THE RATIONAL POINTS OF A DEFINABLE SET

J. PILA and A. J. WILKIE

Abstract

Let $X \subset \mathbb{R}^n$ be a set that is definable in an o-minimal structure over $\mathbb{R}$. This article shows that in a suitable sense, there are very few rational points of X which do not lie on some connected semialgebraic subset of X of positive dimension.

1. Introduction

This article is concerned with the distribution of rational and integer points on certain nonalgebraic sets in $\mathbb{R}^n$. To contextualize the kind of results sought and, in particular, to motivate the present setting of definable sets in o-minimal structures over $\mathbb{R}$ (see Definition 1.7), we begin by describing earlier results.

The ideas pursued here grew from the article [4] of Bombieri and Pila, where a technique using elementary real-variable methods and elementary algebraic geometry was used to establish upper bounds for the number of integer points on the graphs of functions $y = f(x)$ under various natural smoothness and convexity hypotheses. Results were obtained for f variously assumed to be (sufficiently) smooth, algebraic, or real analytic. Several results concerned the *homothetic dilation* of a fixed graph $X : y = f(x)$.

Definition 1.1

Let $X \subset \mathbb{R}^n$. For a real number $t \geq 1$ (which is always tacitly assumed), the *homothetic dilation* of X by t is the set $tX = \{\langle tx_1, \ldots, tx_n\rangle : \langle x_1, \ldots, x_n\rangle \in X\}$. By $X(\mathbb{Z})$ we denote the subset of X comprising the points with integer coordinates.

Suppose now that X is the graph of a function $f : [0, 1] \to \mathbb{R}$. Trivially, one has $\#(tX)(\mathbb{Z}) \leq t + 1$ (with equality, e.g., for $f(x) = x$ and positive integral t). According to Jarník [15], a strictly convex arc $\Gamma : y = g(x)$ of length ℓ contains at most $3(4\pi)^{-1/3}\ell^{2/3} + O(\ell^{1/3})$ integer points. (And, moreover, the exponent and constant are the best possible.) So if X is strictly convex, one infers that

$$\#(tX)(\mathbb{Z}) \leq c(X)t^{2/3}.$$

DUKE MATHEMATICAL JOURNAL
Vol. 133, No. 3, © 2006
Received 29 November 2004. Revision received 10 October 2005.
2000 *Mathematics Subject Classification*. Primary 11G99, 03C64.
Pila's work supported in part by McGill University and the Natural Sciences and Engineering Research Council of Canada.

However, Swinnerton-Dyer [32] showed that a substantially better estimate may be obtained if f is assumed to be C^3 and strictly convex; namely,

$$\#(tX)(\mathbb{Z}) \leq c(X,\epsilon)t^{3/5+\epsilon}$$

for all positive ϵ. (Regarding this circle of limited smoothness problems, also see [30], [21].)

Counting integer points on tX is of course the same as counting points $\langle m/t, n/t\rangle$ on X, and in this guise such questions arose in work of Sarnak [29] on Betti numbers of abelian covers. He conjectured that if f is C^∞ and strictly convex, then in fact,

$$\#(tX)(\mathbb{Z}) \leq c(X,\epsilon)t^{1/2+\epsilon}$$

for all positive ϵ. The exponent $1/2$ is the best possible here in view of $f(x) = x^2$. This conjecture was the starting point of [4], where it is affirmed.

If f is assumed to be transcendental analytic, however, then the exponent may be reduced to ϵ, and this result of [4] is the prototype for the results presented here.

THEOREM 1.2 ([4, Theorem 1])
Let $f : [0,1] \to \mathbb{R}$ be a transcendental real-analytic function. Let X be the graph of f, and let $\epsilon > 0$. There is a constant $c(X,\epsilon)$ such that

$$\#(tX)(\mathbb{Z}) \leq c(X,\epsilon)t^{\epsilon}.$$

Theorem 1.2 answers a question also raised by Sarnak [29] and has found application in that context (see [20], [28]). It can be adapted to a result on rational points of bounded height, which we proceed to state.

Definition 1.3
Let $H : \mathbb{Q} \to \mathbb{R}$ denote the usual height function $H(a/b) = \max(|a|, b)$, where $a, b \in \mathbb{Z}$, $b > 0$, and $\gcd(a,b) = 1$. Extend to $H : \mathbb{Q}^n \to \mathbb{R}$ by setting $H\langle\alpha_1,\ldots,\alpha_n\rangle = \max(H(\alpha_j))$. If $X \subset \mathbb{R}^n$, let $X(\mathbb{Q})$ denote the subset of points with rational coordinates. Set (for $T \geq 1$)

$$X(\mathbb{Q},T) = \left\{P \in X(\mathbb{Q}), H(P) \leq T\right\},$$

and define the *density function* of X to be

$$N(X,T) = \#X(\mathbb{Q},T).$$

This is not the usual projective height H_{proj}, which is not especially natural in the context of nonalgebraic sets, although this makes no difference to our results. The

density function $N(X, T)$ is a natural function to study in situations where $X(\mathbb{Q})$ may be infinite and is the primary object of study here. The aim is to establish upper bounds for $N(X, T)$ under natural geometric conditions on X and with the guiding idea that transcendental sets should contain few rational points, in an appropriate sense.

