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The elementary theory of restricted analytic fields with exponentiation

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Introduction

In [16] and [17] Wilkie proved the remarkable result that the field of real numbers with exponentiation is model complete. When we combine this with Hovanskii's finiteness theorem [9], it follows that the real exponential field is o-minimal. In o-minimal expansions of the real field the definable subsets of $\mathbf{R}^n$ share many of the nice structural properties of semialgebraic sets. For example, definable subsets have only finitely many connected components, definable sets can be stratified and triangulated, and continuous definable maps are piecewise trivial (see [5]).

In this paper we will prove a quantifier elimination result for the real field augmented by exponentiation and all restricted analytic functions, and use this result to obtain o-minimality. We were led to this while studying work of Ressayre [13] and several of his ideas emerge here in simplified form. However, our treatment is formally independent of the results of [16], [17], [9], and [13].

An essential part of our approach is a re-examination of the model theory of the reals with restricted analytic functions. Let $\mathbf{R}\{X_1, \ldots, X_m\}$ denote the ring of all real power series in $X_1, \ldots, X_m$ that converge in a neighborhood of I^m, with $\mathrm{I} = [-1, 1]$. For $f \in \mathbf{R}\{X_1, \ldots, X_m\}$ we let $\widetilde{f}\colon \mathbf{R}^m \to \mathbf{R}$ be given by:

$$\widetilde{f}(x) = \begin{cases} f(x), & \text{for } x \in \mathrm{I}^m, \\ 0, & \text{for } x \notin \mathrm{I}^m. \end{cases}$$

We call the $\widetilde{f}$'s *restricted analytic functions*. Let L_{an} be the language of ordered rings $\{<, 0, 1, +, -, \cdot\}$ augmented by a new function symbol for each function $\widetilde{f}$. We let $\mathbf{R}_{\text{an}}$ be the reals with its natural L_{an}-structure and let T_{an} be the theory of $\mathbf{R}_{\text{an}}$.

In [4] van den Dries observed that T_{an} is model complete and o-minimal, as a consequence of Gabrielov's theorem [8] that the complement of a subanalytic

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set is subanalytic. Denef and van den Dries [2] proved the stronger result that $\mathbf{R}_{\mathrm{an}}$ admits quantifier elimination if we add a function symbol $^{-1}$ for $x \mapsto \frac{1}{x}$, where $0^{-1} = 0$ by convention.

With hindsight, we can say that some rather basic questions on the elementary theory of $\mathbf{R}_{\mathrm{an}}$ went unasked (and hence unanswered) until recently, when these questions became urgent after the work of Wilkie and Ressayre. In Section 2 we fill these gaps. In particular we give a complete axiomatization of T_{an}. This will allow us to show that certain generalized power series fields can naturally be expanded to models of T_{an}. In Section 3 we use these results to prove a key valuation theoretic fact about models of T_{an}.

Let $L_{\mathrm{an}}(\exp)$ be the language L_{an} with a new unary function symbol exp. Let $T_{\mathrm{an}}(\exp)$ be the theory obtained by adding to T_{an} the universal closures of the following axioms:

E1) $\exp(x+y) = \exp(x)\exp(y)$.
E2) $x < y \to \exp(x) < \exp(y)$.
E3) $x > 0 \to \exists y \exp(y) = x$.
E4$_n$) $x > n^2 \to \exp(x) > x^n$; for each natural number $n > 0$.
E5) $-1 \leq x \leq 1 \to \exp(x) = \mathrm{E}(x)$; where E is the function symbol of L_{an} corresponding to the exponential power series $\sum \frac{1}{n!} X^n \in \mathbf{R}\{X\}$.

Let log be a further unary function symbol and let $T_{\mathrm{an}}(\exp, \log)$ be the extension of $T_{\mathrm{an}}(\exp)$ given by the following defining axiom.

L) $(x > 0 \to \exp(\log x) = x) \wedge (x \leq 0 \to \log(x) = 0)$.

In Section 4 we will prove our main results:

1) $T_{\mathrm{an}}(\exp, \log)$ admits quantifier elimination;
2) $T_{\mathrm{an}}(\exp)$ is complete;
3) any function $f\colon \mathbf{R}^n \to \mathbf{R}$ definable in $(\mathbf{R}_{\mathrm{an}}, \exp)$ is given piecewise by terms in the language $L_{\mathrm{an}}(\exp, \log)$.

In Section 5 we use the quantifier elimination to give a very elementary proof that $(\mathbf{R}_{\mathrm{an}}, \exp)$ is o-minimal. This result was obtained earlier in [6] using an extension of Hovanskii's theorem, results of Wilkie [16] on noetherian differential rings of $\mathcal{C}^\infty$-functions, and the nontrivial fact that the rings $\mathbf{R}\{X_1, \ldots, X_m\}$ are noetherian. The present proof bypasses all this, and is based instead on generalities about Hardy fields.

1. Preliminaries

We begin with some preliminaries on valuations and power series.

(1.1) *The standard real valuation.* Let K be an ordered field. Let $\mathrm{Fin}(K)$ be the ring of elements bounded in absolute value by a rational number. Then

$\mathrm{Fin}(K)$ is a valuation ring with maximal ideal $\mu(K)$, the infinitesimals of K, and units

$$\mathrm{Un}(K) = \mathrm{Fin}(K) \setminus \mu(K).$$

We let v denote the associated valuation given by the quotient map $K^\times \to K^\times/\,\mathrm{Un}(K)$. We let $v(K^\times)$ denote the value group $K^\times/\,\mathrm{Un}(K)$ written additively.

Note that $v(x) = v(y)$ if and only if x and y are in the same archimedean class of K. The valuation reflects the ordering of K in that $0 < x \leq y$ implies $v(x) \geq v(y)$. The infinitesimals of K have positive value and the elements of K with infinite absolute value have negative value.

If $F \subseteq K$, we identify $v(F^\times)$ with a subgroup of $v(K^\times)$ in the obvious way.

(1.2) *Power series fields.* Fix a field k and an ordered abelian group Γ. The *power series field* $K = k((t^\Gamma))$ consists of all formal power series

$$x = \sum_{\gamma \in \Gamma} a_\gamma t^\gamma$$

with "exponents" $\gamma \in \Gamma$ and "coefficients" $a_\gamma \in k$, such that the *support* of x, $\mathrm{supp}(x) = \{\gamma \in \Gamma\colon\ a_\gamma \neq 0\}$, is a well-ordered subset of Γ. These series are added and multiplied in the usual way (with $t^\gamma \cdot t^\delta = t^{\gamma+\delta}$) and form a field (see [7] for proofs of elementary facts of this sort). We consider k as a subfield of K by identifying $c \in k$ with ct^0. Note that $\mathrm{ord}\colon K^\times \to \Gamma$ given by $\mathrm{ord}(x) = \min(\mathrm{supp}(x))$ is a valuation of K with valuation ring

$$k[[t^\Gamma]] = \{x \in k((t^\Gamma))\colon \quad \text{if } \gamma \in \mathrm{supp}(x) \text{ then } \gamma \geq 0\},$$

maximal ideal

$$\mu = \{x \in k((t^\Gamma))\colon \quad \text{if } \gamma \in \mathrm{supp}(x) \text{ then } \gamma > 0\},$$

and residue field k. This valuation is henselian (see for example [14]).

In fact K is maximal with value group Γ and residue field k (we will discuss this further in §3).

If k is an ordered field, then K can be ordered. Let $x = \sum a_\gamma t^\gamma$ and $y = \sum b_\gamma t^\gamma$. Then $x < y$ if $a_g < b_g$ where g is least such that $a_g \neq b_g$. We will often consider the special case when $k = \mathbf{R}$. In this case the valuation ord is equivalent to the valuation v given above.

We will also use the fact that a henselian valued field with real closed residue field and divisible value group is real closed (see [12]). Thus if k is real closed and Γ is divisible, then $K = k((t^\Gamma))$ is itself real closed.

Let $X = (X_1, \ldots, X_n)$ and let $k[[X]]$ denote the ring of power series $\sum c_i X^i$, where $i = (i_1, \ldots, i_n)$ ranges over $\mathbf{N}^n$, $c_i \in k$ and $X^i = X_1^{i_1} \cdots X_n^{i_n}$.

Let $\mu^n = \mu \times \ldots \times \mu \subseteq K^n$. If $a = (a_1, \ldots, a_n) \in \mu^n$ and $f = \sum c_i X^i \in k[[X]]$, then $f(a) = \sum c_i a^i$ is a well-defined element of K since only finitely many terms $c_i a^i$ contribute a nonzero coefficient to a given monomial t^γ and the union of the supports of the elements $c_i a^i$ is well-ordered. (The case $n = 1$ is treated in [7], and the general case is similar.) One checks easily that the map $f(X) \mapsto f(a)$ is a k-algebra homomorphism from $k[[X]]$ into $k[[t^\Gamma]]$.

2. An axiomatization of T_{an}

We let $\mathbf{R}\langle X_1, \ldots, X_n \rangle$ denote the ring of power series in $X_1, \ldots, X_n$ over $\mathbf{R}$ which converge in a neighborhood of 0 and we let $\mathbf{R}\{X_1, \ldots, X_n\}$ denote the subring of power series which converge on a neighborhood of I^n, $\mathrm{I} = [-1, 1]$. Note that for $n = 0$ both rings are just $\mathbf{R}$.

Let $K \supseteq \mathbf{R}$ be an ordered field. We follow the notation from (1.1) but write μ for $\mu(K)$. If $U \subseteq \mathbf{R}^n$ is open we let

$$U(K) = \{x \in K^n \colon\ x - a \in \mu^n \text{ for some } a \in U\} = \bigcup_{a \in U} a + \mu^n.$$

Note that $U \subseteq U(K)$ and $\mathbf{R}^n(K) = \mathrm{Fin}(K)^n$. We also set $I(K) = \{x \in K \colon\ -1 \leq x \leq 1\}$.

