Introduction to Model theory

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These notes aim at giving the basic definitions and results from model theory. My intention in writing them, is that they should provide the reader with many examples, even with some proofs, and contain most of the definitions. Differences in vocabulary can be quite an obstacle to mutual understanding, and having the terminology written up somewhere should prove helpful. An index is included at the end.

People interested in reading more should consult standard model theory books. For instance: C.C. Chang, H.J. Keisler, *Model Theory*, North-Holland Publishing Company, Amsterdam 1973; W. Hodges, *A shorter model theory*, Cambridge University Press, 1997.

The notes are organised as follows. Chapter 1 gives the very basic definitions of languages, structures and satisfaction. Chapter 2 gives more definitions and some important theorems, such as the compactness theorem. Section 3 introduces ultraproducts and their properties. In section 4, we give a short proof of the decidability of the theory of the field of real numbers.

1. Languages, structures, satisfaction

(1.1) **Languages**. A language is a collection $\mathcal{L}$, finite or infinite, of symbols. These symbols are of three kinds:

– *function* symbols

– *relation* symbols

– *constant* symbols

To each function symbol f is associated a number $n(f) \in \mathbb{N}^{>0}$, and to each relation symbol R a number $n(R) \in \mathbb{N}$. The numbers $n(f)$ and $n(R)$ are called the *arities* of the function f, resp., the relation R.

(1.2) **$\mathcal{L}$-structures**. We fix a language $\mathcal{L} = \{f_i, R_j, c_k \mid i \in I, j \in J, k \in K\}$, where the f_i's are function symbols, the R_j's are relation symbols, and the c_k's are constant symbols.

An *$\mathcal{L}$-structure* $\mathcal{M}$ is then given by

– A set M, called the *universe of* $\mathcal{M}$,

– For each function symbol $f \in \mathcal{L}$, a function $f^{\mathcal{M}} : M^{n(f)} \to M$, called the *interpretation of f in* $\mathcal{M}$,

– For each relation symbol $R \in \mathcal{M}$, a subset $R^{\mathcal{M}}$ of $M^{n(R)}$, called the *interpretation of R in* $\mathcal{M}$,

– For each constant symbol $c \in \mathcal{L}$, an element $c^{\mathcal{M}} \in M$, called the *interpretation of c in* $\mathcal{M}$.

The structure $\mathcal{M}$ is then denoted by

$$\mathcal{M} = (M, f_i^{\mathcal{M}}, R_j^{\mathcal{M}}, c_k^{\mathcal{M}} \mid i \in I, j \in J, k \in K).$$

In fact, most of the time the superscript $\mathcal{M}$ disappears, and the structure and its universe are denoted by the same letter. This is when **no confusion is possible**, for instance when there is only one type of structure on $\mathcal{M}$.

(1.3) **Substructures**. Let M be an $\mathcal{L}$-structure. An $\mathcal{L}$-*substructure* of M, or simply a substructure of M if not confusion is likely, is an $\mathcal{L}$-structure N, with universe contained in the universe of M, and such that the interpretations of the symbols of $\mathcal{L}$ in N are restrictions of the interpretation of these symbols in M, i.e.:

– If f is a function symbol of $\mathcal{L}$, then the interpretation of f in N is the restriction of f^M to $N^{n(f)}$,

– If R is a relation symbol of $\mathcal{L}$, then $R^N = R^M \cap N^{n(R)}$,

– If c is a constant symbol of $\mathcal{L}$, then $c^M = c^N$.

Hence a subset of M is the universe of a substructure of M if and only if it contains all the (elements interpreting the) constants of $\mathcal{L}$, and is closed under the (interpretation in M of the) functions of $\mathcal{L}$. Note that **if the language has no constant symbol**, then the empty set is the universe of a substructure of M.

(1.4) **Examples of languages, structures, and substructures**. The concrete structures considered in model theory all come from standard algebraic examples, and so the examples given below will be very familiar to you.

Example 1 - The language of groups. The language of groups, $\mathcal{L}_G$, is the language $\{\cdot, {}^{-1}, 1\}$, where $\cdot$ is a 2-ary function symbol, ${}^{-1}$ is a unary function symbol, and 1 is a constant symbol.

Any group G has a natural $\mathcal{L}_G$-structure, obtained by interpreting $\cdot$ as the group multiplication, ${}^{-1}$ as the group inverse, and 1 as the unit element of the group.

A substructure of the group G is then a subset containing 1, closed under multiplication and inverse: it is simply a subgroup of G.

This is a good place to remark that the notion of substructure is sensitive to the language. While the inverse function and the identity element of the group G are retrievable (definable) from the group multiplication of G, the notion of "substructure" heavily depends on them. For instance, a $\{\cdot, e\}$-substructure of G is simply a submonoid of G containing e, while a $\{\cdot\}$-substructure of G can be empty.

Example 2 - The language of graphs. The language consists of a binary relation symbol, E. Graphs which have at most one edge between two vertices are the $\{E\}$-structures: simply interpret $E(x, y)$ if and only there is an edge going from x to y. Graphs in which there can be several edges between two vertices need a more sophisticated language, see below in (2.15).

Example 3 - The language of rings. The language of rings, $\mathcal{L}_R$, is the language $\{+, -, \cdot, 0, 1\}$, where $+, -$ and $\cdot$ are binary functions, 0 and 1 are constants.

A (unitary) ring S has a natural $\mathcal{L}_R$-structure, obtained by interpreting $+, -, \cdot$ as the usual ring operations of addition, subtraction and multiplication, 0 as the identity element of $+$, and 1 as the unit element of S.

A substructure of the $\mathcal{L}_R$-structure S is then simply a subring of S. Note that it will in particular contain the subring of S generated by 1, i.e., a copy of $\mathbb{Z}$ or of $\mathbb{Z}/p\mathbb{Z}$.

When one deals with fields, it is sometimes convenient to add a symbol for the multiplicative inverse (denoted ${}^{-1}$). By convention $0^{-1} = 0$.

Example 4 - The language of ordered groups, of ordered rings.

One simply adds to $\mathcal{L}_G$, resp. $\mathcal{L}_R$, a binary relation symbol, $\leq$.

(1.5) **Morphisms, embeddings, isomorphisms, automorphisms**. Let M and N be two $\mathcal{L}$-structures. A map $s : M \to N$ is an ($\mathcal{L}$)-*morphism* if for all relation symbol $R \in \mathcal{L}$, function symbol $f \in \mathcal{L}$, and tuples $\bar{a}, \bar{b}$ in M, we have:

$$\text{if } \bar{a} \in R, \text{ then } s(\bar{a}) \in R; \qquad s(f(\bar{b})) = f(s(\bar{b})).$$

An *embedding* is an injective morphism $s : M \to N$, which satisfies in addition for all relation $R \in \mathcal{L}$ and tuple $\bar{a}$ in M, that

$$\bar{a} \in R \iff s(\bar{a}) \in R.$$

An *isomorphism* between M and N is a bijective morphism, whose inverse is also a morphism. Finally, an *automorphism* of M is an isomorphism $M \to M$.

(1.6) **Terms**. We can start using the symbols of $\mathcal{L}$ to express properties of a given $\mathcal{L}$-structure. In addition to the symbols of $\mathcal{L}$, we will consider a set of symbols (which we suppose disjoint from $\mathcal{L}$), called the *set of logical symbols.* It consists of
– logical connectives $\wedge$, $\vee$, $\neg$, and sometimes also (for convenience) $\to$ and $\leftrightarrow$,
– parentheses (and),
– a (binary relation) symbol $=$ for equality,
– infinitely many variable symbols, usually denoted x, y, x_i, etc ...
– the quantifiers $\forall$ (for all) and $\exists$ (there exists).

Fix a language $\mathcal{L}$. An $\mathcal{L}$-formula will then be a string of symbols from $\mathcal{L}$ and logical symbols, obeying certain rules. We start by defining *$\mathcal{L}$-terms* (or simply, terms). Roughly speaking, terms are expressions obtained from constants and variables by applying functions. In any $\mathcal{L}$-structure M, a term t will then define uniquely a function from a certain cartesian power of M to M. Terms are defined by induction, as follows:
– a variable x, or a constant c, are terms.
– if $t_1, \dots, t_n$ are terms, and f is an n-ary function, then $f(t_1, \dots, t_n)$ is a term.

Given a term $t(x_1, \dots, x_m)$, the notation indicating that the variables occurring in t are among $x_1, \dots, x_m$, and an $\mathcal{L}$-structure M, we get a function $F_t : M^m \to M$. Again this function is defined by induction on the complexity of the term:
– if c is a a constant symbol, then $F_c : M^0 \to M$ is the function $\emptyset \mapsto c^M$,
– if x is a variable, then $F_x : M \to M$ is the identity,
– if $t_1, \dots, t_n$ are terms in the variables $x_1, \dots, x_m$ and f is an n-ary function symbol, then $F_{f(t_1,\dots,t_n)} : (x_1, \dots, x_m) \mapsto f(F_{t_1}(\bar{x}), \dots, F_{t_n}(\bar{x}))$ ($\bar{x} = (x_1, \dots, x_m)$).

(1.7) **Formulas**. We are now ready to define formulas. Again they are defined by induction.

An *atomic formula* is a formula of the form $t_1(\bar{x}) = t_2(\bar{x})$ or $R(t_1(\bar{x}), \dots, t_n(\bar{x}))$, where $\bar{x} = (x_1, \dots, x_m)$ is a tuple of variables, $t_1, \dots, t_n$ are terms (of the language $\mathcal{L}$, in the variables $\bar{x}$), and R is an n-ary relation symbol of $\mathcal{L}$.

The set of *quantifier-free formulas* is the set of *Boolean combinations* of atomic formulas, i.e., is the closure of the set of atomic formulas under the operations of $\wedge$ (and), $\vee$ (or) and

$\neg$ (negation, or not). So, if $\varphi_1(\bar{x})$, $\varphi_2(\bar{x})$ are quantifier-free formulas, so are $(\varphi_1(\bar{x})\wedge\varphi_2(\bar{x}))$, $(\varphi_1(\bar{x})\vee\varphi_2(\bar{x}))$, and $(\neg\varphi_1(\bar{x}))$.

One often uses $(\varphi_1 \to \varphi_2)$ as an abbreviation for $((\neg\varphi_1)\vee\varphi_2)$, and $(\varphi_1 \leftrightarrow \varphi_2)$ as an abbreviation for $((\varphi_1 \to \varphi_2)\wedge(\varphi_2\to\varphi_1))$.

A *formula* ψ is then a string of symbols of the form

$$Q_1x_1Q_2x_2\ldots Q_mx_m\,\varphi(x_1,\ldots,x_n) \tag{1}$$

where $\varphi(\bar{x})$ is a quantifier-free formula, with variables among $\bar{x}=(x_1,\ldots,x_n)$, and $Q_1,\ldots,Q_m$ are quantifiers, i.e., belong to $\{\forall,\exists\}$. We may assume $m\le n$.

Important: the variables $x_1,\ldots,x_n$ are supposed distinct: $\forall x_1\exists x_1\ldots$ is not allowed. If $m\le n$, the variables $x_{m+1},\ldots,x_n$ are called the *free variables of the formula* ψ. One usually writes $\psi(x_{m+1},\ldots,x_n)$ to indicate that the free variables of ψ are among $(x_{m+1},\ldots,x_n)$. The variables $x_1,\ldots,x_m$ are called the *bound variables of* ψ. If $n=m$, then ψ has no free variables and is called a *sentence*.

