Sel. math., New ser. 7 (2001) 409 – 445
1022–1824/01/030409–37$1.50 + 0.20/0



**Selecta Mathematica, New Series**

# Expansions of algebraically closed fields in o-minimal structures

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**Abstract.** We develop a notion of differentiability over an algebraically closed field $K$ of characteristic zero with respect to a maximal real closed subfield $R$. We work in the context of an o-minimal expansion $\mathcal{R}$ of the field $R$ and obtain many of the standard results in complex analysis in this setting. In doing so we use the topological approach to complex analysis developed by Whyburn and others. We then prove a model theoretic theorem that states that the field $R$ is definable in every proper expansion of the field $K$ all of whose atomic relations are definable in $\mathcal{R}$. One corollary of this result is the classical theorem of Chow on projective analytic sets.

**Mathematics Subject Classification (2000).** Primary 03C65, 03H05; Secondary 30G06.

**Key words.** o-minimal, nonarchimedean fields, compact complex manifolds.

## 1. Introduction

Algebraically closed fields and their various expansions are studied extensively in model theory. In contrast to differentially closed fields (DCF), or fields with automorphisms (ACFA), which carry no uniformly definable topology of interest (these are so-called simple structures), we study here expansions of algebraically closed fields of characteristic zero within the topological environment of an o-minimal structure. We develop in this setting a notion of differentiability that is analogous to the classical one, over $\mathbb{C}$.

Let $K$ be an algebraically closed field of characteristic zero. Then $K = R(\sqrt{-1})$ for some real closed subfield $R$. Note that $R$ is not unique in any way. For example $\mathbb{C}$, the algebraic closure of $\mathbb{R}$, is also the algebraic closure of a nonarchimedean real closed field (since there is only one algebraically closed field of size $2^{\aleph_0}$).

We fix one such real closed $R$ and $i = \sqrt{-1}$. Since $K = R(i)$ it can be identified, as in the classical case, with $R^2$. Every subset of $K^n$ is identified with a subset of $R^{2n}$ and every function from $K^n$ into $K$ is identified with a function from $R^{2n}$

The second author was partially supported by NSF

into $R^2$. Since $R$ has a natural ordering and topology, it induces the product topology on $R^2$ and hence on $K$, making it into a topological field. Moreover, just like the metric on $\mathbb{C}$, $K$ has a notion of distance, evaluated in $R$.

Since $R$ need not be Dedekind complete, as an ordered set, nor even archimedean, the topology which it induces on $K$ is, in general, far from being locally compact or connected. However, we will restrict ourselves to the category of definable sets in some o-minimal expansion $\mathcal{R}$ of $R$. This category has the following "tameness" properties (see [D1] for a general reference on o-minimality):

1. $R^n$ is (definably) connected and locally (definably) connected. (An $\mathcal{R}$-definable subset of $R^n$ is *definably connected* if it has no decomposition into $\mathcal{R}$-definable, relatively open, nonempty subsets).
2. $R^n$ is locally (definably) compact. (An $\mathcal{R}$-definable set $X \subseteq R^n$ is called *definably compact* if every $\mathcal{R}$-definable function $f(t)$ from an interval $(a, b) \subseteq R$ into $X$ has a limit point in $X$, as $t$ approaches $b$).
3. Every $\mathcal{R}$-definable subset of $R^n$ can be partitioned into finitely many definably connected sets.

We are going to consider in this context a notion analogous to analyticity of functions in the classical setting. Classically, the theory of complex differentiability is developed via the notions of integrals and converging power series, but neither of these is available in our setting. As we discovered while working on this paper, the question of developing the basic theory of differentiation over $\mathbb{C}$ through integration-free methods, was studied extensively, mainly by Whyburn in his book Topological Analysis [W]. This approach, which replaces the use of integrals with winding numbers, turns out to be compatible with o-minimality and allows us to prove analogues of most of the classical results in complex analysis, with the exception of the theory of analytic continuation.

## The structure of the paper — main results

The basic theory of differentiation over $K$ is developed in Section 2. In Sections 2.1–2.3 the topological notion of *winding number* is treated in the o-minimal context. In Sections 2.4–2.10 we define $K$-differentiability and prove its basic properties, such as infinite differentiability, maximum principle, removable singularity, identity theorem, Liouville's theorem. Some of these are stronger than the corresponding classical results, as for example, the following theorem on removable singularities.

**Theorem 1.1.** *Let $U \subseteq K$ be a definable open set, $f : U \to K$ a definable continuous function, which is $K$-differentiable on $U \setminus L$, where $\dim(L) = 1$. Then $f$ is $K$-differentiable on all of $U$.*

O-minimality puts strong restrictions on the definable $K$-differentiable functions over $K$. As we show, isolated singularities of such functions are either removable or poles, and as a result we prove:

**Theorem 1.2.** *Assume that $f : K \to K$ is a definable function in an o-minimal expansion of $R$, such that $f$ is $K$-differentiable outside a finite set $A$. Then $f$ is a rational function over $K$ outside $A$.*

Note (see 2.28) that if we allow functions with restricted domains then there are many transcendental $\mathbb{C}$-differentiable functions which are definable in o-minimal expansions of $\mathbb{R}$.

In Section 2.11 we show that, in the o-minimal context, there are strong restrictions on the behavior of $K$-differentiable functions at the boundary of their domain.

In Section 3 we apply results from Section 2 and prove the following model theoretic corollary.

**Theorem 1.3.** *Let $\mathcal{R}$ be an o-minimal expansion of a real closed field $R$, $\mathcal{K}$ an expansion of the algebraic closure $K$ of $R$. Assume that all atomic relations of $\mathcal{K}$ are definable in $\mathcal{R}$. If $\mathcal{K}$ is a proper expansion of $\langle K, +, \cdot \rangle$ then $R \subseteq K$ is definable in $\mathcal{K}$.*

When we translate this theorem to the classical setting of compact complex manifolds we obtain in Section 3 a proof to the theorem of Chow on the algebraicity of projective analytic sets.

We assume familiarity with basic o-minimality and suggest as a reference [D1] and [D2]. As a reference for the classical material in complex analysis we mainly used [R].

The work in this paper covers only $K$-differentiability in one variable. The multivariable case will be treated in a subsequent paper.

Most of this work was conducted while the authors were visiting the Mathematical Science Research Institute during the spring 1998. We thank the Logic seminar at the University of Illinois, Urbana-Champaign (especially Matthias Aschenbrenner), and Margarita Otero for commenting on earlier versions of this paper.

## 2. Differentiability

### 2.1. Four topological facts

In several places in the paper we are going to use the facts below. We assume here that we work in an o-minimal expansion of a real closed field $R$.

To simplify the arguments we let

$$\overline{R} = \{-\infty\} \cup R \cup \{+\infty\},$$

and identify it (definably) with some closed and bounded interval in $R$, in the obvious way. $\overline{R}$, with this induced topology, has all the topological properties of closed and bounded intervals. We equip $(\overline{R})^n$ with the product topology.

**Fact 2.1.** *Given $\epsilon > 0$, let $\{F_t : t \in (0,\epsilon)\}$ be a definable family of definably connected closed subsets of $\overline{R}^n$, such that for every $t_1 < t_2$ we have $F_{t_1} \subseteq F_{t_2}$. Then*

$$F = \bigcap_{t\in(0,\epsilon)} F_t$$

*is definably connected as well.*

*Proof.* Since $\overline{R}$ can be identified with a closed interval in $R$, we may assume that all the $F_t$'s are contained in a closed and bounded box in $R^n$.

Let $I = (0,\epsilon)$. Assume for contradiction that $F$ is not definably connected. Then there is a definable open set $U \subseteq R^n$ such that $U \cap F$ and $F \cap (R^n \setminus U)$ are nonempty, and such that $(cl(U) \setminus U) \cap F = \emptyset$ (i.e., $U \cap F$ is closed in $F$). We may assume that $U$ is bounded. $cl(U) \setminus U$ is called the *frontier of* $U$ and denoted by $fr(U)$.

Since each $F_t$ is definably connected, $fr(U) \cap F_t \neq \emptyset$ for every $t \in I$. By curve selection there is a definable $\sigma : I \to R^n$ such that for every $t \in I$, $\sigma(t) \in fr(U) \cap F_t$. By o-minimality, since $fr(U)$ is closed and bounded, there is $c \in fr(U)$ such that $\lim_{t\to 0} \sigma(t) = c$. Since the $F_t$'s form a decreasing family of closed sets $c$ lies in every $F_t$ and therefore $c \in F$. Contradiction. □

We say that a definable $U \subseteq R^n$ is *locally definably connected* at $u_0 \in R^n$ if for all sufficiently small $\epsilon > 0$ the set

$$\{u \in U : |u - u_0| < \epsilon\}$$

is definably connected. (Note that $u_0$ need not be in $U$!).

**Fact 2.2.** *Let $U \subseteq R^n$ be a definable open set, locally definably connected at $u_0 \in R^n$, and assume that $f : U \to R^k$ is definable and continuous. Let $\Omega \subseteq (\overline{R})^{n+k}$ be the topological closure of the graph of $f$, as a subset of $(\overline{R})^{n+k}$. Let*

$$\Omega_{u_0} = \{v \in (\overline{R})^k : (u_0, v) \in \Omega\}.$$

*Then $\Omega_{u_0}$ is definably connected.*

*Proof.* For $t > 0$ let $B_t = \{z \in U : |z - u_0| < t\}$. Fix $\epsilon > 0$ such that for $t \in (0,\epsilon)$ the set $B_t$ is definably connected. By continuity, the set $f(B_t)$ is definably connected, and hence also its topological closure in $(\overline{R})^k$, which we denote by $F_t$. The family $\{F_t : t \in (0,\epsilon)\}$ satisfies the assumptions of Fact 2.1, thus $F = \bigcap_{t\in(0,\epsilon)} F_t$ is definably connected. It is not difficult to see that $F = \Omega_{u_0}$. □

Note that $\Omega_{u_0}$, in the above, is the set of limit points of $f(u)$ in $\overline{R}^k$ as $u$ approaches $u_0$.

The facts below are true in any o-minimal structure with definable choice.

**Fact 2.3.** *Let $X \subseteq M^n$ be a definable, definably compact set, $Y \subseteq M^k$ a definable set, and assume that $V$ is a definable, relatively open subset of $X \times Y$. Fix $y_0 \in Y$ and assume that $X \times \{y_0\} \subseteq V$. Then there is a definable relatively open neighborhood $W \subseteq Y$ of $y_0$ such that $X \times W \subseteq V$.*

*Proof.* Assume that there is no such $W$. Then, by definable choice, there is a definable path $\gamma = (\gamma_1, \gamma_2)$ from some interval $(0, \delta)$ into $X \times Y$ such that for every $t \in (0, \delta)$, $\gamma(t) \notin V$ and $\gamma_2(t)$ tends to $y_0$ as $t$ tends to 0. Since $X$ is definably compact, $\gamma_1(t)$ tends to some $x_0 \in X$, as $t$ approaches 0. Since $V$ is open and $X \times \{y_0\} \subseteq V$, there is an open $V_1 \subseteq V$ containing $(x_0, y_0)$. But then, there is a $t$ such that $\gamma(t) \in V$. Contradiction. $\square$

**Fact 2.4.** *Let $Z, X, Y$ be definable sets, $X$ definably compact. Let $F : X \times Y \to Z$ be a definable continuous map, $y_0 \in Y$. Then for every $\epsilon > 0$ there is a relatively open neighborhood $W \subseteq Y$ of $y_0$ such that $|F(x, y) - F(x, y_0)| < \epsilon$ for every $y \in W$ and every $x \in X$.*

*Proof.* Let
$$V = \big\{(x, y) \in X \times Y : |F(x, y) - F(x, y_0)| < \epsilon\big\}.$$
Then $X \times \{y_0\} \subseteq V$ and we can apply the last fact. $\square$

### 2.2. Winding numbers on the unit circle

We fix $\mathcal{R}$ an o-minimal expansion of $R$, $i = \sqrt{-1}$ and $K = R(i)$ the algebraic closure of $R$. Every element of $K$ can be uniquely written as $x + iy$, for $x, y \in R$ and thus $K$ is identified with $R^2$, equipped with its topology. We use the letters $z, w$ to denote elements of $K$ and, depending on the context, we sometimes think of these as elements of $R^2$. Addition and multiplication of these is done with respect to the field operations of $K$. For $z = x + iy \in K$, we let, as usual,
$$|z| = \sqrt{x^2 + y^2}$$
and take $S^1 \subseteq K$ to be the unit circle, i.e.,
$$S^1 = \big\{z \in K : |z| = 1\big\},$$
a subgroup of $K^\times$.

We are going to define here the notion of a winding number for definable (partial) maps from $R^2$ into $R^2$. This is the main ingredient in developing the basic theory of differentiability in $K$ without integration.

We now fix a parameterization of $S^1$ as follows: We take a definable continuous $\sigma : [0, 1] \to S^1$ such that $\sigma(0) = \sigma(1) = e$ the identity of $S^1$ and such that $\sigma|[0, 1)$ is a bijection of $[0, 1)$ and $S^1$. We do so in a “counterclockwise” manner (it is easy to see that this makes sense in our context just like in the classical setting). The ordering of $[0, 1)$ induces a linear ordering on $S^1$ via $\sigma|[0, 1)$.

We define a "universal covering" of $S^1$ as follows. Let $\mathcal{H} = \mathbb{Z} \times S^1$ and define a group operation on it:

$$(n, s) + (m, t) = \begin{cases} (m + n, st) & \text{if } s, t \leq st \\ (m + n + 1, st) & \text{otherwise.} \end{cases}$$

The identity element of $\mathcal{H}$ is $(0, e)$ and the group inverse of an element $(m, s) \in \mathcal{H}$ is $(-m - 1, s^{-1})$ for $s \neq e$ and $(-m, e)$ for $s = e$.

$\mathcal{H}$ can be linearly ordered lexicographically, using the above linear ordering of $S^1$ and the standard ordering of $\mathbb{Z}$. The following is easy to verify.

