Model theory of fields with operators - a survey

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For Jouko, on the occasion of his 60th birthday

Introduction

The model theory of fields with operators has proven to be very useful in applications of model theory to problems outside logic. Differentially closed fields and separably closed fields were instrumental in Hrushovski's proof of the conjecture of Mordell-Lang ([39]), and in work of Hrushovski and Pillay on counting the number of transcendental points on certain varieties ([44]). Let me also mention Ax's work ([1]) on connections between transcendence theory and differential algebra, which has been recently elaborated on by Bertrand and Pillay ([6]) and Kowalski ([50]) in positive characteristic. The model theory of difference fields also provided various achievements: Hrushovski's new proof of the conjecture of Manin-Mumford ([40]), Scanlon's approach to p-adic abc conjectures and proximity questions ([87], [88], [89]), and his solution to Denis' conjecture ([90]); more recently, applications to algebraic dynamics by Hrushovski and the author ([21], [22], [16]), and by Medvedev and Scanlon ([59]). Model theory also provides some insight on the Galois theory of systems of differential (or difference) equations and of Picard-Vessiot extensions, or of strongly normal extensions of Kolchin, see e.g. [72].

The aim of this article is not to present these applications, but to give a survey of what is known of the model theory of these enriched fields. We will discuss such issues as existence of model companions and their various axiomatisations, elimination of quantifiers and decidability; we will also investigate stability theoretic properties and mention some open problems.

Another notable omission of this survey is that of exponential fields, which have seen many extraordinary developments in the past twenty years: in the context of the field of real numbers, or in the context of the field of complex numbers. I do not consider myself an expert in this subject, parts of which are still in full progress.

1 Basic results

1.1. Differential fields of characteristic 0.
Recall that a *differential field* is a field K endowed with a derivation D, i.e., an additive map

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$D : K \to K$ satisfying the Leibniz rule $D(xy) = xD(y) + D(x)y$. We study them in the language of rings augmented by a function symbol for the derivation, $\{+, -. \cdot .0, 1, D\}$.

The existence of a model completion (commonly called DCF_0) of the theory of differential fields of characteristic 0 was first shown by Robinson ([82]); hence the theory eliminates quantifiers and is complete ([81]). The models of DCF_0 are called *differentially closed fields*. Since every derivation of a field extends to its algebraic closure, a differentially closed field is algebraically closed as a field. Blum gave a simple axiomatisation of DCF_0 in her PhD thesis ([8], see also [9]), and showed that DCF_0 is ω-stable. A consequence of this result is the existence and uniqueness of the differential closure of a differential field: any totally transcendental theory has a unique prime model (Shelah). Poizat showed ([77]) that DCF_0 eliminates imaginaries.

1.2. Several derivations.
These results were extended to several commuting derivations by McGrail ([58]): the theory $\mathrm{DCF}_{0,n}$ of differentially closed fields of characteristic 0 with n commuting derivations is complete, eliminates quantifiers (in the language of rings augmented by n function symbols for the derivations), is ω-stable of rank ω^n and eliminates imaginaries.

The case of fields with non-commuting derivations but which have a finite dimensional commutator Lie subalgebra was first studied by Yaffe ([101]); Singer ([93]) then showed how to reduce the study of their existentially closed models to the case of of models of DCF_0. The case of arbitrary non-commuting derivations is a consequence of a result of Moosa and Scanlon, and will be discussed later.

1.3. Geometric axiomatisation.
The axiomatisation given by Blum is very simple and uses differential polynomials in one variable. If $F(X) \in K[X]$, $X = (X_0, \dots, X_n)$, involves non-trivially the variable X_n, then the associated differential polynomial $f(X) = F(X, Dx, \dots, D^n x)$ is said to have *order n*. Besides the usual axioms for differential fields of characteristic 0, Blum adds the scheme of axioms for algebraically closed fields, and the following scheme of axioms: for every pair of differential polynomials f and g, with the order of f strictly greater than the order of g, the axiom

$$\exists x \ f(x) = 0 \wedge g(x) \neq 0.$$

[Note that these axioms imply that the derivation is non-trivial: $\exists x \ Dx = 1$.]

A different kind of axiomatisation of DCF_0 (so called *geometric axiomatisation*) was given by Pierce and Pillay ([70]). Its formulation uses a very limited amount of differential algebra, but nevertheless allows to describe several variable differential varieties. It contains the usual axioms for differential fields of characteristic 0; for the remaining axioms we need first a definition.

Let $f_1, \dots, f_m \in K[X]$, $X = (X_1, \dots, X_n)$. Suppose that they generate a prime ideal of $K[X]$ and let $V \subset K^n$ be the irreducible affine variety defined by the equations $f_1(x) = \dots = f_m(x) = 0$. Define the *twisted tangent bundle of* V, $\tau(V)$, by the following equations:

$$(x, y) \in \tau(V) \iff \bigwedge_{i=1}^{m} f_i(x) = 0 \wedge \sum_{j=1}^{n} \frac{\partial f_i}{\partial X_j}(x) y_j + f_i^D(x) = 0.$$

Here f_i^D denotes the polynomial obtained by applying D to the coefficienst of f_i. Note that if $a \in V(K)$, then $(a, Da) \in \tau(V)$, and for $a \in V$, the algebraic set $\tau(V)(a) = \{b \mid (a,b) \in \tau(V)\}$ coincides with the algebraic set defined by "$y - D(a)$ belongs to the tangent space of V at a". There is a natural onto map $\pi : \tau(V) \to V$. The geometric axiomatisation of DCF_0 is then obtained by adding tie the theory of algebraically closed differential fields the scheme of axioms: For every irreducible varieties $V \subset K^n$ and $W \subset \tau(V)$ such that $\pi(W)$ is Zariski dense in V, there is a point a such that $(a, D(a)) \in W$.