If all quantifiers $Q_1,\ldots,Q_m$ are $\exists$, then ψ is called an *existential formula*; if they are all $\forall$, then ψ is called a *universal formula*. One can define a hierarchy of complexity of formulas, by counting the number of alternances of quantifiers in the string $Q_1,\ldots,Q_n$. Let us simply say that a Π_2*-formula*, also called a $\forall\exists$*-formula*, is one in which $Q_1\ldots Q_n$ is a block of $\forall$ followed by a block of $\exists$, that a Σ_2*-formula*, also called a $\exists\forall$*-formula*, is one in which $Q_1\ldots Q_n$ is a block of $\exists$ followed by a block of $\forall$. In these definitions, either block is allowed to be empty, so that an existential formula is both a Π_2 and a Σ_2-formula. Let us also mention that a *positive formula* is one of the form $Q_1x_1\ldots Q_mx_m\varphi(x_1,\ldots,x_n)$, where $\varphi(\bar{x})$ is a finite disjunction of finite conjunctions of **atomic** formulas.

(1.8) **Warning**. I lied, this is not the usual definition of a formula. A formula as in (1) is said to be in *prenex form*. The set of formulas in prenex form is **not closed** under Boolean operations. One has however a notion of "*logical equivalence*", under which for instance the formulas $Q_1x_1Q_2x_2\ldots Q_mx_m\,\varphi(x_1,\ldots,x_m,x_{m+1},\ldots,x_n)$ and $Q_1y_1Q_2y_2\ldots Q_my_m\,\varphi(y_1,\ldots,y_m,x_{m+1},\ldots,x_n)$ are logically equivalent. Then it is quite easy to see that a Boolean combination of formulas in prenex form is logically equivalent to a formula in prenex form. E.g,

$$(Q_1x_1\ldots Q_mx_m\,\varphi_1(x_1,\ldots,x_n))\wedge(Q'_1x_1\ldots Q'_mx_m\,\varphi_2(x_1,\ldots,x_n))$$

is logically equivalent to

$$Q_1x_1Q'_1y_1\ldots Q_mx_mQ'_my_m(\varphi_1(x_1,\ldots,x_n)\wedge\varphi_2(y_1,\ldots,y_m,x_{m+1},\ldots,x_n)).$$

If one wants to be economical about the number of quantifiers, one notes that in general $\forall x\,\varphi_1(x,\ldots)\wedge\forall x\,\varphi_2(x,\ldots)$ is logically equivalent to $\forall x\,(\varphi_1(x,\ldots)\wedge\varphi_2(x,\ldots))$, and $\exists x\,\varphi_1(x,\ldots)\vee\exists x\,\varphi_2(x,\ldots)$ is logically equivalent to $\exists x\,(\varphi_1(x,\ldots)\vee\varphi_2(x,\ldots))$. For negations, one uses the logical equivalence of $\neg(Q_1x_1\ldots Q_mx_m\,\varphi(x_1,\ldots,x_n))$ with $Q'_1x_1\ldots Q'_mx_m\,\neg(\varphi(x_1,\ldots,x_n))$, where $Q'_i=\exists$ if $Q_i=\forall$, $Q'_i=\forall$ if $Q_i=\exists$. Thus the negation of a Π_2-formula is a Σ_2-formula, etc.

Logical equivalence can also be used to rewrite Boolean combinations, and one can show that any quantifier-free formula $\varphi(\bar{x})$ is logically equivalent to one of the form $\bigvee_i \bigwedge_j \varphi_{i,j}(\bar{x})$, where the $\varphi_{i,j}$ are atomic formulas or negations of atomic formulas.

(1.9) **Comments and examples**. The definitions given above are completely formal. When considering concrete examples, they get very much simplified, to agree with current usage. The first thing to note is that the formula $(\neg(x = y)$ is abbreviated by $x \neq y$.

Example 1. $\mathcal{L}_G = \{\cdot, {}^{-1}, 1\}$. A term is built up from 1, $\cdot$, ${}^{-1}$. and some variables. E.g., $\cdot(1, {}^{-1}(\cdot(x_1, {}^{-1}(x_1))))$ is a term, in the variable x_1. If we work in an arbitrary $\mathcal{L}_G$-structure, i.e., not necessarily a group, this expression cannot be simplified. If we work in a group, then we will first of all switch to the usual notation of xy instead of $\cdot(x, y)$ and x^{-1} instead of ${}^{-1}(x)$; then allow ourselves to use the associativity of the group law to get rid of extraneous parentheses. The term above then becomes $1(x_1x_1^{-1})^{-1}$, which can be further simplified to 1 (now using the defining properties of ${}^{-1}$ and of 1). From now on, we will assume that our $\mathcal{L}_G$-structures are **groups**.

A term in the variables $x_1, \ldots, x_n$ is then simply a word in the symbols $x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}$. One can do a further reduction: replace occurrences of the formula $w_1(\bar{x}) = w_2(\bar{x})$ by $w_1(\bar{x})(w_2(\bar{x}))^{-1} = 1$. An atomic formula will then be a finite disjunction of finite conjunctions of equations $w(\bar{x}) = 1$ and inequations $w(\bar{x}) \neq 1$. The following formula is a Π_2-formula:

$$\forall \bar{x}\, \exists \bar{y}\, (w_1(\bar{x}, \bar{y}, \bar{z}) = 1 \wedge w_2(\bar{x}, \bar{z}) \neq 1),$$

where $w_1(\bar{x}, \bar{y}, \bar{z})$ is a word in the elements of the tuple $(\bar{x}, \bar{y}, \bar{z})$ and their inverses, and $w_2(\bar{x}, \bar{z})$ is a word in the elements of $(\bar{x}, \bar{z})$ and their inverses. The free variables of this formula are the elements of the tuple $\bar{z}$, while the elements of $(\bar{x}, \bar{y})$ are the bound variables of the formula.

In case all structures considered are free groups, containing two non-commuting elements a, b, then a quantifier-free formula can be written as finite disjunction of formulas of the form

$$w(\bar{x}, a, b) = 1 \wedge w'(\bar{x}, a, b) \neq 1$$

for some words w, w'.

Example 2. The language of graphs $\{E\}$. The only terms are variables (since there are no function or constant symbols). Thus an atomic formula is of the form $E(x, y)$ or $(x = y)$. An example of formula in this language is e.g.,

$$\exists y_1, \ldots, y_m (\bigwedge_{i=1}^{m-1} E(y_i, y_{i+1}) \wedge E(x_1, y_1) \wedge E(y_n, x_2) \wedge \bigwedge_{1 \leq i < j \leq m} y_i \neq y_j).$$

Note the use of $x \neq y$ instead of $\neg(x = y)$. The free variables of this formula are x_1, x_2.

Example 3. $\mathcal{L}_R = \{+, -, \cdot, 0, 1\}$. Again, terms as defined formally, are extremely ugly. But, in case all $\mathcal{L}_R$-structures considered are rings, they can be rewritten in a more natural fashion. From now on, **all $\mathcal{L}_R$-structures are commutative rings**.

If $n \in \mathbb{N}^{>1}$ the term $1 + 1 + \cdots + 1$ (n times) will simply be denoted by n. Similarly $x + x + \cdots + x$ (n times) is denoted by nx, and $x \cdot \ldots \cdot x$ (n times) by x^n. An arbitrary term will then be of the form $f(x_1, \ldots, x_n)$, where $f(X_1, \ldots, X_n) \in \mathbb{Z}[X_1, \ldots, X_n]$.

Quantifier-free formulas are finite disjunctions of finite conjunctions of equations and inequations. Thus, in the ring $\mathbb{C}$, they will define the usual *constructible sets*.

If one adds $\leq$ to the language, and assumes that our structures are **ordered rings**, then quantifier-free formulas can be rewritten as finite conjunctions of finite disjunctions of formulas of the form

$$f(\bar{x}) = 0, \quad g(\bar{x}) > 0, \tag{2}$$

where f, g are polynomials over $\mathbb{Z}$. Here, $x < y$ stands for $x \leq y \wedge \neg(x \neq y)$, and one uses the equivalences $x \neq 0 \iff x < 0 \vee x > 0$, $x > 0 \iff (-x) < 0$. If M is an ordered ring, then M-quantifier-free formulas will be as above, except that f and g are polynomials over M. In case M is the ordered field $\mathbb{R}$, one gets the usual semi-algebraic sets.

(1.10) **Satisfaction**. Let M be an $\mathcal{L}$-structure, $\varphi(\bar{x})$ an $\mathcal{L}$-formula, where $\bar{x} = (x_1, \ldots, x_n)$ is a tuple of variables occurring freely in φ, and $\bar{a} = (a_1, \ldots, a_n)$ an n-tuple of elements of M. We wish to define the notion M *satisfies* $\varphi(\bar{a})$, (or $\bar{a}$ *satisfies* φ *in* M, or $\varphi(\bar{a})$ *holds in* M, *is true in* M), denoted by

$$M \models \varphi(\bar{a}).$$

(The negation of $M \models \varphi(\bar{a})$ is denoted by $M \not\models \varphi(\bar{a})$.) This is done by induction on the complexity of the formulas:

– If $\varphi(\bar{x})$ is the formula $t_1(\bar{x}) = t_2(\bar{x})$, where t_1, t_2 are $\mathcal{L}$-terms in the variable $\bar{x}$, then

$$M \models t_1(\bar{a}) = t_2(\bar{a}) \text{ if and only if } F_{t_1}(\bar{a}) = F_{t_2}(\bar{a}).$$

– If $\varphi(\bar{x})$ is the formula $R(t_1(\bar{x}), \ldots, t_m(\bar{x}))$, where $t_1, \ldots, t_m$ are terms and R is an m-ary relation, then

$$M \models R(t_1(\bar{a}), \ldots, t_m(\bar{a})) \text{ if and only if } (F_{t_1}(\bar{a}), \ldots, F_{t_m}(\bar{a})) \in R^M.$$

– If $\varphi(\bar{x}) = \varphi_1(\bar{x}) \vee \varphi_2(\bar{x})$, then

$$M \models \varphi(\bar{a}) \text{ if and only if } M \models \varphi_1(\bar{a}) \textbf{ or } M \models \varphi_2(\bar{a}).$$

– If $\varphi(\bar{x}) = \varphi_1(\bar{x}) \wedge \varphi_2(\bar{x})$, then

$$M \models \varphi(\bar{a}) \text{ if and only if } M \models \varphi_1(\bar{a}) \textbf{ and } M \models \varphi_2(\bar{a}).$$

– If $\varphi(\bar{x}) = \neg\varphi_1(\bar{x})$, then

$$M \models \varphi(\bar{a}) \text{ if and only if } M \not\models \varphi_1(\bar{a}).$$

– If $\varphi(\bar{x}) = \exists y\, \psi(\bar{x}, y)$, where the free variables of ψ are among $\bar{x}, y$, then

$$M \models \varphi(\bar{a}) \text{ if and only if there is } b \in M \text{ such that } M \models \psi(\bar{a}, b).$$

– if $\varphi(\bar{x}) = \forall y\, \psi(\bar{x}, y)$, then

$$\begin{aligned} M \models \varphi(\bar{a}) &\text{ if and only if } M \models \neg(\exists y\, \neg\psi(\bar{a}, y)) \\ &\text{ if and only if for all } b \text{ in } M, \quad M \models \varphi(\bar{a}, b). \end{aligned}$$

(1.11) **Parameters, definable sets**. Let M be an $\mathcal{L}$-structure, $\varphi(\bar{x}, \bar{y})$ a formula ($\bar{x}$ an n-tuple of variables, $\bar{y}$ an m-tuple of variables), and $\bar{a} \in M^n$. Then the set $\{\bar{b} \in M^m \mid M \models \varphi(\bar{a}, \bar{b})\}$ is called a *definable set*. We also say that it is *defined over* $\bar{a}$ by the formula $\varphi(\bar{a}, \bar{y})$, or that it is $\bar{a}$*-definable*. The tuple $\bar{a}$ is a *parameter of the formula* $\varphi(\bar{a}, \bar{y})$.