**Claim 2.5.** *$\mathcal{H}$ is a linearly ordered, abelian, divisible group. The projection map $\pi : \mathcal{H} \to S^1$ is a group homomorphism.*

$\langle \mathcal{H}, + \rangle$ is clearly not first-order definable. It is what we call in [PS] a $\bigvee$-definable group. As such it is equipped with a group topology which in this case is also the weakest topology on $\mathcal{H}$ making $\pi$ continuous, and it coincides with the order topology. At a neighborhood of every point which is not in $\pi^{-1}(e)$, this topology is just the subset topology induced on $\mathcal{H}$ by $R^3$. It is not difficult to show that it is definably connected, i.e., for every definable set $V \subseteq R^3$, if $V \cap \mathcal{H}$ is clopen in $\mathcal{H}$ (with respect to the group topology), then $V \cap \mathcal{H}$ is either empty or equals $\mathcal{H}$.

**Definition 2.6.** Let $f : [0, 1] \to S^1$ be a definable continuous map. A map $\hat{f} : [0, 1] \to \mathcal{H}$ is called *an extension of $f$* if it is continuous and for every $x \in [0, 1]$ we have $\pi \hat{f}(x) = f(x)$.

We are mainly interested in the following universal property of $\mathcal{H}$.

**Proposition 2.7.** *Every definable continuous map $f : [0, 1] \to S^1$ has an extension $\hat{f} : [0, 1] \to \mathcal{H}$ which is definable (as a map into $R^3$) and in particular sends $[0, 1]$ onto a definable set.*

*The extension is unique up to a constant in the following sense. If $g_1$ and $g_2$ are two definable extensions, then there is an element in $\mathcal{H}$ of the form $(m, e)$ such that $g_1 = g_2 + (m, e)$.*

*Proof.* The map $\pi : \mathcal{H} \to S^1$ induces a *local* ordering on $S^1$. Namely, given $p \in S^1$ we fix a small definable open interval $I$ in $\mathcal{H}$ such that $\pi$ is one-to-one on $I$ and $p \in \pi(I)$; $\pi$ induces an linear ordering on $\pi(I)$, which is independent of our choice of $I$. We use the notions "right of $p$" and "left of $p$" with respect to this notion. (except for the point $e \in S^1$, this is the same ordering which $S^1$ inherits from $[0, 1)$).

**Existence**

We assume first that $f$ is not locally constant at any point of $[0,1]$. Let $s_0 < s_1 < \cdots < s_N$ be the collection of all points $s \in [0,1]$ such that $f(s) = e$. If $s_N < 1$, then let $s_{N+1} = 1$.

We define $\hat{f} : [0,1] \to \mathcal{H}$ as follows. For $x \in [0, s_0)$ we let $\hat{f}(x) = (0, f(x))$, and define
$$\hat{f}(s_0) = \lim_{x \to s_0^-} \hat{f}(x).$$

Then, by induction, we define $\hat{f}$ on the interval $(s_i, s_{i+1})$, depending on the local behavior of $f$ at $s_i^+$:

**Case 1:** $f$ is strictly increasing at $s_i^+$ (with respect to the linear ordering of $[0,1]$ and the local ordering of $S^1$).

Given $x \in (s_i, s_{i+1})$, we then let
$$\hat{f}(x) = \hat{f}(s_i) + \big(0, f(x)\big).$$

**Case 2:** $f$ is strictly decreasing at $s_i^+$ (with respect to the ordering above).

Given $x \in (s_i, s_{i+1})$ we then let
$$\hat{f}(x) = \hat{f}(s_i) + \big(-1, f(x)\big).$$

In both cases we let
$$\hat{f}(s_{i+1}) = \lim_{x \to s_{i+1}^-} \hat{f}(x).$$

Note that if $i < N$, then $\hat{f}(s_{i+1})$ is of the form $(n, e)$ for some $n \in \mathbb{Z}$.

Clearly, $\hat{f}$ satisfies $\pi \hat{f} = f$. For continuity, it is sufficient to check that $\hat{f}$ is continuous at $s_i^+$ for $i = 0, \ldots, N$.

Assume first that $f$ is increasing at $s_i^+$. We have that,
$$\lim_{x \to s_i^+} \hat{f}(s_i) + \big(0, f(x)\big) = \hat{f}(s_i) + \lim_{x \to s_i^+} \big(0, f(x)\big),$$
with the limits taken in $\mathcal{H}$.

As $x$ tends to $s_i$ from the right, $f(x)$ tends, in $S^1$, to $e$ from the right. Hence, $(0, f(x))$ tends to $(0, e)$, and so $\hat{f}(x)$ tends to $\hat{f}(s_i)$. The second case is similarly proved.

If $f$ is constant on some interval (at most finitely many, by o-minimality), then we can replace in the above argument each $s_i$ by that interval, say $[s_i^0, s_i^1]$, and perform a similar process.

Note that since $N$ is finite this process produces a definable $\hat{f}$, whose image is contained in the interval $[(-N, e), (N, e)] \subseteq \mathcal{H}$.

**Uniqueness**

Assume that $g_1, g_2$ are two definable continuous functions from $[0,1]$ into $\mathcal{H}$ which satisfy $\pi g_1(x) = \pi g_2(x)$ for all $x \in [0,1]$.

For every $x \in [0,1]$, we have $\pi(g_1(x) - g_2(x)) = e$, and therefore $g_1(x) - g_2(x)$ is of the form $(n, e)$ for some $n \in \mathbb{Z}$. Since $g_1 - g_2$ is a definable continuous function on a definably connected set, its image must be definably connected and therefore contains a single point. Hence, $g_1 - g_2 = (n, e)$. □

**Remark 2.8.**

1. Even though $\mathcal{H}$ is not definable, every bounded part of it is definable. Therefore, for every definable continuous function $f : [0,1] \to S^1$, the image of $[0,1]$ under $\hat{f}$ is a (bounded) definably connected set, with respect to the group topology.
2. Assume that $\{f_a : a\}$ is a definable family of continuous functions from $[0,1]$ into $S^1$. Then, by o-minimality there is an $N$ such that $f_a^{-1}(e)$ has at most $N$ definably connected components. One can therefore carry out the above construction uniformly in $a$, and thus the family $\{\hat{f}_a : a\}$ is definable as well.
3. If $f : [0,1] \to S^1$ is a nonsurjective definable continuous function, then there is $x \in S^1$ such that no element of the form $(m, x)$ lies in the image of $\hat{f}$. Since this image is definably connected, it follows that in $\mathcal{H}$ we have $|\hat{f}(1) - \hat{f}(0)| < (1, e)$.

**Definition 2.9.**

1. Let $f : [0,1] \to S^1$ be a definable continuous function and let $\hat{f} : [0,1] \to \mathcal{H}$ be some definable extension of $f$. We define *the winding number of* $f$ to be

$$W(f) = \hat{f}(1) - \hat{f}(0).$$

2. Let $f : S^1 \to S^1$ be a definable continuous map. Then *the winding number of* $f$ is defined to be $W(f\sigma)$. We still write $W(f)$ for it.

By the uniqueness part of 2.7, the definition of $W(f)$ does not depend on our choice of $\hat{f}$. Moreover, if $\{f_a : a\}$ is a definable family of continuous functions from $[0,1]$ into $S^1$ then, by Remark (2), the family $\{W(f_a) : a\}$ is definable as well.

The notion of winding numbers is particularly interesting when $f : [0,1] \to S^1$ satisfies $f(1) = f(0)$. We call such an $f$ *a circular map*. We say that $f$ is *a circular bijection* if $f$ is circular and $f|[0,1)$ is injective. We use the notions of circular maps and circular bijections more generally, for functions from $[0,1]$ into $R^2$ that satisfy the same conditions. The winding number for circular maps are integers by the following claim.

**Claim 2.10.** *If $f : [0,1] \to S^1$ is a definable continuous circular map, then $W(f) = (m, e)$, for some $m \in \mathbb{Z}$. We write $W(f) = m$.*

*Proof.* Let $\hat{f}$ be an extension of $f$. Then $\pi\hat{f}(1) = \pi\hat{f}(0)$ and therefore $\hat{f}(1) - \hat{f}(0) = (m, e)$ for some $m \in \mathbb{Z}$. □

As the following fact shows, the definition of winding numbers for circular maps does not depend on our initial choice of $\sigma$, only on its "orientation".

**Fact 2.11.** *If $\sigma' : [0,1] \to [0,1]$ is a definable order-preserving bijection and $f : [0,1] \to S^1$ is a definable continuous circular map, then $W(f\sigma') = W(f)$. If $\sigma'$ is order-reversing, then $W(f\sigma') = -W(f)$.*

*Proof.* Going through the proof of 2.7, we replace everywhere $s_i$ by $s'_i = {\sigma'}^{-1}(s_i)$. Since $\sigma'$ is order-preserving, the behavior of $f\sigma'$ at ${s'_i}^+$ is the same as that of $f$ at $s_i^+$. Hence, $\widehat{(f\sigma')}(s'_i) = \hat{f}(s_i)$, and $\lim_{x \to s'_N} \widehat{(f\sigma')} = \lim_{x \to s_N} \hat{f}$. It follows that $W(f\sigma') = W(f)$.

We leave the order-reversing case for the reader. □

**Definition 2.12.** Let $\sigma' : [0,1] \to S^1$ be a definable continuous circular bijection. We say that $\sigma'$ has *positive orientation* if $\sigma^{-1}\sigma' : [0,1] \to [0,1]$ is order-preserving (i.e, $\sigma'$ is "counterclockwise"). Otherwise, we say that $\sigma'$ has *negative orientation.*

We can now list some of the main properties of $W(f)$.

**Lemma 2.13.**

(1) *Let $\sigma' : [0,1] \to S^1$ be a definable circular bijection. If $\sigma'$ has positive orientation, then $W(\sigma') = 1$. If $\sigma'$ has negative orientation, then $W(\sigma') = -1$.*
(2) *If $f : [0,1] \to S^1$ is a definable continuous circular map which is not surjective, then $W(f) = 0$.*
(3) *If $f, g : [0,1] \to S^1$ are definable continuous maps, then $W(f \cdot g) = W(f) + W(g)$, where multiplication and addition are taken in $K$ and $\mathcal{H}$, respectively. In particular, if $g(x) = f(x)^{-1}$ for all $x \in [0,1]$, then $W(g) = -W(f)$.*
(4) *Let $D$ be a definable, definably connected set, and $F(x, y)$ a definable continuous function from $[0,1] \times D \to S^1$, such that for every $d \in D$, $F(-, d)$ is circular. Then $W(F(-, d))$ is constant as $d$ varies over $D$.*
(5) *Let $f : S^1 \to S^1$ be of the form $f(z) = z_0 \cdot z$ for some fixed $z_0 \in S^1$. Then $W(f) = 1$.*
(6) *Let $f_1, f_2$ be definable continuous maps from $[0,1]$ into $S^1$ such that $f_1(1) = f_2(0)$. Let $f : [0,1] \to S^1$ be the concatenation of the two maps. Namely,*

$$f(t) = \begin{cases} f_1(2t) & t \in [0, 1/2) \\ f_2(2t-1) & t \in [1/2, 1]. \end{cases}$$

*Then $f$ is continuous and $W(f) = W(f_1) + W(f_2)$.*

*Proof.* (1) is immediate from 2.11. (2) follows from Remark (3) above.

For (3), let $h_1 = \widehat{f \cdot g}$ and note that by definition of addition in $\mathcal{H}$, $h_2 = \hat{f} + \hat{g}$ is also an extension of $f \cdot g$. But then

$$W(f \cdot g) = \hat{f}(1) + \hat{g}(1) - (\hat{f}(0) + \hat{g}(0)) = W(f) + W(g).$$

(4). By Remark 2.8 (2), the map $d \mapsto W(F(-,d))$ is a definable map from $D$ into $\mathbb{Z}$. It is thus sufficient to show that the map is locally constant.

Assume that $W(F(-,d_0)) = m$, and consider $H(x,d) = F(x,d) \cdot F(x,d_0)^{-1}$. $H$ is still a continuous function, and for every $d \in D$, $H(-,d)$ is circular. Moreover, $H(x,d_0) = e$ for all $x \in [0,1]$.

By 2.4, since $[0,1]$ is definably compact, there is an open neighborhood $U \subseteq D$ of $d_0$ such that for all $x \in [0,1]$ and $d \in U$, $|H(x,d) - H(x,d_0)| = |H(x,d) - e| < 1$. But then for all $d \in U$, $H(-,d)$ is a nonsurjective circular map from $[0,1]$ into $S^1$, hence $W(H(-,d)) = 0$. It follows from (3) that $W(F(-,d)) = W(F(-,d_0)) = m$ and therefore that the winding number of $F(-,d)$ remains constant as $d$ varies over $U$.

(5) is an immediate consequence of (3) together with the fact that the winding number of the identity map is 1 and of any constant map is 0.

(6) follows from the definition of $\hat{f}$. □

**Definition 2.14.** Let $f : [0,1] \to R^2$ be a definable continuous function, $w \notin f([0,1])$. *The winding number of $f$ with respect to $w$*, denoted by $W(f,w)$, is defined as $W(f^*)$ where $f^*(z) = (f(z) - w)/|f(z) - w|$.

**Fact 2.15.** *Let $f : [0,1] \to R^2$ be a definable continuous circular map, and let $W$ be a definably connected component of $R^2 \setminus f([0,1])$. If $w_1, w_2 \in W$, then $W(f,w_1) = W(f,w_2)$.*

*Proof.* Consider the map $H : [0,1] \times W \to S^1$ defined by $H(x,w) = (f(x) - w)/|f(x) - w|$.

For every $w \in W$, $H(-,w)$ is a definable continuous circular map. Hence, by 2.13 (4), $W(H(-,w_1)) = W(f,w_1) = W(f,w_2) = W(H(-,w_2))$. □

The following lemma will be used later on.