(2.1). Let $K \supseteq \mathbf{R}$ be an ordered field. We assume that for all f in $\mathbf{R}\langle X_1, \ldots, X_n \rangle$, $n \in \mathbf{N}$, K is equipped with a function $f_K \colon\ \mu^n \to K$, such that the following conditions are satisfied:

C1) $(f + g)_K = f_K + g_K$ and $(f \cdot g)_K = f_K \cdot g_K$ for $f, g \in \mathbf{R}\langle X_1, \ldots, X_n \rangle$, and c_K is the constant function $x \mapsto c$, for $c \in \mathbf{R} \subseteq \mathbf{R}\langle X_1, \ldots, X_n \rangle$;

C2) $(X_i)_K \colon\ \mu^n \to K$ is the i^{th} coordinate function $(x_1, \ldots, x_n) \mapsto x_i$, for $X_i \in \mathbf{R}\langle X_1, \ldots, X_n \rangle$;

C3) If $f \in \mathbf{R}\langle X_1, \ldots, X_n \rangle$ and $g_1, \ldots, g_n \in (X_1, \ldots, X_m)\mathbf{R}[X_1, \ldots, X_m]$ (i.e., the g_i have constant term zero), then

$$f(g_1, \ldots, g_n)_K(x) = f_K(g_1(x), \ldots, g_n(x))$$

for all $x \in \mu^m$.

(2.2). Condition C1) says that $f \mapsto f_K$ is an $\mathbf{R}$-algebra homomorphism from $\mathbf{R}\langle X_1, \ldots, X_n \rangle$ into K^{μ^n}.

Given $f \in \mathbf{R}\langle X_1, \ldots, X_n \rangle$ we may also regard f as an element of $\mathbf{R}\langle X_1, \ldots, X_{n+1} \rangle$ in which the variable X_{n+1} happens to be absent. In this way f_K becomes a function on μ^{n+1}. Fortunately, there is no real conflict, since the conditions easily imply $f_K(x_1, \ldots, x_{n+1}) = f_K(x_1, \ldots, x_n)$.

Note that $f_K(0) = f(0) \in \mathbf{R}$ for $f \in \mathbf{R}\langle X_1, \ldots, X_n \rangle$, since $f = f(0) + \sum_{i=1}^n X_i h_i$ for suitable $h_i \in \mathbf{R}\langle X_1, \ldots, X_n \rangle$.

(2.3) *Main Example.* Let Γ be an ordered abelian group and let $K = \mathbf{R}((t^{\Gamma}))$. As noted in (1.2), each $f \in \mathbf{R}[[X_1, \ldots, X_n]]$, and in particular each convergent $f \in \mathbf{R}\langle X_1, \ldots, X_n\rangle$, defines a function $a \mapsto f(a)$ from μ^n into K. We denote this function f_K. One verifies easily that conditions C1), C2) and C3) are satisfied.

Of course if $\Gamma = \{0\}$, we get $K = \mathbf{R}$, $\mu = \{0\}$ and $f_K(0) = f(0)$, for $f \in \mathbf{R}\langle X_1, \ldots, X_n\rangle$.

We now derive some useful consequences of C1)-C3).

LEMMA 2.4. *Let $f \in \mathbf{R}\langle X_1, \ldots, X_n\rangle$. Then*

$$f_K(\mu^n) \subseteq f(0) + \mu \subseteq \mathrm{Fin}(K).$$

Proof. Let $0 < \epsilon \in \mathbf{R}$. Note that $f - f(0) + \epsilon$ has constant term $\epsilon > 0$. Thus there is $g \in \mathbf{R}\langle X_1, \ldots, X_n\rangle$ such that $g^2 = f - f(0) + \epsilon$. Thus $f = f(0) - \epsilon + g^2$ and $f_K(x) = f(0) - \epsilon + (g_K(x))^2$. Thus $f_K(x) \geq f(0) - \epsilon$, for all $\epsilon > 0$ in $\mathbf{R}$ and $x \in \mu^n$. In the same way we get $f_K(x) \leq f(0) + \epsilon$ for all $\epsilon > 0$ in $\mathbf{R}$ and $x \in \mu^n$. $\square$

LEMMA 2.5. *The valued field K is henselian.*

Proof. If suffices to show that if $a_1, \ldots, a_n \in \mu$ and $q(T) = 1 + T + a_1T^2 + \ldots + a_nT^{n+1}$, then $q(T)$ has a zero in the valuation ring $\mathrm{Fin}(K)$.

Consider the polynomial $p(X_1, \ldots, X_n, T) = 1 + T + X_1T^2 + \ldots + X_nT^{n+1}$ in $\mathbf{R}\langle X\rangle[T]$, where $X = (X_1, \ldots, X_n)$. Since $p(0, -1) = 0$ and $\frac{\partial p}{\partial T}(0, -1) = 1$, the implicit function theorem gives us a power series $\alpha(X) \in \mathbf{R}\langle X\rangle$ with $\alpha(0) = -1$ and $p(X, \alpha(X)) = 0$. Substituting $a = (a_1, \ldots, a_n) \in \mu^n$, $\alpha_K(a)$ is a zero of $q(T)$. Moreover, $\alpha_K(a) \in \mathrm{Fin}(K)$, by the previous lemma. $\square$

COROLLARY 2.6. *If each positive element of K has an n^{th} root, for $n = 2,3,\ldots$, then K is real closed.*

Proof. Since every positive element of K has an n^{th} root, the value group is divisible. Since K has residue field $\mathbf{R}$ and K is henselian, it follows that K itself is real closed. $\square$

(2.7). For open, nonempty $U \subseteq \mathbf{R}^n$, let $\mathrm{An}(U)$ be the $\mathbf{R}$-algebra of real analytic functions $f\colon U \to \mathbf{R}$. We assign to each $f \in \mathrm{An}(U)$ a function $f_K\colon U(K) \to K$ as follows:

Given $a \in U$, let

$$f_a(X) = \sum \frac{1}{i!}\frac{\partial^{|i|} f}{\partial X^i}(a)X^i \in \mathbf{R}\langle X\rangle$$

be the Taylor series of f at a. Then we put

$$f_K(a+x) = (f_a)_K(x)$$

for $x \in \mu^n$.

The following analogues of C1)-C3) hold:

C1)$_U$ $(f+g)_K = f_K + g_K$, $(f \cdot g)_K = f_K \cdot g_K$ for $f, g \in \mathrm{An}(U)$ and c_K is the constant function $x \mapsto c$ for $c \in \mathbf{R} \subseteq \mathrm{An}(U)$.

C2)$_U$ If X_i denotes the i^{th} coordinate function $(x_1, \dots, x_n) \mapsto x_i$ from U to $\mathbf{R}$, then $(\mathrm{X}_i)_K$ is the i^{th} coordinate function from $U(K)$ to K.

C3)$_{U,V}$ Given $f \in \mathrm{An}(U)$, polynomials $g_1, \dots, g_n \in \mathbf{R}[X_1, \dots, X_m]$, and a nonempty open $V \subseteq \mathbf{R}^m$ with $g(V) \subseteq U$ for $g = (g_1, \dots, g_n)\colon V \to \mathbf{R}^n$, the function $f \circ g \in \mathrm{An}(V)$ satisfies $(f \circ g)_K(x) = f_K(g_1(x), \dots, g_n(x))$, for $x \in V(K)$.

To check C3)$_{U,V}$, use the fact that $(f \circ g)_a = f_{g(a)}(g_a - g(a))$ for $a \in V$.

Note that for $f \in \mathrm{An}(U)$ and $V \subseteq U$ open and nonempty, we have $f_K \mid U = f$ and $(f \mid V)_K = f_K \mid V(K)$. If $K = \mathbf{R}$, then $f_K = f$.

(2.8). We now assign to each series $f \in \mathbf{R}\{X_1, \dots, X_n\}$ a function $\widetilde{f}_K$ from K^n into K:

Take a real analytic $\widehat{f} \in \mathrm{An}(U)$ for some open neighborhood $U \subseteq \mathbf{R}^n$ of I^n such that $\widehat{f}(x) = f(x)$ for all $x \in \mathrm{I}^n$, and put

$$\widetilde{f}_K(x) = \begin{cases} \widehat{f}_K(x) & \text{for } x \in \mathrm{I}(K)^n, \\ 0 & \text{for } x \notin \mathrm{I}(K)^n. \end{cases}$$

Note that $\widetilde{f}_K$ does not depend on the choice of $\widehat{f}$.

By associating to each $f \in \mathbf{R}\{X_1, \dots, X_n\}$ the function $\widetilde{f}_K\colon K^n \to K$, we make K into an L_{an}-structure such that $\mathbf{R}_{\mathrm{an}} \subseteq K$. We will also consider K occasionally as an $L_{\mathrm{an}}(^{-1})$-structure, where $^{-1}$ is a unary function symbol interpreted as multiplicative inverse, with $0^{-1} = 0$.

In particular for any ordered abelian group Γ, the ordered power series field $K = \mathbf{R}((t^\Gamma))$ has a natural expansion to an L_{an}-structure. We denote this expansion $\mathbf{R}((t^\Gamma))_{\mathrm{an}}$.