Let M be an $\mathcal{L}$-structure. The set of M-definable subsets of M^n is clearly closed under unions, intersections and complements (corresponding to the use of the logical connectives $\vee$, $\wedge$ and $\neg$). If $S \subseteq M^{n+1}$ is defined by the formula $\varphi(\bar{x}, \bar{a})$, $\bar{x} = (x_1, \ldots, x_{n+1})$, and $\pi : M^{n+1} \to M$ is the projection on the first n coordinates, then $\pi(S)$ is defined by the formula $\exists x_{n+1}\, \varphi(\bar{x}, \bar{a})$, and the complement of $\pi(S)$ by the formula $\forall x_{n+1}\, \neg\varphi(\bar{x})$.

(1.12) **The examples, revisited**. (1) $\mathcal{L}_G = \{\cdot, {}^{-1}, 1)$. Consider the formula $\varphi(x, y)$: $xy = yx$. Let G be a group (endowed with its natural $\mathcal{L}_G$-structure), and $g \in G$. Then the formula $\varphi(x, g)$ defines in G the centraliser of g in G, while the formula $\psi(y) = \forall x\, \varphi(x, y)$ will define the centre of G. The sentence $\forall x, y\, (xy = yx)$ will only be satisfied if G is abelian.

Other examples of definable subsets of a group G are: The conjugacy class of an element g (by $\exists y\; y^{-1}gy = x$); the set of commutators ($\exists y, z\, (y^{-1}z^{-1}yz = x)$); the set of squares ($\exists y\, (y^2 = x)$), or more generaly of n-th powers ($\exists y\, y^n = x$); the set of elements of order $\leq n$ ($x^n = 1$).

Are usually **not** definable: the commutator subgroup; the set of torsion elements; the subgroup generated by the squares.

(2) $\mathcal{L} = \{E\}$. The formula $\varphi(x_1, x_2) = \exists y_1, \ldots, y_m(\bigwedge_{i=1}^{m-1} E(y_i, y_{i+1}) \wedge E(x_1, y_1) \wedge E(y_n, x_2) \wedge \bigwedge_{1 \leq i < j \leq m} y_i \neq y_j)$ will define in a graph Γ the set of pairs (x_1, x_2) for which there is a path of length exactly $m + 1$ going from x_1 to x_2.

Other examples of definable sets: the set of elements connected by an edge to at least two distinct elements ($\exists y_1, y_2\, (y_1 \neq y_2 \wedge E(x, y_1) \wedge E(x, y_2))$); the set of elements at distance $\leq n$ of a given element a ($\exists y_1, \ldots, y_{n-1}\, (\bigwedge_{i=1}^{n-2} E(y_i, y_{i+1}) \wedge E(a, y_1) \wedge E(y_{n-1}, x))$.

Are usually **not** definable: the connected component of an element a (unless all of its elements are at bounded distance of a); the set of pairs contained in a loop.

2. Theories, and some big theorems

In this section we will introduce many definitions and important concepts. We will also mention the very important *Compactness theorem*, one of the crucial tools of model theorists.

(2.1) **Theories, models of theories, etc.**. Let $\mathcal{L}$ be a language. A $\mathcal{L}$*-theory* (or simply, a theory), is a set of sentences of the language $\mathcal{L}$. A *model of a theory* T is an $\mathcal{L}$-structure M which satisfies all sentences of T, denoted by $M \models T$. The class of all models of T is denoted $Mod(T)$. If $\mathcal{K}$ is a class of $\mathcal{L}$-structures, then $Th(\mathcal{K})$ denotes the set of all sentences true in all elements of $\mathcal{K}$, and $Th(\{M\})$ is denote by $Th(M)$.

A theory T is *consistent* iff it has a model. If φ is a sentence which holds in all models of T, this is denoted by $T \models \varphi$. Two $\mathcal{L}$-structures M and N are *elementarily equivalent*, denoted $M \equiv N$, iff they satisfy the same sentences, iff $Th(M) = Th(N)$. A theory is *complete* iff given a sentence φ, either $T \models \varphi$ or $T \models \neg\varphi$. Equivalently, if any two models of T are elementarily equivalent. (Observe that if M is an $\mathcal{L}$-structure, then necessarily $Th(M)$ is complete).

Elementary equivalence is an equivalence relation between $\mathcal{L}$-structures. Two isomorphic $\mathcal{L}$-structures are clearly elementarily equivalent, however the converse only holds **for finite $\mathcal{L}$-structures**. A famous theorem (of Shelah) states that two structures are elementarily equivalent if and only if they have isomorphic ultrapowers, see definition in Section 3.

(2.2) **Elementary substructures, extensions, embeddings, etc.** Let $M \subseteq N$ be $\mathcal{L}$-structures. We say that M *is an elementary substructure of* N, or that N *is an elementary extension of* M, denoted by $M \prec N$, iff for any formula $\varphi(\bar{x})$ and tuple $\bar{a}$ from M,

$$M \models \varphi(\bar{a}) \iff N \models \varphi(\bar{a}).$$

A map $f : M \to N$ is an *elementary embedding* iff it is an embedding, and if $f(M) \prec N$. In other words, if for any formula $\varphi(\bar{x})$ and tuple $\bar{a}$ from M, $M \models \varphi(\bar{a})$ if and only $N \models \varphi(f(\bar{a}))$.

Using induction on the complexity of formulas, one can show

Tarski's test. *Let M be a substructure of N. Then $M \prec N$ if and only if, for every formula $\varphi(\bar{x}, y)$ and tuple $\bar{a}$ in M, if $N \models \exists y\, \varphi(\bar{a}, y)$, then there exists $b \in M$ such that $N \models \varphi(\bar{a}, b)$.*

Note that while the element b is in M, the satisfaction is taken in N.

(2.3) **Some useful facts**. (1) If $M_1 \prec M_2$ and $M_2 \prec M_3$, then $M_1 \prec M_3$.

(2) If $M_1 \subseteq M_2 \subseteq M_3$, $M_1 \prec M_3$ and $M_2 \prec M_3$, then $M_1 \prec M_2$.

(3) Let M_n, $n \in \mathbb{N}$, be a *chain* of $\mathcal{L}$-structures (i.e., $M_n \subseteq M_{n+1}$ for all n). Then $M_\omega = \bigcup_{n\in\mathbb{N}} M_n$ has a unique $\mathcal{L}$-structure which makes the M_n's substructures of M_ω: the interpretation of the constants is the obvious one, $R^{M_\omega} = \bigcup_{n\in\mathbb{N}} R^{M_n}$, and $f^{M_\omega} = \cup_{n\in\mathbb{N}} f^{M_n}$, for R a relation symbol, and f a function symbol.

Assume that $M_n \prec M_{n+1}$ for each $n \in \mathbb{N}$. Then $M_n \prec M_\omega$ for each $n \in \mathbb{N}$.

(2.4) **Elimination of quantifiers**. Formulas with more than two alternances of quantifiers are fairly awkward, and usually difficult to decide the truth of. One therefore tries to "eliminate quantifiers".

Definition. A theory T *eliminates quantifiers* if for any formula $\varphi(\bar{x})$ there is a quantifier-free formula $\psi(\bar{x})$ which is *equivalent to* $\varphi(\bar{x})$ *modulo* T, i.e., is such that

$$T \models \forall \bar{x}(\varphi(\bar{x}) \leftrightarrow \psi(\bar{x})).$$

Note that the set of free variables in φ and ψ are the same. Thus if φ is a sentence, so is ψ. (If the language has no constant symbol, then one allows ψ to be either $\top$ (true) or $\perp$ (false); if the language contains a constant symbol c, then one can use instead the formulas $c = c$ or $c \neq c$).

Definition. A theory T is *model complete* iff it is consistent and whenever $M \subseteq N$ are models of T, then $M \prec N$.

Remarks. Clearly, a theory which eliminates quantifiers is also model complete. An important result concerning model complete theories is the following:
A consistent theory T is model complete if and only, for any formula $\varphi(\bar{x})$, there is an existential formula $\psi(\bar{x})$ such that

$$T \models \forall \bar{x}(\varphi(\bar{x}) \leftrightarrow \psi(\bar{x})).$$

Note that this is equivalent to every formula being equivalent modulo T to a universal formula: if $\neg\varphi(\bar{x})$ is equivalent modulo T to the existential formula $\psi(\bar{x})$, then $\varphi(\bar{x})$ is equivalent modulo T to the universal formula $\neg\psi(\bar{x})$.

One can show that a theory T is model complete, if whenever $M \subseteq N$ are models of T, then M is *existentially closed in* N, i.e., if $\varphi(\bar{x})$ is an existential formula, and $\bar{a}$ a tuple in M such that $N \models \varphi(\bar{a})$, then $N \models \varphi(\bar{a})$.

(2.5) **Examples**. (1) The theory of algebraically closed fields, denoted ACF, is axiomatised by the axioms for fields, plus, for each $n > 1$, the axiom $\forall x_1, \ldots, x_n \exists y\, (y^n + x_1 y^{n-1} + \cdots + x_n = 0)$. ACF eliminates quantifiers, see (2.14).

(2) Consider the field of real numbers, first with its natural $\mathcal{L}_R$-structure, then as an $\mathcal{L}_R \cup \{\leq\}$-structure. Let T_0 be its theory in $\mathcal{L}_R$, T_1 its theory in $\mathcal{L}_R \cup \{\leq\}$. Then one can show that T_0 is model complete, but does not eliminate quantifiers (in $\mathcal{L}_R$), while T_1 eliminates quantifiers. The quantifier one cannot eliminate in $\mathcal{L}_R$ is the existential quantifier of $\exists y(y^2 = x)$. This can be seen as follows: consider the substructure $\mathbb{Q}(\sqrt{2})$ of $\mathbb{R}$. As a field (i.e., as an $\mathcal{L}_R$-structure), it has an automorphism sending $\sqrt{2}$ to $-\sqrt{2}$. This automorphism however does not respect the ordering. See chapter 4 for a proof of quantifier-elimination.

(2.6) **Soundness and completeness theorem**. Given a set of sentences, there is a notion of proof, i.e., which statements are deducible from the given statements using some formal rules of deduction, such as modus ponens (from A and $A \to B$ deduce B), and some substitution rules (from a sentence of the form $\varphi(c)$ where c is a constant, deduce $\exists x\, \varphi(x)$). A proof can be thought of therefore as a finite sequence of sentences, each being obtained from the previous ones by applying some deduction rules. The first result, the *soundness theorem*, tells us that our notion of satisfaction is well-defined: *If a theory T has a model, then one cannot derive a contradiction from T, i.e., one cannot prove from T the sentence $\forall x(x \neq x)$.*

Gödel's completeness theorem then states the converse: *If from a given theory T, one cannot derive the sentence $\forall x(x \neq x)$, then the theory T has a model.*

Another way of stating this result is by saying that *the set of sentences deducible from a given theory T is* **exactly** *the set of sentences true in* **all models of** *T, i.e., in the notation introduced above, that it coincides with $Th(Mod(T))$.*

(2.7) **Decidability**. A theory T is *decidable*, if there is an algorithm allowing to decide whether a sentence φ holds in all models of T or not. (Here, the difficulty lies in being able to tell when φ **does not hold** in all models of T). If one can enumerate a theory T and one knows (somehow) that T is complete, then T is decidable: given a sentence φ, start enumerating the proofs from T; eventually you reach a proof of either φ or $\neg\varphi$.

