pre-positive cone of the ring R is *prime* if $P \cup (-P) = R$ and $P \cap -P$ is a prime ideal of R. Let X be an indeterminate.

(a) Describe the pre-positive prime cones of $\mathbb{R}[X]$.
(b) Let $\bar{\mathbb{Q}} \subset \mathbb{R}$ be the real closure of $\mathbb{Q}$. Describe the pre-positive prime cones of $\bar{\mathbb{Q}}[X]$.
(c) Describe the pre-positive prime cones of $\mathbb{Q}[X]$.

Problem 2.
The set of pre-positive prime cones of a ring A is also called the *real spectrum* of the ring, and denoted $\mathrm{Spec}_R(A)$ or $\mathrm{Sper}(A)$. It is endowed with a topology, with basic open sets of the form

$$D(a_1, \dots, a_n) = \{P \in \mathrm{Sper}(A) \mid a_1, \dots, a_n \in P \setminus (-P)\}.$$

Equivalently, $D(a_1, \dots, a_n\} = \{P \in \mathrm{Sper}(A) \mid -a_1, \dots, -a_n \notin P\}$. So, the cones for which $a_1, \dots, a_n$ are strictly positive.

(a) Consider the natural map $\mathrm{Sper}(A) \to \mathrm{Spec}(A)$, $P \mapsto P \cap (-P)$, where $\mathrm{Spec}(A)$ is the set of prime ideals of A, endowed with the Zariski topology (a basis of open sets is given by $U(a) = \{Q \in \mathrm{Spec}(A) \mid a \notin Q\}$, for $a \in A$).
Is this map continuus?
(b) Show that $\mathrm{Sper}(A)$ is *quasi-compact* (Every cover of $\mathrm{Sper}(A)$ by open sets contains a finite subcover).
(c) Show that if $P, Q \in \mathrm{Sper}(A)$, then $P \in \mathrm{cl}(\{Q\})$ iff $Q \subseteq P$ (P is a *specialization* of Q).
(d) Show that the irreducible closed sets of $\mathrm{Sper}(A)$ are of the form $\mathrm{cl}(\{P\})$, for some $P \in \mathrm{Sper}(A)$.
(e) If $P \subseteq Q$ et $P \subseteq R$, then $Q \subseteq R$ or $R \subseteq Q$, for $P, Q, R \in \mathrm{Sper}(A)$.

Problem 3.
We work in an o-minimal structure $(R, <, \dots)$.

(a) Show that a cell $C \subseteq R^n$ is open iff $\dim(C) = n$.
(b) Show that a cell $C \subseteq R^n$ is definably connected.
(c) Show that if $C \subseteq R^n$ is a cell, and p is the projection on the first m coordinates ($m < n$) then $p(C)$ is a cell.

(d) Let $C \subseteq R^n$ be a cell, with associated sequence $(i_1, \ldots, i_n)$. Let $\lambda_1 < \lambda_2 < \cdots < \lambda_r$ be the indices j such that $i_j = 1$.
(d-1) What is $\dim(C)$?
(d-2) Let $p : R^n \to R^r$ be the projection $(x_1, x_2, \ldots, x_n) \mapsto (x_{\lambda_1}, x_{\lambda_2}, \ldots, x_{\lambda_r})$. Show that $p(C)$ is an open cell of R^r, and that p defines a homeomorphism between C and $p(C)$.

(e) Show that if $S \subseteq R^m$ is definable and definably connected, and if $f : S \to R^n$ is continuous deinable, then $f(S)$ is definably connected.

(f) (Recall: $(f, g) = \{(x, y) \mid x \in (a, b), f(x) < y < g(x)\}$). Let $C \subset R^2$ be a cell given by $C = (f, g)$, where $f, g : (a, b) \to R$ are strictly increasing definable continuous functions, and for all $x \in (a, b)$, $f(x) < g(x)$.
Let $C^* = \{(u, v) \mid (v, u) \in C\}$; describe a cell decomposition of R^2 which partitions C^*.

Problem 4.
Let $(G, \cdot, 1, <, \ldots)$ be an o-minimal ordered group (non-trivial, with maybe extra structure). We will show that G is commutative and divisible.

(a) Show that if H is a definable subgroup of G then $H = (1)$ or $H = G$.

(b) If $g \in G$, let $C(g) = \{h \in G \mid hg = gh\}$. What can you say about $C(g)$? Show that G is commutative.

(c) Show that G is divisible.

Problem 5.
Let $(R, +, -, \cdot, 0, 1, < \ldots)$ be an ordered o-minimal field (with maybe extra structure). We will show it is real closed.

(a) Show that every positive element of R is a square.

(b) Let $p(X) \in R[\bar{X}]$, $\bar{X} = (X_1, \ldots, X_n)$. Explain in a few words why the map $R^n \to R$, $\bar{a} \mapsto p(\bar{a})$, is continuous.

(c) Let $p(X) \in R[X]$ (X a single variable), and $a < b \in R$ such that $p(a) < 0 < p(b)$. Show that there is $c \in (a, b)$ such that $p(c) = 0$.

(d) Show that (c) implies that R is real closed.