An easy variant of Theorem 4.6 in [2] says that $(\mathbf{R}_{\mathrm{an}}, ^{-1})$ admits quantifier elimination. By going carefully through the proof of this theorem one checks that it goes through with minor modifications for the L_{an}-structure K, provided K is real closed. More precisely:

PROPOSITION 2.9. *Let $\theta(X_1, \dots, X_m)$ be an $L_{\mathrm{an}}(^{-1})$-formula. There is a quantifier-free $L_{\mathrm{an}}(^{-1})$-formula $\theta^*(X_1, \dots, X_m)$ depending on θ but not on K, such that if K is real closed, then $K \models \theta \leftrightarrow \theta^*$.*

(2.10). The key step in [2] is Basic Lemma 4.10. Its proof uses at certain points that I and its powers I^n are compact. Of course, $\mathrm{I}(K^n)$ is not compact (if $K \neq \mathbf{R}$), but we do not need this. The compactness of I^n is enough since if I^n is covered by finitely many open balls $U_\lambda = \{x \in \mathbf{R}^n : \|x - c_\lambda\| < \epsilon_\lambda\}$ ($\lambda \in \Lambda$, where Λ is finite), then $\mathrm{I}(K)^n$ is covered by the corresponding open sets $U_\lambda(K) = \{x \in K^n : \|x - c_\lambda\| \leq \delta$ for some $\delta \in \mathbf{R}$, $0 < \delta < \epsilon_\lambda\}$.

COROLLARY 2.11. *If K is real closed, then $\mathbf{R}_{\mathrm{an}} \preceq K$. In particular $\mathbf{R}_{\mathrm{an}} \preceq \mathbf{R}((t^\Gamma))_{\mathrm{an}}$, if Γ is divisible.*

Proof. We have $\mathbf{R}_{\mathrm{an}} \subseteq K$ as $L_{\mathrm{an}}(^{-1})$-structures, and by (2.10) they are models of a common $L_{\mathrm{an}}(^{-1})$-theory with quantifier elimination. □

(2.12). We now give the promised axiomatization of T_{an} and begin with three axiom schemes corresponding to the conditions C1)-C3). As usual omitting initial universal quantifiers, we let $\bar{x} = (x_1, \ldots, x_m)$.

AC 1) For $f, g \in \mathbf{R}\{X_1, \ldots, X_m\}$, $m \in \mathbf{N}$

$$\widetilde{f+g}(\bar{x}) = \widetilde{f}(\bar{x}) + \widetilde{g}(\bar{x});$$
$$\widetilde{fg}(\bar{x}) = \widetilde{f}(\bar{x}) \cdot \widetilde{g}(\bar{x});$$
$$\bigwedge_{i=1}^{m} |x_i| \leq 1 \to \widetilde{0}(\bar{x}) = 0 \wedge \widetilde{1}(\bar{x}) = 1;$$

and

$$\bigvee_{i=1}^{m} |x_i| > 1 \to \widetilde{0}(\bar{x}) = \widetilde{1}(\bar{x}) = 0,$$

where $\widetilde{0}$ and $\widetilde{1}$ are the function symbols corresponding to the elements 0 and 1 of $\mathbf{R}\{X_1, \ldots, X_m\}$.

AC 2)

$$\bigwedge_{i=1}^{m} |x_i| \leq 1 \to \widetilde{X_j}(\bar{x}) = x_j,$$
$$\bigvee_{i=1}^{m} |x_i| > 1 \to \widetilde{X_j}(\bar{x}) = 0,$$

where X_j is considered as an element of $\mathbf{R}\{X_1, \ldots, X_m\}$, $1 \leq j \leq m$.

AC3) For $f \in \mathbf{R}\{X_1, \ldots, X_n\}$ and $g_1, \ldots, g_n \in \mathbf{R}[X_1, \ldots, X_m]$ such that $g_i(0, \ldots, 0) = 0$, $f(g_1, \ldots, g_n) \in \mathbf{R}\{X_1, \ldots, X_m\}$, and $g(\mathrm{I}^m) \subseteq \mathrm{I}^n$, where $g = (g_1, \ldots, g_n) \colon \mathbf{R}^m \to \mathbf{R}^n$:

$$\bigwedge_{i=1}^{m} |x_i| \leq 1 \to \widetilde{f(g_1, \ldots, g_n)}(\bar{x}) = \widetilde{f}(g_1(\bar{x}), \ldots, g_n(\bar{x})).$$

(2.13). Let K be an L_{an}-structure which is an ordered field and a model of AC1), AC2), AC3). We let $\widetilde{f}_K$ be the interpretation of $\widetilde{f}$ in K. If $c \in \mathbf{R}$ and $\widetilde{c}$ is the constant symbol associated to c viewed as an element of $\mathbf{R}\{X_1, \dots, X_m\}$ for $m = 0$, then $c \mapsto \widetilde{c}_K$ defines an ordered field embedding of $\mathbf{R}$ into K. Identifying $\mathbf{R}$ with its image under this embedding, we consider K as an ordered field extension of $\mathbf{R}$.

We next associate to each $f \in \mathbf{R}\langle X_1, \dots, X_m\rangle$ a function $f_K \colon \mu^m \to K$. Choose $\varepsilon > 0$ such that $f_\varepsilon \colon = f(\varepsilon X_1, \dots, \varepsilon X_m) \in \mathbf{R}\{X_1, \dots, X_m\}$. For $a \in \mu^m$, let $f_K(a) = (\widetilde{f_\varepsilon})_K(\frac{a}{\varepsilon})$.

It is routine to check that f_K does not depend on the choice of ε, and that the assignment $f \mapsto f_K$ satisfies C1)-C3). This assignment and the construction from (2.7) and (2.8) gives us a way of associating to each f in $\mathbf{R}\{X_1, \dots, X_m\}$ a function $f_K^\sharp \colon K^m \to K$. We would like to have $f_K^\sharp = \widetilde{f}_K$. To get this we need a further set of universal axioms.

AC4) For $f, g \in \mathbf{R}\{X_1, \dots, X_m\}$, $0 < \varepsilon \in \mathbf{R}$, $a = (a_1, \dots, a_m) \in \mathrm{I}^m$, such that $g = f_a(\varepsilon X_1, \dots, \varepsilon X_m)$, where $f_a = \sum \frac{1}{i!} \frac{\partial^{|i|} f}{\partial X^i}(a) X^i \in \mathbf{R}\langle X_1, \dots, X_m\rangle$ is the Taylor series of f at a:

$$\left(\bigwedge_{i=1}^{m} |x_i| \leq 1 \;\&\; \bigwedge_{i=1}^{m} |\widetilde{a}_i + \widetilde{\varepsilon} x_i| \leq 1\right) \to \widetilde{f}(\widetilde{a}_1 + \widetilde{\varepsilon} x_1, \dots, \widetilde{a}_m + \widetilde{\varepsilon} x_m) = \widetilde{g}(\bar{x}),$$

where the $\widetilde{a}_i$'s and $\widetilde{\varepsilon}$ are the constant symbols associated to the a_i and ε.

Clearly $\mathbf{R}_{\mathrm{an}}$ satisfies AC4), and if K also satisfies AC4), then clearly $f_K^\sharp = \widetilde{f}_K$.

Combining (2.12) and (2.13) with (2.6), (2.9) and (2.11), we can draw the following conclusion:

THEOREM 2.14. *The theory T_{an} is axiomatized by*

1) *the axioms for ordered fields,*

2) *the universal axioms* AC1)-AC4),

3) *for $n = 2,3, \dots$ the axiom saying that each positive element has an n^{th} root.*

In particular if $K \models T_{\mathrm{an}}$, then any substructure of K which is closed under n^{th}-roots and division is itself a model of T_{an}.

By [2, Section 4], we have quantifier elimination in the language $L_{\mathrm{an}}(^{-1})$ for the theory T_{an} extended by the defining axiom for $^{-1}$. Adding also the unary function symbols $\sqrt[n]{\ }$ for $n = 2, 3, \dots$ and the defining axioms

$$(x > 0 \to ((\sqrt[n]{x})^n = x \wedge \sqrt[n]{x} > 0)) \wedge (x \leq 0 \to \sqrt[n]{x} = 0)$$

we see that the extension by definitions of T_{an} obtained in this way has a universal axiomatization.

Combining this with quantifier elimination gives:

COROLLARY 2.15. *For each function* $f\colon \mathbf{R}^n \to \mathbf{R}$ *definable in* $\mathbf{R}_{\mathrm{an}}$ *there are* $L_{\mathrm{an}}({}^{-1},(\sqrt[n]{\ })_{n=2,3,\ldots})$*-terms* $t_1(x_1,\ldots,x_n),\ldots,t_k(x_1,\ldots,x_n)$ *such that for all* $a \in \mathbf{R}^n$ *there is an* i *with* $f(a) = t_i(a)$. *(We say that* f *is piecewise given by the terms* $t_1,\ldots,t_k$.*)*

Proof. Suppose not. Let $\phi(\bar{x},y)$ be the formula defining "$f(\bar{x}) = y$". Then the type $\Sigma(\bar{x}) = \{\neg\phi(\bar{x},t(\bar{x}))\colon t \text{ a term}\}$ is consistent. Let $M \models T_{\mathrm{an}}$ be a model containing a realization $\bar{a}$ of $\Sigma(\bar{x})$. Let N be the $L_{\mathrm{an}}({}^{-1},(\sqrt[n]{\ })_{n=2,3,\ldots})$-substructure of M generated by $\bar{a}$. Since T_{an} has a universal axiomatization in this language, $N \models T_{\mathrm{an}}$. By quantifier elimination $N \preceq M$. But N models $\forall y\ \neg\phi(\bar{a},y)$ and M models $\forall\bar{x}\exists y\phi(\bar{x},y)$, a contradiction. □

3. Valuation theoretic properties of models of T_{an}

(3.1). Let $M \subseteq N$ be models of T_{an}. If $y \in N \setminus M$, we let $M\langle y\rangle \preceq N$ denote the definable closure of $M \cup \{y\}$ in N. By *o*-minimality ([11]), the type of y over M is determined by the cut y makes in the ordering of M. Thus if N' is a second elementary extension of M, $z \in N' \setminus M$, such that for all $m \in M$ we have $m < y$ if and only if $m < z$, and $M\langle z\rangle$ is the definable closure of $M \cup \{z\}$, then there is an L_{an}-isomorphism of $M\langle y\rangle$ onto $M\langle z\rangle$ fixing M and sending y to z.