How does one prove that a theory T is decidable? One tries to find a nice set of sentences Σ contained in T, and show that any other sentence is equivalent, modulo Σ, to a sentence which one can decide. For instance, by the results of quantifier elimination for algebraically closed fields given above, one shows that the theory ACF of algebraically closed fields is decidable: the set of axioms of ACF is clearly enumerable, any sentence is equivalent to a quantifier-free sentence, and there are very few of those. So, given a sentence φ, one starts enumerating all the proofs from ACF. Eventually one will reach a proof of $(\varphi \leftrightarrow \psi)$ where ψ is some quantifier-free sentence. Then, one again enumerates proofs to decide whether ψ is equivalent to $\top$, $\perp$, or whether it implies a finite disjunction of sentences of the form $p = 0$ or $p \neq 0$. The procedure will eventually terminate and say whether φ holds in all algebraically closed fields or not.

Similarly for the theory ORCF of real closed fields (axioms for ordered fields, any odd degree polynomial has a root, and every positive element has a square root). From which it follows that the theory of real closed fields in the language of rings is also decidable, since the ordering can be defined by "the non-negative elements are the squares".

How does one prove that a theory is undecidable? Usually one tries to code the ring $(\mathbb{Z}, +, -\cdot, 0, 1)$ in some model of the theory. Or to show that any finite graph is codable in some model of the theory.

(2.8) **Compactness theorem**. *Let T be a set of sentences in a language $\mathcal{L}$. If every finite subset of T has a model, then T has a model.*

We will present later a proof of this theorem using ultraproducts. Note that it is also a consequence of the completeness theorem, since any proof involves only finitely many elements of T. It also has for consequence the first half of the next theorem.

(2.9) **Löwenheim-Skolem Theorems**. *Let $\mathcal{L}$ be a language. T a theory, and let M be an infinite model of T.*

(1) Let κ be an infinite cardinal, $\kappa \geq |M| + |\mathcal{L}|$. Then M has an elementary extension N with $|N| = \kappa$.

(2) Let X be a subset of M. Then M has an elementary substructure N containing X, with $|N| \leq |X| + |\mathcal{L}| + \aleph_0$.

(2.10) **Comments**. These results allow us to use large models with good properties. For instance, assume that we have a set $\Sigma(x_1, \dots, x_n)$ of formulas in the variables $(x_1, \dots, x_n)$, and that we know that every finite fragment of $\Sigma(x_1, \dots, x_n)$ is satisfiable in some model M of T, i.e., there is a tuple $\bar{a}$ of M which satisfies all formulas of this finite fragment. Then there is a model N of T containing a tuple $\bar{b}$ which satisfies simultaneously all formulas of $\Sigma(\bar{x})$.

Using other techniques, one can show that if $\bar{a}$ and $\bar{b}$ are tuples of an $\mathcal{L}$-structure M, which satisfy the same formulas in M, then M has an elementary extension M^*, in which there is an automorphism which sends $\bar{a}$ to $\bar{b}$.

(2.11) **Application of the compactness theorem**. *Let T be a theory in a language $\mathcal{L}$, and Δ a set of formulas in the (free) variables $(x_1, \dots, x_n)$, closed under finite disjunctions.*

Let $\Sigma(x_1, \dots, x_n)$ be a set of formulas in the free variables $(x_1, \dots, x_n)$, such that every finite fragment of $\Sigma(x_1, \dots, x_n)$ is satisfiable in a model of T. The following conditions are equivalent:

(1) There is a subset $\Gamma(\bar{x})$ of Δ such that, if $\bar{c} = (c_1, \dots, c_n)$ are new constant symbols, then

$$T \cup \Gamma(\bar{c}) \models \Sigma(\bar{c}), \qquad T \cup \Sigma(\bar{c}) \models \Gamma(\bar{c}).$$

(2) For all models M and N of T, and n-tuples $\bar{a}$ in M and $\bar{b}$ in N, if $N \models \Sigma(\bar{b})$ and $\bar{a}$ satisfies (in M) all formulas of Δ that are satisfied by $\bar{b}$ (in N), then $M \models \Sigma(\bar{a})$.

Remark. If the set $\Sigma(\bar{x})$ is finite, then so is $\Gamma(\bar{x})$. Hence, taking $\varphi(\bar{x})$ to be the conjunction of the formulas of $\Sigma(\bar{x})$, one obtains that $\varphi(\bar{x})$ is equivalent, modulo T, to a finite conjunction of formulas of Δ.

(2.12) **Preservation theorems**. This has for consequences several preservation theorems. One direction is trivial, the other one not.

Let T be a theory in a language $\mathcal{L}$.

(1) The following conditions are equivalent:
- *(a) Whenever $M \subseteq N$ are $\mathcal{L}$-structures, and $M \models T$, then $N \models T$.*
- *(b) T has an axiomatisation given by existential sentences.*

(2) The following conditions are equivalent:
- *(a) Whenever $M \subseteq N$ are $\mathcal{L}$-structures, and $N \models T$, then $M \models T$.*
- *(b) T has an axiomatisation given by universal sentences.*

(3) The following conditions are equivalent:
- *(a) Whenever M_n, $n \in \mathbb{N}$, is a chain of models of T, then $\bigcup_{n \in \mathbb{N}} M_n$ is a model of T.*
- *(b) T has an axiomatisation given by $\forall\exists$ sentences.*

(4) The following conditions are equivalent:
- *(a) Whenever M is a model of T, and $f : M \to N$ is an morphism, then $f(M) \models T$.*
- *(b) T has an axiomatisation given by positive sentences.*

Note that (3) implies that a model complete theory has an axiomatisation given by $\forall\exists$ sentences.

(2.13) **Craig's interpolation theorem**. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two languages. Let φ be a sentence of $\mathcal{L}_1$ and ψ a sentence of $\mathcal{L}_2$. If $\varphi \models \psi$, then there is a sentence θ of $\mathcal{L}_1 \cap \mathcal{L}_2$ such that $\varphi \models \theta$ and $\theta \models \psi$.*

A somewhat different interpolation theorem is given by Robinson: *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two languages, and $\mathcal{L}_0 = \mathcal{L}_1 \cap \mathcal{L}_2$. Assume that T_1 and T_2 are theories in $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively, such that $T_0 = T_1 \cap T_2$ is complete. Then $T_1 \cup T_2$ is consistent.*

(2.14) **Application of the Löwenheim-Skolem theorem to the theory of algebraically closed fields**. This will allow us to show that the theory of algebraically closed fields (denoted ACF) eliminates quantifiers. From that we will deduce that the *completions* of ACF (i.e., the complete theories extending ACF) are obtained by specifying the characteristic. This will actually be obvious from the proof.

Notation. Let M be an $\mathcal{L}$-structure, and $\bar{a}$ an n-tuple in M. We denote by $tp_M(\bar{a})$ the set of formulas satisfied by $\bar{a}$ in M.

Theorem. *Let $T = ACF$ be the theory of algebraically closed fields. Then T eliminates quantifiers. Moreover, any two models of T of the same characteristic are elementarily equivalent.*

Proof. By (2.11), it suffices to show that if M and N are algebraically closed, and if $\bar{a}$ and $\bar{b}$ are n-tuples from M and N respectively, which satisfy the same quantifier-free formulas, then they satisfy the same formulas, i.e., $tp_M(\bar{a}) = tp_N(\bar{b})$.

First note that if $\bar{a}$ and $\bar{b}$ satisfy the same formulas, then the fields M and N have the same characteristic. Indeed, if $char(M) = p > 0$, then $\bar{a}$ satisfies (the sentence) $p = 0$ (where p is an abbreviation for the term $1 + 1 + \cdots + 1$ p times). Hence the prime subfield of M is isomorphic to the prime subfield of N, and we will denote this field by k. By assumption $\bar{a}$ and $\bar{b}$ satisfy the same quantifier-free formulas, and this implies that there is a field-isomorphism $f : k(\bar{a}) \to k(\bar{b})$ sending $\bar{a}$ to $\bar{b}$.

Let κ be a cardinal, $\kappa > |M|, |N|$. By Löwenheim-Skolem, M has an elementary extension M^* of cardinality κ, and N has an elementary extension N^* of cardinality κ. Select transcendence bases X of M^* over $k(\bar{a})$ and Y of N^* over $k(\bar{b})$. Then $|X| = |Y| = \kappa$, so that there is a bijection $g : X \to Y$. Then $f \cup g$ extends to an isomorphism $h : M^* \to N^*$. As isomorphisms preserve formulas, this implies that $tp_{M^*}(\bar{a}) = tp_{N^*}(\bar{b})$. As $M \prec M^*$ and $N \prec N^*$, this shows that $tp_M(\bar{a}) = tp_N(\bar{b})$.

Moreover the proof shows that any two algebraically closed fields of the same characteristic are elementarily equivalent, as they have isomorphic elementary extensions.

We will see later an important consequence of this, the Lefshetz principle.

(2.15) **Many-sorted structures**. Many-sorted structures are like ordinary structures, except that there are now several universes, usually disjoint (but not always), with associated sorts. The language will have sorts, relation symbols will have attached to them, not only an arity, but also a tuple of sorts, and similarly for functions. Formulas are built in the usual way, the only restriction being that variables now have **a sort attached to them**. In case one has finitely many sorts, say n, one can reduce to the usual case, by adding for instance to the language n new unary relation symbols $R_1, \ldots, R_n$, with intended interpretation the universes of the sorts.

A formal definition is a bit awkward, and we will rather give four natural examples. Most classical results hold in many-sorted logic, with sometimes the appropriate adaptation. In particular, the compactness theorem holds.

Example 1. Let G be a finitely generated group, say by $a_1, \ldots, a_n$. We consider the language with two sorts: one sort is the group sort, the other one is the "length" sort. The language is:

– $\{\cdot, ^{-1}, 1, a_1, \ldots, a_n\}$, applied to the group sort. Here we have added to the language of groups n new constant symbols for the elements $a_1, \ldots, a_n$ (we denote, abusively maybe, the constant and the element by the same symbol).

– Any structure you want on the length sort. E.g., a constant symbol 0, and a binary relation $\leq$. Maybe also a symbol for addition and subtraction.

– A binary function symbol d, with domain the group sort squared, and range the length sort.

The two-sorted structure we have in mind is the structure

$$\mathcal{G} = ((G, \cdot, ^{-1}, 1, a_1, \dots, a_n), (\mathbb{N}, +, 0, \leq), d),$$

where $(G, \cdot, ^{-1}, 1, a_1, \dots, a_n)$ is our group with the distinguished elements $a_1, \dots, a_n$, $(\mathbb{N}, +, 0, \leq)$ is the non-negative integers with their natural addition, subtraction and ordering, and $d : G^2 \to \mathbb{N}$ is the distance function (on the Cayley graph of G with respect to the set $\{a_1, \dots, a_n\}$ of generators). Note that for each n, to be at distance $\leq n$ is expressible by a first order formula. But there is no formula expressing that every element is at finite distance from 1, unless G is finite.

Note also that an elementary extension of $(G, \mathbb{N}, d)$ will be a two-sorted structure $(G^*, \mathbb{N}^*, d^*)$, in which the distance function will still be onto, and satisfy certain ultrametric inequalities; but its range will (in general) be a non-standard model of the integers, so that distance between two elements of G^* may be infinite. One can also replace $\mathbb{N}$ by any additive ordered subgroup of the reals, but the map d will then not be onto. We will come back to this example in asymptotic cones.

Example 2. Consider the field $\mathbb{Q}_p$, and its valuation $v : \mathbb{Q}_p \to \mathbb{Z}$, residue map $\mathbb{Z}_p \to \mathbb{F}_p$. it is customary to view this valued field as a 3-sorted structure $\mathcal{Q}_p$: the sorts are the field sort, the value group sort, and the residue field sort. We have two sets of field operations, one for the field sort, and one for the residue field sort. We also have the language of ordered groups for the residue sort, together with a distinguished constant ∞. Finally, we have the valuation map $\mathbb{Q}_p \to \mathbb{Z} \cup \{\infty\}$, and the residue map $\pi : \mathbb{Z}_p \to \mathbb{F}_p$ (if one wants this map to be defined on the whole field sort, one assigns the elements of $\mathbb{Q}_p \setminus \mathbb{Z}_p$ to 0, for instance).