Several attempts were made to find geometric axiomatisations for $DCF_{0,n}$, see [69], [84]; the problem is however much harder than in the one derivation case, and the axiomatisations which were obtained are in my opinion much less elegant.

1.4. Other theories of differential fields of characteristic 0.
All the fields discussed above are algebraically closed. One can however impose conditions on the pure field, and ask that it be existentially closed (as a differential field) with respect to those conditions. This was done by Singer ([92]) for ordered differential fields (thus obtaining a notion of closed ordered differential field), and by Tressl for the class of differential fields which are *large*, and whose theory in the language of rings is model-complete[1]. More work on closed ordered differential fields was done by Brihaye, Michaux, Point, and Rivière ([62], [76], [78], [79], [80]). The "uniform" axiomatisation of Tressl was generalised by Guzy in [34].

Itai and Hrushovski ([42]) showed the existence of many model-complete theories of differential fields of characteristic 0 which are superstable, and are not DCF_0. Chatzidakis and Hrushovski ([19]) studied the limit theory of the differential fields $(\mathbb{F}_p(t)^s, d/dt)$ as p goes to ∞, and showed that it interprets arithmetic[2].

One can also consider valued fields with operators. Valued differential fields have been studied extensively by Guzy, Michaux, Point and Rivière in [13], [30], [31], [32], [33], [35], [36]. Valued difference fields were studied by Hrushovski ([41]), by Durhan and Van den Dries ([2], [3]), by Bélair, Macintyre and Scanlon ([4]), by Pal ([68]), ...

1.5. More on DCF_0.
Shelah ([91]) showed that differential closures are not necessarily minimal: it is possible to find a differential field K, with differential closure $\hat{K}$ containing a strictly smaller K-isomorphic copy of $\hat{K}$. (This result was also proved independently by Rosenlicht and by Kolchin). In the same paper Shelah also showed that there are 2^κ non-isomorphic differentially closed fields of cardinality κ. The question on the number of non-isomorphic countable differentially closed fields stayed open for a long time, and was finally solved by Hrushovski and Sokolovich, who showed there are $2^{\aleph_0}$ many of them ([45], see [56] for a published proof).

1.6. Differential fields of positive characteristic.
Let K be a differential field of positive characteristic p, with derivation D. The Frobenius map

[1] A field K is *large* if it is existentially closed in the field of power series $K((t))$.
[2] $\mathbb{F}_p(t)^s$ denotes the separable closure of $\mathbb{F}_p(t)$.

$x \mapsto x^p$ is then a ring embedding, and we denote the image of K by K^p. It is a subfield of K, and D vanishes on K^p. Thus, differential fields of positive characteristic have very large subfields of constants. We say that K is *differentially perfect*, or *strict*, if K^p coincides with the constant subfield, i.e.:

$$\forall x \, Dx = 0 \rightarrow \exists y \, y^p = x.$$

Existentially closed differential fields of characteristic $p > 0$ are called *differentially closed*. Their theory, denoted DCF_p, was axiomatised by Wood ([97]), who showed that it is complete and model complete, and eliminates quantifiers in the language of differential rings augmented by a function symbol r which is interpreted as the inverse of the Frobenius map $x \mapsto x^p$: one adds the axiom $\forall x \, (Dx \neq 0 \wedge r(x) = 0) \vee r(x)^p = x$. Note that this axiom implies that the field is differentially perfect, and thus DCF_p is the model completion of the theory of differentially perfect differential fields of characteristic p. Wood and Shelah showed that DCF_p is stable not superstable, and that every differential field K has a differential closure (see [99], where Wood also gives Shelah's proof that the differential closure is unique). The underlying subfield of a differentially closed field of positive characteristic is of course separably closed, and of infinite degree of imperfection. I think one does not know a natural language in which differentially closed fields of positive characteristic eliminate imaginaries. See also the next section on separably closed fields.

1.7. Separably closed fields of positive characteristic.
Separably closed fields are fields K with the following property: if $f(X) \in K[X]$ is such that $f'(X)$ is not identically 0 (such polynomials are called separable), then $f(X)$ has a zero in K. If the characteristic of the field is 0, then all polynomials are separable. In positive characteristic p, non-separable polynomials are of the form $f(X) = g(X^p)$, where $g \in K[X]$.

Let us fix a field K of characteristic $p > 0$. Then K^p is a subfield of K, and K is a K^p-vector space. A *p-basis of* K is a set $B \subset K$ such that $K = K^p[B]$, and B is minimal such. The size of B (an integer or ∞) is called the *degree of imperfection* of K (also called its *Ershov invariant*). If $|B| = e \in \mathbb{N}$, then $[K : K^p] = p^e$. For more details on p-bases, see [10].