Our main goal in this section is to show that the value group $v(M\langle y\rangle^\times)$ is equal to the value group of the real closure of $M(y)$ (the field generated by M and y), which is the divisible hull of $v(M(y)^\times)$, where v is the standard real valuation of (1.1).

We will prove this by examining embeddings of models of T_{an} into power series models. We begin by proving several lemmas on extending embeddings. Our argument is essentially the proof that $k((t^\Gamma))$ is maximal with residue field k and value group Γ (see [14]).

(3.2). If K is an ordered field and $\Gamma = v(K^\times)$, we call $s\colon \Gamma \to K^\times$ a *section* if s is a homomorphism from Γ to the multiplicative group of K and $v(s(g)) = g$ for all $g \in \Gamma$. The following argument shows that if every positive element of K has an n^{th} root for all n, then there is always a section s. Let $(g_j)_{j\in J}$ be a basis for Γ as a $\mathbf{Q}$-vector space. For each $j \in J$, let $b_j \in K$ such that $v(b_j) = g_j$ and $b_j > 0$. Then define s by $s(\sum q_j g_j) = \prod b_j^{q_j}$ for $q_j \in \mathbf{Q}$.

LEMMA 3.3. *Let* $M,N \models T_{\mathrm{an}}$ *with* $M \subseteq N$. *Let* Γ *be the value group of* M. *Let* $s\colon \Gamma \to M^\times$ *be a section and suppose we have an* L_{an}*-embedding* $\tau\colon M \to \mathbf{R}((t^\Gamma))$, *with* $\tau(s(g)) = t^g$ *for all* $g \in \Gamma$. *If* $y \in N \setminus M$ *and*

$v(M(y)^{\times}) = \Gamma$, *then we can extend* τ *to an* L_{an}*-embedding from* $M\langle y\rangle$ *into* $\mathbf{R}((t^{\Gamma}))$.

Proof. We view M as an L_{an}-substructure of $\mathbf{R}((t^{\Gamma}))_{\mathrm{an}}$ by identifying M with its image under τ. By remark (3.1) it suffices to find an element of $\mathbf{R}((t^{\Gamma}))$ in the cut of y over M.

We construct a sequence $(x_\alpha\colon \alpha < \delta)$ of approximations to y, where δ is an ordinal which is yet to be determined. Let $g_\alpha = v(x_\alpha - y)$. We choose δ and $(x_\alpha\colon \alpha < \delta)$ such that $g_\alpha < g_\beta$ for $\alpha < \beta$, and for all $z \in M$, there is an $\alpha < \delta$ such that $v(z - y) < g_\alpha$.

Let $x_0 = 0$.

Given $x_\alpha \in M$, let $g_\alpha = v(y - x_\alpha)$ and let a_α be the residue of $\frac{y - x_\alpha}{t^{g_\alpha}}$. We let $x_{\alpha+1} = x_\alpha + a_\alpha t^{g_\alpha}$.

Let α be a limit ordinal such that we have constructed x_β for $\beta < \alpha$. There are two cases to consider.

Case 1. There is $z \in M$ such that for all $\beta < \alpha$, $v(y - z) > g_\beta$.
In this case pick some such z, let $x_\alpha = z$, and continue.

Case 2. There is no such z.
In this case let $\delta = \alpha$. This completes the construction of $(x_\alpha\colon \alpha < \delta)$.

For $\alpha < \delta$, let $x_\alpha = \sum a_{\alpha,g} t^g$. If $\alpha < \beta$, then $v(x_\beta - x_\alpha) = g_\alpha$. Thus $a_{\alpha,g} = a_{\beta,g}$ for all $g < g_\alpha$. If $g \in \Gamma$ and $g < g_\alpha$ for some $\alpha < \delta$, then let $b_g = a_{\alpha,g}$. Otherwise let $b_g = 0$.

Let $w = \sum b_g t^g$. It is easy to check that $\mathrm{supp}(w)$ is well-ordered and for all $\alpha < \delta$, $v(w - x_\alpha) = g_\alpha$. We claim that w and y realize the same cut over M. Suppose not. Without loss of generality assume there is $m \in M$ with $w < m < y$; then $v(y - m) > g_\alpha$ for all $\alpha < \delta$, a contradiction. $\square$

We next examine the case when the value group does extend. The next claim is a general fact about real closed fields.

LEMMA 3.4. *Let* K *and* F *be real closed fields with* $K \subseteq F$. *Let* $y \in F \setminus K$. *If* $v(K(y)^{\times}) \neq v(K^{\times})$, *then for some* $a \in K$, $v(y - a) \notin v(K^{\times})$.

Proof. We know that for some monic polynomial $p(X) \in K[X]$, $v(p(y))$ is not in $v(K^{\times})$. Since K is a real closed field we can find $b_1, \dots, b_m, c_1, \dots, c_n, d_1, \dots, d_n \in K$ such that:

$$p(X) = \prod_{i=1}^{m} (X - b_i) \prod_{j=1}^{n} ((X - c_j)^2 + d_j^2).$$

Suppose that for all i and j, $v(y - b_i)$ and $v(y - c_j) \in v(K^{\times})$. For some j we must have $v((y - c_j)^2 + d_j^2) \notin v(K^{\times})$. In this case $v(y - c_j) = v(d_j)$ and

$v((y-c_j)^2+d_j^2) > v(d_j^2)$. On the other hand $(y-c_j)^2+d_j^2 > d_j^2 > 0$. Thus $v((y-c_j)^2+d_j^2) \le v(d_j^2)$, a contradiction. □

(3.5). Let M, N, Γ, s and τ be as in (3.3). Let $y \in N \setminus M$ be such that $v(M(y)^\times) \neq \Gamma$. We wish to extend our embedding to $M\langle y\rangle$. By (3.4) there is an $a \in M$ such that $v(y-a) \notin v(M^\times)$. Thus without loss of generality (by replacing y with $y-a$) we may assume $v(y) = g \notin \Gamma$. We may also assume $y > 0$. Let Γ_1 be the divisible subgroup of $v(N^\times)$ generated by Γ and g. We can extend s to $s_1 \colon \Gamma_1 \to M\langle y\rangle^\times$ by $s_1(\gamma+qg) = s(\gamma)y^q$ for $\gamma \in \Gamma$ and $q \in \mathbf{Q}$. For $m \in M$ with $m > 0$, $m < y$ if and only if $v(m) > g$ if and only if $m < t^g$. Thus we can extend τ to an L_{an}-embedding from $M\langle y\rangle$ into $\mathbf{R}((t^{\Gamma_1}))$ such that $\tau(s(h)) = t^h$ for all $h \in \Gamma_1$.

COROLLARY 3.6. *If $M \models T_{\text{an}}$ and $v(M^\times) = \Gamma$, then there is an L_{an}-embedding τ of M into $\mathbf{R}((t^\Gamma))_{\text{an}}$. Moreover for any section $s \colon \Gamma \to M^\times$, there is a τ such that $\tau(s(g)) = t^g$ for all $g \in \Gamma$.*

Proof. Start with $M_0 = \mathbf{R}_{\text{an}}$, $\Gamma_0 = \{0\}$, $s(0) = 1$ and τ the identity; then iterate (3.3) and (3.5). Note that we need only apply (3.5) to elements $y = s(g)$. □

We can now prove the main result of this section.

COROLLARY 3.7. *Suppose $M, N \models T_{\text{an}}$ and $y \in N \setminus M$. Then $v(M\langle y\rangle^\times)$ is the divisible hull of the value group of $M(y)$.*

Proof. Let $\Gamma = v(M^\times)$. By (3.6) there are an L_{an}-embedding of M into $\mathbf{R}((t^\Gamma))_{\text{an}}$ and a section s with $\tau(s(g)) = t^g$. If $v(M(y)^\times) = \Gamma$, then, by (3.3), we can extend τ to an embedding of $M\langle y\rangle$ into $\mathbf{R}((t^\Gamma))_{\text{an}}$. Thus $v(M\langle y\rangle^\times) = \Gamma$. Otherwise, let Γ_1 be the divisible hull of $v(M(y)^\times)$. Then (3.5) allows us to extend τ to an embedding of $M\langle y\rangle$ into $\mathbf{R}((t^{\Gamma_1}))_{\text{an}}$. In this case $v(M\langle y\rangle^\times) = \Gamma_1$. □

(3.8). Since the $\mathbf{Q}$-linear dimension of the divisible hull of $v(M(y)^\times)$ over $v(M^\times)$ is at most one, (3.7) implies that the $\mathbf{Q}$-linear dimension of $v(M\langle y\rangle^\times)$ over $v(M^\times)$ is at most 1. This result also follows from a theorem of Wilkie [17] on "smooth" theories in view of the fact that T_{an} is smooth (cf. [6]).

(3.9). We say that $\mathcal{R}$ is a *restricted analytic expansion* of $\mathbf{R}$ if

$$\mathcal{R} = (\mathbf{R}, <, +, -, \cdot, (f_j)_{j \in J})$$

for some index set J and functions $f_j \colon \mathbf{R}^{n_j} \to \mathbf{R}$ definable in $\mathbf{R}_{\text{an}}$. Let T be the theory of $\mathcal{R}$, a restricted analytic expansion of $\mathbf{R}$. If Γ is a divisible ordered abelian group, we can naturally view $\mathbf{R}((t^\Gamma))$ as a model of T.

The following version of (3.7) holds for T:

If $N \models T$, $M \preceq N$ and $y \in N \setminus M$, then the value group of the model of T generated by $M \cup \{y\}$ in N is equal to the divisible hull of the value group of the field $M(y)$.