Example 3. Let K be a field, and fix an integer $m > 0$. Consider the polynomial ring $R = K[X_1, \dots, X_m]$. We will define an ω-sorted structure (indexed by the integers) on R. For $d \in \mathbb{N}$, the elements of sort d are the polynomials of total degree $\leq d$, and we denote the corresponding set by R_d. We add an addition, subtraction and multiplication, in the natural way. Then $+$ for instance will send $R_d \times R_e$ to $R_{\sup\{d,e\}}$, while $\cdot$ will send $R_d \times R_e$ to R_{d+e}.

Elementary statements about the ω-sorted structure R are expressible by ordinary $\mathcal{L}_R$-formulas in the field K: as all variables have a sort, quantification over polynomials of degree $\leq d$ is achieved by quantifying over their coefficients, i.e., over N-tuples for a certain N computable from m, d.

A similar construction can be done for $K[V]$, for V an algebraic set defined over K.

Example 4. The 2-sorted language used for graphs in which there can be several edges between two vertices, is simply the language with two sorts, the vertice sort, and the edge sort. It has three unary function symbols, σ, τ and $^{-}$. The domains of σ and τ are the edge sort, and their range the vertice sort. The domain and range of $^{-}$ is the edge sort.

A graph will be given by $(V, E, \sigma, \tau, ^{-})$, where V is the set of vertices, E is the set of edges, $\sigma : E \to V$ assigns to an edge e its starting point $\sigma(e)$, and $\tau : V \to E$ assigns to an edge e its endpoint $\tau(e)$. The map $^{-}$ reverses the direction of edges. Hence we have $\bar{\bar{e}} = e$ for every edge e, and $\sigma(\bar{e}) = \tau(e)$, $\tau(\bar{e}) = \sigma(e)$.

One can put additional structure on the graph, for instance by colouring the edges. This is done by adding unary predicates on the edge sort. Bipartitite graphs can be treated in a similar manner.

(2.16) **Additional remarks on quantifier-elimination**. So, what does one do if a theory T in a language $\mathcal{L}$ does not eliminate quantifiers? One possibility is to form the Skolemization of T: for each formula $\varphi(x_1, \dots, x_n)$ one adds a new n-ary relation symbol R_φ, and adds to T an axiom saying that R_φ defines precisely the set of n-tuples satisfying φ. The resulting theory eliminates quantifiers. However, this does not bring any information about our theory T.

The hope is that one does not need to add much to the language to obtain quantifier-elimination. For instance we saw that to get elimination of quantifiers of the theory ORCF of real closed fields, it is enough to add the ordering to $\mathcal{L}_R$.

Another interesting example is the theory of the field of p-adic numbers $\mathbb{Q}_p$, now viewed as an ordinary $\mathcal{L}_R$-structure. A beautiful result of Macintyre states that it is enough to add to the language a unary predicate P_n for each $n > 1$, as well as the inverse function $^{-1}$. The interpretation of the predicates P_n in $\mathbb{Q}_p$ is the set of non-zero n-th powers, i.e.,

$$\mathbb{Q}_p \models \forall x\, (P_n(x) \leftrightarrow (x \neq 0 \wedge \exists y\, y^n = x)).$$

Observe that the elements of positive valuation are then definable: $v(x) > 0 \iff P_2(1 + px^2)$. The first-order theory of the expansion of $\mathbb{Q}_p$ to this enlarged language then eliminates quantifiers. This allows one to give a good description of the definable sets.

(2.17) **Diagrams**. Diagrams are meant to give a logical formulation of the following properties: the structure N contains a copy of the structure M; there is a morphism $f : M \to N$; there is an elementary embedding $f : M \to N$.

Definitions. Let M be an $\mathcal{L}$-structure, $A \subseteq M$. We let $\mathcal{L}(A)$ be the language obtained by adding to the language new symbols of constants c_a for each $a \in A$. If $\bar{a}$ is a tuple of elements of A, we let $\bar{c}_{\bar{a}}$ denote the tuple of constants corresponding to the elements of the tuple $\bar{a}$. $\mathcal{L}(A)$ is then a bona fide language, but may be much larger than $\mathcal{L}$. Note that M becomes naturally an $\mathcal{L}(A)$-structure, when one interprets the constant c_a by a, for $a \in A$. This structure is usually denoted by $(M, c_a)_{a \in A}$, or $(M, a)_{a \in A}$ (i.e., we denote the constant and the element by the same symbol).

(1) The *quantifier-free diagram of A in M*, $\Delta(A)$, is the set of all quantifier-free sentences $\varphi(\bar{c}_{\bar{a}}) \in \mathcal{L}(A)$ which hold in the $\mathcal{L}(A)$-structure M (i.e., such that $M \models \varphi(\bar{a})$).
(2) The *positive diagram of A in M*, $\Delta^+(A)$, is the set of all positive quantifier-free sentences $\varphi(\bar{c}_{\bar{a}}) \in \mathcal{L}(A)$ which hold in the $\mathcal{L}(A)$-structure M.
(3) The *elementary diagram of A in M*, *$Diag(A)$*, is the set of all sentences $\varphi(\bar{c}_{\bar{a}}) \in \mathcal{L}(A)$ which hold in the $\mathcal{L}(A)$-structure M.
(4) Let $\mathcal{L}'$ be a language containing $\mathcal{L}$. An *expansion of the $\mathcal{L}$-structure M to $\mathcal{L}'$* is an $\mathcal{L}'$-structure M', with same universe as M, and such that the interpretation of the symbols of $\mathcal{L}$ in M and in M' coincide. M is then called a *reduct of M' to $\mathcal{L}$*. For instance, $(M, c_a)_{a \in A}$ is an expansion of M to $\mathcal{L}(A)$.

Fact. *Let M and N be two $\mathcal{L}$-structures. The following are immediate consequences of the definition:*

(1) N can be expanded to an $\mathcal{L}(M)$-structure which is a model of $\Delta(M)$ if and only if there is an embedding $f : M \to N$.

(2) N can be expanded to an $\mathcal{L}(M)$-structure which is a model of $\Delta^+(M)$ if and only there is a morphism $f : M \to N$.

(3) N can be expanded to an $\mathcal{L}(M)$-structure which is a model of $Diag(M)$ if and only if there is an elementary embedding $f : M \to N$.

Note also

(5) An $\mathcal{L}$-theory T is model complete if and only if, for every model M of T, $T \cup \Delta(M)$ is complete (in $\mathcal{L}(M)$).

(6) An $\mathcal{L}$-theory T eliminates quantifiers if and only if, for every $M \models T$ and subset A of M, $T \cup \Delta(A)$ is complete (in $\mathcal{L}(A)$).

(2.18) **Questions and hopes on free groups**. Sela has shown that the theory T of all non-abelian free groups (in the language $\mathcal{L}_G$) is complete, and moreover, that every formula is equivalent, modulo T, to a Boolean combination of $\forall\exists$-formulas. From what I understand, he does not yet have an axiomatisation of the theory T, nor does he knows whether it is decidable.

One of the outstanding open questions for model theorists, is whether the theory T is stable. Stability is a certain combinatorial property of a theory, which allows one to use a huge machinery to study models and possible interactions between definable subsets. One definition of stability is the following: we say that a formula $\varphi(\bar{x}, \bar{y})$ is *stable* (for T) if there is exists an integer n such that in a model M of T, any two sequences of tuples $(\bar{a}_i)$ and $(\bar{b}_j)$ indexed by some initial segment of $\mathbb{N}$, and satisfying

$$M \models \varphi(\bar{a}_i, \bar{b}_j) \qquad \iff i < j,$$

have length $\leq n$. The (complete) theory T is *stable* iff all formulas are stable. Stability forbids in particular the existence of an infinite (definable) linear ordering on models of T.

It is believed that the theory T of non-abelian free groups is stable. One question is to understand better the sets definable in free groups. An arbitrary Boolean combination of $\forall\exists$-definable sets is quite an intractable object. The hope is that the complication originates from certain definable sets. I.e., that there are *a few* families of $\forall\exists$-definable sets, such that if one adds suitable relation symbols for these definable sets, then one can eliminate quantifiers to a lower level. Ideally to existential formulas, but one shouldn't be too optimistic.

There are also questions related to induced structure. For instance, let $g \neq 1$, and consider the stabiliser $C(g)$ of g in the free group F. We know that $C(g)$ is isomorphic to the abelian group $\mathbb{Z}$. One question is: what is the structure induced by F on $C(g)$? If one knows that T is stable, this reduces to the following: let $D \subset F^m$ be definable (without parameters). Describe $D \cap C(g)^m$. Is it definable in the structure $(C(g), \cdot, ^{-1}, 1)$? (Maybe the answer to this question is trivially no, but one can ask analogous questions for any definable subset of F^n). If it were, then the induced structure would be very simple: indeed the theory of $\mathbb{Z}$ in the language of groups enlarged by adding unary predicates S_n for each of the subgroups $n\mathbb{Z}$ of $\mathbb{Z}$, $n > 1$, eliminates quantifiers.

Here we are using the following property of a stable theory: *let M be a model of a stable theory T, let S be a 0-definable subset of M^n, and let D be an M-definable subset of M^{mn}. Then $D \cap S^m$ is definable with parameters from S.* (Warning: the formula may change).

the formula with parameters in S defining $D \cap S^m$ is not necessarily the same as the formula defining D.)

3. Ultraproducts, Łos theorem

In this section we will introduce an important tool: ultraproducts. They are at the centre of many applications, within and outside model theory.

(3.1) **Filters and ultrafilters**. Let I be a set. A *filter on* I is a subset $\mathcal{F}$ of $\mathcal{P}(I)$ (the set of subsets of I), satisfying the following properties:
(1) $I \in \mathcal{F}$, $\emptyset \notin \mathcal{F}$.
(2) If $U \in \mathcal{F}$ and $V \supseteq U$, then $V \in \mathcal{F}$.
(3) If $U, V \in \mathcal{F}$, then $U \cap V \in \mathcal{F}$.

A *ultrafilter on* I is a filter on I which is maximal for inclusion. Equivalently, it is filter $\mathcal{F}$ such that for any $U \in \mathcal{P}(I)$, either $U \in \mathcal{F}$ or $I \setminus U \in \mathcal{F}$.

(3.2) **Remarks**. (1) Note that condition (1) forbids that both U and $I \setminus U$ belong to the same filter on I.

(2) Using Zorn's lemma (and therefore the axiom of choice), every filter on I is contained in a ultrafilter.

(3) If $\mathcal{G} \subset \mathcal{P}(I)$ has the *finite intersection property* (i.e., the intersection of finitely many elements of $\mathcal{G}$ is never empty), then $\mathcal{G}$ is contained in a filter. The *filter generated by* $\mathcal{G}$ is then the set of elements of $\mathcal{P}(I)$ containing some finite intersection of elements of $\mathcal{G}$.

(3.3) **Principal and non-principal ultrafilters, Fréchet filter**. Let I be a set. A ultrafilter $\mathcal{F}$ on I is *principal* if there is $i \in I$ such that $\{i\} \in \mathcal{F}$ (and then we will have: $U \in \mathcal{F} \iff i \in U$). A ultrafilter is *non-principal* if it is not principal. Note that if I is finite, then every ultrafilter on I is principal.

Let I be infinite. The *Fréchet filter on* I is the filter $\mathcal{F}_0$ consisting of all cofinite subsets of I. A ultrafilter $\mathcal{F}$ on I is then non-principal if and only if contains the Fréchet filter on I. Note that if $S \subseteq I$ is infinite, then $\mathcal{F}_0 \cup \{S\}$ has the finite intersection property, so that it is contained in a ultrafilter.