Ershov ([28]) showed that given a prime p and $e \in \mathbb{N} \cup \{\infty\}$, the theory $\mathrm{SCF}_{p,e}$ of separably closed fields of characteristic p and degree of imperfection e is complete (and therefore decidable). He proved it through a model completion result: expand the language by adding n-ary predicates Q_n, $n \in \mathbb{N}$, and add to the theory of fields of characteristic p an axiom saying that $Q_n(x_1, \dots, x_n)$ holds if and only if the elements $x_1, \dots, x_n$ are p-independent. Consider the theory T_e obtained from the theory of fields of characteristic p by adding the defining axioms for the predicates Q_n and the axiom $\forall x_1, \dots, x_{e+1} \neg Q_{e+1}(x_1, \dots, x_{e+1})$ [actually, only adding to the language $Q_1, \dots, Q_{e+1}$ would suffice]. Then the model companion of T_e is precisely $T_e \cup \mathrm{SCF}_{p,e}$.

Assume that K is a separably closed field of characteristic p, and degree of imperfection $e \in \mathbb{N}$. Fix a p-basis $b_1, \dots, b_e$, and let $m_i(b)$, $0 \leq i < p^e$ enumerate the so-called *p-monomials in* b, i.e. the elements $b_1^{i_1} \cdots b_e^{i_e}$ where $0 \leq i_1, \dots, i_e < p$. Observe that because $K = K^p[b_1, \dots, b_e]$, the p-monomials $m_i(b)$ form precisely a basis of the K^p-vector space K. The λ-functions for

$i < p^e$ are then defined uniquely by the equation

$$x = \sum_{i<p^e} \lambda_i(x)^p m_i(b),$$

for $x \in K$. Delon showed in [26] that separably closed fields eliminate quantifiers and imaginaries in this extended language $\{+, -, \cdot, 1, b_1, \ldots, b_e, \lambda_i\ (i < p^e)\}$.

In the case of infinite Ershov invariant, the language can be adapted by fixing for each n an enumeration $m_{i,n}(x)$ of the p-monomials in $x = (x_1, \ldots, x_n)$ and setting

– $\lambda_{i,n}(a; b_1, \ldots, b_n) = 0$ for all $i < p^n$ if $a \notin K^p[b_1, \ldots, b_n]$,
– if $a \in K^p[b_1, \ldots, b_n]$, then the $\lambda_{i,n}$ are defined by $a = \sum_{i<p^n} \lambda_{i,n}(a; b)^p m_{i,n}(b)$.

Delon showed quantifier elimination of the expanded theory of separably closed fields of infinite Ershov invariant, in the language $\{+, -, \cdot, \lambda_{i,n}\ (n \in \mathbb{N}, i < p^n)\}$. No natural language in which separably closed fields of infinite Ershov invariant eliminate imaginaries is known, this has been an open problem for a long time. Note that in the finite invariant case, it was necessary to add to the language new symbols of constants for the elements of a p-basis, but of course one cannot do that in the infinite invariant case.

1.8. Hasse-Schmidt derivations. We fix a field K of characteristic $p > 0$. An *iterative Hasse-Schmidt derivation* on K is a stack of operators $D_i : K \to K$, $i \in \mathbb{N}$, which are additive, and satisfy

– $D_0(x) = x$,
– $D_n(xy) = \sum_{i+j=n} D_i(x)D_j(y)$,
– $D_nD_m(x) = \binom{n+m}{m} D_{n+m}$.

Note that D_0 is the identity map, that D_1 is just an ordinary derivation, that for $i < p$, D_i is a rational multiple of $D_1^{(i)}$ (the i-th iterate of D_1), and that $D_1^{(p)} = 0$. One can also show that D_{p^n} satisfies $D_{p^n}(x^{p^n}) = D_1(x)^{p^n}$ for $n \geq 1$, so that D_{p^n} defines a derivation on K^{p^n}. The theory of fields with a Hasse-Schmidt derivation is axiomatisable (in the language $\{+, -, \cdot, 0, 1, D_n\ (n \in \mathbb{N})\}$), and has a model companion (see Ziegler [102]) which is obtained from the original theory by adjoining to it $\mathrm{SCF}_{p,1}$ together with the axiom expressing strictness (i.e., $\ker D_1 = K^p$). The resulting theory $\mathrm{SCH}_{p,1}$ eliminates imaginaries and quantifiers. Note that we did not have to add constant symbols for the elements of a p-basis to eliminate imaginaries.

The results can be extended to arbitrary finite invariant e, by adding e stacks of commuting operators to get the theory $\mathrm{SCH}_{p,e}$ of e commuting iterative Hasse-Schmidt derivations.
Bi-interpretability with SCF_e.
Let K be a separably closed field of characteristic p and finite degree of imperfection e, fix a p-basis $b_1, \ldots, b_e$ of K. One can then define e Hasse-Schmidt derivations $\mathcal{D}^i$, $i = 1, \ldots, e$ on K by setting:

– $D_0^i = id$ for all i;
– $D_1^i(b_j) = 1$ if $j = i$, 0 otherwise;
– $D_{p^n}^i(b_j) = 0$ for $n \geq 1$ and all i, j.

These conditions, together with the axioms for commuting iterative Hasse-Schmidt derivations, define uniquely the stacks $\mathcal{D}^i$ on K for $i = 1, \dots, e$. Ziegler ([102]) shows that this expansion makes K into a model of $\mathrm{SCH}_{p,e}$, and that the above operators are definable in the language of rings augmented by constants for the elements of the p-basis $\{b_1, \dots, b_e\}$.