For example this applies to $\mathbf{R}_e = (\mathbf{R}, <, +, -, \cdot, 0, 1, e)$ where

$$e(x) = \begin{cases} \exp(x) & -1 \leq x \leq 1, \\ 0, & \text{otherwise} \end{cases}.$$

The proof follows the same lines as the arguments above, though some care is needed if the residue field of M is not all of $\mathbf{R}$. Let $M \models T$. Since the prime model of T is archimedean, M has a maximal archimedean elementary submodel K. One easily checks the following two facts:

K is isomorphic to an elementary submodel of $\mathcal{R}$.

Every element of $\mathrm{Fin}(M)$ is infinitely close to an element of K. Thus K is isomorphic to the residue field of M.

The arguments adapt to prove that if Γ is the value group of M, then there is an elementary embedding $\tau\colon M \to \mathbf{R}((t^\Gamma))$ of T-models. If K is a maximal archimedean elementary submodel of M and $s\colon \Gamma \to M^\times$ is a section, we can choose τ such that $\tau(K) \subseteq \mathbf{R}$ and $\tau(s(g)) = t^g$ for all $g \in \Gamma$. Moreover, if $M \subseteq N$, $L \supseteq K$ is a maximal archimedean elementary submodel of N, $\Gamma_1 \supseteq \Gamma$ is the value group of N, and $s_1\colon \Gamma_1 \to N^\times$ is a section extending s, then we can extend τ to an embedding of T-models $\tau_1\colon N \to \mathbf{R}((t^{\Gamma_1}))$ where $\tau_1(L) \subseteq \mathbf{R}$ and $\tau_1(s_1(g)) = t^g$ for all $g \in \Gamma_1$.

4. The theory of $(\mathbf{R}_{\mathrm{an}}, \exp)$

Let $T_{\mathrm{an}}(\exp)$ be the $L_{\mathrm{an}}(\exp)$ theory described in the introduction. If $K \models T_{\mathrm{an}}(\exp)$, we define log on K by $\exp(\log x) = x$ for $x > 0$, while $\log(x) = 0$ for $x \leq 0$. In this section we will prove that $T_{\mathrm{an}}(\exp)$ has quantifier elimination in the language $L_{\mathrm{an}}(\exp, \log)$.

Let $K \models T_{\mathrm{an}}(\exp)$. We write $F \subseteq_{\mathrm{an}} K$ to indicate that F is an L_{an}-substructure of K, and in this case we say that F is *log-closed* if $\log(x) \in F$ for all $x \in F$. If $L \models T_{\mathrm{an}}(\exp)$, $F \subseteq_{\mathrm{an}} K$, and F is log-closed, then we say that $\sigma\colon F \to L$ is a *log-preserving embedding* if σ is an L_{an}-embedding and $\log(\sigma(x)) = \sigma(\log(x))$ for all $x \in F$.

We will prove the following embedding theorem.

THEOREM 4.1. *Suppose $K \models T_{\mathrm{an}}(\exp)$, $F_0 \subseteq_{\mathrm{an}} K$ is log-closed and $F_0 \models T_{\mathrm{an}}$. If L is a $|K|^+$-saturated model of $T_{\mathrm{an}}(\exp)$ and $\sigma_0\colon F_0 \to L$ is a log-preserving embedding, then σ_0 can be extended to a log-preserving embedding of K into L.*

Theorem 4.1 will follow from three lemmas on extensions of embeddings. These lemmas were inspired by the ideas of Ressayre [13].

For $F \subseteq_{\text{an}} K$ and $y \in K \setminus F$, we let $F\langle y\rangle$ denote the T_{an}-definable closure of $F \cup \{y\}$ in K.

LEMMA 4.2. *Let* K,L,F_0 *and* σ_0 *be as in* (4.1). *Suppose* $x \in K \setminus F_0$ *and* $v(F_0(x)^\times) = v(F_0^\times)$. *Let* $F = F_0\langle x\rangle$. *Then* F *is log-closed and* σ_0 *can be extended to a log-preserving embedding* $\sigma\colon F \to L$.

Proof. By (3.7) we know that $v(F^\times) = v(F_0^\times)$. Let $0 < w \in F$. There are $z \in F_0$ and $\varepsilon \in F$ such that $v(\varepsilon) > 0$ and $w = z(1+\varepsilon)$. Then $\log(w) = \log(z) + \log(1+\varepsilon)$. Since F_0 is log-closed, $\log(z) \in F_0$. Since log is analytic at 1, there is an L_{an}-term l, such that for $v(\delta) > 0$, $l(\delta) = \log(1+\delta)$. Thus $\log(1+\varepsilon) \in F$ and $\log(w) \in F$. Hence F is log-closed.

Let $y \in L$ realize the image under σ_0 of the cut of x over F. By o-minimality, σ_0 extends to an L_{an}-embedding $\sigma\colon F \to L$ with $\sigma(x) = y$. For $w \in F$ choose z and ε as above. Then $\sigma(w) = \sigma(z)(1+\sigma(\varepsilon))$. Since σ_0 is log-preserving and σ is an L_{an}-embedding, $\log\sigma(z) = \sigma(\log z)$ and $\log(1+\sigma(\varepsilon)) = \sigma(\log(1+\varepsilon))$. Thus $\log\sigma(w) = \sigma(\log w)$, so σ is log-preserving. $\square$

Iterating (4.2) allows us to extend our embedding from F_0 to a log-closed model $F \subseteq_{\text{an}} K$ of T_{an} such that $v(F^\times) = v(F_0^\times)$ and for all $y \in K \setminus F$, $v(F(y)^\times) \neq v(F^\times)$. Once we have done this we close under exponentiation.

LEMMA 4.3. *Let* K,L,F_0 *and* σ_0 *be as in* (4.1). *Suppose that* $v(F_0(x)^\times) \neq v(F_0^\times)$ *for all* $x \in K \setminus F_0$. *Suppose* $x \in F_0$ *and* $\exp x \notin F_0$. *Let* $F = F_0\langle \exp x\rangle$. *Then* F *is log-closed and* σ_0 *may be extended to a log-preserving embedding* $\sigma\colon F \to L$ *with* $\sigma(\exp x) = \exp\sigma(x)$.

Proof. We first claim that $v(\exp x) \notin v(F_0^\times)$. Otherwise, there are $u \in F_0$ and $\varepsilon \in F$, with $v(\varepsilon) > 0$ and $\exp x = u(1+\varepsilon)$. Let $w = \log u \in F_0$. Then $\exp(x-w) = 1+\varepsilon$, and $v(x-w) > 0$. But then, since F_0 is closed under restricted analytic functions, $\exp(x-w) \in F_0$ and $\exp x \in F_0$, a contradiction.

Let $g = v(\exp x) \notin v(F_0^\times)$. By (3.7), $v(F^\times) = v(F_0^\times) \oplus \mathbf{Q}g$. Let $0 < w \in F$. There are $a \in F_0$, $q \in \mathbf{Q}$ and $\varepsilon \in F$ such that $v(\varepsilon) > 0$ and $w = a(1+\varepsilon)\exp(qx)$. Then $\log w = \log a + \log(1+\varepsilon) + qx \in F$.

We claim that $\exp\sigma_0(x)$ realizes the image under σ_0 of the cut of $\exp x$ over F_0. Let $w \in F_0$, with $w > 0$. Then

$$\begin{aligned} w < \exp x &\Leftrightarrow \log w < x \\ &\Leftrightarrow \sigma_0(\log w) < \sigma_0(x) \\ &\Leftrightarrow \log(\sigma_0(w)) < \sigma_0(x) \\ &\Leftrightarrow \sigma_0(w) < \exp\sigma_0(x). \end{aligned}$$

Hence we can extend σ_0 to an L_{an}-embedding $\sigma\colon F \to L$ with $\sigma(\exp x) = \exp(\sigma_0(x))$. It is easy to see that σ is log-preserving. $\square$

By iterating (4.2) and (4.3) we can keep extending the embedding until we reach a situation where 1) there is no way to extend F_0 without extending the value group and 2) F_0 is closed under exponentiation.

LEMMA 4.4. *Let K,L,F_0 and σ_0 be as in* (4.1). *Suppose that F_0 is closed under exponentiation and $v(F_0(x)^\times) \neq v(F^\times)$ for all $x \in K \setminus F_0$. Let $x \in K \setminus F_0$. There is a log-closed $F \models T_{\text{an}}$ such that $F_0(x) \subseteq F \subseteq_{\text{an}} K$ and a log-preserving embedding $\sigma\colon F \to L$ extending σ_0.*

Proof. Without loss of generality $x > \mathbf{R}$ and (by (3.4)) $v(x) \notin v(F_0^\times)$. We build sequences $\beta_0, \beta_1, \ldots \in F_0$ and $x_0, x_1, \ldots \in K$. For all $n \in \mathbf{N}$ we will have $x_n > \mathbf{R}$ and $v(x_n) \notin v(F_0^\times)$.

Let $x_0 = x$. Given x_n we see that $v(F_0(\log x_n)^\times) \not\subseteq v(F_0^\times)$; for otherwise, since F_0 is maximal with this value group, $\log x_n \in F_0$ and, since F_0 is closed under exponentiation, $x_n \in F_0$, a contradiction.

By (3.4), there is a $\beta_n \in F_0$ such that $v(\log x_n - \beta_n) \notin v(F_0^\times)$. Let $x_{n+1} = |\log x_n - \beta_n|$, so $\log x_n = \beta_n + \varepsilon_n x_{n+1}$, where $\varepsilon_n = \pm 1$. Note that $v(x_n) < v(\log x_n) \leq v(x_{n+1}) \leq 0$.

CLAIM 1. $v(x_{n+1}) < 0$.