THEOREM 1.4 ([21, Theorem 9])
Let $f : [0, 1] \to \mathbb{R}$ be a transcendental analytic function with graph X and $\epsilon > 0$. There is a constant $c(X, \epsilon)$ such that

$$N(X, T) \leq c(X, \epsilon)T^{\epsilon}.$$

Now if f is of special form (e.g., $f(x) = e^x$ or, say, f is a G-function), then one may have much stronger results (or at least conjectures) on the scarcity of rational (or even algebraic) points (see, e.g., [1]). At the other extreme, constructions going back to Weierstrass (see, e.g., [19], [26]) show that an entire transcendental f may take rational values at every rational argument. These constructions do not take much care of the height density of points.

However, given any function $\epsilon : [1, \infty) \to \mathbb{R}$, strictly decreasing with $\epsilon(t) \to 0$ as $t \to \infty$, it is possible (see [23, Section 7.5]) to construct a transcendental analytic function f on $[0, 1]$ and a (rather lacunary) sequence of positive integers $T_j \to \infty$ such that

$$N(X, T_j) \geq T_j^{\epsilon(T_j)}.$$

Thus the above result cannot be much improved in general (e.g., taking $\epsilon(t) = (\log t)^{-1/2}$ shows that for certain X, no bound of form $N(X, T) \leq C(\log T)^K$ holds, and so forth).

Let us now turn to higher-dimensional sets. The idea of applying the methods of [4] to the density function of a higher-dimensional transcendental analytic set—for example, the graph X of a transcendental analytic function $f : [0, 1]^2 \to \mathbb{R}$—was prompted by a question of Bourgain (see [23]). Those methods entail studying the intersections of X with algebraic hypersurfaces of high degree. Such intersections may be highly singular, though they are still semianalytic. One is further led to consider images under projections, and thus one is led to the class of (bounded) *subanalytic sets* (see [3]) as the natural (and more general) class of sets to consider. The fundamental properties of subanalytic sets such as the uniformization theorem, Gabrielov's theorem, and Tamm's theorem (see [3]) enabled Pila to establish analogues of Theorems 1.2 and 1.4 for compact subanalytic sets of dimension 2 in $\mathbb{R}^n$.

To frame this result, it is necessary to consider a new feature that arises for sets X of dimension 2 and higher. Namely, X may contain semialgebraic subsets of positive dimension even if X itself is not semialgebraic, and these semialgebraic subsets may

contain many, that is, $\gg T^{\delta}$ for some positive δ, rational points of height at most T. For example, if $X = \{\langle x, y, z\rangle : z = x^y, x, y \in [1, 2]\}$, then each rational y gives rise to a semialgebraic curve in X. To accommodate this feature, we make the following definition.

Definition 1.5
Let $X \subset \mathbb{R}^n$. The *algebraic part* of X, denoted X^{alg}, is the union of all connected semialgebraic subsets of X of positive dimension. The *transcendental part* of X is the complement $X - X^{\mathrm{alg}}$.

The algebraic part of a set may be hard to identify in general, and it may be a complicated set. In the previous example, it consists of infinitely many connected components (one for each $y \in \mathbb{Q} \cap [1, 2]$), and so it is not subanalytic; this point is discussed further after Theorem 1.9. However, if the algebraic part is excluded from the count of rational points, a result of the same quality as Theorem 1.4 may be obtained.

THEOREM 1.6 ([24, Theorem 1.1])
Let $X \subset \mathbb{R}^n$ be a compact subanalytic set of dimension 2, and let $\epsilon > 0$. There is a constant $c(X, \epsilon)$ such that

$$N(X - X^{\mathrm{alg}}, T) \leq c(X, \epsilon)T^{\epsilon}.$$

The corresponding result for integer points on the dilation of such a set is in [23], where Theorems 1.2 and 1.4 are also upgraded to apply to any 1-dimensional compact subanalytic set $X \subset \mathbb{R}^n$.

The exclusion of X^{alg}, where rational points may well accumulate, is weakly analogous to (and suggested by) the notion of the *special set* in diophantine geometry and the philosophy of "geometry governs arithmetic" [14, Section F.5]. The conjecture of Lang [17, Chapter I, Section 3] (also see [14, Section F.5]) asserts that an algebraic variety V is *Mordellic* (i.e., has only finitely many rational points) outside its special set. When V is a curve, this is the Mordell conjecture, proved by Faltings [10]. In higher dimensions, it remains very much open. The special set is the Zariski closure of the union of all nonconstant rational images of morphisms from projective spaces and abelian varieties. Excluding the algebraic part from our count is much coarser but seems to be the appropriate way of separating out the essentially transcendental part of the problem.

For results on integer points on surfaces and higher-dimensional hypersurfaces under hypotheses of a differential-geometric nature, see [2], [30], and [31].

The generalization of Theorem 1.6 to arbitrary compact subanalytic sets in $\mathbb{R}^n$ was conjectured in [23]. Such sets are *globally subanalytic* (meaning subanalytic as

subsets of the compact real-analytic manifold $\mathbb{P}^n(\mathbb{R})$). The class of globally subanalytic sets is an example of an o-minimal structure, and this turns out to be the natural setting for our arguments.

The notion of an o-minimal structure arose in the study of model-theoretic questions about $\mathbb{R}$ ("Tarski's problem"; see [6]). Here is a definition, following [35]. (However, readers need to be familiar with the theory as developed, e.g., in [7] to follow our proofs.)