Note also that there are many ways of expanding K, but any one of them will make K into a model of $\mathrm{SCH}_{p,e}$. The elimination of quantifiers in the language with the Hasse-Schmidt derivations implies that the λ-functions $\lambda_i(x)$ over a given p-basis b are terms in (x, b) (in the language of $\mathrm{SCH}_{p,e}$).

1.9. Derivations of the Frobenius.
Let K be a field of characteristic $p > 0$, and $q = p^n$ a power of p. A derivation of the q-Frobenius on K is an additive endomorphism of K which satisfies a twisted Leibnitz rule

$$\delta(ab) = a^q\delta(b) + b^q\delta(a).$$

When $q = 1$, this is an ordinary derivation on K. Kowalski shows in [48] that the theory of fields endowed with a derivation of the q-Frobenius has a model companion, which is complete, eliminates quantifiers in the language $\{+, -, \cdot, \delta, \lambda_{i,m}\ m \in \mathbb{N}, i < p^m\}$, is stable not superstable. He gives a geometric axiomatisation of the theory, after introducing the appropriate version of the twisted tangent bundle. Strictness of K is defined as with ordinary differential fields: K is strict iff $\ker\delta = K^p$. It is unknown whether it suffices to add the inverse of the Frobenius map to the language of differential rings to get quantifier elimination. Kowalski's paper leaves open several other questions, some of them quite intriguing.

1.10. Difference fields.
A difference field is a field K with a distinguished endomorphism, usually denoted by σ. Macintyre showed that the theory of difference fields has a model companion, ACFA ([55]). Its models are inversive difference fields, i.e., the endomorphism σ is onto. It admits a geometric axiomatisation: one says that σ is an automorphism of the algebraically closed difference field σ, and one takes the following scheme of axioms:

For every irreducible varieties U and V, with $V \subset U \times U^\sigma$, such that V projects dominantly onto U and U^σ, there is a tuple a such that $(a, \sigma(a)) \in V$.

Here U^σ denotes the variety image of U under σ. We now list some properties of ACFA: its completions are obtained by describing the behaviour of σ on the algebraic closure of the prime field; ACFA is decidable; it does not eliminate quantifiers in the language of difference fields, the reason being that an automorphism of a field E may have non-isomorphic extensions to the algebraic closure of E. Macintyre gives in [55] a language in which ACFA eliminates quantifiers.

Chatzidakis and Hrushovski start an in-depth study of ACFA in [18]. Among other things, they show that all completions of ACFA eliminate imaginaries, are unstable, but are supersimple and of SU-rank ω.

A field which plays an important role is the *fixed field*, $\mathrm{Fix}(\sigma) = \{a \in K \mid \sigma(a) = a\}$. In characteristic $p > 0$, if $q = p^n$ for some $n \in \mathbb{Z}$, we denote by Frob_q the map $x \mapsto x^q$. Then the automorphisms $\tau = \sigma^n \mathrm{Frob}^m$ are also definable ($n \in \mathbb{N}, m \in \mathbb{Z}$), as are their fixed fields. These fields are pseudo-finite, and therefore unstable (they have the independence property); but they are *stably embedded*, i.e., every K-definable subset of a cartesian power of $\mathrm{Fix}(\tau)$ is definable with parameters from $\mathrm{Fix}(\tau)$. One can also show that if K is a model of ACFA, then so is the difference field (K, τ).

1.11. Mixing operators: the commutative case.
We already talked about several commuting derivations. The other examples all involve an endomorphism or automorphism. The difficulty in all these cases is to show that the class of existentially closed models is elementary. Once it is done, results by Chatzidakis and Pillay ([25]) give much information on the behaviour of the models, definable sets, algebraic closure, independence, etc. Usually, the proofs done for ACFA generalise almost instantly. Below we will describe what is known.

Separably closed fields
The first new example is probably the theory of separably closed fields of fixed degree of imperfection (in the language with the λ-functions) with a distinguished endomorphism σ. The model companion SCFA of this theory exists ([15]), is simple but not supersimple nor stable, and eliminates imaginaries if the degree of imperfection is finite. As is the case with ACFA, it does not eliminate quantifiers.

Difference differential fields
The next example was investigated by Bustamante-Medina in [14]. Consider the theory of differential fields of characteristic 0 endowed with an endomorphisms σ. It admits a model companion, denoted DCFA (this was first proved by Hrushovski in the late 90's; for an axiomatisation, see [14]). The theory is supersimple of SU-rank ω^2, eliminates imaginaries. There are three important subfields: the fixed field, the field of constants, and their intersection. Recently, the results of Bustamante-Medina were generalised by León Sánchez, to DCFA_n, the theory of existentially closed models of the theory of fields of characteristic 0 with n commuting derivations and an endomorphism σ of the differential field. It appears in [85], and uses the techniques of [84].

Commuting automorphisms
Hrushovski showed in the 90's that the theory of fields with two commuting automorphisms does not have a model companion. The example appears in a paper of Kikyo ([47]), I will give a slightly different one here. We let $\mathcal{K}$ be the class of all difference fields with commuting automorphisms σ, τ.
We let K be an algebraically closed field containing $\mathbb{Q}$, on which τ is the identity, and let σ be any automorphism of K such that if ω is a primitive cubic root of 1, then $\sigma(\omega) = \omega^2$, and such that K has an element b_n with $\langle\sigma\rangle$-orbit of size exactly n for each $n > 2$, and an element b with infinite $\langle\sigma\rangle$-orbit.