Otherwise, $v(x_n) = v(\exp \beta_n) \in v(F_0^\times)$, since $v(x_n) = v(\exp \beta_n) + v(\exp \varepsilon_n x_{n+1})$ and $v(\exp \varepsilon_n x_{n+1}) = 0$.

CLAIM 2. *$v(x_0)$,$v(x_1), \ldots$ are $\mathbf{Q}$-linearly independent over $v(F_0^\times)$.*

Suppose

$$v(x_m) = \sum_{i=m+1}^{n} q_i v(x_i) + v(w),$$

$q_i \in \mathbf{Q}$ and $w \in F_0$. There is $c \in K$ with $v(c) = 0$ such that

$$x_m = cw \prod_{i=m+1}^{n} x_i^{q_i}.$$

Hence,

$$\varepsilon_m x_{m+1} = \log c + \log w - \beta_m + \sum q_i \log x_i.$$

For $i \geq m+1$ and $n \in \mathbf{N}$, $(\log x_i)^n < x_{m+1}$. Thus $v(\log x_i) > v(x_{m+1})$ for all $i \geq m+1$. Also, $v(\log c) = 0 > v(x_{m+1})$. Since

$$v(x_{m+1}) \geq \min(v(\log c), v(\log w - \beta_n), v(\log x_{m+1}), \ldots, v(\log x_n)),$$

we have $v(x_{m+1}) = v(\log w - \beta_n) \in v(F_0^\times)$, a contradiction.

Let $F_{n+1} = F_n\langle x_n\rangle$ and $F = \bigcup F_n$. Let $y \in L$ realize the image under σ_0 of the cut of x over F_0. Define a sequence (y_n) in L by: $y_0 = y$, and $y_{n+1} = \frac{\log y_n - \sigma_0(\beta_n)}{\varepsilon_n}$.

CLAIM 3. *For all i, y_i realizes the image under σ_0 of the cut of x_i over F_0.*

We prove this by induction on i. Suppose it is true for y_n (and without loss of generality we assume $\varepsilon_n = 1$). Let $w \in F_0$. Then

$$\begin{aligned} w < x_{n+1} &\Leftrightarrow w + \beta_n < x_{n+1} + \beta_n \\ &\Leftrightarrow (\exp w)(\exp \beta_n) < x_n \\ &\Leftrightarrow (\exp \sigma_0(w))(\exp \sigma_0(\beta_n)) < y_n \\ &\Leftrightarrow \sigma_0(w) < y_{n+1}. \end{aligned}$$

CLAIM 4. *We can extend σ_0 to an L_{an}-embedding $\sigma_n \colon F_n \to L$, by sending x_i to y_i for $i < n$.*

For $n = 1$, this follows immediately from Claim 3. Assume we have $\sigma_n \colon F_n \to L$ with $\sigma_n(x_i) = y_i$ for $i < n$. We must show that y_n realizes the image under σ_n of the cut of x_n over F_n.

Let $w > 0$, $w \in F_n$. There are $z \in F_0$ and $\varepsilon \in K$ with $v(\varepsilon) > 0$ and $q_0, \dots, q_{n-1} \in \mathbf{Q}$ such that

$$w = z(1+\varepsilon)\prod_{i=0}^{n-1} x_i^{q_i}.$$

Let m be least such that $q_m \neq 0$. Let $u = z^{-1/q_m}$, $r_j = -q_j/q_m$, for $j < n$, $r_n = 1/q_m$ and $\delta = (1+\varepsilon)^{-1/q_m} - 1$. Assume $q_m < 0$. (If $q_m > 0$ the argument is similar.)

Then

$$\begin{aligned} x_n < w \Leftrightarrow x_m &< u(1+\delta)\prod_{j=m+1}^{n} x_j^{r_j} \\ &\Leftrightarrow \varepsilon_m x_{m+1} < \log u - \beta_m + \log(1+\delta) + \sum r_j \log x_j. \end{aligned}$$

As in Claim 2, $v(x_{m+1}) < v(\log x_j) < v(\log(1+\delta))$. Thus $x_n < w$ if and only if $\varepsilon_m x_{m+1} < \log u - \beta_m$ if and only if $\varepsilon_m y_{m+1} < \log \sigma_0(u) - \sigma_0(\beta_m)$ if and only if $y_n < \sigma_n(w)$.

We let $\sigma = \bigcup \sigma_n \colon F \to L$.

CLAIM 5. *F is log-closed and $\sigma \colon F \to L$ is log-preserving.*

Let $0 < w \in F_{n+1}$. By (3.6) and Claim 2,

$$v(F_{n+1}^{\times}) = v(F_0^{\times}) \oplus \mathbf{Q}v(x_0) \cdots \oplus \mathbf{Q}v(x_n).$$

Thus there are $u \in F_0$, $q_0, \dots, q_n \in \mathbf{Q}$ and $\varepsilon \in F_{n+1}$ with $v(\varepsilon) > 0$ such that $w = u(1+\varepsilon)\prod x_i^{q_i}$. Then

$$\begin{aligned}\log w &= \log u + \log(1+\varepsilon) + \sum q_i \log x_i \\ &= \log u + \log(1+\varepsilon) + \sum q_i(\varepsilon_i x_{i+1} + \beta_i) \in F_{n+2}.\end{aligned}$$

Also $\sigma(\log w) = \log\sigma(u) + \log(1+\sigma(\varepsilon)) + \sum q_i(\varepsilon_i y_{i+1} + \sigma(\beta_i)) = \log\sigma(w)$. $\square$

Theorem 4.1 is now easily obtained by iterating lemmas (4.2), (4.3) and (4.4). We can now prove our main results.

COROLLARY 4.5. $T_{\text{an}}(\exp)$ *admits quantifier elimination in the language* $T_{\text{an}}(\exp,\log)$.

Proof. We use the following test for quantifier elimination (see for example [15]).

Let T be an L-theory. Suppose that whenever $M, N \models T$, N is $|M|^+$-saturated, A is an L-substructure of M and $\sigma\colon A \to N$ is an L-embedding, then σ extends to an L-embedding of M into N. Then T admits quantifier elimination in the language L.

We can apply this test to $T_{\text{an}}(\exp,\log)$ using (4.1). We need only show that an $L_{\text{an}}(\exp,\log)$-substructure A of a model of $T_{\text{an}}(\exp,\log)$ is a model of T_{an}. By (2.14) it suffices to show A is closed under $^{-1}$ and $\sqrt[n]{\ }$. But for $x > 0$, $x^{-1} = \exp(-\log x)$ and $\sqrt[n]{x} = \exp(\frac{\log x}{n})$, while for $x < 0$, $x^{-1} = -\exp(-\log(-x))$. $\square$

COROLLARY 4.6. $T_{\text{an}}(\exp)$ *is a complete axiomatization of* $\mathrm{Th}(\mathbf{R}_{\text{an}},\exp)$ *and admits a universal axiomatization in the language* $L_{\text{an}}(\exp,\log)$.

Proof. Let $M \models T_{\text{an}}(\exp)$. We can view M as an extension of $(\mathbf{R}_{\text{an}},\exp)$. By quantifier elimination, $(\mathbf{R}_{\text{an}},\exp) \preceq M$. Since T_{an} has a universal axiomatization and E3) can be re-written as

$$\forall x > 0 \quad (\exp(\log x) = x),$$

$T_{\text{an}}(\exp,\log)$ can be universally axiomatized. $\square$

COROLLARY 4.7. *If* $f\colon \mathbf{R}^n \to \mathbf{R}$ *is* $L_{\text{an}}(\exp)$*-definable, then there are terms* $t_1(y_1,\dots,y_n),\dots,t_k(y_1,\dots,y_n)$ *in the language* $L_{\text{an}}(\exp,\log)$ *such that for all* $x_1,\dots,x_n$ *there is an* i *such that* $f(\bar{x}) = t_i(\bar{x})$.

Proof. Since $T_{\text{an}}(\exp,\log)$ is universally axiomatized and admits quantifier elimination, the proof is as in (2.15). $\square$

(4.8). Let $e_1(x) = \exp x$ and let $e_{n+1}(x) = \exp(e_n(x))$. In [6] it was shown that if the function $f\colon \mathbf{R} \to \mathbf{R}$ is definable in $(\mathbf{R}_{\text{an}},\exp)$, then there are an n and an M such that $f(x) < e_n(x)$ for $x > M$. A simpler proof of this fact can be obtained from (4.7) and results in the next section.

(4.9). As an odd footnote to (4.5), note that since in (4.1) we do not need the substructure to be closed under exp, we can also eliminate quantifiers in the language $L_{\mathrm{an}}({}^{-1}, (\sqrt[n]{})_{n=2,3,\ldots}, \log)$ without exp. We see no real applications of this observation yet.

(4.10). Suppose $\mathcal{R}$ is a restricted analytic expansion of $\mathbf{R}_e$. The arguments of this section and (3.9) combine to show that $(\mathcal{R}, \exp)$ admits quantifier elimination in the language where we add a function symbol for log and function symbols for functions $f\colon \mathbf{R}^N \to \mathbf{R}$ that are definable without parameters in $\mathcal{R}$. From [16] we know that $\mathbf{R}_e$ is model complete. Combining this with quantifier elimination, we can conclude Wilkie's result that the ordered exponential field of real numbers is model complete.

The arguments above also show that the theory of $(\mathcal{R}, \exp)$ can be axiomatized by the theory of $\mathcal{R}$ and the axioms E1)–E5). For $\mathcal{R} = \mathbf{R}_e$ this is Ressayre's theorem ([13]).

5. *o*-minimality and Hardy fields

In this final section we will apply the results of Section 4 to show that $(\mathbf{R}_{\mathrm{an}}, \exp)$ is *o*-minimal.