Definition 1.7

A *prestructure* is a sequence $\mathscr{S} = (\mathscr{S}_n : n \geq 1)$, where each $\mathscr{S}_n$ is a collection of subsets of $\mathbb{R}^n$. A prestructure $\mathscr{S}$ is called a *structure (over the real field)* if, for all $n, m \geq 1$, the following conditions are satisfied.

(1) $\mathscr{S}_n$ is a boolean algebra (under the usual set-theoretic operations).

(2) $\mathscr{S}_n$ contains every semialgebraic subset of $\mathbb{R}^n$.

(3) If $A \in \mathscr{S}_n$ and $B \in \mathscr{S}_m$, then $A \times B \in \mathscr{S}_{n+m}$.

(4) If $m \geq n$ and $A \in \mathscr{S}_m$, then $\pi[A] \in \mathscr{S}_n$, where $\pi : \mathbb{R}^m \to \mathbb{R}^n$ is projection onto the first n coordinates.

If $\mathscr{S}$ is a structure and $X \subset \mathbb{R}^n$, we say that X is *definable* in $\mathscr{S}$ if $X \in \mathscr{S}_n$. If $\mathscr{S}$ is a structure and, in addition,

(5) the boundary of every set in $\mathscr{S}_1$ is finite,

then $\mathscr{S}$ is called an *o-minimal* structure (over the real field).

The paradigm example of an o-minimal structure is provided by the semialgebraic sets; closure under projections is the content of the Tarski-Seidenberg theorem (see [7]).

The class of semianalytic sets is not closed under (proper) projections (see the classical example due to Osgood in [3]). But, as mentioned above, the globally subanalytic sets form an o-minimal structure, denoted $\mathbb{R}_{\text{an}}$. It is the closure under *complementation* that is hardest to establish; the aforementioned theorem of Gabrielov is the key to his proof of this fact (see [11]).

Another example of an o-minimal structure is the structure $\mathbb{R}_{\exp}$ generated by e^x. Here $\mathscr{S}_n$ is the collection of subsets of $\mathbb{R}^n$ of the form $X = \pi(f^{-1}(0))$, where, for some $m \geq n$, $f : \mathbb{R}^m \to \mathbb{R}$ is an *exponential polynomial* (i.e., $f(x_1, \ldots, x_m) = Q(x_1, \ldots, x_m, e^{x_1}, \ldots, e^{x_m})$ for some polynomial $Q \in \mathbb{R}[X_1, \ldots, X_{2m}]$) and $\pi : \mathbb{R}^m \to \mathbb{R}^n$ is the projection on the first n coordinates. The requisite finiteness property (5) follows from Khovanskiĭ's theorem [16, Chapter 1, Theorem 2]. In showing that $\mathbb{R}_{\exp}$ is a structure, the major difficulty is again the "theorem of the complement," which was established by Wilkie in [34]. Note that $\mathbb{R}_{\exp}$ contains (e.g., see [9]) sets such as $\{\langle x, x^r\rangle, x > 0\}$ for positive irrational r or $\{\langle x, e^{-1/x}\rangle, x > 0\}$, which do not belong to $\mathbb{R}_{\text{an}}$. (They are not subanalytic at the origin.)

In fact, the structure $\mathbb{R}_{\text{an,exp}}$ generated by the union of $\mathbb{R}_{\text{an}}$ and $\mathbb{R}_{\text{exp}}$ is o-minimal (see [8]). However, it is known (see [27]) that there is no largest o-minimal structure and that there are o-minimal structures containing nowhere analytic functions.

Thus o-minimal structures provide rich and flexible categories to work in and, at the same time, a natural setting for our methods.

Let $\mathscr{S}$ be an o-minimal structure (over $\mathbb{R}$). A *definable* set $X \subset \mathbb{R}^n$ means a set definable in $\mathscr{S}$. Now we can state a preliminary version of our main result.

THEOREM 1.8 (First version)
Let $X \subset \mathbb{R}^n$ be a definable set, and let $\epsilon > 0$. There is a constant $c(X, \epsilon)$ such that

$$N(X - X^{\text{alg}}, T) \leq c(X, \epsilon)\, T^{\epsilon}.$$

The proof of the theorem begins by showing that the points in question reside on few (i.e., $O_{X,\epsilon}(T^{\epsilon})$) hypersurfaces of suitable degree $d(\epsilon)$; it then proceeds by induction on the dimension of X. Thus it is necessary to have an estimate of the same form as in Theorem 1.8 for those hypersurface intersections but in which the constant is uniform over all intersections of X with hypersurfaces of fixed degree, that is, a result for a definable *family* of sets. The following convention is adopted. In considering subsets $Z = \{\langle x, y\rangle\} \subset \mathbb{R}^n \times \mathbb{R}^m$, projection on the first factor is denoted π_1, and on the second, it is π_2. Put $Y = Y_Z = \pi_2(Z)$, and for $y \in Y$, put $Z_y = \{z \in Z : \pi_2(z) = y\}$, and let $X_y = X_{Z,y} = \pi_1(Z_y)$ be its image in $\mathbb{R}^n$. A *family* $Z \subset \mathbb{R}^n \times \mathbb{R}^m$ of sets means the collection of *fibres* $\{X_y : y \in Y_Z\}$. A family Z is *definable* if the set Z is. The result to be proved is then the following.