**Lemma 2.16.** *Suppose that $\pi_1, \pi_2 : [0,1] \to R^2$ are definable continuous, circular maps whose image does not contain 0, and moreover for every $t \in [0,1]$, the line segment between $\pi_1(t)$ and $\pi_2(t)$ does not contain 0. Then*

$$W(\pi_1, 0) = W(\pi_2, 0).$$

*Proof.* Consider $H : [0,1] \times [0,1] \to R^2$ defined by

$$H(t,s) = (1-s)\pi_1(t) + s\pi_2(t).$$

$H$ is definable and continuous. The assumption ensures that $H(t,s)$ is never 0, hence we can apply 2.13(4). □

### 2.3. Winding numbers and simple closed curves

**Definition 2.17.** A definable set $C \subseteq R^2$ is called a *simple closed curve* if there is a continuous definable circular bijection $\pi : [0,1] \to C$.

The following theorem is the o-minimal version of the Jordan curve lemma. It is due to Woerheide [Wo].

**Theorem 2.18.** *If $C \subseteq R^2$ is a simple closed curve, then $R^2 \setminus C$ is a disjoint union of two definably connected open sets, one of which is bounded and the other is unbounded. We call the bounded component the interior of $C$,* $\mathrm{Int}(C)$*.*

**Lemma 2.19.** *Let $\pi : [0,1] \to R^2$ be a continuous, definable circular bijection, whose image is a simple closed curve $C$. If $w$ lies in the unbounded component of $R^2 \setminus C$, then $W(\pi, w) = 0$. If $w$ lies in the interior of $C$, then either $W(\pi, w) = 1$ or $W(\pi, w) = -1$.*

*Proof.* Assume that $w$ lies in the unbounded component of $R^2 \setminus C$. Then, using 2.15, we may assume that $w$ lies very far from $C$. It easily follows from 2.13(2) that $W(\pi, w) = 0$.

In the case that $w$ lies in the interior of $C$ we offer two proofs. For one proof, see V.2.2 in [W]. The proof there works almost verbatim in our context. For another proof, we observe first that there is $w_1$ in the interior of $C$ such that the horizontal line through $w_1$ crosses $C$ exactly once, to the right of $w_1$ (we omit the proof here). Now, the map $\pi^*(t) = (\pi(t) - w_1)/|\pi(t) - w_1|$ from $[0,1]$ onto $S^1$ takes the value $e$ exactly once, and moreover $\pi^*$ is either locally increasing or decreasing at this point. It follows that $W(\pi^*, w_1)$ is either 1 or or $-1$. By 2.15, $W(\pi^*, w) = \pm 1$ as well. $\square$

**Definition 2.20.** Let $C$ be a definable simple closed curve, given by the map $\pi$. We say that $C$ *has positive orientation* if $W(\pi, w) = 1$ for $w$ in the interior of $C$. If $W(\pi, w) = -1$, we say that $C$ has *negative orientation*.

*Given a definable simple closed curve $C$, we always will assume that $C$ has positive orientation (clearly, we can always find such $\pi$).*

**Definition 2.21.** Let $C \subseteq R^2$ be a definable, simple closed curve, $f : C \to R^2$ a definable continuous function, $w \in R^2 \setminus f(C)$. Then $W_C(f, w)$ is defined to be $W(f\pi, w)$ for any definable parameterization $\pi$ of $C$ with positive orientation.

**Definition 2.22.** A definable simple closed curve is called *star-shaped* if there is $p$ in the interior of $C$ such that, for every $z \in C$, the line segment between $p$ and $z$ lies entirely within $\mathrm{Int}(C)$.

**Fact 2.23.** *Let $C \subseteq R^2$ be a definable, star-shaped, simple closed curve with interior $D$. Let $f : D \cup C \to R^2$ be a definable continuous map. If $w \notin f(D \cup C)$, then $W_C(f, w) = 0$.*

*Proof.* Take $p$ as in the definition. Without loss of generality $p = 0$. For $r \in R$, let $C_r = \{rz : z \in C\}$. Since $C$ is star-shaped, the simple closed curve $C_r$ is contained in $D$ for every $r \in (0, 1)$.

Since $w \notin f(D \cup C)$, the map $f^*(z) = (f(z) - w)/|f(z) - w|$ is well defined on $D \cup C$. By continuity, there is $r > 0$ sufficiently small such that $f^*|C_r$ is a nonsurjective map from $C_r$ into $S^1$. It follows that $W_{C_r}(f^*) = 0$.

Since $w \notin f(D \cup C)$ there is a continuous function $H : C \times [r, 1] \to S^1$ such that $H(x, 1) = (f(x) - w)/|f(x) - w|$ and $H(x, r) = f^*(rx)$. By 2.13 (4), $W_C(f(x), w) = W_C(f^*(rx), w) = W_{C_r}(f^*, w) = 0$. □

**Lemma 2.24.** *Let $C$ be a definable, star-shaped, simple closed curve whose interior is $D$. Let $f : D \cup C \to R^2$ be a definable continuous function and let $W$ be a definably connected component of $R^2 \setminus f(C)$. If there exists $w \in W$ such that $W_C(f, w) \neq 0$, then $W \subseteq \operatorname{Int}(f(D))$.*

*Proof.* Note first that since $C$ is a closed and bounded set, so is $f(C)$ and hence $R^2 \setminus f(C)$ is a finite union of definable open sets that are definably connected. In particular, $W$ is an open set and hence it is sufficient to show that $W \subseteq f(D)$.

Assume that there is $w' \in W$ that is not in $f(D)$. By 2.23, $W_C(f, w') = 0$, and therefore, by 2.15, $W_C(f, w) = 0$. □

We are going to need the following lemma.

**Lemma 2.25.** *Let $C$ be a definable simple closed curve whose interior is $D$, $A \subseteq D$ definable and let $f : (D \cup C) \setminus A \to R^2$ be a definable continuous function. Assume that for $w \in R^2$, there are definable simple closed curves $C_1, \dots, C_k \subseteq D$ such that their interiors $U_1, \dots, U_k \subseteq D$ are pairwise disjoint and $f^{-1}(w)$ is contained in the union of the $U_i$'s. Assume also that $A \subseteq \bigcup_i U_i$.*

*Then*

$$W_C(f, w) = \Sigma_i W_{C_i}(f, w).$$

*Proof.* We will prove the lemma for the case $k = 2$; the general case is proved similarly. Assume that $\pi_i : [0, 1] \to C_i$, $i = 1, 2$, are definable continuous circular bijection of positive orientation. Since $D$ is definably connected, it is not hard to see that one can define now a continuous map $h : [0, 1] \to D$ with the following properties:

$$h(t) = \begin{cases} \pi_1(4t) & t \in [0, 1/4) \\ \pi_2(4(t - 1/2)) & t \in [1/2, 3/4), \end{cases}$$

and for $t \in [1/4, 1/2) \cup [3/4, 1]$, $h(t) = h(5/4 - t)$, and $h(t) \notin C_1 \cup C_2$.

The image of $h$, call it $C_3$, consists of $C_1 \cup C_2$ together with a curve which is repeated twice, along opposite directions, and connects $C_1$ and $C_2$.

It is easy to see that

$$W(fh, w) = W(f\pi_1, w) + W(f\pi_2, w) = W_{C_1}(f, w) + W_{C_2}(f, w).$$

Now, let $\gamma$ be any definable simple path connecting $C_3$ to $C$ such that $\gamma$ lies entirely in $D \setminus (U_1 \cup U_2)$. We can find a definable map $g : [0,1] \to C \cup \gamma \cup C_3$, which goes back and forth along $\gamma$ and concatenates $\pi$ with a map of negative orientation along $C_3$.

We then have $W(fg, w) = W(f\pi, w) - W(fh, w)$.

It is left to see that $W(fg, w) = 0$. For that, we need to continuously separate the two copies of $\gamma$ such that we get now a simple closed curve whose interior is contained in $D_1 = D \setminus U_1 \cup U_2$. Since $w$ does not lie in $f(D_1)$, the winding number of $f$ with respect to $w$, along boundary of $D_1$ is zero. $\square$

## 2.4. Differentiability in $R$ and in $K$

First we recall some facts on the notion of $R$-differentiability (see Section 7 of [D1]). Given a definable open $U \subseteq R^m$ and definable $f : U \to R^n$, $f$ is called *$R$-differentiable at $c \in U$* if there is a linear map $T : R^m \to R^n$ such that

$$\lim_{|h| \to 0} \frac{|f(c+h) - f(c) - T(h)|}{|h|} = 0.$$

$T$ is then denoted by $d_c f$ and is called the $R$-differential of $f$ at $c$.

If $n = 1$ the partial $R$-derivatives, $\frac{\partial f}{\partial x_j}$, of $f$ are defined just as in the classical case. If $f$ is $R$-differentiable, then all the partial derivatives exist, but the converse is not true in general. However, if all the partial derivatives exist and continuous on some neighborhood of $c$, then $f$ is $R$-differentiable on this neighborhood and the map $c \mapsto d_c f$ is also continuous. The converse of the last statement is true as well.

For an open set $V \subseteq R^m$ and $f : V \to R^n$, we write $f = (f_1, \ldots, f_n)$. If $f$ is $R$-differentiable at $c$, then so are all the $f_i$'s and then the matrix of the partial derivatives is just the matrix of $d_c f$ with respect to the standard basis.

We define differentiability with respect to $K$ just as in the classical case. The basic definitions depend on the definability of the functions and sets in question, but we will use them here only for definable objects.

**Definition 2.26.** Let $U \subseteq K$ be a definable open set, $z_0 \in U$. A (definable) function $f : U \to K$ is *$K$-differentiable at $z_0$* if

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

exists in $K$ (where the limit is taken with respect to the topology of $R^2$). This limit, if it exists, is called the *$K$-derivative of $f$ at $z_0$* and is denoted by $f'(z_0)$.

Given $\alpha \in K$, the map $z \mapsto \alpha z$ is a $K$-linear endomorphism of $K$, as a vector space over the field $K$. We denote this map by $\lambda_\alpha$ when it is viewed as an endomorphism of the $R$-vector space $\langle R^2, + \rangle$. Given any $R$-linear map $\lambda : R^2 \to R^2$, it is easy to see that $\lambda = \lambda_\alpha$ for some $\alpha \in K$ if and only if its matrix, with respect to the standard basis, is of the form

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix},$$

for $a, b \in R$.

The following fact is easy to prove, just as in the classical case.

**Fact 2.27.** *Let $f : U \to K$ be a (definable) function on an open subset of $K$, $z_0 \in U$. Then the following are equivalent.*

(1) *$f$ is $K$-differentiable at $z_0$.*
(2) *$f$ is $R$-differentiable at $z_0$, as a function from $R^2$ to $R^2$, and $df_{z_0}$ equals to $\lambda_\alpha$ for some $\alpha \in K$.*
(3) *$f = (f_1, f_2)$, as a function from $R^2$ into $R^2$, is $R$-differentiable at $z_0$ and*

$$\frac{\partial f_1}{\partial x_1}(z_0) = \frac{\partial f_2}{\partial x_2}(z_0) \qquad \frac{\partial f_1}{\partial x_2}(z_0) = -\frac{\partial f_2}{\partial x_1}(z_0).$$

The basic rules of differentiation (for addition, multiplication and composition of functions) can be proved here just like in the classical case.

In particular, if $f \in K[x]$ is a polynomial over $K$, then the notion of $K$-derivative which we have defined above agrees with the formal definition of derivatives for polynomials. Thus, for polynomial maps, the notion of derivatives is independent of our initial choice of the real closed field $R$, even though the topology on $K$ and hence the general definition of derivative depend on $R$.

The known examples for $K$-differentiable functions in o-minimal structures all arise from the classical setting:

**Example 2.28.**

(1) Consider $\mathcal{R} = \mathbb{R}_{an}$, the expansion of the field of real numbers by functions $f : [0,1]^n \to \mathbb{R}$, where $f$ is real analytic on an open set containing $[0,1]^n$. Let $B \subseteq \mathbb{C}$ be the closed complex unit disc. If $F$ is a complex analytic function on an open subset of $\mathbb{C}$, containing $B$, then the real and imaginary parts of $F|B$ are given by restricted analytic functions and therefore are definable in $\mathbb{R}_{an}$. Thus $F|B$ is definable there as well.
(2) It is easy to see that the complex exponential function is not definable in any o-minimal expansion of $\mathbb{R}$. But its restriction to the set $\mathbb{R} \times [-1,1] \subseteq \mathbb{R}^2$, considered as a subset of $\mathbb{C}$, is definable in $\mathbb{R}_{an,\exp}$, the expansion of $\mathbb{R}_{an}$ by the real exponential function $\exp : \mathbb{R} \to \mathbb{R}$ (see [DMM] for a

reference to the o-minimality of $\mathbb{R}_{an}$ and $\mathbb{R}_{an,\exp}$). If $\mathcal{R}$ is a nonarchimedean elementary extension of $\mathbb{R}_{an,\exp}$, then we can get in $\mathcal{R}$ new $K$-differentiable functions using the classical ones. For example, if $\alpha \in R$ is an infinitesimal element, then the map $z \mapsto \exp(\alpha z)$ is definable in $\mathcal{R}$ and $K$-differentiable on $R \times [-1/\alpha, 1/\alpha] \subseteq K$.

### 2.5. $K$-differentiability and winding numbers

*The arguments in Sections* 2.5-2.10 *adapt ideas of Whyburn and Connell, mainly from* [W], [C1], [C2] *to the context of o-minimal structures. In many cases, the arguments are practically identical to the original ones and we include them here only for the sake of completeness. In other cases, o-minimality allows us to strengthen these results (e.g.* 2.31 *and* 2.37*), or simplify the arguments.*

**Fact 2.29.** *Let $U \subseteq K$ be a definable open set, $f : U \to K$ a definable map. If $f$ is $K$-differentiable at $z_0 \in U$ and if $f'(z_0) \neq 0$, then there is an $\epsilon > 0$ such that for every $r < \epsilon$ and every circle $C$ around $z_0$ with radius $r$, $W_C(f, f(z_0))$ is well defined and equals* 1.

*Proof.* Since $f'(z_0) \neq 0$ the function $f(z)$ does not take the value $f(z_0)$ at $z \neq z_0$ sufficiently close to $z_0$.