(3.4) **Cartesian products of $\mathcal{L}$-structures**. Fix a language $\mathcal{L}$. Let I be an index set, and (M_i), $i \in I$, a family of $\mathcal{L}$-structures. We define the $\mathcal{L}$-structure $M = \prod_{i \in I} M_i$ as follows:

— The universe of M is simply the cartesian product of the M_i's, i.e., the set of sequences $(a_i)_{i \in I}$ such that $a_i \in M_i$ for each $i \in I$.

— If c is a constant symbol of $\mathcal{L}$, then $c^M = (c^{M_i})_{i \in I}$.

— If R is an n-ary relation symbol, then $R^M = \prod_{i \in I} R^{M_i}$.

— If f is an n-ary function symbol and $((a_{1,i})_i, \dots, (a_{n,i})_i) \in M^n$, then

$$f^M((a_{1,i})_i, \dots, (a_{n,i})_i) = (f^{M_i}(a_{1,i}, \dots, a_{n,i}))_{i \in I}.$$

(3.5) **Reduced products of $\mathcal{L}$-structures**. Let I be a set, $\mathcal{F}$ a filter on I, and (M_i), $i \in I$, a family of $\mathcal{L}$-structures. The *reduced product of the M_i's over $\mathcal{F}$*, denoted by $\prod_{i\in I} M_i/\mathcal{F}$, is the $\mathcal{L}$-structure defined as follows:

— The universe of $\prod_{i\in I} M_i/\mathcal{F}$ is the quotient of $\prod_{i\in I} M_i$ by the equivalence relation $\equiv_{\mathcal{F}}$ defined by

$$(a_i)_i \equiv_{\mathcal{F}} (b_i)_i \iff \{i \in I \mid a_i = b_i\} \in \mathcal{F}.$$

We denote by $(a_i)_{\mathcal{F}}$ the equivalence class of the element $(a_i)_i$ for this equivalence relation. The structure on $\prod_{i\in I} M_i/\mathcal{F}$ is then simply the "quotient structure", i.e.,

— The interpretation of c is $(c^{M_i})_{\mathcal{F}}$, for c a constant symbol of $\mathcal{L}$.

— If R is an n-ary relation symbol, and if $a_1, \dots, a_n \in \prod_{i\in I} M_i/\mathcal{F}$ are represented by $(a_{1,i})_i, \dots, (a_{n,i})_i \in \prod_{i\in I} M_i$, then we set

$$\prod_{i\in I} M_i/\mathcal{F} \models R(a_1, \dots, a_n) \iff \{i \in I \mid (a_{1,i}, \dots, a_{n,i}) \in R^{M_i}\} \in \mathcal{F}.$$

— If f is an n-ary function symbol and if $a_1, \dots, a_n \in \prod_{i\in I} M_i/\mathcal{F}$ are represented by $(a_{1,i})_i, \dots, (a_{n,i})_i \in \prod_{i\in I} M_i$, then we set

$$f^M(a_1, \dots, a_n) = (f^{M_i}(a_{1,i}, \dots, a_{n,i}))_{\mathcal{F}}.$$

The properties of filters guarantee that the quotient structure is well-defined. Note that the quotient map : $\prod_{i\in I} M_i \to \prod_{i\in I} M_i/\mathcal{F}$, $(a_i)_i \mapsto (a_i)_{\mathcal{F}}$, is a morphism of $\mathcal{L}$-structures.

Definitions. If all structures M_i are equal to a structure M, then we write $M^I/\mathcal{F}$ instead of $\prod_i M_i/\mathcal{F}$, and the structure is called a *reduced power of M*. If the filter $\mathcal{F}$ is an ultrafilter, then $\prod_i M_i/\mathcal{F}$ is called the *ultraproduct of the M_i's* (with respect to $\mathcal{F}$), and $M^I/\mathcal{F}$ the *ultrapower of M* (with respect to $\mathcal{F}$).

(3.6) **Łos Theorem**. *Let I be a set, $\mathcal{F}$ a ultrafilter on I, and (M_i), $i \in I$, a family of $\mathcal{L}$-structures. Let $\varphi(x_1, \dots, x_n)$ be an $\mathcal{L}$-formula, and let $a_1, \dots, a_n \in \prod_{i\in I} M_i/\mathcal{F}$ be represented by $(a_{1,i})_i, \dots, (a_{n,i})_i \in \prod_{i\in I} M_i$. Then*

$$\prod_{i\in I} M_i/\mathcal{F} \models \varphi(a_1, \dots, a_n) \iff \{i \in I \mid M_i \models \varphi(a_{1,i}, \dots, a_{n,i})\} \in \mathcal{F}.$$

(3.7) **Corollary**. *Let I be a set, $\mathcal{F}$ an ultrafilter on I, and M an $\mathcal{L}$-structure. Then the natural map $M \to M^I/\mathcal{F}$, $a \mapsto (a)_{\mathcal{F}}$, is an elementary embedding. (Here $(a)_{\mathcal{F}}$ is the equivalence class of the sequence with all terms equal to a).*

(3.8) **Remarks and comments**. Let I be an infinite index set, and $\mathcal{F}$ a ultrafilter on I.

(1) If $\mathcal{F}$ is principal, say $\{j\} \in \mathcal{F}$, then $\prod_{i\in I} M_i/\mathcal{F} \simeq M_j$ for any family of $\mathcal{L}$-structures M_i, $i \in I$.

(2) Suppose that the M_i's are fields, with maybe additional structure (e.g., an ordering, new functions, etc.). Consider the ideal $\mathcal{M}$ of $\prod_i M_i$ generated by all elements $(a_i)_i$ such that $\{i \in I \mid a_i = 0\} \in \mathcal{F}$. Then $\mathcal{M}$ is a maximal ideal of $\prod_i M_i$, and quotienting

by the equivalence relation $\equiv_{\mathcal{F}}$ is equivalent to quotienting by the maximal ideal $\mathcal{M}$. The strength of Łos theorem is to tell you that the elementary properties of the M_i's, including the ones depending on the additional structure, are preserved. E.g., that $\mathbb{R}^I/\mathcal{F}$ is a real closed field.

(3.9) **Shelah's isomorphism theorem**. *Let M and N be two $\mathcal{L}$-structures. Then $M \equiv N$ if and only if there is a ultrafilter $\mathcal{F}$ on a set I such that $M^I/\mathcal{F} \simeq N^I/\mathcal{F}$.*

Note the following immediate consequence: if $M \equiv N$, then there is M^* in which both M and N embed elementarily.

(3.10) **Application 1: proof of the compactness theorem**. *Let T be a theory in a language $\mathcal{L}$, and assume that every finite subset s of T has a model M_s. Then T has a model.*

Proof. If T is finite, there is nothing to prove, so we will assume that T is infinite. Let I be the set of all finite subsets of T. For every $\varphi \in T$, let $S(\varphi) = \{s \in I \mid \varphi \in s\}$. Then the family $\mathcal{G} = \{S(\varphi) \mid \varphi \in T\}$ has the finite intersection property, and therefore is contained in a ultrafilter $\mathcal{F}$. We claim that $\prod_{s\in I} M_s/\mathcal{F}$ is a model of T: let $\varphi \in T$. Then, by assumption, $\{s \in I \mid M_s \models \varphi\}$ contains $S(\varphi)$, and therefore belongs to $\mathcal{F}$. By Łos's theorem, $\prod_{s\in I} M_s/\mathcal{F} \models \varphi$.

(3.11) **Application 2: Lefshetz principle**. *Let φ be a sentence of the language $\mathcal{L}_R$. The following conditions are equivalent:*
(1) $\mathbb{C} \models \varphi$.
(2) If K is an algebraically closed field of characteristic 0, then $K \models \varphi$.
(3) There is an integer n, such that if p is a prime number $> n$, then $\tilde{\mathbb{F}}_p \models \varphi$.
(4) There is an integer n, such that if p is a prime number $> n$ and K is an algebraically closed field of characteristic p, then $K \models \varphi$.

Proof. The equivalences of (1) and (2), and of (3) and (4) are clear, by (2.14). Assume (2). Then ACF$\cup\{p \neq 0 \mid p$ a prime $\} \models \varphi$. By compactness, there are finitely many prime numbers $p_1, \dots, p_m$ such that ACF$\cup\{p_1 \neq 0, \dots, p_m \neq 0\} \models \varphi$. Take $n > \sup\{p_1, \dots, p_m\}$. This shows that (2) implies (3).

Assume (3), and let $\mathcal{F}$ be a non-principal ultrafilter on the set P of primes. For each prime p, choose an algebraically closed field K_p of characteristic p. By assumption, $\{p \in P \mid K_p \models \varphi\} \in \mathcal{F}$, and therefore $\prod_{p\in P} K_p/\mathcal{F} \models \varphi$. For each p, the set $\{q \in P \mid q \neq p\}$ is also in the ultrafilter. By Łos's Theorem, $\prod_{p\in P} K_p/\mathcal{F}$ is an algebraically closed field of characteristic 0, and therefore shows (2).

(3.12) **Application 3: Orderable groups**. *Let G be a group, and assume that every finitely generated subgroup of G is orderable. Then G is orderable.*

Proof. Let I be the set of finite subsets of G. For each $s \in I$, let G_s be the subgroup of G generated by s, and fix an ordering $\leq$ on G_s. Let $\mathcal{F}$ be a ultrafilter on I containing all sets $S(g) = \{s \in I \mid g \in s\}$ for $g \in G$, and consider the $\mathcal{L}_G \cup \{\leq\}$-structure

$$(G^*, \cdot, ^{-1}, 1, \leq) = \prod_{s\in I} (G_s, \cdot, ^{-1}, 1, \leq)/\mathcal{F}.$$

Then G^* is an ordered group, by Łos theorem. We embed G in G^* in the following fashion: for $g \in G$, we set

$$f(g) = (g_s)_{\mathcal{F}}, \qquad \text{where } g_s = \begin{cases} g & \text{if } g \in G_s \ , \\ 1 & \text{otherwise.} \end{cases}$$

One checks that f is a group embedding, and $f(G)$ is therefore an ordered group.

(3.13) **Types**. Let M be an $\mathcal{L}$-structure, $A \subseteq M$, and $\bar{b} \in M^m$. The *type of* $\bar{b}$ *over* A, denoted by $tp(\bar{b}/A)$ (or $tp_M(\bar{b}/A)$), is the set of all formulas $\varphi(\bar{a}, \bar{y})$ satisfied by $\bar{b}$ in M, where $\bar{a}$ is a finite tuple of elements of A.

One has the following result: if $\bar{b}$, $\bar{c}$ realise the same type over A in M, i.e., if $tp(\bar{b}/A) = tp(\bar{c}/A)$, then there is an elementary extension M^* of M, and an automorphism of M^*, which fixes the elements of A and sends $\bar{b}$ to $\bar{c}$.

More generally, an n-type over a subset A of M is a set $\Sigma(\bar{y})$ of formulas $\varphi(\bar{a}, \bar{y})$ in the n-tuple $\bar{y}$ of variables and with $\bar{a} \in A$, which is *finitely realisable in* M, i.e., is such that if $s \subset \Sigma$ is finite, then there is $\bar{b} \in M^n$ satisfying all the formulas of s.

(3.14) **Application 4: ω_1-saturation**. *Let I be a countable set, $\mathcal{F}$ a non-principal ultrafilter on I, and (M_i), $i \in I$, a family of $\mathcal{L}$-structures, where $|\mathcal{L}| \leq \aleph_0$. Let $A \subset M^* = \prod_{i\in I} M_i/\mathcal{F}$ be countable, and let $\Sigma(\bar{y})$ be a type over A. Then $\Sigma(\bar{y})$ is realised in M^*, i.e., there is $\bar{b} \in M^*$ which satisfies all the formulas of $\Sigma(\bar{y})$.*

Another way of stating this property is to say that if (S_n), $n \in \mathbb{N}$, is a countable family of definable subsets of M^{*m} with the finite intersection property, then $\bigcap_n S_n \neq \emptyset$.