THEOREM 1.9 (Second version)
Let $Z \subset \mathbb{R}^n \times \mathbb{R}^m$ be a definable family, and let $\epsilon > 0$. There is a constant $c(Z, \epsilon)$ with the following property. Let X be a fibre of Z. Then

$$N(X - X^{\text{alg}}, T) \leq c(Z, \epsilon)\, T^{\epsilon}.$$

The example $X = \{\langle x, y, z\rangle \in \mathbb{R}^3 : z = x^y, x, y \in [2, 3]\}$, for which $X^{\text{alg}} = \{\langle x, y, z\rangle \in X : y \in \mathbb{Q}\}$, shows that X^{alg} is not, in general, semialgebraic (or even definable; a definable set has only finitely many connected components). Nevertheless, it might be supposed that for any X and ϵ, there are a semialgebraic set $X_\epsilon \subset X$ and a constant $c(X, \epsilon)$ such that $N(X - X_\epsilon, T) \leq c(X, \epsilon)T^{\epsilon}$. This is not the case; consider $X = \{\langle x, y\rangle : 0 < x < 1,\ 0 < y < e^x\}$. Then $X^{\text{alg}} = X$, but X is not semialgebraic. (Otherwise, its bounding graph $y = e^x, x \in [0, 1]$ would be semialgebraic.) So $N(X - X_\epsilon, T) \gg T^4$ for any semialgebraic $X_\epsilon \subset X$. However, it is possible to find a *definable* $X_\epsilon \subset X^{\text{alg}}$ with the desired property; indeed, for a definable family Z, the

sets X_ϵ may be taken to be fibres of a definable family $W(Z, \epsilon) \subset Z$, and this is the final version of the result to be proved.

THEOREM 1.10 (Final version)
Let $Z \subset \mathbb{R}^n \times \mathbb{R}^m$ be a definable family, and let $\epsilon > 0$. There is a definable family $W = W(Z, \epsilon) \subset Z$ and a constant $c(Z, \epsilon)$ with the following property. Let $y \in Y$. Put $X = X_{Z,y}$, and put $X_\epsilon = X_{W,y}$. Then $X_\epsilon \subset X^{\mathrm{alg}}$, and

$$N(X - X_\epsilon, T) \leq c(Z, \epsilon)T^\epsilon.$$

Note that this version makes a nontrivial assertion in situations, like the example above, in which $(X - X^{\mathrm{alg}})(\mathbb{Q})$ is empty but X^{alg} is not definable.

The diophantine part of the proof follows the strategy going back to [4]. The heart of the analytic part of the proof is the possibility of a certain uniform parameterization of the fibres X in a definable family. The uniformity required is in the number of $C^{(r)}$ maps $(0, 1)^{\dim(X)} \to X$ required to cover X and, at the same time, in bounds on the sizes of all their partial derivatives up to some prescribed finite order r. This is achieved in Sections 2–5 by establishing an o-minimal version of Gromov's algebraic reparameterization lemma (see [12, page 232], itself a refinement of a method of Yomdin [37]) for obtaining such parameterizations of closed semialgebraic sets. Our o-minimal version of Gromov's reparameterization lemma may well find other applications. (Note: Gromov's proof is very brief; a careful proof has been given by Burguet [5].)

In [23], a conjecture is also made about integer points on the dilation of a compact subanalytic set. That conjecture is essentially (though not strictly) weaker than the corresponding statement about rational points and is also affirmed here, for (bounded) definable sets, in Section 8. Other results on integer points on definable curves are obtained in [36].

While, as indicated, the estimate $N(X - X^{\mathrm{alg}}, T) = O_{X,\epsilon}(T^\epsilon)$ cannot be improved for globally subanalytic sets, a much better estimate might be anticipated for other o-minimal structures where we have more control over the definable sets. For example, we have the following.

Conjecture 1.11
Let X be definable in $\mathbb{R}_{\exp}$. Then there are constants $c_1(X)$, $c_2(X)$ such that (for $T \geq e$)

$$N(X - X^{\mathrm{alg}}, T) \leq c_1(X)(\log T)^{c_2(X)}.$$

It should be observed that the results here give, even in the original situation of 1-dimensional sets $X : y = f(x)$ in $\mathbb{R}^2$, an extension of the previous results; curves

definable in o-minimal structures may have derivatives that degenerate at an endpoint, or they may not be analytic at any point (e.g., see the examples in [27]). The previous results may be inapplicable in such situations, and thus the present results may well find further application in the circle of problems (see [29], [20], [28]) which provided their original motivation.

In this article, $A \subset B$ means that A is a subset of (and is possibly equal to) B. The cardinality of a set A is denoted $\#A$, and $\mathbb{N}$ denotes the set of nonnegative integers. The letters $i, j, k, \ell, m, n, r, d$ are reserved exclusively to range over $\mathbb{N}$.

2. Reparameterization (after Yomdin and Gromov)

For Sections 2–5, we fix an o-minimal structure $\mathscr{S}$ over a real closed field M; definable means definable in $\mathscr{S}$ (see [7]). Although we are ultimately interested only in $\mathbb{R}$, the greater generality actually simplifies the arguments here because it guarantees a certain uniformity in parameters which would be absent if we restricted our attention to structures over $\mathbb{R}$.

Recall that an element $a \in M$ is called *finite* if $|a| \leq c$ for some $c \in \mathbb{N}$. (We assume that $\mathbb{Q}$ is identified with the prime subfield of M.) A finite element of M is also called *strongly bounded*. An n-tuple of elements of M is *strongly bounded* if all its coordinates are, and a definable subset of M^n is *strongly bounded* if there is a fixed finite bound for all the coordinates of all its elements. Furthermore, a definable function is *strongly bounded* if its graph is (equivalently, if its domain and range are).