Consider the following formula:

$$\varphi(x, y, z) := y^3 = x + z \wedge \tau(y) = \omega y \wedge \sigma(x) = \tau(x) = x.$$

Consider the field $K(t)$, t an indeterminate, and define $\sigma(t) = \tau(t) = t$. Because the elements $\sigma^i(b)$, $b \in \mathbb{Z}$, are all distinct, the algebraic extensions of K obtained by adjoining to $K(t)$ a cubic root α_i of $t + \sigma^i(b)$ are linearly disjoint over $K(t)$. So, setting L to be the algebraic extension of $K(t)$ generated by the α_i's, $i \in \mathbb{Z}$, and defining $\sigma(\alpha_i) = \alpha_{i+1}$, and $\tau(\alpha_i) = \sigma^i(\omega)\alpha_i$ for all i, the automorphisms σ and τ of L commute, and (t, α_0) is a solution of $\varphi(x, y, b)$.

Assume now that $\varphi(a, \alpha, b_n)$ holds in some difference field in $\mathcal{K}$ extending K. Then $\sigma^n(\alpha) = \omega^j \alpha$ for some j. On the other hand, the fact that σ and τ commute implies

$$\tau\sigma^n(\alpha) = \tau(\omega)^j \tau(\alpha) = \omega^{j+1}\alpha = \sigma^n(\omega\alpha) = \omega^{2^n+j}\alpha,$$

so that necessarily $2^n \equiv 1$ modulo 3, i.e., n is even. So, when n is odd, the formula $\varphi(x, y, b_n)$ has no solution in any extension of K in $\mathcal{K}$. [Note that if n is even, then the construction done for b can be easily adapted to yield a solution of the formula $\varphi(x, y, b_n)$.]

Take a non-principal ultrapower M of K on the set I of odd integers. If $b^* = (b_n)_{n\in I}/\mathcal{U}$, then the $\langle\sigma\rangle$-orbit of b^* is infinite, so that the formula $\varphi(x, y, b^*)$ has a solution in some extension $M' \in \mathcal{K}$ of M. However, any formula $\psi(z)$ satisfied by b^* will be satisfied by infinitely many b_i's, for which we know that the formula $\varphi(x, y, b_i)$ has no solution. This shows that the model companion of the theory of fields with two commuting automorphisms does not exist.

1.12. A remark. Note that this example does not provide an answer to the following two questions:

Let T be a theory which eliminates quantifiers in a language $\mathcal{L}$, consider the class $\mathcal{K}'$ of models of T with a distinguished automorphism σ. The questions are whether the class of existentially closed elements of $\mathcal{K}'$ is an elementary class, when

(i) - T is a complete theory extending ACFA, or

(ii) - T is a complete theory of pseudo-finite field.

The reason is that it is difficult to show that a given system of difference equations has a solution in a member of $\mathcal{K}'$, since the underlying field of this extension must be a model of ACFA or a pseudo-finite field.

For instance, take the difference field $K(t)$ described above. We want to show that the formula $\varphi(x, y, b)$ has a solution in some extension of $K(t)$ in $\mathcal{K}'$. Consider the field $L = K(t, \alpha_i)_{i\in\mathbb{Z}}$.

In case (i), the satisfaction of $\varphi(x, y, b)$ is equivalent to the fact that (L, σ, τ) embeds in a member of $\mathcal{K}'$; hence we need to extend σ and τ to the algebraic closure L^{alg} of L, in such a fashion that σ and τ commute. This seems hard to achieve.

In case (ii), let L_0 be the subfield of L fixed by τ; then σ restricts to an automorphism of L_0, and we consider the difference field (L_0, σ), which we wish to extend to a member of $\mathcal{K}'$. The theory of pseudo-finite fields reduces the problem to the following: extend σ_0 and τ to the algebraic closure L^{alg} of L_0 in such a fashion that σ_0 normalises $\langle\tau\rangle$. Again, this seems hard to achieve.

1.13. A general framework for fields with operators
The following is a particular case of a setup introduced in [66] by Moosa and Scanlon, and which allows to treat some of the above considered cases in a uniform fashion. Consider a $\mathbb{Q}$-algebra D which is finite-dimensional over $\mathbb{Q}$ and has basis $\{\varepsilon_0,\dots,\varepsilon_{\ell-1}\}$ (as a $\mathbb{Q}$-vector space). We assume that the projection π of D onto the first coordinate is a $\mathbb{Q}$-algebra homomorphism. We call ψ the isomorphism $D\to\mathbb{Q}^\ell$ associated to the basis. If R is a $\mathbb{Q}$-algebra, one sets $\mathcal{D}(R)=D\otimes_{\mathbb{Q}}R$.

A $\mathcal{D}$-ring is a $\mathbb{Q}$-algebra R, together with a sequence $\partial=(\partial_1,\dots,\partial_{\ell-1})$ of operators on R, such that the map

$$e:R\to\mathcal{D}(R),\qquad x\mapsto x\varepsilon_0+\partial_1(x)\varepsilon_1+\cdots+\partial_{\ell-1}\varepsilon_{\ell-1}$$

is a homomorphism of $\mathbb{Q}$-algebras.

The class of $\mathcal{D}$-rings is axiomatisable in the language of rings augmented by symbols for the operators. There is a unique $\mathcal{D}$-ring structure on $\mathbb{Q}$, obtained by forcing $1=\varepsilon_0+\partial_1(1)\varepsilon_1+\cdots+\partial_{\ell-1}\varepsilon_{\ell-1}$.