(5.1). Let $L = \{<, 0, 1, +, -, \cdot, \ldots\}$ be an expansion of the language of ordered rings where we add no new relation symbols. Let

$$\mathcal{R} = (\mathbf{R}, +, -, \cdot, <, 0, 1, \ldots)$$

be an L-structure expanding the ordered field of real numbers and let $T = \mathrm{Th}(\mathcal{R})$.

We refer to L-terms with parameters from $\mathbf{R}$ as $\mathbf{R}$-terms.

Our first lemma gives an equivalent characterization of *o*-minimality if T has quantifier elimination.

LEMMA 5.2. *Suppose T has quantifier elimination. Then T is o-minimal if and only if for each $\mathbf{R}$-term $t(X)$ in one variable X there is $m \in \mathbf{R}$ such that either $t(x) > 0$ for all $x > m$, or $t(x) = 0$ for all $x > m$, or $t(x) < 0$ for all $x > m$.*

Proof. ($\Rightarrow$) This implication is clear.

($\Leftarrow$) We assume that all $\mathbf{R}$-terms have the property described above. Let $S \subseteq \mathbf{R}$ be definable with parameters. By quantifier elimination either there is an m such that $(m, +\infty) \subseteq S$ or there is an m such that $(m, +\infty) \cap S = \emptyset$.

With the use of fractional linear transformations a similar property holds at $-\infty$, and to the right and left of each $r \in \mathbf{R}$. Thus there is an $m > 0$ such that

i) either $(m, +\infty) \subseteq S$ or $(m, +\infty) \cap S = \emptyset$,

ii) either $(-\infty, -m) \subseteq S$ or $(-\infty, -m) \cap S = \emptyset$,

and for each $r \in \mathbf{R}$ there is $\varepsilon > 0$ such that

iii) either $(r, r + \varepsilon) \subseteq S$ or $(r, r + \varepsilon) \cap S = \emptyset$, and

iv) either $(r - \varepsilon, r) \subseteq S$ or $(r - \varepsilon, r) \cap S = \emptyset$.

Hence the boundary $\partial(S)$ of S in $\mathbf{R}$ is closed, bounded, and contains only isolated points. Thus $\partial(S)$ is finite, and S is a finite union of points and intervals. □

(5.3). Lemma 5.2 suggests that Hardy field methods might be useful in proving *o*-minimality.

If $f, g \colon \mathbf{R} \to \mathbf{R}$, we say that f and g have the same *germ* at $+\infty$ if there is $m \in \mathbf{R}$ such that $f(x) = g(x)$ for all $x > m$. We let $\mathcal{G}$ be the ring of germs at $+\infty$ of (not necessarily continuous) functions $f \colon \mathbf{R} \to \mathbf{R}$. We will usually not distinguish notationally between a function and its germ. We use the term "ultimately" to abbreviate "for all sufficiently large real numbers".

A subring A of $\mathcal{G}$ is called a *$\mathcal{G}$-domain* if for each $f \in A$ either ultimately $f(x) > 0$, ultimately $f(x) < 0$, or ultimately $f(x) = 0$. If A is a $\mathcal{G}$-domain, then A is an integral domain and A has a natural ordering given by $f > 0$ if and only if ultimately $f(x) > 0$. A $\mathcal{G}$-field is a $\mathcal{G}$-domain that happens to be a field. If A is a $\mathcal{G}$-domain then A has a (unique) fraction field in $\mathcal{G}$, and this fraction field is a $\mathcal{G}$-field.

We identify $\mathbf{R}$ with the $\mathcal{G}$-field of germs of constant functions. If we view x as the identity function, then $\mathbf{R}[x]$ is a $\mathcal{G}$-domain and its fraction field $\mathbf{R}(x)$ is a $\mathcal{G}$-field.

(5.4). If F is an n-ary function symbol of L we define $F_{\mathcal{G}} \colon \mathcal{G}^n \to \mathcal{G}$ by $F_{\mathcal{G}}(f_1, \dots, f_n) =$ the germ at $+\infty$ of the function $x \mapsto F(f_1(x), \dots, f_n(x))$. Similarly for each term $t(x_1, \dots, x_n)$ we define $t_{\mathcal{G}} \colon \mathcal{G}^n \to \mathcal{G}$ by letting $t_{\mathcal{G}}(f_1, \dots, f_n)$ be the germ of the function $x \mapsto t(f_1(x), \dots, f_n(x))$. When F is the function symbol for addition (respectively multiplication) $F_{\mathcal{G}}$ is the usual addition (respectively multiplication) on $\mathcal{G}$.

An *$\mathcal{R}$-field* is a $\mathcal{G}$-field that is closed under $F_{\mathcal{G}}$ for all function symbols F in L. Lemma 5.2 immediately implies:

LEMMA 5.5. *If T has quantifier elimination and there is an $\mathcal{R}$-field containing $\mathbf{R}(x)$, then $\mathcal{R}$ is o-minimal.*

(5.6). From now on we make two assumptions on T:

i) T admits quantifier elimination.
ii) T has a universal axiomatization.

By the results of Section 2 and Section 4, these assumptions hold for $T_{\mathrm{an}}({}^{-1}, (\sqrt[n]{\ })_{n=2,3,\ldots})$ and $T_{\mathrm{an}}(\exp, \log)$.

If $N \models T$ and M is an L-substructure of N, then by i) and ii) we have $M \preceq N$.

(5.7). If K is an $\mathcal{R}$-field then we can naturally expand K to make it an L-structure. For each n-ary function symbol F of L, interpret F as $F_K : K^n \to K$ where $F_K = F_{\mathcal{G}} \mid K^n$.

LEMMA 5.8. *If K is an $\mathcal{R}$-field, then as viewed with its natural L-structure, $K \models T$.*

Proof. Let $\mathcal{M}$ be a proper elementary extension of the expansion $(\mathcal{R}, (f)_{f:\,\mathbf{R}\to\mathbf{R}})$. So for each $f : \mathbf{R} \to \mathbf{R}$ we have a unary function symbol whose interpretation in $\mathcal{M}$ is a function $f_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}$.

Fix $a \in \mathcal{M}$ positive infinite. If f and g have the same germ, then there is $m \in \mathbf{R}$ such that $\mathcal{M} \models \forall x > m(f(x) = g(x))$. Hence $f_{\mathcal{M}}(a) = g_{\mathcal{M}}(a)$, so the value of $f_{\mathcal{M}}(a)$ depends only on the germ of f and a. Define the map $i_a : \mathcal{G} \to \mathcal{M}$ by $f \mapsto f_{\mathcal{M}}(a)$. One checks immediately that i_a is a ring homomorphism such that $i_a(F_{\mathcal{G}}(f_1, \ldots, f_n)) = F_{\mathcal{M}}(i_a(f_1), \ldots, i_a(f_n))$ for each n-ary function symbol F of L and $f_1, \ldots, f_n \in \mathcal{G}$.

In particular, $i_a \mid K : K \to \mathcal{M}$ is an ordered field embedding and $i_a(K)$ is an L-substructure of $\mathcal{M}$. Since T has a universal axiomatization, $i_a(K)$ is the underlying set of a model of T. For any n-ary function symbol F, if $f_1, \ldots, f_n \in K$, then

$$i_a(F_K(f_1, \ldots, f_n)) = i_a(F_{\mathcal{G}}(f_1, \ldots f_n)) = F_{\mathcal{M}}(i_a(f_1), \ldots, i_a(f_n)).$$

Thus i_a is an L-structure isomorphism between K and $i_a(K)$. Thus $K \models T$. $\square$

If K is an $\mathcal{R}$-field and $g \in \mathcal{G}$, we say that g is *comparable* to K if for each $f \in K$ either ultimately $g(x) < f(x)$, or ultimately $g(x) > f(x)$, or ultimately $g(x) = f(x)$.

LEMMA 5.9. *Suppose T is o-minimal. Let K be an $\mathcal{R}$-field. If $g \in \mathcal{G}$ is comparable to K, then*

$$K\langle g\rangle := \{t_{\mathcal{G}}(f_1, \ldots, f_n, g) : t(x_1, \ldots, x_{n+1}) \text{ is a term and } f_1, \ldots, f_n \in K\}$$

is an $\mathcal{R}$-field. Clearly it is the smallest $\mathcal{R}$-field containing $K \cup \{g\}$.

Proof. We let $\mathcal{M}, a$ and i_a be as in the proof of (5.8). By (5.8) we can view K as a model of the o-minimal theory T.

We may of course assume that $g \notin K$. Since g is comparable to K, g determines a cut in K. Let $f \in K$ and suppose that $f(x)$ is ultimately less than $g(x)$. There is an $m \in \mathbf{R}$ such that $\mathcal{M} \models \forall x > m \ \ (f(x) < g(x))$. Thus $i_a(f) < i_a(g)$. Similarly, if $f(x)$ is ultimately greater than $g(x)$, then $i_a(f) > i_a(g)$. Thus $i_a(g)$ realizes the image under i_a of the cut of g in K.

Fix a term $t(x_1, \ldots, x_{n+1})$ and $f_1, \ldots, f_n \in K$. Since T is o-minimal, there are $h_0, h_1 \in K \cup \{\pm\infty\}$ such that $h_0 < g < h_1$ and the sign of $t(f_1, \ldots, f_n, y)$ is constant for $y \in (h_0, h_1) \cap K$. Assume that $t(f_1, \ldots, f_n, y) > 0$ for all $y \in (h_0, h_1)$. We will show that the term $t(f_1(x), \ldots, f_n(x), g(x))$ is ultimately positive. In the other cases, similar arguments will show that the term $t(f_1(x), \ldots, f_n(x), g(x))$ is either ultimately zero or ultimately negative.