Definition 2.1

Let $X \subset M^n$ be definable. A definable function $\phi : (0,1)^\ell \to X$, where $\ell = \dim X$, is called a *partial parameterization* of X. A finite set S of partial parameterizations of X is called a *parameterization* of X if $\bigcup_{\phi \in S} \operatorname{range}(\phi) = X$. (Of course, standard notation like "$(0, 1)$" refers to its natural interpretation in M.)

We are interested in various extra conditions on the functions in such an S. In particular, it is not hard to show, using the $C^{(r)}$-cell decomposition theorem (see [7, Chapter 3, Section 2]), that every bounded set has a $C^{(r)}$-parameterization. We are interested in bounding the derivatives.

Definition 2.2

A parameterization S (of some definable set X) is called an *r-parameterization* if every $\phi \in S$ is of class $C^{(r)}$ and has the property that $\phi^{(\alpha)}$ is strongly bounded for each $\alpha \in \mathbb{N}^{\dim X}$ with $|\alpha| \leq r$, where $|\alpha|$ is the sum of the coordinates of α.

THEOREM 2.3 (Reparameterization theorem (after Gromov))

For any $r \in \mathbb{N}$ and any strongly bounded, definable set X, there exists an r-parameterization of X.

There is also a version for functions.

Definition 2.4
Suppose that S is an r-parameterization of the definable set $X \subset M^m$ and that $F : X \to M^n$ is a definable function. Then we say that S is an *r-reparameterization* of F if, for each $\phi \in S$, $F \circ \phi$ is of class $C^{(r)}$ and $(F \circ \phi)^{(\alpha)}$ is strongly bounded for all $\alpha \in \mathbb{N}^{\dim X}$ with $|\alpha| \leq r$.

THEOREM 2.5
For any $r \in \mathbb{N}$ and any strongly bounded, definable function F, there exists an r-reparameterization of F.

Sections 3–5 are devoted to the proofs of Theorems 2.3 and 2.5. It is convenient to assume that M is $\aleph_0$-saturated, in particular, non-Archimedean, in these sections. This can always be assumed by taking a suitable elementary extension and noting that the statements of Theorems 2.3 and 2.5 pull back to the original structure.

3. The unary function case

There is a very simple but crucial analytic trick at the heart of the proof of Theorem 2.5 which we now state and prove. Indeed, the rest of the argument is just a case of organizing the induction carefully.

LEMMA 3.1
Let $r \geq 2$, and suppose that $f : (0, 1) \to M$ is a definable function of class $C^{(r)}$ with $f^{(j)}$ strongly bounded for $0 \leq j \leq r - 1$. Suppose further that $|f^{(r)}|$ is (weakly) decreasing. Define $g : (0, 1) \to M$ by

$$g(x) = f(x^2).$$

Then $g^{(j)}$ is strongly bounded for $0 \leq j \leq r$.

Proof
By the chain rule (applied in M), $g^{(i)}(x) = \sum_{j=0}^{i} \rho_{i,j}(x) \cdot f^{(j)}(x^2)$ for each $i = 0, 1, \ldots, r$ and $x \in (0, 1)$, where each $\rho_{i,j}$ is a polynomial with integer coefficients (of degree j, in fact).

Now, by our hypothesis on f, all summands are strongly bounded except, possibly, the one with $i = j = r$. One easily checks that this summand is $2^r x^r f^{(r)}(x^2)$. Let c be a positive integer strongly bounding the function $f^{(r-1)}$, and suppose, for a contradiction, that there is a some $x_0 \in (0, 1)$ with $|f^{(r)}(x_0)| > 4c/x_0$. By the mean value theorem (applied in M), there is some $\xi \in [x_0/2, x_0]$ such that $f^{(r-1)}(x_0) - f^{(r-1)}(x_0/2) = f^{(r)}(\xi) \cdot (x_0 - x_0/2)$. But by our hypothesis on $f^{(r)}$, we have $|f^{(r)}(\xi)| \geq |f^{(r)}(x_0)| >$

$4c/x_0$. Hence

$$2c \geq \left| f^{(r-1)}(x_0) - f^{(r-1)}\left(\frac{x_0}{2}\right)\right| > \frac{4c}{x_0}\left(x_0 - \frac{x_0}{2}\right) = 2c.$$

This contradiction shows that

$$|2^r x^r f^{(r)}(x^2)| \leq 2^r x^r \frac{4c}{x^2}$$

for all $x \in (0, 1)$, and the right-hand side here is bounded by $2^{r+2}c$ since $r \geq 2$. Thus $g^{(i)}$ is strongly bounded for $i = 0, 1, \ldots, r$, and the lemma is proved. □

LEMMA 3.2

Let $F : (0, 1) \to M$ be a definable, strongly bounded function. Then F has a 1-reparameterization, say, S, with the additional property that for each $\phi \in S$, either ϕ or $F \circ \phi$ is a polynomial (restricted to $(0, 1)$) with strongly bounded coefficients.

Proof

By o-minimality, choose elements $a_0 = 0 < a_1 < \cdots < a_p < a_{p+1} = 1$ of M, such that for each $i = 0, 1, \ldots, p$, F is of class $C^{(1)}$ and satisfies either $|F'| \leq 1$ throughout (a_i, a_{i+1}) or $|F'| \geq 1$ throughout (a_i, a_{i+1}).