Let $d = f'(z_0)$. Then for $C$ a circle around $z_0$ sufficiently small, $(f(z) - f(z_0))/d(z - z_0)$ is close to 1, for $z \in C$. But then-

$$\frac{f(z) - f(z_0)}{|f(z) - f(z_0)|} \cdot \frac{|d||z - z_0|}{d(z - z_0)}$$

is close to 1 as well. Hence, by 2.13 (2), (3) the maps $z \mapsto (f(z)–f(z_0))/|f(z)–f(z_0)|$ and $z \mapsto d(z - z_0)/|d||z - z_0|$ have the same winding number on $C$. By 2.13 (5), the winding number of the latter function is 1. □

The following technical lemma is the main ingredient for the results in this section.

**Lemma 2.30.** *Let $C$ be a definable, star-shaped, simple closed curve whose interior is $D$. Let $f : D \cup C \to K$ be a definable continuous function, which is $K$-differentiable on $D \setminus L$, where $L$ is definable and* $\dim(L) = 1$. *Assume that $W$ is a definably connected component of $K \setminus f(C)$. Then the following are equivalent.*

(1) *$W \cap f(D) \neq \emptyset$,*
(2) *$W \subseteq f(D)$.*
(3) *There exists $w \in W$ such that $W_C(f, w) \neq 0$.*
(4) *For all $w \in W$, $W_C(f, w) \neq 0$.*

*Proof.* It is immediate that (4) implies (3). By 2.24, (3) implies (2), and again, it is immediate that (2) implies (1). It is thus left to show that (1) implies (4).

By replacing $L$ with its topological closure we may assume that $L$ is a closed set. Note that $W \cap f(D)$ must be infinite since $f(D)$ is definably connected, and by (1), $f$ is not a constant map. If we let $U = f^{-1}(W)$, then $U$ is a nonempty open subset of $D$, and $f$ takes infinitely many values on $U$.

Since the notion of a winding number is uniformly definable we may work in an $\aleph_1$-saturated structure. We assume that $f$ and $C$ are $\emptyset$-definable. Now, if $z$ is generic in $U$ over $\emptyset$, then $z \notin L$ and therefore $f$ is $K$-differentiable at $z$. Moreover, by o-minimality, $f'$ is continuous there. We claim that there is such a $z$ for which $f'(z) \neq 0$. If not, then $f'(z) = 0$ for every generic $z$ in $U$. But then $f$ is locally constant on $U \setminus L$ and since this set is dense in $U$, $f$ is locally constant on $U$, thus taking only finitely many values there, contradicting our previous observation.

Take $z_1 \in U \setminus L$ such that $f$ is continuously differentiable at $z_1$ and $f'(z_1) \neq 0$. Then, as a function from $R^2$ to $R^2$, $f$ is a $C^1$ function at $z_1$, whose differential at $z_1$ is invertible. By the Inverse Function Theorem (see (7.2.11) in [D1]), $f$ is a local homeomorphism at $z_1$. In particular, $W \cap f(D)$ contains a nonempty open set.

Let $w_0$ be a generic element of $W \cap f(D)$ over $\emptyset$. Since $\dim(w_0/\emptyset) = 2$, $f^{-1}(w_0)$ is finite and hence for every $z \in f^{-1}(w_0)$, $\dim(z/\emptyset) = 2$. In particular, $f$ is a $C^1$ function at $z$ in the sense of $R$, and $z \notin L$. Moreover, for every such $z$ we have $f'(z) \neq 0$, since otherwise there is an open, definably connected, neighborhood $V$ of $z$ such that $f'$ is zero on $V$, contradicting the fact that $f^{-1}(w_0)$ is finite.

Let $f^{-1}(w_0) = \{z_1, \dots, z_k\}$. By 2.29, around each $z_i$ there is a closed disc $D_i$, with boundary $C_i$ such that $W_{C_i}(f, w_0) = 1$ and such that they are all pairwise disjoint. Moreover, for every $z \in D \setminus \bigcup_i D_i$, $f(z) \neq w_0$.

We now can apply 2.25 and obtain

$$W_C(f, w_0) = \Sigma_i W_{C_i}(f, w_0) = k \neq 0.$$

By 2.15, for every $w \in W$, $W_C(f, w) = k \neq 0$. □

### 2.6. The Maximum principle, Identity theorem and "Liouville's theorem"

**Theorem 2.31** (The Maximum Principle)**.** *Let $C \subseteq K$ be a definable, star-shaped simple closed curve whose interior is $D$. Let $f : D \cup C \to K$ be a definable continuous function which is $K$-differentiable on $D \setminus L$, where* $\dim(L) = 1$*. If $z_0$ is any point in $D$, then either $f(z_0) \in f(C)$ or $f(z_0) \in \mathrm{Int}(f(D))$. In particular,*

$$|f(z_0)| \leq \mathrm{Max}_{z \in C} |f(z)|.$$

*Proof.* Assume that $f(z_0) \notin f(C)$ and let $W$ be the definably connected component of $K \setminus f(C)$ containing $f(z_0)$. By 2.30, $W \subseteq f(D)$, hence $f(z_0) \in \mathrm{Int}(f(D))$.

The rest easily follows: Indeed, let $M$ be the maximum of $|f|$ on $D \cup C$. If $|f(z)| = M$ for $z \in D$, then clearly, $f(z) \notin \mathrm{Int}(f(D))$. But then, there is $c \in C$ such that $|f(z)| = |f(c)| = M$. $\square$

We derive many of the basic results in complex analysis from the maximum principle. Below are several such theorems.

**Definition 2.32.** For $f : U \to K$, $z_0 \in U$, we write

$$\lim_{z \to z_0} f(z) = \infty$$

if $\lim_{z \to z_0} |f(z)| = +\infty$ in $R$.

Part (1) of the theorem below is sometimes called, in the classical setting, the identity theorem. Classically, the assumption on $f$ is that its zero set has an accumulation point. In the o-minimal case this is the same as being infinite. (2) is a strengthening of (1) and it handles boundary points of $\mathrm{dom}(f)$.

**Theorem 2.33.** *Let $U \subseteq K$ be a definably connected open set, $f : U \to K$ a definable function which is $K$-differentiable on $U$.*

1. *If $f(z) = w_0$ for infinitely many $z \in U$, then $f(z) = w_0$ for all $z \in U$.*
2. *Let $\hat{U}$ be the set of points $z_0$ in the topological closure of $U$ such that the limit of $f(z)$, as $z$ tends to $z_0$ in $U$, exists in $K$. We denote this limit, if it exists, by $f(z_0)$. If $f(z) = w_0$ for infinitely many $z \in \hat{U}$, then $f(z) = w_0$ for all $z \in \hat{U}$.*

*Proof.* Clearly, it is sufficient to prove (2). We may assume that $w_0 = 0$.

Let $Z = \{z \in \hat{U} : f(z) = 0\}$ and assume that $Z$ is infinite and different than $\hat{U}$.

Our plan is to use $f$ in order define a new $K$-differentiable function on some open set which is constant along a star-shaped simple closed curve, and hence should be constant on its interior.

Let $\partial Z$ denote the boundary of $Z$ in $\hat{U}$ (namely, $Cl(Z) \setminus \mathrm{Int}(Z)$). We claim that $\partial Z$ is infinite. If not, then $Z \cap (\hat{U} \setminus \partial Z)$ is nonempty and has no boundary in $\hat{U} \setminus \partial Z$, therefore $\hat{U} \setminus \partial Z$ is definably disconnected. However, this is impossible because $\hat{U}$ is contained in the closure of an open definably connected set $U$, and $\partial Z$ is finite.

Using rotation and translation, we may assume that $0 \in K$ lies on $\partial Z$ and that $\partial Z$ is the graph of a $C^1$ function $h : I \to R$, for $I \subseteq R$ an open interval around $0 \in R$, and that $h$ has a local extremum at 0 (and $h(0) = 0$). Moreover, we may assume that $Z$ lies "below $\partial Z$". More precisely, for some $\epsilon > 0$, if $x \in I$ and $h(x) < t < \epsilon$, then the point $x + it$ lies in $U \setminus Z$.

Let $0 < r < \epsilon$. For $t \in [-1, 1]$, consider the $R$-line $y = tx + r$ through $ir$ and let $\gamma_r(t) \in K$ be the first intersection point (if any) for $x > 0$, of this line with $\partial Z$. Note that if $r$ is sufficiently small, then $ir \in U \setminus Z$ and for all $t \in [-1, 1]$, $\gamma_r(t)$ is well defined. It follows that the line segment $I_r(t)$ between $ir$ and $\gamma_r(t)$ lies entirely in $U \setminus Z$.

Let $\gamma_r^1 \subseteq \partial Z$ be the curve drawn by $\gamma_r(t)$ as $t$ varies in $[-1, 1]$, and let $\gamma_r^2, \gamma_r^3, \gamma_r^4$ be the results of rotating $\gamma_r^1$ around $ir$ in angles of $\pi/2, \pi, 3\pi/2$, respectively. The union of these four curves forms a definable simple closed curve which we denote by $\gamma_r$, and its interior is $D_r$. This is a star-shaped curve (as witnessed by $ir$), and if we choose $r$ sufficiently small, then $D_r$ is contained in $U$.

We let $z_0 = ir$ and define the function $g(z)$ on $D_r \cup \gamma_r$ by the following formula:

$$g(z) = f(z) \cdot f\big(z_0 + i(z - z_0)\big) \cdot f\big(z_0 - (z - z_0)\big) \cdot f\big(z_0 - i(z - z_0)\big).$$

$g(z)$ is $K$-differentiable on $D_r$ and continuous on the closure of $D_r$. Since $\gamma_r^1$ is contained in $Z$, $g$ vanishes on the boundary $\gamma_r$ of $D$.

By the maximum principle for star-shaped simple closed curves, $g(z_0) = f(z_0) = 0$, contradicting our assumption that $z_0 \notin Z$. □

We can now prove that $K$-differentiable maps are open.

**Corollary 2.34.** *Let $U \subseteq K$ be a definably connected open set, $f : U \to K$ a definable function which is $K$-differentiable on $U$. If $f$ is not a constant map on $U$, then it maps every open subset of $U$ onto an open subset of $K$.*

*Proof.* Let $z_0$ be in $U$. Then, by 2.33, there is an open disc $D$ centered at $z_0$, such that $D$ and its boundary $C$ are contained in $U$, and for every $z \in C$ we have $f(z) \neq f(z_0)$. By 2.31, $f(z_0)$ lies in the interior of $f(D)$. □

**Remark 2.35.** Notice that the openness of $K$-differentiable maps immediately implies the maximum principle not only for star-shaped simple closed curves, but for all simple closed curves.

The following theorem is the direct analogue of Liouville's theorem.

**Theorem 2.36.** *Let $f : K \to K$ be a definable $K$-differentiable function. If $|f|$ is bounded on $K$, then it is constant.*

*Proof.* Take some $z_0 \in K$ and let $h(z) = (f(z) - f(z_0))/(z - z_0)$ for $z \neq z_0$, $h(z_0) = f'(z_0)$. By the differentiability of $f$, $h(z)$ is differentiable on $K \setminus \{z_0\}$ and continuous at $z_0$. By the maximum principle, for any circle $C$ of radius $r$ around $z_0$,

$$|h(z_0)| \leq \max_{z \in C} \left| \frac{f(z) - f(z_0)}{r} \right|.$$

It follows that $h(z_0) = 0$. Since $z_0$ is arbitrary $f'$ vanishes at every point in $K$, hence $f(z)$ is a constant function. □

### 2.7. Removable singularities

The theorem below is stronger than the standard removable singularity theorem in that we are able to remove not only isolated singularities but a one dimensional set of them. Similar theorems, in the classical setting, can be found in the more advanced literature on the subject.

**Theorem 2.37.** *Let $U \subseteq K$ be a definable open set, $f : U \to K$ a definable continuous function, which is $K$-differentiable on $U \setminus L$, where $L$ is definable,* $\dim(L) \leq 1$. *Then $f$ is $K$-differentiable on $U$.*

*Proof.* We fix $z_0 \in L$ and show that $f$ is $K$-differentiable at $z_0$. Let $D \subseteq U$ be a small open disc centered at $z_0$, whose boundary is $C \subseteq U$.

For $y \in U \setminus L$, consider the function $h_y(z) = (f(z) - f(y))/(z - y)$. It is continuous, as a function of $z$, at every $z \in D \cup C \setminus \{y\}$, and moreover, since $f$ is differentiable at $y$, $h_y(z)$ has a continuous extension to $z = y$. As a function of $z$ again, $h_y(z)$ is differentiable on $D \setminus (L \cup \{y\})$.

By the maximum principle, $|h_y(z)|$ attains its maximum on $C$. Hence (now varying $y$ also), for every $y \in D \setminus L$ and every $z' \in D$, we have

$$\left|\frac{f(z') - f(y)}{z' - y}\right| \leq \max_{z \in C}\left|\frac{f(z) - f(y)}{z - y}\right|.$$

In particular, for every $y \in D \setminus L$

$$\left|\frac{f(z_0) - f(y)}{z_0 - y}\right| \leq \max_{z \in C}\left|\frac{f(z) - f(y)}{z - y}\right|.$$

Consider now the expression on the right hand side of the last inequality. If we let $y$ vary over a closed and bounded disc $D_1 \subseteq D$ around $z_0$, which does not intersect $C$, then this expression is bounded, and hence also is the expression on the left.

By o-minimality, it follows that $(f(z_0) - f(y))/(z_0 - y)$ approaches a limit in $K$ as $y$ approaches $z_0$ along any definable path. It remains to show that this limit is independent of the path.