Since K is isomorphic to $i_a(K)$,

$$i_a(K) \models \forall y \quad (i_a(h_0) < y < i_a(h_1) \rightarrow t(i_a(f_1), \ldots, i_a(f_n), y) > 0).$$

By quantifier elimination $i_a(K)$ is an elementary submodel of the L-reduct of $\mathcal{M}$, so

$$\mathcal{M} \models \forall y \quad (i_a(h_0) < y < i_a(h_1) \rightarrow t(i_a(f_1), \ldots, i_a(f_n), y) > 0).$$

In particular $\mathcal{M} \models t(i_a(f_1), \ldots, i_a(f_n), i_a(g)) > 0$. Thus

$$t(f_1(a), \ldots, f_n(a), g(a)) > 0.$$

If we choose another positive infinite $b \in \mathcal{M}$ and consider the associated embedding $i_b \colon \mathcal{G} \rightarrow \mathcal{M}$, then we conclude by the same reasoning that $t(f_1(b), \ldots, f_n(b), g(b)) > 0$. Since $(\mathcal{R}, (f)_{f \colon \mathbf{R} \to \mathbf{R}}) \preceq \mathcal{M}$, there is an $m \in \mathbf{R}$ such that $t(f_1(y), \ldots, f_n(y), g(y)) > 0$ for all $y > m$. Thus we see that $t(f_1(x), \ldots, f_n(x), g(x))$ is ultimately positive.

We have shown that $K\langle g \rangle$ is an $\mathcal{R}$-field. □

(5.10). A $\mathcal{C}^1$-*germ* is an element $g \in \mathcal{G}$ which is the germ at $+\infty$ of a $\mathcal{C}^1$-function defined on an interval $(m, +\infty)$. In that case g has a derivative $g' \in \mathcal{G}$, defined as the germ at $+\infty$ of the derivative of such a function.

A *Hardy field* is a $\mathcal{G}$-field K such that for all $f \in K$, f is a $\mathcal{C}^1$-germ and $f' \in K$. We call K an $\mathcal{R}$-Hardy field if it is an $\mathcal{R}$-field which is also a Hardy field.

Identifying $\mathbf{R}$ with a subring of $\mathcal{G}$ in the usual way, we see that $\mathbf{R}$ is an $\mathcal{R}$-Hardy field.

If $\mathbf{R}^*$ is any o-minimal expansion of the ordered field $\mathbf{R}$, then the Hardy field $H(\mathbf{R}^*)$ of germs at $+\infty$ of functions $f \colon \mathbf{R} \rightarrow \mathbf{R}$ that are $\mathbf{R}$-definable in $\mathbf{R}^*$ is an $\mathcal{R}^*$-Hardy field. If the theory of $\mathbf{R}^*$ has quantifier elimination and a universal axiomatization, then $H(\mathbf{R}^*)$ is $\mathbf{R}^*\langle x\rangle$, the smallest $\mathbf{R}^*$-field containing $\mathbf{R}$ and the germ of the identity function.

Next we recall two well known results on Hardy fields. For the reader's convenience we repeat here short proofs by Boshernitzan ([1]).

LEMMA 5.11. *Let K be a Hardy field and $f \in K$.*
a) *$e^f \in \mathcal{G}$ is comparable to K.*
b) *If $f > 0$, then $\log(f)$ is comparable to K.*

Proof. a) We may assume $e^f \notin K$. Let $\Omega = \{h \in \mathcal{G}\colon h$ has arbitrarily large zeros$\}$. Given $g \in K$ we have to show that $e^f - g$ is not in Ω. If $e^f - g \in \Omega$, then $g > 0$ and $h = 1 - ge^{-f} \in \Omega$. Hence $h' = e^{-f}(f'g - g') \in \Omega$. But then $f'g - g' \in \Omega \cap K = \{0\}$. Thus $f' = \frac{g'}{g} = (\log(g))'$ and $e^f = rg$ for some real number r. Since $e^f - g \in \Omega$ we must have $r = 1$ and $e^f = g \in K$.

b) Again we assume $\log(f) \notin K$. If $g \in K$ and $h = \log(f) - g \in \Omega$, then $h' = \frac{f'}{f} - g' \in \Omega \cap K = \{0\}$. Thus $g' = \frac{f'}{f}$ and $g = \log(f) + r$ for some $r \in \mathbf{R}$. But then, arguing as above, $\log(f) \in K$. □

LEMMA 5.12. *Suppose T is o-minimal. Let K be an $\mathcal{R}$-Hardy field and $f \in K$. Then $K\langle e^f\rangle$ is an $\mathcal{R}$-Hardy field, and if $f > 0$, then $K\langle\log(f)\rangle$ is an $\mathcal{R}$-Hardy field. Hence every $\mathcal{R}$-Hardy field can be extended to an $\mathcal{R}$-Hardy field that is closed under exponentiation and under taking logarithms of positive elements.*

Proof. Let $g \in K\langle e^f\rangle$. Choose a term $t(x_1, \ldots, x_{n+1})$ and $f_1, \ldots, f_n \in K$ such that $g = t_{\mathcal{G}}(f_1, \ldots, f_n, e^f)$. We must show that g is a $\mathcal{C}^1$-germ whose derivative belongs to $K\langle e^f\rangle$. We may assume that $e^f \notin K$.

Since T is o-minimal, we can find L-formulas $\phi_1(\bar{x}), \ldots, \phi_m(\bar{x})$ and $\psi_1(\bar{x}, y), \ldots, \psi_m(\bar{x}, y)$ such that in any $\mathcal{N} \models T$, the sets $C_j = \{\bar{x} \in \mathcal{N}^{n+1}\colon \mathcal{N} \models \phi_j(\bar{x})\}$ are cells partitioning $\mathcal{N}^{n+1}$ and $\psi_j(\bar{x}, y)$ defines the graph of a $\mathcal{C}^1$-function h_j on an open neighborhood of C_j such that h_j agrees with the function defined by t on C_j. (This form of cell decomposition is proved in [3] for strongly o-minimal expansions of the real field and by [10] every o-minimal structure is strongly o-minimal.)

By (5.8), $K \models T$. Since e^f is comparable to K, it determines a cut in K. Let C be the single cell of the partition such that "$(f_1, \ldots, f_n, v) \in C$" is in the type of e^f over K and let h be the corresponding $\mathcal{C}^1$-function on an open neighborhood of C. There are $g_0, g_1 \in K \cup \{\pm\infty\}$ such that $g_0 < e^f < g_1$ and $(f_1, \ldots, f_n, y) \in C$ for all $y \in (g_0, g_1) \cap K$.

Let $\mathcal{M}$ be as in (5.8). Let $a \in \mathcal{M}$ be a positive infinite element. Arguing as in (5.8) we see that $(f_1(a), \ldots, f_n(a), e^{f(a)}) \in C$. Since this does not depend on the choice of a, there is an $m \in \mathbf{R}$ such that $(f_1(x), \ldots, f_n(x), e^{f(x)}) \in C$

for all $x > m$. Thus $g(x) = h(f_1(x), \ldots, f_n(x), e^{f(x)})$ for all $x > m$. Changing m if necessary, we may also assume that $f_1, \ldots, f_n$ are $\mathcal{C}^1$ on $(m, +\infty)$. Hence by the chain rule and the fact that $(e^f)' = f'e^f$ we obtain that g is $\mathcal{C}^1$ on $(m, +\infty)$ and $g' \in K\langle e^f \rangle$.

In the same way we can show that $K\langle \log f \rangle$ is an $\mathcal{R}$-Hardy field for $f > 0$. $\square$

COROLLARY 5.13. *($\mathbf{R}_{\text{an}}$,exp) is o-minimal.*

Proof. Let $\mathcal{R} = (\mathbf{R}_{\text{an}}, {}^{-1}, (\sqrt[n]{\ })_{n=2,3,\ldots})$. The structure $\mathcal{R}$ is o-minimal and satisfies the assumptions of (5.6). Let $H(\mathcal{R})$ be the $\mathcal{R}$-Hardy field of germs of $+\infty$ of definable functions. By (5.12) we can extend $H(\mathcal{R})$ to an $\mathcal{R}$-Hardy field K that is closed under exponentiation and under taking logarithms of positive elements. Now consider $(\mathcal{R}, \exp, \log)$ which admits quantifier elimination by (4.5). Clearly K is an $(\mathcal{R}, \exp, \log)$-Hardy field. Thus by (5.5), $(\mathcal{R}, \exp, \log)$ is o-minimal. Hence $(\mathbf{R}_{\text{an}}, \exp)$ is o-minimal. $\square$

(5.14) *Remarks.* Lemma (5.9) was inspired by a similar result of Boshernitzan ([1, Theorem 4.1]) on germs of continuous functions. The paper [1] also gives examples of real closed Hardy fields containing $\mathbf{R}(x)$ and real power series $\sum a_n x^{-n}$ converging for all sufficiently large x, such that K cannot be extended to a Hardy field containing the germ at $+\infty$ of the function defined by this power series for large x. These examples show that $(\mathbf{R}_{\text{an}}, {}^{-1}, (\sqrt[n]{\ })_{n=2,3,\ldots})$-Hardy fields are essential in the proof of (5.11) as real closed Hardy fields cannot always be extended to real closed Hardy fields closed under the restricted analytic operations.

Several results of this section can be generalized somewhat.

While o-minimality is a fundamental assumption in (5.9) and (5.12), the assumption that T is universally axiomatized and admits quantifier elimination is more a matter of convenience, since this can always be achieved by adding function symbols for definable functions. In a similar manner we can also avoid the assumption that L contains no new relation symbols as long as T is o-minimal.

The proof we gave of (5.12) shows the following:

Suppose T is o-minimal, K is an $\mathcal{R}$-Hardy field, h is a $\mathcal{C}^1$-germ comparable to K and $h' \in K\langle h \rangle$. Then $K\langle h \rangle$ is an $\mathcal{R}$-Hardy field.

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