In the first case, define $\phi_i : (0, 1) \to M$ by $x \mapsto (a_{i+1} - a_i)x + a_i$.

In the second case (when F is certainly strictly monotone and continuous on (a_i, a_{i+1})), we set $b_i = \lim_{x \to a_i^+} F(x)$, $b_{i+1} = \lim_{x \to a_{i+1}^-} F(x)$ and define $\phi_i : (0, 1) \to M$ by $x \mapsto F^{-1}((b_{i+1} - b_i)x + b_i)$.

In either case, range$(\phi_i) = (a_i, a_{i+1})$, and both ϕ_i and $F \circ \phi_i$ are of class $C^{(1)}$ throughout $(0, 1)$ with strongly bounded derivatives. Furthermore, at least one of these functions is linear with coefficients in $[-1, 1]$. It is now clear that $S = \{\phi_0, \phi_1, \ldots, \phi_p, \hat{a_1}, \ldots, \hat{a_p}\}$ is a 1-reparameterization of F with the required additional property, where each $\hat{a_i}$ denotes the constant function on $(0, 1)$ with value a_i. □

LEMMA 3.3

Let $r \geq 1$, and suppose that $F : (0, 1) \to M$ is a definable, strongly bounded function. Then F has an r-reparameterization (with the additional property that for each ϕ in it, either ϕ or $F \circ \phi$ is a polynomial (restricted to $(0, 1)$) with strongly bounded coefficients).

Proof

The proof (of the whole statement, including the parenthetical property) is by induction on r. The case $r = 1$ being Lemma 3.2, suppose that $r \geq 2$, and suppose that S is

an $(r-1)$-reparameterization of F with the additional property. Let $\phi \in S$, and write $\{\phi, F \circ \phi\} = \{g, h\}$, where g is a polynomial (restricted to $(0,1)$) with strongly bounded coefficients. Thus, in particular, $g^{(i)}$ exists and is strongly bounded for all i. However, we know only that $h^{(i)}$ exists, is continuous, and is strongly bounded for $i = 0, \ldots, r-1$. In order to apply Lemma 3.1, we use o-minimality to pick elements $0 = a_0 < a_1 < \cdots < a_{p_\phi} < a_{p_\phi+1} = 1$ in M (depending on ϕ), so that for each $i = 0, \ldots, p_\phi$, the function h is of class $C^{(r)}$ on (a_i, a_{i+1}) and $|h^{(r)}|$ is (weakly) monotonic on (a_i, a_{i+1}).

Let $\theta_{\phi,i} : (0,1) \to (0,1)$ be defined by

$$\theta_{\phi,i}(x) = \begin{cases} (a_{i+1} - a_i)x + a_i & \text{if } |h^{(r)}| \text{ is (weakly) decreasing,} \\ (a_i - a_{i+1})x + a_{i+1} & \text{if } |h^{(r)}| \text{ is (weakly) increasing.} \end{cases}$$

(We choose the first option, say, if $|h^{(r)}|$ is constant.)

It is immediate from the inductive hypothesis that $h \circ \theta_{\phi,i} : (0,1) \to M$ is of class $C^{(r)}$ and that $(h \circ \theta_{\phi,i})^{(i)}$ is strongly bounded for $i = 0, \ldots, r-1$. Furthermore, $|(h \circ \theta_{\phi,i})^{(r)}|$ is (weakly) decreasing. Let $\rho : (0,1) \to (0,1)$ be the $C^{(\infty)}$ bijection sending x to x^2. Then, by Lemma 3.1, the function $h \circ \theta_{\phi,i} \circ \rho : (0,1) \to M$ has strongly bounded ith derivative for $i = 0, \ldots, r$. Of course, the function $g \circ \theta_{\phi,i} \circ \rho$ is still a polynomial with strongly bounded coefficients, and $\{h \circ \theta_{\phi,i} \circ \rho, g \circ \theta_{\phi,i} \circ \rho\} = \{\phi \circ \theta_{\phi,i} \circ \rho, F \circ (\phi \circ \theta_{\phi,i} \circ \rho)\}$. Notice also that as i varies from 0 to p_ϕ, range$(\phi \circ \theta_{\phi,i} \circ \rho)$ covers range(ϕ) apart from finitely many points. So we have only to add finitely many constant functions (taking values in $(0,1)$) to the set $\{\phi \circ \theta_{\phi,i} \circ \rho : \phi \in S\}$ in order for it to become an r-reparameterization of F with the required additional property. This completes the induction and the proof of the lemma. □

COROLLARY 3.4

Let X be a strongly bounded subset of M, and let $F : X \to M$ be a strongly bounded function. Then for all $r \geq 1$, F has an r-reparameterization.

Proof

Since X is a (finite) union of strongly bounded intervals and points, it clearly has an r-parameterization, say, S, by linear and constant functions. Now use Lemma 3.3 to r-reparameterize each function $F \circ \phi : (0,1) \to M$ for $\phi \in S$, and take the union of these reparameterizations. □

We now proceed to the case of functions taking values in M^n for $n \geq 2$. However, there is nothing special about unary functions in this process, so we do the general case now.

LEMMA 3.5
Let $m, r \geq 1$, and assume that every definable, strongly bounded function with domain a subset X of M^ℓ (for some $\ell \leq m$) and range a subset of M has an r-reparameterization. Then for any $n \geq 1$, the same is true for such functions having range a subset of M^n (and domain X).