Some examples of $\mathcal{D}$-rings:
1 – Differential rings: Let $\mathcal{D}(R)=R[\eta]/(\eta^2)$, with R-basis $\{1,\eta\}$.
2 – Rings with higher order derivations: let $\mathcal{D}=R[\eta]/(\eta^n)$, with R-basis $\{1,\eta,\dots,\eta^{n-1}\}$. This is a generalisation of ordinary differential rings, since no iterativity condition is imposed.
3 – Difference rings: Let $\mathcal{D}(R)=R\times R$, with R-basis $\{(1,0),(0,1)\}$.
4 – Rings with several (higher order) derivations and endomorphisms: take the tensor product (over R) of various rings as above. The derivations do not commute.
5 – D-rings: $\mathcal{D}(R)=R^2$, with multiplication defined by

$$(x_1,y_1)\cdot(x_2,y_2)=(x_1x_2,x_1y_2+x_2y_1+y_1y_2c).$$

If $c=0$, this is just an ordinary derivation. If $c\neq 0$, the map $\sigma(x)=x+c\partial_1(x)$ defines a ring endomorphism of R, and (because $c\in\mathbb{Q}$) ∂_1 is definable from σ. These structures were considered by Scanlon in [86].
6 – Derivations of an endomorphism σ: $\mathcal{D}(R)=R\times R[\eta]/(\eta^2)$, with R-basis $\{(1,0),(0,1),(1,\eta)\}$. Then the endomorphism σ and the operator δ satisfy $\delta(xy)=\sigma(x)\delta(y)+\sigma(y)\delta(x)$.
7 – Taking fibered products of finite dimensional $\mathbb{Q}$-algebras.

The main result of [66] is that under certain conditions on the algebra $\mathcal{D}$, the theory of $\mathcal{D}$-rings which are integral domains has a model companion, $\mathcal{D}$-CF_0. The axiomatisation is geometric, once one has introduced the appropriate twisted tangent bundle. They then describe completions of the theory, independence, types, and show that they eliminate imaginaries. The theory is simple, and sometimes supersimple, stable, or ω-stable.

2 Stability theoretic notions

Basically the fields we study fall within four (= two times two) classes. The operators considered are variations of derivations or of automorphisms. If some automorphism occurs, the theory of the enriched field will be simple and unstable. If only derivations occur, then it will be stable. One can then refine this analysis. If the underlying field is algebraically closed then we actually get ω-stability in the case of commuting derivations, and supersimplicity when at most one automorphism is present. In the case of an endomorphism of an algebraically closed field, or of an automorphism of a separably closed field, or of several non-commuting automorphisms (of a field or differential field) we only obtain simplicity.

2.1. Algebraic closure, independence.
In all cases where the underlying field is algebraically closed or separably closed, model-theoretic algebraic closure and independence are extremely easy to describe, once one knows algebraic closure and independence in ACF. Basically, to get the model-theoretic algebraic closure, one closes under all obvious definable functions, and then takes the (relative, field-theoretic) algebraic closure. The independence of algebraically closed sets is then the one induced by ACF (or possibly in the case of infinite degree of imperfection, SCF):

$$A \perp_C B \text{ if and only } \mathrm{acl}(AC) \text{ and } \mathrm{acl}(BC) \text{ are independent over } \mathrm{acl}(C).$$

2.2. Zilber's conjecture – the trichotomy.
Let T be a stable theory, M a model of T, and D a strongly minimal definable subset of M. Then acl defines a pregeometry on D, i.e., it satisfies the Steinitz exchange property: if the elements a, b of D and the set $C \subset D$ are such that $a \in \mathrm{acl}(Cb) \setminus \mathrm{acl}(C)$, then $b \in \mathrm{acl}(Ca)$. Quotienting $S = D \setminus (\mathrm{acl}(\emptyset) \cap D)$ by the equivalence relation $a \sim b \iff \mathrm{acl}(a) = \mathrm{acl}(b)$, one obtains a geometry cl on the set $N := S/\sim$, which is of one of the following kinds:
– degenerate or trivial: for any $A \subset N$, $\mathrm{cl}(A) = \bigcup_{a\in A} \mathrm{cl}(\{a\})$;
– modular non-trivial: there are $a_1, \ldots, a_n, b$ with $b \in \mathrm{cl}(a_1, \ldots, a_n) \setminus \bigcup_i \mathrm{cl}(\{a_i\})$; if A, B are closed and have non-empty intersection C, then $A \perp_C B$.
– non-modular: there are closed sets A and B which have non-empty intersection C and are not independent over C.

Zilber's conjecture stated that any geometry originating from a strongly minimal set, coincides with the geometry of one of the following structures:

– (Trivial) An infinite set with no-structure.
– (Modular non-trivial): A vector space $(V, +, -, 0, \alpha)_{\alpha\in F}$ over a fixed division ring F.
– (Non-modular) An algebraically closed field $(F, +, \cdot, c)_{c\in F_0}$ with constants for a subfield F_0.

The conjecture was disproved by Hrushovski in [38] (see also [37]). Let us say that a (stable) structure M *satisfies the trichotomy* if whenever D is a strongly minimal set definable in M, then the pregeometry associated to D satisfies Zilber's conjecture. It turns out that many of the stable fields with operators do satisfy the trichotomy: this was proved by Hrushovski and

Sokolovich for DCF_0 ([45]); to speak about the other examples we first need a discussion.