For $y_1, y_2 \in D \setminus L$ and $z \in D \setminus \{y_1, y_2\}$, consider the function

$$h_{y_1,y_2}(z) = \frac{f(z) - f(y_1)}{z - y_1} - \frac{f(z) - f(y_2)}{z - y_2}.$$

As before, $h_{y_1,y_2}$ can be extended to a continuous function on $D$, which is $K$-differentiable on $D \setminus (L \cup \{y_1, y_2\})$. By the maximum principle,

$$\left|\frac{f(z_0) - f(y_1)}{z_0 - y_1} - \frac{f(z_0) - f(y_2)}{z_0 - y_2}\right| \leq \max_{z \in C}\left|\frac{f(z) - f(y_1)}{z - y_1} - \frac{f(z) - f(y_2)}{z - y_2}\right|.$$

On the other hand, using 2.4 (applied twice to $F(x,y) = h_y(x)$), for every $\epsilon > 0$ there is a closed disc $D_1$ around $z_0$ such that for every $y_1, y_2 \in D_1$ and $z \in C$,

$$\left| \frac{f(z) - f(y_1)}{z - y_1} - \frac{f(z) - f(y_2)}{z - y_2} \right| < \epsilon.$$

It follows that as $y_1$ and $y_2$ approach $z_0$ along definable paths $\gamma_1, \gamma_2$, respectively, which do not intersect $L$, we have

$$\lim_{y_1 \in \gamma_1 \to z_0} \frac{f(z_0) - f(y_1)}{z_0 - y_1} = \lim_{y_2 \in \gamma_2 \to z_0} \frac{f(z_0) - f(y_2)}{z_0 - y_2}.$$

Denote $\frac{f(z_0)-f(y)}{z_0-y}$ by $g(y)$. Since $\dim L \leq 1$, there are at most finitely many limit points (including, possibly $\infty$), for $g(y)$, as $y$ approaches $z_0$ along $L$. It follows that $g(y)$ has at most finitely many limit points (including possibly $\infty$) as $y$ approaches $z_0$ in $U$ and at least one of those lies in $K$. But $U \setminus \{z_0\}$ is locally definably connected at $z_0$, and $g(y)$ is continuous there, so by 2.2 the set of limit points is definably connected, and therefore contains a single point. But then $f$ is $K$-differentiable at $z_0$. □

**Theorem 2.38.** *Let $U \subseteq K$ be an open set, $z_0 \in U$, and assume that $f : U \to K$ is a definable function which is $K$-differentiable on $U \setminus \{z_0\}$, and bounded on some neighborhood of $z_0$. Then $f$ can be extended, as a $K$-differentiable function, to $z_0$.*

*Proof.* The function $h(z) = (z - z_0)f(z)$ is continuous on $U$ and differentiable at $U \setminus \{z_0\}$. By 2.37, $h$ is $K$-differentiable at $z_0$, hence $\lim_{z \to z_0} f(z)$ exists. Thus $f$ can be extended, as a continuous function, to $z_0$. By 2.37 again, $f$ is now differentiable at $z_0$ as well. □

**Remark.** It might seem at first that 2.38 is too weak. Namely that, as in Theorem 2.37, we could have removed not only isolated singularities but a one dimensional set of them. This of course is false, for consider one branch of the function $f(z) = \sqrt{z}$ on $U \setminus \{x \in R : x > 0\}$, where $U \subseteq K$ is the open unit disc. $f$ is $K$-differentiable and bounded but its singularities cannot be removed. As we will show in Section 2.11, we can always, under the assumptions of 2.38, define $f$ on every $z_0 \in L$ such that it is continuous when we approach $z_0$ along a given definably connected neighborhood. But since $U \setminus L$ is not in general locally connected at generic $z_0 \in L$, there will often be more than one way to define $f$ at $z_0$, as in the example.

## 2.8. Infinite differentiability

**Lemma 2.39.** *Assume that $h$ is a definable $K$-differentiable function on a definable open set $U \subseteq K$, $z_0 \in U$. Then for all $\epsilon > 0$ there exists an open disc $D_1$ around $z_0$ such that*

$$\left| \frac{h(y) - h(x)}{y - x} - h'(z_0) \right| < \epsilon,$$

*for all $x \neq y$, both in $D_1$. In particular, $h'(z)$ is continuous at $z_0$.*

*Proof.* Let $\epsilon > 0$. By the $K$-differentiability of $h$ there is an open disc $D$ around $z_0$, with boundary $C \subseteq U$ such that for every $z \in D \cup C$,

$$\left| \frac{h(z) - h(z_0)}{z - z_0} - h'(z_0) \right| < \epsilon/2.$$

As in the proof of 2.37, since $C$ is definably compact, there is another open disc $D_1 \subseteq D$ around $z_0$ such that for every $y \in D_1$ and $z \in C$,

$$\left| \frac{h(z) - h(z_0)}{z - z_0} - \frac{h(z) - h(y)}{z - y} \right| < \epsilon/2.$$

We thus have, for every $z \in C$ and $y \in D_1$,

$$\left| \frac{h(z) - h(y)}{z - y} - h'(z_0) \right| < \epsilon.$$

We now change the roles of the variables and for a fixed $y \in D_1$ consider the function $g(x) = (h(y) - h(x))/(y - x) - h'(z_0)$. This function is (or can be extended to be) continuous at every point of $D_1$. It is also differentiable at $D_1 \setminus \{y\}$. By the maximum principle,

$$\left| \frac{h(y) - h(x)}{y - x} - h'(z_0) \right| \leq \mathrm{Max}_{z \in C} \left| \frac{h(y) - h(z)}{y - z} - h'(z_0) \right|,$$

for every $x, y \in D_1$.

Putting the last two equalities together we get the desired result. □

**Theorem 2.40.** *Let $U \subseteq K$ be a definable open set and assume that $f : U \to K$ is a definable function which is $K$-differentiable on $U$. Then $f$ is infinitely differentiable on $U$.*

*Proof.* It is enough to show that $f'(z)$ is $K$-differentiable on $U$. We fix $z_0 \in U$. By replacing $f(z)$ with $f(z + z_0) - f(z_0) - f'(z_0)z$, we may assume that $z_0 = 0$ and that $f(0) = f'(0) = 0$.

Let $h(z) = f(z)/z$, and $h(0) = 0$. Then $h$ is $K$-differentiable on $U \setminus \{0\}$ and continuous at 0. By 2.38, it is $K$-differentiable at 0 as well and by differentiation rules we have for $z \neq 0$ that

$$h'(z) = f'(z)/z - f(z)/z^2.$$

By the differentiability of $h$, $\lim_{z \to 0} f(z)/z^2$ exists and by 2.29, $\lim_{z \to 0} h'(z)$ exists. Therefore, $\lim_{z \to 0} f'(z)/z$ exists and since $f'(0) = 0$, this implies that $f'(z)$ is $K$-differentiable at 0. □

### 2.9. Taylor-like and Laurent-like series

**Lemma 2.41.** *Let $U \subseteq K$ be an open definable set, $z_0 \in U$, and let $f : U\setminus\{z_0\} \to K$ be a definable $K$-differentiable function. Then the following hold:*

(i) *The limit of $f(z)$ as $z$ tends to $z_0$ exists, (possibly equals $\infty$).*
(ii) *Either $f(z)$ or $1/f(z)$ can be extended, as a $K$-differentiable function, to $z_0$.*
(iii) *Assume that $D \subseteq U$ is an open disc around $z_0$ with boundary $C \subseteq U$, $w_0 \in K \setminus f(C)$, such that $W_C(f, w_0) = 0$. Then $\lim_{z\to z_0} f(z) \neq w_0$.*

*Proof.* (i) Consider $\Gamma(f) \subseteq K \times K$, the graph of $f$. It is a set of dimension two, hence its frontier has at most dimension one. In particular, there is a point $c \in K$ such that $(z_0, c)$ does not lie in the closure of $\Gamma$. The function $1/(f(z)-c)$ is thus a $K$-differentiable function on $U \setminus \{z_0\}$ which is bounded at $z_0$. By 2.38, $1/(f(z)-c)$ has a limit at $z_0$. If this limit is nonzero, then $\lim_{z\to z_0} f(z)$ exists in $K$. Otherwise, $\lim_{z\to z_0} f(z) = \infty$.

(ii) is immediate from (i) and 2.38.

(iii) Assume that $\lim_{z\to z_0} f(z) = w_0$. By 2.38, $f$ can be extended to a differentiable function on $U$ with $f(z_0) = w_0$. But then $w_0 \in f(D)$, contradicting 2.30. □

As the following lemma shows, in the o-minimal context there are no (using the classical terminology) essential singularities, only removable singularities and poles.

**Lemma 2.42.** *Let $U \subseteq K$ be an open definable set, $z_0 \in U$, $f : U \setminus \{z_0\} \to K$ a definable $K$-differentiable function, which is not constantly zero at a neighborhood of $z_0$. (By 2.41 $\lim_{z\to z_0} f(z) \in K \cup \{\infty\}$).*

(i) *If $\lim_{z\to z_0} f(z)$ is in $K$ (hence $f$ is $K$-differentiable at $z_0$) then there is a unique $n \in \mathbb{N}$ such that $\lim_{z\to z_0} f(z)/(z-z_0)^n$ exists in $K$ and is different from zero. We call $n$ the order of $f$ at $z_0$.*
(ii) *If $\lim_{z\to z_0} f(z) = \infty$ then there is a unique $n > 0$ in $\mathbb{N}$ such that $\lim_{z\to z_0} f(z)(z-z_0)^n$ exists in $K$ and is different from zero. We call $z_0$ a pole of $f$ and $-n$ is the order of $f$ at $z_0$.*

*Proof.* (i) If $f(z_0) \neq 0$ then we take $n = 0$. So we assume that $f(z_0) = 0$. Take $D \subseteq U$ an open disc containing $z_0$, whose boundary is $C \subseteq U$ and such that $0 \notin f(D \cup C \setminus \{z_0\})$ (by 2.33, there is such $D$). By 2.41(iii), since $\lim_{z\to z_0} f(z) = 0$, we have $W_C(f, 0) = n \neq 0$. Define $h(z) = f(z)/(z-z_0)^n$. Since $W_C((z-z_0)^n, 0) = n$, we can conclude from 2.13(3) that $W_C(h, 0) = 0$. Therefore, again by 2.41(iii), the limit of $h(z)$ as $z$ approaches $z_0$ exists (possibly $\infty$) and is different than 0. If $\lim_{z\to z_0} h(z) = \infty$, then we consider $1/h$ instead of $h$ and then, by the same argument, we see that $\lim_{z\to z_0} 1/h(z) \neq 0$, contradiction. Therefore, $\lim_{z\to z_0} f(z)/(z-z_0)^n$ exists in $K$. The uniqueness of $n$ is easy to verify.

For (ii), replace $f(z)$ by $1/f(z)$. □

As a corollary of the proofs of the above lemma we obtain.

**Corollary 2.43.** *Let $U \subseteq K$ be an open definable set, $z_0 \in U$, $f : U \setminus \{z_0\} \to K$ a definable $K$-differentiable function, which is not constantly zero at a neighborhood of $z_0$. Then the order of $f$ at $z_0$ equals $W_C(f, 0)$, for all sufficiently small circles $C \subseteq U$ centered at $z_0$.*

The following two theorems give expansions for $f$ at points which are (possibly) isolated singularities. These are analogous to either the Taylor expansion for analytic functions or the Laurent expansions for meromorphic functions.

**Theorem 2.44.** *Let $U \subseteq K$ be a definable open set, $z_0 \in U$, $f : U \to K$ a definable $K$-differentiable function, which is not constantly zero in a neighborhood of $z_0$. Let $n$ ($\geq 0$) be the order of $f$ at $z_0$, as given by 2.42(i).*

*Then there are $a_n, a_{n+1}, \dots, a_i, \dots \in K$, $i \geq n$, such that $a_n \neq 0$ and for every $k \geq n$,*

$$\lim_{z \to z_0} \frac{f(z) - \Sigma_{i=n}^{k-1} a_i (z - z_0)^i}{(z - z_0)^k} = a_k.$$

*Proof.* By 2.42(i), $\lim_{z \to z_0} f(z)/(z - z_0)^n \neq 0$. We let $a_n$ equal this limit. Consider now the function $f_n(z) = f(z) - a_n(z - z_0)^n$. If $f_n$ is constantly zero in a neighborhood of $z_0$, then we take $a_k = 0$ for $k > n$. Otherwise, applying 2.42(i) to $f_n$, we obtain an $m \in \mathbb{N}$ such that

$$\frac{f_n(z)}{(z - z_0)^m} \to c \neq 0$$

as $z$ tends to $z_0$. We have $m > n$, for otherwise, $\lim_{z \to z_0} f_n(z)/(z - z_0)^m = 0$. We then let $a_m$ be this limit and define $a_i = 0$ for $i = n + 1, \dots, m - 1$. We proceed similarly to obtain the whole sequence of the $a_i$'s. □

**Theorem 2.45.** *Let $U \subseteq K$ be a definable open set, $z_0 \in U$, $f : U \setminus \{z_0\} \to K$ a definable $K$-differentiable function. Assume that $\lim_{z \to z_0} f(z) = \infty$. Let $-n$ ($< 0$) be the order of $f$ at $z_0$, as given by 2.42(ii). Then there are $a_{-n}, a_{-n+1}, \dots, a_{-1}, a_0 \in K$, such that $a_{-n} \neq 0$ and for every $-n \leq k \leq 0$,*

$$\lim_{z \to z_0} \left( f(z) - \Sigma_{i=-n}^{k-1} a_i (z - z_0)^i \right) (z - z_0)^{-k} = a_k.$$

*In particular, the function*

$$f(z) - \left( \frac{a_{-n}}{(z - z_0)^n} + \cdots + \frac{a_{-1}}{z - z_0} \right)$$

*has a limit at $z_0$ and hence can be extended, as a $K$-differentiable function, to $z_0$.*

*Proof.* The same as 2.44, with 2.42(ii) replacing 2.42(i). □

Let $f : U \setminus \{z_0\} \to K$ be a definable $K$-differentiable function. If $f$ vanishes on some punctured neighborhood of $z_0$, we let $n(f, z_0) = \infty$ and define $a_n(f, z_0)$ to be 0 for all $n \in \mathbb{Z}$. Otherwise, we let $n(f, z_0)$ denote the order of $f$ at $z_0$ and let $\langle a_k(f, z_0) : k \in \mathbb{Z}\rangle$ be the sequence as given in 2.44 and 2.45, where $a_k(f, z_0)$ is taken to be 0 if $k < n(f, z_0)$. We sometimes omit $z_0$ in the notation and write $n(f)$ and $a_k(f)$ instead.

We list some basic properties of the expansion above.