Proof
It is clearly sufficient (by the obvious inductive argument) to show that if $n \geq 2$ and $F : X \to M^{n-1}$, $f : X \to M$ are definable, strongly bounded functions such that F has an r-reparameterization, then so does the function $\langle F, f\rangle : X \to M^n$, where we may as well suppose that X is a definable (strongly bounded) subset of M^m.

So let S be an r-reparameterization of F, and let $\phi \in S$. Say that $\phi : (0,1)^\ell \to X$, where $\ell = \dim(X) \leq m$. Apply the hypothesis of the lemma to the function $f \circ \phi : (0,1)^\ell \to M$ to obtain an r-reparameterization of it, say, T_ϕ. Then each $\psi \in T_\phi$ has domain $(0,1)^\ell$, and it clearly follows by repeated use of the chain rule that each function $(\phi \circ \psi)^{(\alpha)} : (0,1)^\ell \to M^m$ for $\alpha \in \mathbb{N}^\ell$ with $|\alpha| \leq r$ is strongly bounded. It is now easy to check that $\{\phi \circ \psi : \phi \in S, \psi \in T_\phi\}$ is an r-reparameterization of $\langle F, f\rangle$, as required. □

COROLLARY 3.6
Let $n \geq 1$, and suppose that $F : X \to M^n$ is a strongly bounded function, where X is a (strongly bounded) subset of M. Then for any $r \geq 1$, F has an r-reparameterization.

Proof
This is immediate from Corollary 3.4 and the case $m = 1$ of Lemma 3.5. □

4. Some questions of convergence

In Gromov's proof, things can be arranged, it seems, so that derivatives are a priori bounded, and we need to be able to reduce to this situation. We achieve this by first truncating our given function and finding the reparameterization for the truncation. We then let the truncations converge to the original function. So we require an observation that allows us to conclude that the reparameterizations converge as well. In fact, we lose one level of differentiability here, but this hardly matters. The final proofs of Theorems 2.3 and 2.5 are arranged so that we require only a theory of convergence for unary functions, so we treat only that case here.

So suppose that $N \geq 1$, $N \in \mathbb{N}$, and suppose that $\{F_t : (0,1) \to (0,1)^N : t \in (0,1)\}$ is a definable family of functions (meaning that the map sending $\langle t, x\rangle$ to $F_t(x)$ is a definable function on $(0,1)^2$). Suppose further that $r \geq 1$, that all the functions F_t are of class $C^{(r)}$, and that their derivatives $F_t^{(i)}$ are strongly bounded for $i = 0, \ldots, r$. Since we are assuming that M is non-Archimedean, this clearly implies a uniform finite

bound, say, c. Using o-minimality, we may define a function $F_0 : (0, 1) \to [0, 1]^N$ by $F_0(x) = \lim_{t\to 0^+} F_t(x)$.

Now the fact that $r \geq 1$ implies that F_0 is continuous. To see this, suppose that $x_1, x_2 \in (0, 1)$ are distinct. Choose $t \in (0, 1)$ such that $\|F_0(x_i) - F_t(x_i)\| \leq |x_1 - x_2|$ for $i = 1, 2$. (We use the sup norm $\|\langle u_1, \ldots, u_p\rangle\| = \max\{|u_1|, \ldots, |u_p|\}$ on Cartesian products of M throughout.) By the mean value theorem (in M), we also have $\|F_t(x_1) - F_t(x_2)\| \leq N\, c\, |x_1 - x_2|$ (as c is a bound for $F_t'(x)$ for $x \in [x_1, x_2]$). Thus

$$\begin{aligned}\|F_0(x_1) - F_0(x_2)\| &\leq \|F_0(x_1) - F_t(x_1)\| + \|F_t(x_1) - F_t(x_2)\| + \|F_t(x_2) - F_0(x_2)\| \\ &\leq (Nc + 2)\, |x_1 - x_2|,\end{aligned}$$

whence the continuity of F_0.

One can now go on to show that for each $i = 0, \ldots, r - 1$, F_0 is of class $C^{(i)}$, $F_t^{(i)}$ is strongly bounded, and, indeed, $F_0^{(i)}(x) = \lim_{t\to 0^+} F_t^{(i)}(x)$ for each $x \in (0, 1)$.

Remark 4.1

This result properly belongs to the theory of *definably Banach* spaces (over o-minimal structures) currently being developed by A. J. Wilkie and Margaret Thomas. The simplest example is the set $\Omega^{(r)}$ of all M-definable functions $F : (0, 1) \to M$ with continuous and bounded derivatives up to order r, which is naturally a normed vector space for the field structure on M and with norm $\|F\|^{(r)} := \sup_{x\in(0,1), i=0,\ldots,r} |F^{(i)}(x)|$. If $\sigma = \{F_t : t \in (0, 1)\}$ is a definable family contained in $\Omega^{(r)}$, then it is clear what we should mean by saying that σ is *Cauchy* (as $t \to 0^+$), and it is routine to check that the pointwise limit of σ is indeed the $\|\cdot\|^{(r)}$-limit and lies in $\Omega^{(r)}$ if σ is Cauchy. More important for us here, however, is the fact, borrowed and modified from the classical theory, that $\Omega^{(r)}$ is *definably compactly* contained in $\Omega^{(r-1)}$ for all $r \geq 1$. In other words, *every* $\|\cdot\|^{(r)}$-bounded, definable family σ in $\Omega^{(r)}$ is Cauchy in $\Omega^{(r-1)}$, and hence the pointwise limit of σ lies in $\Omega^{(r-1)}$. The crucial point in the o-minimal setting is that one knows a priori that this limit function is $(r - 1)$-times continuously differentiable at all but finitely many $x \in (0, 1)$.