The concept of satisfying the trichotomy can be generalised in two fashions. First, in the stable case, one can allow ∞-definable sets, as long as they are of U-rank 1, i.e., as long as acl defines a pregeometry. Then the trichotomy is stated as above.
In the unstable case, some care needs to be taken, and the trichotomy becomes a *dichotomy* between the first two cases (*modular case*) on one hand, and the third case (*field case*) on the other. We still work with a definable (or ∞-definable) set D, on which acl defines a pregeometry, but which is not necessarily strongly minimal. We also assume that there is a good independence notion - e.g., that the theory is simple. We say that D is *modular* iff whenever a, b are tuples of elements of D, then they are independent over the intersection of their algebraic closures in D^{eq}. The dichotomy then states:

If D is not modular, then some type p of rank 1 which is realised in D is non-orthogonal to a field.

Recall that a type p of rank 1 over a set A of parameters is non-orthogonal to an ∞-definable set S iff there is some B containing A, a tuple c in S such that $c \downarrow_A B$, and $\mathrm{acl}(Bc)$ contains a realisation a of p which is not in $\mathrm{acl}(B)$.

2.3. Two generalisations of modularity.
1 - The definition of modularity we gave generalises easily to types of arbitrary rank, and in that case it is called *one-basedness*: a (∞-)A-definable set S is *one-based* iff whenever $B \supset A$ and a is a tuple of elements of S, then a and B are independent over the intersection of their algebraic closures in M^{eq}.
Recall that a type p over A is *regular* iff it is orthogonal to any of its forking extensions: if C is a set, a and b are realisations of p, such that $a \downarrow_A C$ and $b \not\downarrow_A C$, then $a \downarrow_C b$. Forking allows to define a pregeometry on the set of realisations of p, i.e., one replaces acl by "forking". A regular type is called modular iff its associated geometry is modular; and non-modular otherwise.

2.4. Properties of one-basedness. The defining property of one-basedness is, in my opinion, quite remarkable. It is extremely desirable in case of a group structure, because of the following

Theorem (Hrushovski-Pillay, [43]) *Let G be a one-based $\mathcal{L}$-structure, which is stable and of finite U-rank, and assume that it is a group (maybe with additional structure). Then for every n, every definable subset of G^n is a finite Boolean combination of cosets of definable subgroups of G^n. Furthermore, these subgroups are defined over* $\mathrm{acl}(\emptyset)$, *and in particular there are at most* $|\mathcal{L}| + \aleph_0$ *many of them.*

Note a consequence of this result: If G satisfies the assumptions of the theorem, then G has a subgroup G_0 of finite index which is commutative. This comes from looking at the graph of multiplication in G^3, or at the graph of the inverse map.

2.5. Some other structures which satisfy the dichotomy.
Separably closed fields of finite degree of imperfection.
If K is a saturated separably closed field, then no definable subset of K is ranked by the U-rank. But there are many ∞-definable sets D which have U-rank 1, for instant the perfect core

$k = \bigcap_n K^{p^n}$ of K. Note that the field k is algebraically closed. Moreover, the induced structure on k is that of a pure field: if $S \subset K^n$ is definable in K, then $S \cap k^n$ is definable in the pure field k. Assume that the degree of imperfection of K is finite. Hrushovski showed in [39] that non-modular *thin*[3] types of U-rank 1 are non-orthogonal to the field k; this result was later extended by Delon in [27] to all U-rank 1 non-modular types.

These results extend to fields with commuting iterative Hasse-Schmidt derivations. I do not know if they extend to any of the following characteristic p fields: separably closed fields of infinite degree of imperfection, differentially closed fields of positive characteristic, or to existentially closed fields with a derivation of the q-Frobenius.

Differentially closed fields with several commuting derivations.
Moosa, Pillay and Scanlon study in [67] the modularity of regular types in $\mathrm{DCF}_{n,0}$. They show that a regular non-modular type is non-orthogonal to the generic of a definable subgroup of $\mathbb{G}_a$. This has for consequence the dichotomy for types of rank 1: a type of rank 1 which is non-modular, is non-orthogonal to the generic type of the field of absolute constants, defined by $D_1x = D_2x = \cdots D_nx = 0$.

The hope is that the result extends to regular types of higher rank: that regular non-modular types are non-orthogonal to a field of constants, i.e., a subfield defined by a conjunction of conditions of the form $\sum a_iD_i(x) = 0$. A partial result in that direction was obtained by Süer ([94]).

Difference fields.
Chatzidakis and Hrushovski, later joined by Peterzil, show in [18], [24] that models of ACFA satisfy the dichotomy: a type of SU-rank 1 which is not modular, is non-orthogonal to a fixed field, i.e., to $\mathrm{Fix}(\sigma)$ if the characteristic is 0; or to some $\mathrm{Fix}(\sigma^n\mathrm{Frob}^m)$ if the characteristic is positive.

Because the theory ACFA is unstable, other issues connected to stability or instability appear. Most types are not stationary. In characteristic 0, a type which is foreign to $\mathrm{Fix}(\sigma)$ will be stable stably embedded. So, the unique source of unstability is $\mathrm{Fix}(\sigma)$. In positive characteristic this is not the case: there are modular subgroups of $\mathbb{G}_a$ which are unstable.