Problem 6. (The statement is very long, but the problem is not difficult.)
We consider an o-minimal structure $\mathcal{R} = (\mathbb{R}, +, -, \cdot, 0, 1, <, \ldots)$ expanding the usual structure of ordered field on $\mathbb{R}$. We will admit the following result:
(**Definable trivialisation**) *Let $f : S \to A \subseteq \mathbb{R}^m$ a definable function (in $\mathcal{R}$) which is continuous. Then there is a partition of S in definable subsets $S_1, \ldots, S_k$ such that the sets $f(S_i)$ are distinct and form a partition of $f(S)$, definable sets F_i ($\subseteq \mathbb{R}^{N_i}$), and definable functions $\lambda_i : S_i \to F_i$, such that for each i, $(f|_{S_i}, \lambda_i) : S_i \to (f(S_i) \times F_i)$ is a homeomorphism.*
The pair (λ_i, F_i) is called a *trivialisation* of the restriction of f to S_i. If $A_1, \ldots, A_r$ are definable subsets of S, one can moreover assume that the partition (S_i) is compatible with the A_j, i.e., that there are $G_{ij} \subset F_i$ such that for every i and j, $(f_i|_{A_j \cap S_i}, \lambda_i|_{A_j \cap S_i})$ defines a homeomorphism between $A_j \cap S_i$ et $f(A_j \cap S_i) \times G_{ij}$
One says that $X, Y \subset \mathbb{R}^m$ have the same *embedded homeomorphism type* if there is a homeomorphism $F : \mathbb{R}^m \to \mathbb{R}^m$ which sends X to Y.

(a) Let $S \subseteq \mathbb{R}^{m+n}$ be definable. Show that the sets $S_x := \{y \in \mathbb{R}^n \mid (x, y) \in S\}$ have a finite number of (embedded) homeomorphism types. Do at least the non-embedded case.

(b) Let $f : U \times V \to \mathbb{R}$ be a definable function ($U \subseteq \mathbb{R}^m$, $V \subseteq \mathbb{R}^n$), and for $c \in V$, write f_c for the function $U \to \mathbb{R}$ defined by $f_c(x) = f(x, c)$, and $Z(f_c) = \{x \in U \mid f_c(x) = 0\}$. Show that the family $Z(f_c)$, $c \in V$, has only finitely many embedded homeomorphism types.

(c) We assume now that the o-minimal structure $\mathcal{R}$ contains the exponential map exp. (It is known that $\mathbb{R}_{\exp} = (\mathbb{R}, +, -, \cdot, \exp, 0, 1, <)$ is o-minimal.) We fix m and n, and consider the family $\mathcal{F} = \mathcal{F}_{m,n}$ of polynomials $f(X) \in \mathbb{R}[X_1, \ldots, X_m]$ having at most n terms. (I.e., of the form $\sum_{i=1}^n a_i X_1^{\alpha_{i1}} X_2^{\alpha_{i2}} \cdots X_m^{\alpha_{in}}$, where $a_i \in \mathbb{R}$, and $\alpha_{ij} \in \mathbb{N}$. Note that the degree is unbounded.). We will (almost) show that there is a finite number of possibilities for the homeomorphism type of $Z(f) := \{x \in \mathbb{R}^m \mid f(x) = 0\}$, $f \in \mathcal{F}$. We first fix for each $1 \leq r \leq n$ a sequence (of parity) $\boldsymbol{\epsilon_r} = (\epsilon_{r,1}, \ldots, \epsilon_{r,m}) \in \{0, 1\}^m$, et consider only the polynomials $f(X)$ satisfying that the exponent of X_j in the r-th monomial is even iff $\epsilon_{r,j} = 0$; this defines a subfamily $\mathcal{F}_{\boldsymbol{\epsilon}}$ of $\mathcal{F}$.

[(c-1)] Assume that all $\epsilon_{r,i}$ are 0. Show that there is a definable function $F : \mathbb{R}^{1+2m} \to \mathbb{R}$, which when restricted to $\mathbb{R} \times (2\mathbb{N})^m \times \mathbb{R}^m$, gives precisely the map

$$(c, \alpha_1, \ldots, \alpha_m, x_1, \ldots, x_m) \mapsto c x_1^{\alpha_1} \cdots x_m^{\alpha_m}.$$

[(c-2)] Show that there are only finitely many embedded homeomorphism type of $Z(f)$, for $f \in \mathcal{F}_{\boldsymbol{\epsilon}}$.

[(c-3)] One can do the same for other sequences of 0 and 1. What is the analogue of the function F (given in (c-1)) for the sequence $\boldsymbol{\epsilon_1} = (1, 1, 0, \ldots, 0)$? (So, we are interested in what happens on $\mathbb{R} \times (2\mathbb{N} + 1)^2 \times (2\mathbb{N})^{m-2} \times \mathbb{R}^m$.)

Problem 7. We assume that $(R, +, -, \cdot, 0, 1, <, \ldots)$ is an o-minimal structure, which expands a real closed field. Let $I \subset R$ an open interval, a an extremity if I (in R or $\pm\infty$). We assume that $g'(x) \neq 0$ for all x in a neighbourhood of a.

(a) If $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$, show that

$$\lim_{x \to a} f(x)/g(x) = \lim_{x \to a} f'(x)/g'(x).$$

(b) We suppose now that $\lim_{x \to a} |f(x)| = \lim_{x \to a} |g(x)| = +\infty$. What can you say about $\lim_{x \to a} f'(x)/g'(x)$?

[Hint: Reduce to the case where $a \in R$ is the left extremity of I.]