**Theorem 2.46.** *Let $U \subseteq K$ be a definable open set, $z_0 \in U$, and let $f : U \setminus \{z_0\} \to K$ be a definable, $K$-differentiable function.*

(1) *If $g : U \setminus \{z_0\} \to K$ is another definable $K$-differentiable map, then $a_k(f + g, z_0) = a_k(f, z_0) + a_k(g, z_0)$.*

(2) *For all $k \in \mathbb{Z}$,*
$$a_k(f', z_0) = ka_{k+1}(f, z_0).$$
*In particular, if $n(f, z_0) \neq 0$, then $n(f', z_0) = n(f, z_0) - 1$, and if $n(f, z_0) = 0$, we have $n(f', z_0) \geq n(f, z_0)$.*

(3) *If $f$ can be extended to a $K$-differentiable function at $z_0$, then*
$$a_k(f, z_0) = \frac{f^{(k)}(z_0)}{k!},$$
*for $k \geq 0$.*

*Proof.* (1) is immediate from the definition of the $a_i$'s and the distributivity of the limit through addition.

For (2), let $n = n(f)$ and note that $g(z) = f(z)(z - z_0)^{-n}$ approaches $a_n(f) \neq 0$, as $z$ approaches $z_0$. Hence, $g(z)$ is (or can be extended to be) $K$-differentiable at $z_0$. It follows that $g'(z)$ has a limit at $z_0$ and therefore $g'(z)(z - z_0)$ tends to zero as $z$ tends to $z_0$. But $g'(z)(z - z_0) = f'(z)(z - z_0)^{-(n-1)} - nf(z)(z - z_0)^{-n}$ (even when $n = 0$), therefore
$$\lim_{z \to z_0} \frac{f'(z)}{(z - z_0)^{n-1}} = \lim_{z \to z_0} \frac{nf(z)}{(z - z_0)^n} = na_n(f, z_0),$$
and in particular, the order of $f'(z)$ is $n - 1$ if $n \neq 0$. Replacing now $f(z)$ by $f(z) - a_n(z - z_0)^n$, we may repeat the argument to obtain the desired result.

(3) follows from (2) by induction, using the fact that if $g$ is $K$-differentiable at $z_0$, then $a_0(g, z_0) = g(z_0)$. □

As the following theorem shows, there are no entire functions definable in an o-minimal structure other than the polynomial ones, and no meromorphic functions on all of $K$ other than rational ones. We are going to make model theoretic use of this theorem in Section 3.

**Theorem 2.47.** *Let $A \subseteq K$ be a finite set (possibly empty), $f : K \setminus A \to K$ a definable, $K$-differentiable function. Then $f(z)$ is a rational function.*

*Proof.* We may assume that $f$ has a pole at every element of $A = \{z_1, \ldots, z_k\}$. Then, by 2.45, there are $P_1(z), \ldots, P_k(z)$, rational functions, with poles at $z_1, \ldots, z_k$, respectively, such that $g(z) = f(z) - \Sigma_i P_i(z)$ can be extended to a $K$-differentiable function on all of $K$.

If $g$ is bounded on $K$, then, by 2.36, it must be constant and therefore $f$ is a rational function. If not, we consider the function $g(1/z)$, which must have a single pole at 0 and is bounded as $|z|$ tends to $\infty$. By 2.45, there is $n > 0$ and a polynomial $p(z)$ of degree at most $n$ such that $h(z) = g(1/z) - p(z)/z^n$ is a $K$-differentiable function on all of $K$. Since $p(z)/z^n$ is bounded as $|z|$ tends to $\infty$, $h(z)$ is bounded on $K$ and therefore it is a constant function. Hence, $f(z)$ is rational. □

We conclude this section with several other properties of the expansions. In the classical setting these are usually proved using the convergence of the power series expansion. As we already pointed out, we cannot hope for any such convergence in our setting, but it turns out that these properties are still true here.

**Theorem 2.48.** *Let $U \subseteq K$ be a definably connected open set, $z_0 \in U$. Let $f, g : U \setminus \{z_0\} \to K$ be definable, $K$-differentiable functions. If $a_k(f, z_0) = a_k(g, z_0)$ for all $k \in \mathbb{Z}$, then $f(z) = g(z)$ for all $z \in U$. In particular, if $f$ and $g$ are $K$-differentiable at $z_0$ and $f^{(k)}(z_0) = g^{(k)}(z_0)$ for all $k \geq 0$, then $f = g$ on $U$.*

*Proof.* By the additivity of the $a_i$'s, $a_k(f - g, z_0) = 0$ for all $k \in \mathbb{Z}$. But then, by 2.44 and 2.45, $f - g$ is identically zero on $U$. □

As in the classical case, we identify $R$ with the $x$-axis in $R^2$.

**Corollary 2.49.** *Let $U \subseteq K$ be a definable open set containing* 0, *and let $f : U \setminus \{0\} \to K$ be a definable $K$-differentiable function. Then the following are equivalent.*

(1) *There is a definable open set $V$ around* 0 *such that $f(z) \in R$ for all $z \in V \cap R$.*
(2) *For every $k \in \mathbb{Z}$, we have $a_k(f, 0) \in R$.*

*Proof.* (1) $\Rightarrow$ (2) because $R$ is a closed subset of $R^2$.

(2) $\Rightarrow$ (1): For $z = x + y\sqrt{-1}$, we define $\bar{z} = x - y\sqrt{-1}$. Let $V \subseteq U$ be a definably connected open set containing 0, which is symmetric with respect to the $x$-axis, and let $g(z) = \bar{f}(\bar{z})$ for $z \in V$. It is not hard to see that $g(z)$ is $K$-differentiable on $V \setminus \{0\}$, and that $a_k(g, 0) = a_k(f, 0)$ for all $k$. By 2.48, it follows that $f(z) = g(z)$ for all $z \in V$, which implies that $\bar{f}(z) = f(z)$ for all $z \in V \cap R$, therefore $f(z) \in R$. □

**Theorem 2.50.** *Let $D \subseteq K$ be the open unit disc, $C$ its boundary, and let $f : D \cup C \to K$ be a definable continuous function which is $K$-differentiable on $D$. Assume also that $|f(z)| \leq 1$ for all $z \in D$. Then for all $k \geq 0$,*

$$\frac{f^{(k)}(0)}{k!} \leq 2^k,$$

*and for all $z \in C \cup D$,*

$$\left| f(z) - \Sigma_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} z^k \right| \leq |2z|^n .$$

*Proof.* For $n \geq 1$, let

$$h_n(z) = f(z) - \Sigma_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} z^k.$$

Then, by our assumptions, $|h_1(z)| \leq 2$ on $D$ and by 2.46,

$$f^{(n)}(0)/n! = \lim_{z \to 0} h_n(z)/z^n.$$

By the maximum principle, for every $\rho \in R$ such that $0 < \rho \leq 1$,

$$\left| \frac{f^{(n)}(0)}{n!} \right| \leq \max_{|z|=\rho} \left| \frac{h_n(z)}{z^n} \right| \leq \max_{z \in C} \left| \frac{h_n(z)}{z^n} \right| = \left| \max_{z \in C} h_n(z) \right| .$$

By induction, the right hand size is no more than $1 + 1 + 2 + \cdots + 2^{n-1} = 2^n$, which implies the desired estimates. $\square$

As we already pointed out, the winding number of a uniformly definable family of functions with respect to a uniformly definable simple closed curves and points is uniformly definable as well. This, together with 2.43, gives us the following.

**Theorem 2.51.** *Let $U \subseteq K$, $W \subseteq R^n$, be definable open sets. $F : U \times W \to K$ is a definable function such that for every $w \in W$, $F(-, w)$ is $K$-differentiable on $U$ outside, possibly, a finite set.*

*Then there is $N \in \mathbb{N}$ such that for every $z \in U$ and every $w \in W$, either $F(-, w)$ is constantly zero at some neighborhood of $z$, or the order of $F(-, w)$ at $z$ is between $-N$ and $N$.*

### 2.10. Counting zeroes and poles; residues

**Definition 2.52.** Let $U \subseteq K$ be a definable open set, $A \subseteq U$ finite, and assume that $f : U \setminus A \to K$ is a definable $K$-differentiable function. For $z_0 \in U$ we define *the residue of $f$ at $z_0$* to be

$$\mathrm{res}_{z_0}(f) = a_{-1}(f, z_0).$$

Assume that $A$ is contained in the interior of a definable, simple closed curve $C$. We define

$$\mathrm{RES}_C(f) = \Sigma_{c\in \mathrm{Int}(C)} \mathrm{res}_c(f).$$

Note that since $\mathrm{res}_c(f) = 0$ for $c \notin A$, this is well defined. When $K = \mathbb{C}$ and $R = \mathbb{R}$,

$$\mathrm{RES}_C(f) = \frac{1}{2\pi i}\int_C f(z)dz.$$

If $f$ is $K$-differentiable on the interior of $C$, then for every $z_0 \in \mathrm{Int}(C)$ we clearly have

$$f(z_0) = \mathrm{RES}_C \frac{f(z)}{z-z_0}.$$

We can obtain an analogue of classical results on counting the zeroes and poles of a definable function.

**Proposition 2.53.** *Let $U \subseteq K$ be a definable open set containing a definable, simple closed curve $C$ together with its interior. Assume that, for a finite $A$, $f : U \setminus A \to K$ is a definable, $K$-differentiable function, such that no zero or pole of $f$ lies on $C$. Then*

$$\mathrm{RES}_C \frac{f'}{f} = N - Z = W_C(f, 0),$$

*where $N$ is the total number of zeroes of $f$ in* $\mathrm{Int}(C)$ *and $Z$ is the total number of poles of $f$ in* $\mathrm{Int}(C)$*, both counted with multiplicities.*

*Proof.* We will use 2.46(2) and (3). If $n(f, c) = 0$ for $c \in \mathrm{Int}(C)$, then $n(f'/f, z_0) \geq 0$ and therefore $\mathrm{res}_c(f'/f) = 0$. Hence, in order to calculate $\mathrm{RES}_C(f'/f)$, we need only look at the poles and zeroes of $f$. It is sufficient to show that for every such $c \in \mathrm{Int}(C)$, $\mathrm{res}_c(f'/f)$ equals the order of $f$ at $c$ (where, as we recall, this order is taken to be negative if $c$ is a pole and positive if $c$ is a zero).

Assume that $\lim_{z\to c} f(z)(z-c)^{-n} = a_n \neq 0$, $n \neq 0$. Then, by 2.46,

$$\lim_{z\to c} \frac{f'(z)}{f(z)}(z-c) = \lim_{z\to c} \frac{f'(z)(z-c)^{-(n-1)}}{f(z)(z-c)^{-n}} = \frac{na_n}{a_n} = n.$$

The fact that $N - Z = W_C(f, 0)$ follows from 2.43 and 2.25. □

We can now prove analogues of a theorem by Rouché.

**Theorem 2.54.** *Let $U$ be a definable open set, $C \subseteq U$ a definable closed simple curve, whose interior is contained in $U$. If $f, g : U \to K$ are definable, $K$-differential functions, and if $|f(z) - g(z)| < |g(z)|$ for all $z \in C$, then $f$ and $g$ have the same number of zeroes in the interior of $C$.*

*Proof.* Note that for every $z \in C$, the line segment between $f(z)$ and $g(z)$ cannot intersect 0, hence, $f$ and $g$ satisfy the assumptions of 2.16. It follows that $W_C(f, 0) = W_C(g, 0)$. The theorem now follows from 2.53. □

From the results and arguments developed so far (see also 2.43 and 2.51) we again obtain a definability result for uniformly definable families of definable functions.

**Theorem 2.55.** *Let $U \subseteq K$, $W \subseteq R^n$, be definable open sets. $F : U \times W \to K$ a definable function such that for every $w \in W$, $F(-, w)$ is $K$-differentiable on $U$ outside, possibly, a finite set. Let $C \subseteq U$ be a definable closed curve whose interior is contained in $U$.*

*Then the function $G : W \to K$, which is defined by $G(w) = \mathrm{RES}_C\, F(-, w)$, is definable. In particular, if $K = \mathbb{C}$ and $R = \mathbb{R}$, then the function*

$$G(w) = \frac{1}{2\pi i} \int_C F(z, w) dz,$$

*is definable.*

We comment that in the above, one can also vary $U$ and $C$ in some uniformly definable family.

The following is an analogue of a classical theorem of Hurwitz. It plays a crucial role in the subsequent development of the multivariable theory.

**Theorem 2.56.** *Let $W \subseteq R^n$, $U \subseteq K$ be definable open sets, $F : U \times W \to K$ a definable continuous function such that for every $\bar{w} \in W$, $F(-, \bar{w})$ is a $K$-differentiable function on $U$.*

*Take $(z_0, \bar{w}_0) \in U \times W$ and suppose that $z_0$ is a zero of order $m$ of $F(-, \bar{w}_0)$. Then for every definable neighborhood $V$ of $z_0$ there are definable open neighborhoods $U_1 \subseteq V$ of $z_0$ and $W_1 \subseteq W$ of $\bar{w}_0$ such that $F(-, \bar{w})$ has exactly $m$ zeroes in $U_1$ (counted with multiplicity) for every $\bar{w} \in W_1$.*

*Proof.* Let $C \subseteq V$ be a circle centered at $z_0$ such that $\mathrm{Int}(C) \subseteq V$ and the only zero of $F(-, \bar{w}_0)$ on $C \cup \mathrm{Int}(C)$ is at $z_0$.

Since $C$ is definably compact, $|F(-, \bar{w}_0)|$ attains a minimum $\epsilon > 0$ on $C$. By 2.4, there is a definable open $W$ around $\bar{w}_0$ such that for all $z \in C$ and $\bar{w} \in W$,

$$|F(z, \bar{w}) - F(z, \bar{w}_0)| < \epsilon/2 < |F(z, \bar{w}_0)|.$$

For every $\bar{w} \in W$ we now apply 2.54 to $F(z, \bar{w})$ and $F(z, \bar{w}_0)$. □

## 2.11. Behavior at boundary points

As we saw, when $f$ is a definable, $K$-differentiable function on an open set $U \subseteq K$ and $z_0$ is an isolated point of $\partial U$, then $z_0$ is either a removable singularity or a pole of $f$. The following theorem gives a weaker version of that result for boundary points of $U$ that are not necessarily isolated.