So, in characteristic 0, if G is a definable group which is one-based, then the result of Hrushovski-Pillay on stable one-based groups of finite U-rank will apply, and give that any definable subset of G^n is a Boolean combination of cosets of definable subgroups. In characteristic $p > 0$, we cannot apply the result, however, in case G is a definable subgroup of an algebraic group H (so that the group law is quantifier-free definable), then every quantifier-free definable subset of G^n is a finite Boolean combination of cosets of quantifier-free definable subgroups of G^n. This partial result suffices for most applications. If H is a semi-abelian variety, then G is also stable stably embedded, and the full result holds, see [23].

[3] i.e., those such that given a realisation a over a model K, the field $\mathrm{dcl}(Ka)$ has finite transcendence degree over K.

$\mathcal{D}$-fields.
Moosa and Scanlon show in [66] that the dichotomy theorem holds for finite-dimensional types of SU-rank 1, i.e., types over a set A of tuples a such that the transcendence degree over $\mathrm{acl}(A)$ of $\mathrm{acl}(Aa)$ is finite. This brings me to another remark.

2.6. Some words on proofs of the dichotomy. With the exception of ACFA in characteristic 0, the early results of the dichotomy for theories of enriched fields[4] used a very technical tool, Zariski geometries. Zariski geometries were introduced by Hrushovski and Zilber in [46] and provide a set of axioms for a geometry on a structure, which, if satisfied, imply that the structure satisfies the trichotomy. The original setting is intended for strongly minimal sets. In [39] a version for sets of realisations of minimal types appears, see also [57] and [103]. This was unsatisfactory in many respects, as checking the axioms usually is very long and involved.

A new way of showing the dichotomy: the CBP
Inspired by results of Fujiki and Campana on compact complex manifolds, Pillay introduced in [73] a property of types of canonical bases of types of finite rank, and which implies the dichotomy. Later, Pillay and Ziegler ([75]) somewhat generalised this property to a wider context. This property is called the *Canonical Base Property*, *CBP* for short. Pillay and Ziegler then proceed to show the CBP for certain algebraic examples, in a very algebraic and elementary (and ingenious) fashion. They show it for the following theories and families of types:

– In characteristic 0, types of finite rank in the theories DCF_0 and ACFA.
– In separably closed fields of finite invariant, the very thin types[5] have the CBP.

By the above mentioned result of Pillay, their result gives in particular a direct proof of the dichotomy for DCF_0 and ACFA_0. Their proof can be (and was) adapted to the so-called finite-dimensional types of DCFA and of $\mathrm{DCF}_{0,n}$, $\mathcal{D}$-CF_0, DCFA_n, see [14] and [66], [85]. In the case of $\mathrm{DCF}_{0,n}$, and later of DCFA, the method of arcspaces introduced in [67] allows to show the full dichotomy. It is at the moment still open for arbitrary rank 1 types in $\mathcal{D}$-CF_0.

Let me also mention that the dichotomy holds for types of transformally algebraic elements in SCFA and in SCFE: one reduces the problem to the case of ACFA. Here SCFE denotes the model companion of the theory of difference fields with σ not surjective. It can also be described as the limit theory of the structures $(\mathbb{F}_p(t)^s, x \mapsto x^{p^n})$ when p^n goes to ∞.

2.7. A few words on the CBP. The CBP has other interesting consequences, which can be found in [75], in [96], in [63] and in [16]. It is still open for minimal types in SCF_e, and more generally in most fields of positive characteristic (maybe with additional structure) which are not algebraically closed. Chatzidakis shows in [16] that types of finite rank in existentially closed difference fields of characteristic $p > 0$ have the CBP.

[4]that is: for DCF_0 in [45], for minimal types in separably closed fields of finite degree of imperfection [27], for ACFA in positive characteristic [24].

[5]i.e., those such that given a realisation a over a model K, the structure $K\langle a\rangle$ obtained by closing $K(a)$ under the p-th powers of the λ-functions, has a finite separating transcendence basis over K.

2.8. Definable groups. A classical result says that a group definable in an algebraically closed field K is isomorphic to $G(K)$ for some algebraic group G.
The intended generalisation to a field with operators K is:
$(*)$ *If H is definable in K, then there is a group homomorphism $f : H' \to G(K)$, where H' is a definable subgroup of H of finite index, G is an algebraic group defined over K, and* $\ker(f)$ *is finite.*

This generalisation holds for most theories we described above (and maybe all). They appear in papers by Messmer, and by Bouscaren and Delon on separably closed fields of finite invariant ([60], [11]), by Benoist [5] on fields with Hasse-Schmidt derivations, by Pillay on differential groups ([71], [74]), by Kowalski and Pillay ([51], [52], [53], [54]), by Bustamante-Medina [14], Freitag [29], ... (to name a few).

In a recent paper [7], Blossier, Martin-Pizarro and Wagner introduce the notion of a theory being one-based over another, and describe definable groups in models of the first one as almost embeddable in definable groups in the smaller language of the second one. An application of their result gives then that $(*)$ holds in all the theories we have so far considered, provided that
– the theory eliminates imaginaries,
– independence is given by ACF-independence of the algebraic closures in the sense of the theory.

Thus it seems to me that among the theories I introduced above, the only theories which do not satisfy the hypotheses of the result of Blossier, Martin-Pizarro and Wagner, are those of fields of positive characteristic and which have infinite degree of imperfection.

One should note that this is not quite the optimal result in case of differential fields, since there $(*)$ can be strengthened to conclude that f is injective.

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