(3.15) **A constructions of** $\mathbb{R}$. We will present a construction allowing one to construct $\mathbb{R}$. This construction come from non-standard analysis. Let $\mathcal{L} = \{+, -, \leq, 0)$ be the language of abelian ordered groups, and endow $\mathbb{R}$ with its natural $\mathcal{L}$-structure.

We fix a non-principal ultrafilter $\mathcal{F}$ on $I = \mathbb{N}$. Assume that for each i we are given an $\mathcal{L}$-substructure Γ_i of $\mathbb{R}$ (i.e., an additive subgroup of $\mathbb{R}$ with the induced ordering).

Consider the ordered groups $\Gamma^* = \prod_{i\in I} \Gamma_i/\mathcal{F}$ and $\mathbb{R}^* = \mathbb{R}^I/\mathcal{F}$. We have a natural embedding $\Gamma^* \to \mathbb{R}^*$ induced by each of the inclusions $\Gamma_i \subseteq \mathbb{R}$. Note that $\mathbb{R}^*$ also has a ring structure, and therefore contains a copy of (the ring) $\mathbb{Z}$. Define

$$\begin{gathered} R^{fin} = \{g \in \mathbb{R}^* \mid \exists c \in \mathbb{N} \ -c \leq g \leq c\} \\ \mu = \{g \in \mathbb{R}^* \mid \forall c \in \mathbb{N} \ -1 \leq cg \leq 1\} \\ \Gamma^{fin} = \Gamma^* \cap R^{fin} \qquad \mu_\Gamma = \Gamma^* \cap \mu. \end{gathered}$$

Then R^{fin} is the convex hull of $\mathbb{Z}$ in $\mathbb{R}^*$. Both Γ^{fin} and μ_Γ are convex subgroups of Γ^*. Hence, $\Gamma_{\mathcal{F}} = \Gamma^{fin}/\mu_\Gamma$ is an ordered group, which is clearly archimedean. There are three possibilities for this group: one possibility is that it is trivial: this is the case for instance if each $\Gamma_i = i\mathbb{Z}$. The second possibility isFthat it is discrete, i.e., has a smallest positive element. It is then isomorphic to $\mathbb{Z}$. The third case is when $\Gamma_{\mathcal{F}}$ has no smallest positive element. It is then isomorphic to a dense additive subgroup of $\mathbb{R}$, and we claim that it equals $\mathbb{R}$. We know that it is archimedean, and it therefore suffices to show that it is complete. Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be sequences of elements of $\Gamma_{\mathcal{F}}$ such that $a_n < b_m$ for all n, m. Lift these sequences to sequences $(a'_n)_{n\in\mathbb{N}}$ and $(b'_n)_{n\in\mathbb{N}}$ in Γ^{fin}. Then $a'_n < b'_m$

for all m, n. By ω_1-saturation of Γ^* (see (3.14)), there is an element $c' \in \Gamma^{fin}$ such that $a'_n < c' < b'_m$ for all m, n. The image c of c' in $\Gamma_{\mathcal{F}}$ then satisfies $a_n \leq c \leq b_m$ for all n, m, and this shows the completeness of $\Gamma_{\mathcal{F}}$.

The third case occurs in particular when the groups Γ_i are of the form $r_i\mathbb{Z}$, where $r_i \in \mathbb{R}$, and $(r_i)_{\mathcal{F}} \in \mu$. Or if they are dense in $\mathbb{R}$.

This construction is used in non-standard analysis, with all Γ_i equal to $\mathbb{R}$. The structure R^{fin} is then a valuation ring, with maximal ideal μ (if $a \in R^{fin}$, $a \notin \mu$, then there is $c \in \mathbb{N}$ such that $1/c < |a| < c$, so that $1/a$ also belongs to R^{fin}). The elements of μ are called *infinitesimal*, and the ring homorphism $R^{fin} \to \mathbb{R} = R^{fin}/\mu$ is called the *standard part*.

Comments. One can do the same construction replacing 1 by an arbitrary element $a \in \mathbb{R}^*$. Then one defines $\Gamma^{fin,a}$ to be the intersection of Γ^* with the convex hull of $a\mathbb{Z}$ in $\mathbb{R}^*$, and $\mu_{\Gamma,a}$ to be the elements of Γ^*, whose archimedean class is smaller than the archimedean class of a. Then again one obtains an archimedean ordered group, which is either trivial, or isomorphic to $\mathbb{Z}$, or to $\mathbb{R}$. The two constructions are really similar: if a is represented by the sequence $(a_i)_i$, the first construction applied to the ordered groups $\Gamma'_i = a_i\Gamma_i$ gives exactly the same group.

(3.16) **The theorem of Ax and Kochen**. *Consider the field $\mathbb{F}_p((t))$ of power series over $\mathbb{F}_p$. This is a valued field (where $v(t) = 1$), with value group $\mathbb{Z}$ and residue field $\mathbb{F}_p$. The theorem of Ax and Kochen states that if $\mathcal{F}$ is a non-principal ultraproduct on the set p of primes, then there is an isomorphism of valued fields between the fields $\prod_{p\in P} \mathbb{Q}_p/\mathcal{F}$ and $\prod_{p\in P} \mathbb{F}_p((t))/\mathcal{F}$.*

This implies in particular that, given a sentence φ which holds in all but finitely many of the $\mathbb{Q}_p$'s [resp., in all but finitely many of the $\mathbb{F}_p((t))'s$], the sentence φ will hold in all but finitely many of the $\mathbb{F}_p((t))$'s [resp., all but finitely many of the $\mathbb{Q}_p$'s].

(3.17) **Ultraproducts of many-sorted structures**. The ultraproduct construction carries over to many-sorted structures without any trouble. If s is a sort, then the universe of sort s of the ultraproduct will simply be the ultraproduct of the universes of sort s. Functions between tuples of sorts and relations will be defined similarly. Maybe this is better seen with examples.

Example 1. See Example 1 of (2.15) for the notation. Let I be an index set, $\mathcal{F}$ a ultrafilter on I, and $\mathcal{G}_i$, $i \in I$, a family of 2-sorted structures, where $\mathcal{G}_i = ((G_i, \cdot, ^{-1}, 1), (\Gamma_i, +, 0, \leq), d_i)$.

Then $\mathcal{G}^* = \prod_{i\in I} \mathcal{G}_i/\mathcal{F}$ will be the 2-sorted structure

$$(\prod_{i\in I}(G_i, \cdot, ^{-1}, 1)/\mathcal{F}, \prod_{i\in I}(\Gamma_i, +, 0, \leq)/\mathcal{F}, d^*)$$

where

$$d^*((g_i)_{\mathcal{F}}, (h_i)_{\mathcal{F}}) = (d(g_i, h_i))_{\mathcal{F}}.$$

Example 2. Let P be the set of primes, $\mathcal{F}$ a ultrafilter on P, and consider the 3-sorted structures $\mathcal{Q}_p$ introduced in the example 2 of (2.15).

Then $\prod_p \mathcal{Q}_p/\mathcal{F} = (\prod_p \mathbb{Q}_p/\mathcal{F}, \prod_p \mathbb{F}_p/\mathcal{F}, (\mathbb{Z}\cup\{\infty\})^P/\mathcal{F}, v^*, \pi^*)$,

where $\prod_p \mathbb{Q}_p/\mathcal{F}$ and $\prod_p \mathbb{F}_p/\mathcal{F}$ have their natural field structure, $\mathbb{Z}^P/\mathcal{F}$ the structure of an ordered group, with a distinguished constant ∞, and the maps v^* and π^* are defined by: $v^*((a_p)_\mathcal{F}) = (v(a_p))_\mathcal{F}$, $\pi^*((a_p)_\mathcal{F}) = (v(a_p))_\mathcal{F}$.

Example 3. Let I be an index set, $\mathcal{F}$ a ultrafilter on I, K_i, $i \in I$, a family of fields, and for each i, $R_i = (R_{n,i})$ the ω-sorted structure defined in (2.15).

Then $R^* = \prod_{i\in I} R_i/\mathcal{F}$ is the ω-sorted structure, with universe of sort n $R_n^* = \prod_{i\in I} R_{n,i}/\mathcal{F}$. The addition an multiplication are defined naturally. Observe that R^* is simply the ω-sorted structure associated to the polynomial ring in $X_1,\dots,X_m$ over $K^* = \prod_{i\in I} K_i/\mathcal{F}$.

Example 4 - Asymptotic cones. We start with a family of 2-sorted structures $\mathcal{X}_i = ((X_i, x_i, \cdots), (\Gamma_i, +, -, \leq, 0), d_i)$, where (X_i, d_i) is a metric space with a distinguished point x_i ($i \in I$). The values of d_i are in the subgroup Γ_i of $\mathbb{R}$. We allow X_i to have extra structure, for instance a group structure. Fix a non-principal ultrafilter $\mathcal{F}$ on I. Consider

$$\prod_{i\in I} \mathcal{X}_i/\mathcal{F} = (X^*, \Gamma^*, d^*),$$

where $(X^*, x^*, \cdots) = \prod_{i\in I}(X_i, x_i, \cdots)/\mathcal{F}$, $\Gamma^* = \prod_{i\in I}\Gamma_i/\mathcal{F}$ and $d^*((y_i)_\mathcal{F}, (z_i)_\mathcal{F}) = (d_i(y_i, z_i))_\mathcal{F}$. We consider the ordered subgroup Γ^{fin} and μ_Γ defined in (3.15), set

$$X^{fin} = \{y \in X^* \mid d^*(x^*, y) \in \Gamma^{fin}\},$$

and quotient X^{fin} by the equivalence relation

$$E(y, z) \iff d^*(y, z) \in \mu_\Gamma$$

to obtain a set $X_\mathcal{F}$. We call the map $X^{fin} \to X_\mathcal{F}$ the standard part, and denote it st. Then d^* induces a map $d_\mathcal{F} : X_\mathcal{F} \to \Gamma_\mathcal{F} = \Gamma^{fin}/\mu_\Gamma \subseteq R^{fin}/\mu = \mathbb{R}$. Observe that the structure $(X_\mathcal{F}, \mathbb{R}, d_\mathcal{F})$, being a quotient of the substructure $(X^{fin}, \Gamma^{fin}, d^*)$ of (X^*, Γ^*, d^*), will satisfy all quantifier free formulas (maybe with parameters) satisfied by (X^*, Γ^*, d^*). In particular, the map $d_\mathcal{F}$ will satisfy the axioms of a distance, so that $(X, d_\mathcal{F})$ is a metric space. If each (X_i, d_i) is δ_i-hyperbolic, then $(X_\mathcal{F}, d_\mathcal{F})$ will be $(\delta_i)_\mathcal{F}$-hyperbolic. If for every i, and $y, z \in X_i$, there is a distance preserving embedding $[0, d(y, z)] \cap \Gamma_i \to X_i$ which sends 0 to x and $d(y, z)$ to y, the same will be true for $(X_\mathcal{F}, d_\mathcal{F})$. We will give the proof, as this shows how flexible ultraproducts are: one can add structure if one needs it. Let $a, b \in X_\mathcal{F}$, be represented by $(a_i)_\mathcal{F}, (b_i)_\mathcal{F} \in X^*$ respectively. For each i, fix $f_i : [0, d_i(a_i, b_i)] \cap \Gamma_i \to X_i$ which is distance preserving and sends 0 to a_i, $d_i(a_i, b_i)$ to b_i. Then consider $f^* : [0, d^*((a_i)_\mathcal{F}, (b_i)_\mathcal{F}) \cap \Gamma^* \to X^*$ defined by $f^*((\gamma_i)_\mathcal{F}) = (f(\gamma_i))_\mathcal{F}$ for $(\gamma_i)_\mathcal{F} \in [0, d^*((a_i)_\mathcal{F}, (b_i)_\mathcal{F}) \cap \Gamma^*$. This map is distance preserving. Hence, as $(a_i)_\mathcal{F}, (b_i)_\mathcal{F} \in X^{fin}$, we may pass to the quotient, and get a map $f_\mathcal{F}$, distance preserving, and sending some segment $[0, \gamma]$ of $\Gamma_\mathcal{F}$ to $X_\mathcal{F}$, and with $f_\mathcal{F}(0) = a$, $f(\gamma) = b$.