For $f : U \to K$ a definable function, $z_0 \in K$, we define $\mathrm{Lim}^U_{z_0} f$ to be the set of all limit points of $f(z)$ in $K \cup \{\infty\}$, as $z$ approaches $z_0$ inside $U$. It is easy to see that for $w \in K$ we have,

$$w \in \mathrm{Lim}^U_{z_0} f \Leftrightarrow \forall \epsilon > 0 \exists z \in U\big(|z - z_0| + |f(z) - w| < \epsilon\big). \tag{1}$$

Since $\mathrm{Lim}^U_{z_0} f$ is a subset of the frontier of Graph$(f)$, together possibly with $f(z_0)$, its dimension is at most one.

**Theorem 2.57.** *Let $U \subseteq K$ be a definable open set, $f : U \to K$ a definable $K$-differentiable function.*

(1) *For every $z_0 \in \partial U$, $\mathrm{Lim}^U_{z_0} f$ is a finite subset of $K \cup \{\infty\}$.*
(2) *Let $L$ be the set of points in $\partial U$ at which $U$ is locally definably connected. Then for every $z_0 \in L$, $\mathrm{Lim}^U_{z_0} f$ contains a single element and for all but finitely many $z_0 \in L$ this element is in $K$.*

*Proof.* To prove (1), we assume, towards contradiction, that for some $z_0 \in \partial U$, the set $\mathrm{Lim}^U_{z_0} f \cap K$ is infinite. If $U$ is not definably connected, then for one of its connected components, $V$, the set $\mathrm{Lim}^V_{z_0} f \cap K$ is infinite. By arguing now for $V$ instead of $U$, we may assume that $U$ itself is definably connected.

We may also assume that $f'(z) \neq 0$ for all $z \in U$. Indeed, either $f$ is constant on a neighborhood $U$ of $z_0$ in which case the result is immediate, or there is such a neighborhood $U$ where $f'(z)$ does not vanish.

We assume also that $\mathcal{R}$ is $\aleph_1$-saturated and that $U$ and $f$ are $\emptyset$-definable. We let $W = f(U)$. By 2.34, $W$ is open.

**A reduction step:** We may assume that $f$ is one-to-one on $U$ and that $f^{-1}$ is $K$-differentiable on $W$.

Indeed, for every $w \in W$, there are finitely many elements in $f^{-1}(w)$. We define $g(w)$ to be the nearest such element to $z_0$ among these (if there are several such we choose one definably). We can partition $W$ into definably connected open sets $W_1, \ldots, W_k$ and a set $S$ of smaller dimension such that $g$ is $R$-differentiable on each $W_i$.

For each $i$, by the chain rule, the $R$-differential of $g|W_i$ is nonsingular at every point and moreover, since it as the inverse of the $R$-differential of $f$, it is $K$-linear. Therefore $g|W_i$ is $K$-differentiable.

For each $W_i$, we let $U_i = g(W_i)$. Note that $f$ is one-to-one on each $U_i$ and, by 2.34, $U_i$ is open.

We fix $w_0$ generic in $\mathrm{Lim}^U_{z_0} f \cap K$ over $z_0$. Since $\bigcup W_i$ is dense in $W$, $w_0$ lies in the topological closure of $W_i$ for some $i$. By our definition of $g$ and o-minimality, there is an $i_0$ which satisfies the following:

$$\forall \epsilon \exists w \in W_{i_0}(|g(w) - z_0| + |w - w_0| < \epsilon).$$

But then, it easily follows that $z_0 \in \partial U_{i_0}$ and $w_0 \in \mathrm{Lim}_{z_0}^{U_{i_0}} f$. Since $w_0$ is generic over $z_0$, the set $\mathrm{Lim}_{z_0}^{U_{i_0}} f$ is infinite.

By replacing $U$ with $U_{i_0}$ we may assume that $f$ is one-to-one on $U$ and that $f^{-1}$ is $K$-differentiable, thus establishing the reduction step.

**Claim 2.58.** *For every $w \in K$, if $\dim(w/z_0) \geq 1$, then the set*

$$L(w) = \left\{z \in Cl(U) : w \in \mathrm{Lim}_z^U f\right\}$$

*is finite.*

*Proof.* Assume towards contradiction that $L(w)$ is infinite.

Note that for every $z \in U$, we have $\mathrm{Lim}_z^U f = \{f(z)\}$. Therefore, $L(w) \cap U$ is finite (or else $f$ is constant on $U$) and $L(w) \cap \partial U$ is infinite. Take $z$ generic in $L(w) \cap \partial U$ over $\{z_0, w\}$. Then $\dim(z) = \dim(z/w) = 1$ and therefore $\dim(w/z) = 1$. It follows that $\mathrm{Lim}_z^U f$ is infinite which implies that there are infinitely many $z \in \partial U$ for which $\mathrm{Lim}_z^U f$ is infinite. This is impossible since it implies that the frontier of the graph of $f$ in $K \times K$ has dimension two. We thus proved the claim.

For $\epsilon > 0$ let

$$U_\epsilon = \{z \in U : |z - z_0| < \epsilon\}.$$

We take $w_0 \in K$ generic in $\mathrm{Lim}_{z_0}^U f$ over $z_0$. By the above claim, there is $\epsilon > 0$ sufficiently small such that the only $z \in L(w_0) \cap Cl(U_\epsilon)$ is $z_0$ itself. Moreover, we may choose it such that $w_0$ is still generic over $\{\epsilon, z_0\}$.

Since $w_0$ is generic, there are infinitely many $w \in \mathrm{Lim}_{z_0}^U f$ for which $L(w) \cap Cl(U_\epsilon) = \{z_0\}$. We now replace $U$ by $U_\epsilon$ and thus have:

(i) $f$ is one-to-one on $U$ and $f'(z) \neq 0$ for all $z \in U$.

(ii) There are infinitely many $w \in \mathrm{Lim}_{z_0}^U f$ for which $z_0$ is the unique $z \in Cl(U)$ such that $w \in \mathrm{Lim}_z^U f$. We call the set of these points $L$.

We let $g$ be the inverse function of $f$ on the open set $W = f(U)$. Notice that for $z, w \in K$ we have

$$w \in \mathrm{Lim}_z^U f \Leftrightarrow z \in \mathrm{Lim}_w^W g.$$

Notice also that $L \cap W = \emptyset$ and hence $L \subseteq \partial W$. For $w \in L$ we have $\mathrm{Lim}_w^W g = \{z_0\}$ and therefore $lim_{w' \to w} g(w') = z_0$. In particular, we can extend $g$ continuously to every $w \in L$ by defining $g(w) = z_0$.

Since $g$ is $K$-differentiable on $W$ we can apply to it 2.33(2), thus deducing that $g$ is constant on $W$. Contradiction. This ends the proof of (1).

(2). We let $\bar{R}$ be as in Section 2.1 and let $\bar{K} = \bar{R} \times \bar{R}$.

Let $\Omega$ be the closure of the graph of $f$, as a subset of $\bar{K} \times \bar{K}$. By 2.2, for every $z \in \partial U$, if $U$ is locally definably connected at $z$, then $\Omega_z$ is definably connected.

By (1), $\Omega_z \cap K$ is at most finite, hence for $z \in L$, either $\Omega_z \cap K = \emptyset$ (in which case $lim_{z' \to z} f(z') = \infty$) or $\lim_{z' \to z} f(z')$ exists in $K$. We want to show that for all but finitely many $z$ in $L$ the latter is true.

We may assume that for all $z \in U$, $f(z) \neq 0$. We now replace the function $f(z)$ with $g(z) = 1/f(z)$. $g(z)$ is again $K$-differentiable on $U$. By 2.33(2), there are at most finitely many $z_0 \in L$ for which $\lim_{z \to z_0} g(z) = 0$. Hence, there are at most finitely many $z_0 \in L$ for which $\lim_{z \to z_0} f(z) = \infty$. □

One corollary of the above theorem is the following: Take $U \subseteq K$ to be a definable open set, $f : U \to K$ definable and $K$-differentiable. We consider the case where $\partial U$ is infinite, and take $z_0$ to be a generic point of $\partial U$ such that $U$ is locally definably connected at $z_0$. Then for all $n \in \mathbb{N}$, $\lim_{z \to z_0} f^{(n)}(z)$ exists in $K$, call it $a_n$.

**Question.** Does the sequence $\langle a_n : n \in \mathbb{N} \rangle$ give an expansion for $f$ at $z_0$ in the sense of 2.44? In particular, can we prove that

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = a_1 = \lim_{z \to z_0} f'(z)?$$

(See Section C.9.6 in [R] for a treatment of similar questions in the classical setting).

The behavior of $f$ at points $z_0$ on $\partial U$ that are not generic can differ. It is possible that $f(z)$ and all its derivatives have a limit as $z$ approach $z_0$, in which case the above question makes sense there. It is possible also that only some of the derivatives have a limit in $K$ while others have $\infty$ as a limit. For example, consider a branch of the function $f(z) = z^{1/2}$ on the upper right quadrant in $\mathbb{C}$. $f(z)$ has a limit at 0 but none of its derivatives have.

Unlike the case with isolated singularities, it is possible also to have an essential singularity at the boundary. For example, consider the function $e^z$, defined on the horizontal strip $\mathbb{R} \times [-1, 1]$ (see 2.28). The function $h(z) = e^{-1/z}$ is then definable on some open subset of $K$, with 0 on its boundary. The function $h(z)z^n$ tends to infinity as $z$ tends to 0, for every $n \in \mathbb{N}$.

## 3. An application to model theory and a theorem of Chow

### 3.1. The setting

Let us recall our setting. $R$ is a real closed field, $i = \sqrt{-1}$ fixed and $K = R(i)$ the algebraic closure of $R$, identified with $R^2$. As before, we identify $R$ with the subset of $R^2$ given by $\{(r, 0) : r \in R\}$. Every subset of $K^n$ is identified with a subset of $R^{2n}$. This identification sends every $\bar{a}$ in $K^n$ to a tuple in $R^{2n}$. To avoid complications we use the same notation for a subset of $K^n$ and its identification with a subset of $R^{2n}$. Similarly for tuples in $K^n$ and functions from $K^n$ into $K$.

We now fix $\mathcal{R}$, an o-minimal expansion of $R$ and let $\mathcal{K} = \langle K, +, \cdot, \ldots \rangle$ be an expansion of $K$ which is defined in $\mathcal{R}$. Namely, all the atomic relations of $\mathcal{K}$ are definable sets in $\mathcal{R}$, possibly with parameters. A subset $S \subseteq K^n$ is called $\mathcal{K}$-definable if it is definable in $\mathcal{K}$, with or without parameters. We say that $S \subseteq K^n$ is $\mathcal{R}$-definable if its identification with a subset of $R^{2n}$ is definable in $\mathcal{R}$, with or without parameters.

The main theorem of this section is the following.

**Theorem 3.1.** *Assume that $\mathcal{K}$ is a proper expansion of $\langle K, +, \cdot \rangle$ (i.e., some $\mathcal{K}$-definable set is not definable in $\langle K, +, \cdot \rangle$). Then the set $R$ is definable in $\mathcal{K}$. In particular, $\mathcal{K}$ has the order property and hence it is not a structure of a simple theory.*

**Remark 3.2.**

(i) The above theorem was proved by D. Marker in [M] in the case where $\mathcal{R} = \langle \mathbb{R}, +, \cdot \rangle$ and the same proof goes through when $\mathcal{R}$ is any real closed field. We are going to make use of the first step in Marker's proof but the rest of his arguments rest heavily on the fact that the definable sets are semialgebraic, so we cannot use them here.

(ii) The theorem is related to Zil'ber's Conjecture ([Z]). This conjecture consisted of two parts, both refuted by Hrushovski in their generality ([H1] and [H2]). One part suggested that a strongly minimal structure $\mathcal{M}$ which is not locally modular interprets an algebraically closed field $K$; the other, that this field carries no definable structure other than the field operations. Our theorem shows that if $\mathcal{M}$ is strongly minimal and definable inside an o-minimal structure, then the second part of Zil'ber's Conjecture holds in it. (It is still unknown whether the first part of the conjecture holds for such an $\mathcal{M}$.)

Before proving the theorem, let us note how it can be used, together with some earlier model theoretic analysis of compact complex manifolds by Hrushovski and Zilber [HZ], [Z1], to yield a "model theoretic" proof of a classical theorem of Chow. An analogous result was similarly used in [HZ] to yield the same theorem.

**Chow's Theorem.** *Every analytic subset of $\mathbb{P}_n(\mathbb{C})$ is algebraic.*

*Proof.* To every compact complex manifold $C$ one can associate a first order structure $\mathcal{N}_C$ whose universe is $C$ and whose atomic relations are all analytic subsets of $C^n$, $n \in \mathbb{N}$. These structures were analyzed by Zilber in [Z1], where some basic facts about complex manifolds were shown to imply that $\mathcal{N}_C$ is a stable structure of finite Morley rank, so, in particular, a real closed field cannot be definable in it.

Now, it is not difficult to see that $\mathcal{N}_C$ together with its atomic relations is definable in the o-minimal structure $\mathbb{R}_{an}$ (see Example 2.28).

If we take $C$ to be $\mathbb{P}_n(\mathbb{C})$, then $\mathbb{C}^n$, together with all its algebraic subsets, is definable in $\mathcal{N}_C$. By 3.1, every $\mathcal{N}_C$-definable subset of $\mathbb{C}^n$ is therefore algebraic.

Since $\mathbb{P}_n(\mathbb{C})$ is a union of finitely many such copies of $\mathbb{C}^n$, every $\mathcal{N}_C$-definable subset of it is algebraic. In particular, every analytic subset of $\mathbb{P}_n(\mathbb{C})$ is algebraic. □

We now return to the proof of Theorem 3.1. Our starting point is a very general lemma by Hrushovski.

