

# O-minimal ordered groups and rings

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# Introduction

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- ▶ An ordered group with an o-minimal structure is abelian and divisible.
- ▶ An ordered ring with an o-minimal structure is a real closed field.

# Definition

An ordered group is a group with a linear order which is invariant under multiplication, i.e.

$$x < y \Rightarrow zx < zy, xz < yz$$

# Proposition

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Suppose  $(R, <, \mathcal{S})$  is an o-minimal structure and  $\mathcal{S}$  contains a binary operation  $\cdot$  on  $R$  such that  $(R, <, \cdot)$  is an ordered group. Then  $(R, \cdot)$  is abelian and divisible.

# Proof

Firstly, we prove that a definable subgroup of  $R$  must be  $\{1\}$  or  $R$ . Let  $G$  be a definable subgroup. If  $G$  is not convex, then we have  $g \in G, r \notin G$  such that  $1 < r < g$ . Then we have a sequence

$$1 < r < g < rg < g^2 < rg^2 < g^3 < \dots$$

where  $g^i \in G, rg^i \notin G$ , which contradicts with the fact that  $G$  is definable.

Assume that  $G \neq \{1\}$ , take  $s = \sup G > 1$  and we have  $(1, s) \subseteq G$ . If  $s = +\infty$ ,  $G = R$ ; otherwise we take any  $g \in (1, s)$ , then  $s = g \cdot (g^{-1}s) \in G$  and  $s < gs \in G$ , which is a contradiction!

Back to the proposition, for each  $r \neq 1 \in R$  consider the centralizer  $C_R(r)$  which is a definable subgroup containing  $r \neq 1$ . Hence  $C_R(r) = R$  and  $R$  is abelian as a consequence.

For each  $n > 0$ ,  $\{x^n : x \in R\}$  is a non-trivial definable subgroup. Thus  $R = \{x^n : x \in R\}$ , which means  $R$  is divisible.

# Definition

Recall the definition of ordered rings and fields:

An ordered ring is a ring with a linear order  $<$  satisfying:

- ▶  $0 < 1$ ;
- ▶  $x < y \Rightarrow x + z < y + z$ ;
- ▶  $x < y, z > 0 \Rightarrow xz < yz$ .

The ordered ring is a division ring if for any  $x \neq 0$ , there exists a unique  $y$  such that  $xy = yx = 1$ .

An ordered field is an ordered division ring with commutative multiplication.

# Real closed field

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Real closed field has many equivalent definitions, here we define a real closed field is an ordered field satisfying the intermediate value property: for  $f(T) \in R[T]$  and  $a < b$  such that  $f(a) < 0 < f(b)$ , there is  $c \in (a, b)$  with  $f(c) = 0$ .

# Proposition

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Suppose  $(R, <, \mathcal{S})$  is an o-minimal structure and  $\mathcal{S}$  contains binary operations  $+ : R^2 \rightarrow R$  and  $\cdot : R^2 \rightarrow R$  such that  $(R, <, +, \cdot)$  is an ordered ring. Then  $(R, <, +, \cdot)$  is a real closed field.

For any  $r \neq 0$  in  $R$ , we have a non-trivial definable additive subgroup  $rR$  of  $R$ , hence  $rR = R$  for  $r \neq 0$ . Thus  $(R, <, +, \cdot)$  is an ordered division ring. Let  $\text{Pos}(R)$  be the ordered multiplicative group  $\{r \in R : r > 0\}$ . Restrict the structure  $\mathcal{S}$  to  $\text{Pos}(R)$ , we have an ordered group with an o-minimal structure. By the previous proposition, it is abelian. This tells us the multiplication on  $R$  is commutative. So  $(R, <, +, \cdot)$  is an ordered field.

Notice that the operations  $+$ ,  $-$ ,  $\cdot$ ,  $^{-1}$  are continuous with respect to the interval topology. Thus if we view the polynomial  $f(T) \in R[T]$  as a function  $x \mapsto f(x)$ , it is definable and continuous. Since  $[a, b]$  is definably connected, the image of it under  $f$  is also definably connected, which means  $R$  satisfies the intermediate value property. In conclusion,  $(R, <, +, \cdot)$  is a real closed field.