If for each i, there is a group G_i acting on X_i, then the ultraproduct $G^* = \prod_{i\in I} G_i/\mathcal{F}$ will also act on X^*. But, unless there is an integer N such that for all $i \in I$, $g \in G_i$ and $y \in X_i$, one has $d_i(y, gy) \leq N d_i(x_i, y)$, the group G^* will not act on X^{fin}.

If however all groups G_i are equal, and for every $g \in G$, there is $N = N(g)$ such that for all $i \in I$ and $y \in X_i$, $d_i(y, gy) \leq N(g)d_i(x_i, y)$, then G will act on X^{fin} and also on $X_{\mathcal{F}}$.

Assume now that each X_i is in fact a group $(G, \cdot, ^{-1}, 1, a_1, \ldots, a_n)$, with $x_i = 1$, and that $d_i : G^2 \to \mathbb{R}$ equals $(1/m_i)d$, where d is the distance on the Cayley graph of G with respect to the set $\{a_1, \ldots, a_n\}$ of generators of G (we set $d(x, y) = d(1, yx^{-1})$, so that d is invariant under right multiplication). If the element $N = (m_i)_{\mathcal{F}}$ is infinite, then the image of $d_{\mathcal{F}}$ will be all of $\mathbb{R}$, provided of course our group G is infinite.

The equivalence relation E will correspond to being in the same left-coset of the subgroup $G(0) = \{g \in X^{fin} \mid d^*(1, g) \in \mu\}$ of X^{fin}. Also, both $G(0)$ and X^{fin} are convex subgroups of X^*, but $G(0)$ is not necessarily normal in X^{fin}, so that $X_{\mathcal{F}}$ does not necessarily have a group structure. However, the group X^{fin} acts (transitively) on $X_{\mathcal{F}}$.

4. Real closed fields

In this section we will study the theory ORCF of real closed fields, and show that it is complete, decidable, eliminates quantifiers, etc. We will also mention o-minimality and some quite striking results obtained in the past decade.

(4.1) **The theory ORCF**. This is the theory in the language $\mathcal{L} = \mathcal{L}_R \cup \{\leq\}$, which says the following:
(1) $\leq$ is an ordering, axioms for ordered fields,
(2) every polynomial in one variable of odd order has a root,
(3) every non-negative element is a square.

A model of ORCF is called a *real closed ordered field*. If one replaces axioms (1) and (3) by the following axioms:
(1') axioms for fields of characteristic 0
(3') an axiom expressing that the formula $\varphi(x, y) = \exists z\, (z^2 = y - x)$ defines an ordering $\leq$ on the field then one obtains a theory of the language $\mathcal{L}_R$, denoted RCF, the models of which are called *real closed fields*. Note that every real closed field expands in a unique fashion to a model of ORCF, as the order is definable.

We can also replace the scheme of axioms (2) and (3) by the axiom expressing the sign change property: if $P(X) \in F[X]$, and $a < b \in F$ are such that $P(a)P(b) < 0$, then there is c such that $a < c < b$ and $P(c) = 0$.

Clearly this axiom implies both (2) and (3).

(4.2) **Well-known facts**. To tell the truth, I don't know how well-known these facts are. The first two are easy.

Fact 1. *Let F be a real closed field, and F_0 the smallest subfield of F. Then the ordered field F_0 is isomorphic to the ordered field $\mathbb{Q}$.*

Fact 2. *Let F be a real closed field, F_0 a subfield of F, and assume that F_0 is relatively algebraically closed in F. Then F_0 is real closed.*

Fact 3. *Let F be an ordered field. Then F is contained in a ordered real closed field F' which is algebraic over F. Moreover, if F'' is another algebraic extension of F which is a model of ORCF, then the ordered fields F'' and F' are isomorphic over F.*

Thus the field F' (or F'') will be called the *real closure of the ordered field F*. The proof of this result is quite difficult, and involves Sturm's algorithm.

(4.3) **Tarski's theorem**. *The theory ORCF eliminates quantifiers*

Proof. It suffices, by induction, to show that a formula of the form $\exists y\, \varphi(\bar{x}, y)$, where $\varphi(\bar{x}, y)$ is quantifier-free, is equivalent modulo ORCF, to a quantifier-free formula, Let F_1 and F_2 be two ordered real closed fields, let $\bar{a}$ and $\bar{b}$ be tuples in F_1, F_2, respectively, of the same length, and assume that they satisfy the same quantifier-free formulas. Assume also that $F_2 \models \exists y\, \varphi(\bar{b}, y)$, and let $d \in F_2$ be such that $F_2 \models \varphi(\bar{b}, d)$. We will show that $F_1 \models \exists y\, \varphi(\bar{a}, y)$.

Our assumption that $\bar{a}$ and $\bar{b}$ implies that there is an $\mathcal{L}$-isomorphism f from the ordered subfield $\mathbb{Q}(\bar{a})$ of F_1 to the ordered subfield $\mathbb{Q}(\bar{b})$ of F_2, which sends $\bar{a}$ to $\bar{b}$. By facts 2 and 4, this isomorphism f extends to an isomorphism $f : A_1 \to A_2$, where A_1 is the relative algebraic closure of $\mathbb{Q}(\bar{a})$ in F_1 (and hence its real closure), and A_2 is the relative algebraic closure of $\mathbb{Q}(\bar{b})$ in F_2.

We will now work a little on the formula $\varphi(\bar{x}, y)$. We know that it is a finite disjunction of conjunction of formulas of the form

$$P(\bar{x}, y) = 0, \qquad P(\bar{x}, y) > 0, \tag{1}$$

where P is polynomial with integral coefficients. Since $(\bar{b}, d)$ will necessarily satisfy one of the disjuncts of φ, we may assume that $\varphi(\bar{x}, y)$ is a conjunction of formulas of the form given in (1). Moreover, we may assume that there is only one equation: one uses the fact that in an ordered field, $x_1 = x_2 = \cdots = x_n = 0$ is equivalent to $x_1^2 + x_2^2 + \cdots + x_n^2 = 0$. Hence, we may assume that the formula $\varphi(\bar{x}, y)$ is of the form

$$P(\bar{x}, y) = 0 \wedge \bigwedge_{i=1}^{n} Q_i(\bar{x}, y) > 0,$$

for some polynomials P and $Q_1, \ldots, Q_n$ with integral coefficients. If P is not the 0 polynomial, then d is algebraic over $\mathbb{Q}(\bar{b})$ and therefore belongs to A_2. Hence, as f is an isomorphism of ordered fields, $f^{-1}(d)$ satisfies $\varphi(\bar{a}, y)$, so that $F_1 \models \exists y\, \varphi(\bar{a}, y)$.

This reasoning works equally well if $d \in A_2$ (adding to $\varphi(\bar{b}, y)$ the equation $P'(\bar{b}, y) = 0$, where $P(\bar{b}, Y)$ is the minimal polynomial of d over $\mathbb{Q}(\bar{b})$, so that we may assume that $d \notin A_2$ and that P is the 0 polynomial. Then our formula is a conjunctions of inequalities, and none of the polynomials Q_i is the 0 polynomial.

We consider the set of solutions of the equation $\prod_{i=1}^{n} Q_i(\bar{b}, y) = 0$, and enumerate it as $\beta_1 < \beta_2 < \cdots < \beta_m$. Then $\beta_1, \ldots, \beta_m \in A_2$, and so d is in one of the open intervals $(-\infty, \beta_1), (\beta_1, \beta_2), \ldots, (\beta_m, +\infty)$. Observe that when restricted to any of these open intervals, the functions defined by the $Q_i(\bar{b}, y)$ do not change sign or take the 0-value. Also, $f^{-1}(\beta_1) < f^{-1}(\beta_2) < \cdots < f^{-1}(\beta_m)$ enumerate the roots of $\prod_{i=1}^{n} Q_i(\bar{a}, y) = 0$, and each of the corresponding intervals $(-\infty, f^{-1}(\beta_1)), (f^{-1}(\beta_1), f^{-1}(\beta_2)), \ldots, (f^{-1}(\beta_m), +\infty)$ of A_1 is non-empty. We choose $c \in A_1$ such that $c < f^{-1}(\beta_i) \iff d < \beta_i$. Then one checks that $Q_i(\bar{a}, c)$ and $Q_i(\bar{b}, d)$ have the same sign for $i = 1, \ldots, n$ and are non-zero, so that c satisfies $\varphi(\bar{a}, y)$ in A_1 and also in F_1.

We have therefore shown that the formula $\exists y\varphi(\bar{x}, y)$ and the set Δ of quantifier-free $\mathcal{L}$-formulas in $\bar{x}$ satisfy the criterion given in (2.11). Hence, the formula $\exists y\, \varphi(\bar{x}, y)$ is equivalent modulo ORCF to a quantifier-free formula $\psi(\bar{x})$. This implies that $\forall y\, \neg\varphi(\bar{x}, y)$ is equivalent modulo ORCF to the quantifier-free formula $\neg\psi(\bar{x})$. Applying an easy induction argument (i.e., eliminating quantifiers one at a time) gives that ORCF eliminates quantifiers.

(4.4) **Completeness of ORCF and decidability.** The quantifier-free $\mathcal{L}$-sentences are fairly simple: the only terms without variables are the terms of the form $1 + 1 + \cdots + 1$ and $-(1 + 1 + \cdots + 1)$. Therefore the quantifier-free sentences we have to decide are of the form $n < m$, $n = m$, for $n, m \in \mathbb{Z}$. This is clearly decidable and shows that the theory ORCF is complete (every model of ORCF contains a copy of the ordered field $\mathbb{Q}$). Since the axioms for ORCF are enumerable, the theory ORCF is decidable. Similarly for RCF. This gives

Theorem *(Tarski). The theories RCF and ORCF are complete and decidable.*

(4.5) **O-minimality**. Let $\mathcal{L}$ be a language containing a binary relation $\leq$. An $\mathcal{L}$-structure M is *o-minimal* iff the relation $\leq$ defines an ordering, and every M-definable subset of M is a finite union of points and of open intervals with their endpoints in $M \cup \{-\infty, +\infty\}$.

Thus the real closed field $(\mathbb{R}, +, -, \cdot, \leq, 0, 1)$ is o-minimal: this is essentially immediate using the quantifier-elimination of ORCF.

O-minimality is a very strong property. it is preserved under elementary equivalence, i.e., an $\mathcal{L}$-structure elementarily equivalent to an o-minimal one, is also o-minimal. Thus we will talk of an o-minimal theory, meaning that its models are o-minimal. O-minimality allows one to prove cell-decomposition theorems, which describe definable sets in terms of "basic cells". One of the very exciting developments of the past decade was the discovery of several o-minimal expansions of the ordered field of reals. A result rather simple to state is:

(Wilkie). The theory of the field of real numbers with exponentiation,
$(\mathbb{R}, +, -, \cdot, \exp, \leq, 0, 1)$*, is model complete and o-minimal.*

Macintyre and Wilkie also showed that, *assuming Schanuel's conjecture, the theory of the field of real numbers with exponentiation, is decidable.*