**Lemma 3.3** (Lemma 1 in [H1]). *Assume that $\mathcal{M}$ is an $\aleph_0$-saturated expansion of an algebraically closed field $\langle K, +, \cdot\rangle$. If every unary function definable in $\mathcal{M}$ is definable in $\langle K, +, \cdot\rangle$, then every relation definable in $\mathcal{M}$ is definable in $\langle K, +, \cdot\rangle$.*

The assumption of saturation in the lemma is harmless in our case since, as easily seen, it is sufficient to prove Theorem 3.1 under the assumption that $\mathcal{R}$ is $\kappa$-saturated for some $\kappa$.

The next step in our proof is the first in Marker's proof.

**Lemma 3.4.** *If $R$ is not definable in $\mathcal{K}$, then $\mathcal{K}$ is strongly minimal.*

*Proof.* See the proof of Theorem 3.1 in Marker's paper ([M]). It works verbatim in our context. □

Our plan is to show, assuming strong minimality, that every $\mathcal{K}$-definable function from $K$ into $K$ is $K$-differentiable outside a finite set. By Theorem 2.47, every such function is then a rational function over $K$ and in particular definable in the field language of $K$. By 3.3, it will follow that every $\mathcal{K}$-definable set is definable in the field language of $K$, thus proving Theorem 3.1.

We assume that $\mathcal{R}$ (and therefore $\mathcal{K}$ as well) is $\aleph_1$-saturated. We also assume at times that $\mathcal{K}$ is strongly minimal, in which case we have two notions of dimension for tuples in $K^n$ ($R^{2n}$) and $\mathcal{K}$-definable subsets of $K^n$ ($\mathcal{R}$-definable subsets of $R^{2n}$). Namely, the dimension in the strongly minimal structure $\mathcal{K}$ and the o-minimal structure $\mathcal{R}$. We denote these two notions by $\dim_{\mathcal{K}}$ and $\dim_{\mathcal{R}}$, respectively. Note (see also Lemma 3.5 in [PePiS]) that if $S \subseteq K^n$ is a $\mathcal{K}$-definable set, then $\dim_{\mathcal{R}}(S) = 2\dim_{\mathcal{K}}(S)$.

Given $A \subseteq K$, let $A^* \subseteq R$ be the set obtained from $A$ by replacing each $a \in A$ (as an element of $R^2$) with its two coordinates. We say that $S \subseteq K^n$ is $\mathcal{R}$-definable over the parameter set $A$ if $S$ is definable in $\mathcal{R}$ over $A^*$. Similarly, we take $\dim_{\mathcal{R}}(a/A)$ to mean $\dim_{\mathcal{R}}(a/A^*)$. It follows from the previous remarks that if $\bar{a} \in K^n$ and $A \subseteq K$, then $\dim_{\mathcal{R}}(\bar{a}/A) \le 2\dim_{\mathcal{K}}(\bar{a}/A)$.

Assuming that $S$ is definable over $A$, a point $a \in S$ is called $\mathcal{R}$-generic ($\mathcal{K}$-generic) if $\dim_{\mathcal{R}}(a/A) = \dim_{\mathcal{R}}(S)$ ($\dim_{\mathcal{K}}(a/A) = \dim_{\mathcal{K}}(S)$).

### 3.2. Frontier of definable sets

We fix a $\mathcal{K}$-definable $S \subseteq K^2$ with $\mathcal{R}$-dimension 2 (hence $\mathcal{K}$-dimension 1). We assume that $S$ is $\emptyset$-definable. We call a subset of $K^2$ a $K$-line if it is a solution to an equation $az_2 = bz_1 + c$ for some $a, b, c \in K$. We call $b/a$ the $K$-slope of $L$ (possibly $\infty$). Note that if $\bar{a}$ is $\mathcal{R}$-generic in $S$, then $S$ is a 2-dimensional $R$-manifold in some neighborhood of $\bar{a}$, with tangent plane denoted by $T_{\bar{a}}(S)$, a two-dimensional linear subspace of $R^4$.

**Definition 3.5.** Let $S_1$ and $S_2$ be two definable $R$-submanifolds of $R^n$ such that $\dim_{\mathcal{R}} S_1 + \dim_{\mathcal{R}} S_2 = n$. We say that $S_1$ and $S_2$ intersect *transversally* at $a \in S_1 \cap S_2$ if $T_a(S_1)$ and $T_a(S_2)$ generate together, as vector subspaces, all of $R^n$.

The following lemma does not make any use of the strong minimality of $\mathcal{K}$.

**Lemma 3.6.** *Assume that every $K$-line intersects $S$ at most finitely many times. Take $\bar{a} = \langle a_1, a_2 \rangle \in K^2$ and $\bar{s} \in S$ with $\dim_{\mathcal{R}}(\bar{s}/\bar{a}) = 2$. Then the $\mathcal{K}$-line through $\bar{a}$ and $\bar{s}$ intersects $S$ at $\bar{s}$ transversally.*

*Proof.* For $\bar{b} \neq \bar{a}$ in $K^2$ we denote by $L_{\bar{b}}$ the $K$-line through $\bar{a}$ and $\bar{b}$. Assume towards contradiction that $L_{\bar{s}}$ does not intersect $S$ transversally at $\bar{s}$.

We write $\bar{s} = \langle s_1, s_2 \rangle$ and $\bar{a} = \langle a_1, a_2 \rangle$. Since $K$-lines intersect $S$ finitely many times and since $\dim_{\mathcal{R}}(\bar{s}/\bar{a}) = 2$ the $K$-slope of $L_{\bar{s}}$ is different than $\infty$. Since $\dim_{\mathcal{R}}(\bar{s}/\bar{a}) = 2$ there is an open $\mathcal{R}$-definable $U \subseteq R^4$ containing $\bar{s}$ such that $\bar{a} \notin U$ and, for every $\bar{u} \in U \cap S$, the line $L_{\bar{u}}$ intersects $S$ at $\bar{u}$ not transversally and its slope belongs to $K$.

We define a function $F : U \to K \cup \{\infty\}$ as

$$F((u_1, u_2)) = \frac{u_2 - a_2}{u_1 - a_1}$$

i.e. $F(\bar{u})$ is the $K$-slope of $L_{\bar{u}}$. Notice that each $L_{\bar{u}} \cap U$ is a level curve of $F$ and $F$ is $R$-differentiable on $U$.

Let $\bar{u} \in S \cap U$. By our assumptions, $S$ intersects the level curve of $F$ at $\bar{u}$ non-transversally, which means that the linear spaces $T_{\bar{u}}(S)$ and $ker(dF_{\bar{u}})$ intersects nontrivially. Since $\bar{s}$ is an $\mathcal{R}$-generic point of $S$, there is an open definable $V \subseteq R^2$ and a definable function $g : V \to R^4$ such that $g(V) \subseteq S \cap U$ and $g$ is an immersion.

For any $\bar{v} \in V$, the differential of $g$ at $\bar{v}$ gives a bijection between $R^2$ and $T_{g(\bar{v})}(S)$.

Applying the chain rule to $F \circ g$ we see that its differential has a nontrivial kernel at any point $\bar{v} \in V$. Thus the rank of $d(F \circ g)_{\bar{v}}$ is at most 1 at any point $\bar{v} \in V$ and therefore $\dim_{\mathcal{R}}(F \circ g(V)) \leq 1$.

But then there is $\bar{w} \in K$ such that $F^{-1}(w) \cap S$ is infinite, i.e., some $K$-line intersects $S$ at infinitely many points. Contradiction. □

We denote the frontier of $S$ by $fr(S)$.

**Theorem 3.7.** *Assume that $\mathcal{K}$ is strongly minimal. Then the frontier of $S$ is finite.*

*Proof.* If there is a $K$-line $L$ intersecting $S$ at infinitely many points, then either $S$ coincides with $L$ up to finitely many points, in which case the theorem is obvious or we can replace $S$ with $S \setminus L$. Since this can happen only finitely many times, we may assume that every $K$-line intersects $S$ at most finitely many times.

Assume that theorem fails. Since $S$ has $\mathcal{R}$-dimension 2, $\dim_{\mathcal{R}} fr(S) = 1$. Let $\alpha \in K$ be $\mathcal{R}$-generic and pick a point $a \in fr(S)$ with $\dim_{\mathcal{R}}(a/\alpha) = 1$.

We have $\dim_{\mathcal{R}}(\alpha/a) = 2$. Consider the family of $K$-lines with slope $\alpha$. This family has $\mathcal{K}$-dimension 1 and is $\mathcal{K}$-definable. Thus there is a number $k > 0$ such that all, but maybe finitely many, $K$-lines with slope $\alpha$ intersect $S$ at exactly $k$ points. Since $\dim_{\mathcal{R}}(a/\alpha) = 1$, the line $L$ with slope $\alpha$ through $a$ must intersect $S$ at exactly $k$ points. Let $s_1, \ldots, s_k$ be the points of intersection of $L$ and $S$. Since $\alpha$ is interalgebraic with every $s_i$ over $a$, $\dim_{\mathcal{R}}(s_i/a) = 2$.

By Lemma 3.6, $L$ intersects $S$ transversally at each $s_i$. This implies that small variations of $L$ within the family of all $K$-lines intersect $S$ close to $s_i$. Therefore if $b \in S$ is sufficiently close to $a$, then the $K$-line through $b$ with slope $\alpha$ intersects $S$ at least $k+1$ times: at $b$ and, for each $i$, at some point close to $s_i$. Since $a$ is on the frontier of $S$, there are infinitely many points in $S$ as close to $a$ as we wish. Thus we have infinitely many $K$-lines with slope $\alpha$ intersecting $S$ at more than $k$ points. Contradiction. $\square$

**Corollary 3.8.** *Assume that $\mathcal{K}$ is strongly minimal. If $f : K \to K$ is a $\mathcal{K}$-definable function, then $f$ is continuous everywhere but maybe is finitely many points.*

*Proof.* Consider the graph $G_f$ of $f$. Notice that, by o-minimality, if $z$ is a point of discontinuity of $f$, then there is $w \in K$ such that either $(z, w) \in fr(G_f)$ or $(z, w) \in fr(G_{1/f})$. Now apply 3.7. $\square$

### 3.3. $K$-differentiability of definable functions

Our goal is to show the following theorem

**Theorem 3.9.** *Assume that $\mathcal{K}$ is strongly minimal. If $f : K \to K$ is a $\mathcal{K}$-definable function, then it is $K$-differentiable at all but maybe at finitely many points.*

*Proof.* We still use $\bar{R}$ as in Section 2.1 and $\bar{K} = \bar{R} \times \bar{R}$. By 3.8, $f$ is continuous on an open set $D \subseteq K$ whose complement in $K$ is finite. Let $D_1 \subseteq D$ be the set of points where $f$ is $K$-differentiable, and let $S = D \setminus D_1$. Clearly, $D_1$ is $\mathcal{R}$-definable. Towards getting a contradiction we assume that $S$ is infinite. Thus $\dim_{\mathcal{R}}(S) > 0$.

We assume that $f$, $D$ and $D_1$ are $\emptyset$-definable. Using standard o-minimal arguments, we can find an open interval $I \subseteq R$ and an $R$-differentiable injective function $h : I \to S$. We will also assume that $h$ and $I$ are $\emptyset$-definable. Let $p$ be an $\mathcal{R}$-generic point in $I$ and $u = h(p)$. In order to get a contradiction it is sufficient to show that $f$ is $K$-differentiable at $u$.

Consider the $\mathcal{K}$-definable function $G$ from $D_u = D \setminus \{u\}$ into $K$ defined as

$$G(z) = \frac{F(z) - F(u)}{z - u}$$

We want to show that $\lim_{z\to u} G(z)$ exists.

We let $\Omega$ be the frontier of the graph of $G$ inside $\bar{K} \times \bar{K}$ and let $\Omega_u = \{c \in \bar{K} \mid \langle u, c\rangle \in \Omega\}$.

By o-minimality, $\lim_{z\to u} G(z)$ exists in $\bar{K}$ and equals $l$ if and only if $\Omega_u$ consists of a single element $l$. Thus we need to show that $\Omega_u = \{l\}$ for some $l \in K$.

**Lemma 3.10.** *$\Omega_u \cap K \neq \varnothing$.*

*Proof.* By o-minimality, it is sufficient to show that $|G(h(t))|$ is bounded for $t$ in some neighborhood of $p$.

Since $p$ is an $\mathcal{R}$-generic point on $I$, the function $F(h(t))$ is $R$-differentiable at $p$ (as a function from the interval $(p-\varepsilon, p+\varepsilon)$ into $R^2$). Therefore there is $c_1 \in R$ such that

$$\frac{|F(h(t)) - F(u)|}{|t-p|} < c_1$$

for all $t$ sufficiently close to $p$ (the division is taking place in $R$).

Since $h$ is injective and $p$ is $\mathcal{R}$-generic, $h$ is $R$-differentiable at $p$, and its derivative at $p$ is not zero. Thus there is $c_2 > 0$ such that

$$\frac{|h(t) - u|}{|t-p|} > c_2$$

for all $t$ sufficiently close to $p$. Thus we have, for $t$ sufficiently close to $p$,

$$|G(h(t))| = \frac{|F(h(t)) - F(u)|}{|h(t) - u|} = \frac{|F(h(t)) - F(u)|}{|t-p|}\frac{|t-p|}{|h(t)-u|} < \frac{c_1}{c_2}.$$

$\square$

Now can now finish the proof of Theorem 3.9. By 3.10 and 3.7, the set $\Omega_u \cap K$ is finite and nonempty. Note that $U \setminus \{u\}$ is locally definably connected at $u$, so, by 2.2, $\Omega_u$ is definably connected. It follows that that $\Omega_u \cap K$ contains a single element. $\square$

We now run again through the proof of Theorem 3.1. We showed that, under the assumption of strong minimality, every $\mathcal{K}$-definable $f : K \to K$ is $K$-differentiable outside a finite set. By 2.47, every such function is definable in $\langle K, +, \cdot\rangle$. By 3.3, every definable set in $\mathcal{K}$ is definable in $\langle K, +, \cdot\rangle$. We have then that if $\mathcal{K}$ is strongly minimal, it has the same definable sets as in $\langle K, +, \cdot\rangle$. Theorem 3.1 now follows from 3.4. $\square$

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