We now consider, for each $t \in (0, 1)$, the *set* S_t of coordinate functions of F_t. Let us suppose that it parameterizes $(0, 1)$, so that it is an r-parameterization of $(0, 1)$. We define S_0 to be the set of functions $\phi|_{\phi^{-1}[(0,1)]}$ for ϕ a coordinate function of F_0. Then we have the following.

(A) $\bigcup_{\psi\in S_0} \text{range}(\psi) = (0, 1)\backslash T$ for some finite set $T \subset (0, 1)$. (Otherwise, by o-minimality, there would be a nonempty, open subinterval of $(0, 1)$ missed by each $\psi \in S_0$ and hence missed by each corresponding coordinate function ϕ of F_0. But this easily contradicts the fact that each S_t parameterizes $(0, 1)$ and that $\lim_{t\to 0^+} F_t(x) = F_0(x)$ (for $x \in (0, 1)$), bearing in mind the fact that

as $r \geq 1$, the derivative of each coordinate function of F_t has a uniform finite bound.)

(B) Notice also that each function $\psi \in S_0$ has domain an open subset of $(0, 1)$ (which may have infinite complement in $(0, 1)$), is of class $C^{(r-1)}$, and is such that $\psi^{(i)}$ is strongly bounded for $i = 0, \ldots, r-1$.

We now apply these remarks to set up the inductive process involved in the proofs of Theorems 2.3 and 2.5. We fix $m \geq 1$ in Notation 4.2, Lemma 4.3, and Corollary 4.4.

Notation 4.2

(1) For U a definable, open subset of M^{m+1}, we write $V \Subset U$ to mean that V is a definable, open subset of M^{m+1} with $V \subset U$ and $\dim(U \backslash V) \leq m$.

(2) For $\phi : (0, 1) \to M$ a definable function, we define $I_\phi : (0, 1)^{m+1} \to (0, 1)^m \times M$ by $\langle x_1, \ldots, x_m, x_{m+1} \rangle \mapsto \langle x_1, \ldots, x_m, \phi(x_{m+1}) \rangle$. If $X \subset M^{m+1}$ and $f : X \to M^n$ are definable, f_ϕ denotes $f \circ I_\phi$ (having domain $I_\phi^{-1}[X]$).

LEMMA 4.3

Suppose that $n \geq 1$, that $U \Subset (0, 1)^{m+1}$, and that $f : U \to M^n$ is a definable, strongly bounded function. Suppose further that for each $i = 1, \ldots, m$, $\partial f / \partial x_i$ exists, is continuous, and is strongly bounded (on U).

Then for each $r \geq 2$, there exists an $(r-1)$-parameterization of a cofinite subset of $(0, 1)$, say, S, and a set $V \Subset U$ such that for each $\phi \in S$, $I_\phi[V] \subset U$, f_ϕ is of class $C^{(1)}$ on V, and all of its first partial derivatives $\partial f_\phi / \partial x_i$, $i = 1, \ldots, m+1$, are strongly bounded (on V).

Proof

We treat only the case $n = 1$. The general case follows using an argument similar to that in the proof of Lemma 3.5. Our S is constructed from a certain limit set S_0 (of a suitable family $S_t : t \in (0, 1)$), as described above. (Notice that properties (A) and (B) are not quite the conditions for an $(r-1)$-parameterization of $(0, 1)$, though (A) is precisely what we are asking for here, and (B) can easily be modified by composing with linear functions, as we are to see.)

Now, by o-minimality, let $W \Subset U$ be such that f is of class $C^{(1)}$ on W, and for each $t, y \in (0, 1)$, let $W_t(y)$ denote the set of those $\overline{x} \in (0, 1)^m$ such that the point $\langle \overline{x}, y \rangle$ is at a distance at least t from the set $([0, 1]^m \times \{y\}) \backslash W$. It follows that the map $\overline{x} \mapsto |\partial f / \partial x_{m+1}(\overline{x}, y)|$ is defined and continuous on $W_t(y)$ and hence achieves its maximum value at some point $s_t(y) \in W_t(y)$, provided that this set is nonempty. Since M admits definable Skolem functions, it follows that s may be taken to be a definable function in both t and y (taking the value $\langle 1/2, \ldots, 1/2 \rangle$, say, if $W_t(y) = \emptyset$)

and that (in all cases)

$$\forall t \in (0,1), \forall y \in (0,1), \forall \overline{x} \in W_t(y), \qquad \text{we have } \langle s_t(y), y\rangle \in W$$

and

$$\left|\frac{\partial f}{\partial x_{m+1}}\big(s_t(y), y\big)\right| \geq \left|\frac{\partial f}{\partial x_{m+1}}(\overline{x}, y)\right|. \tag{$*$}$$

Now consider the definable family $\{g_t : (0,1) \to (0,1)^m \times M : t \in (0,1)\}$ given by $g_t(y) := \langle s_t(y), f(s_t(y), y)\rangle$ (where we give f the value 0, say, if $\langle s_t(y), y\rangle \notin U$), and apply Corollary 3.6 to obtain an r-reparameterization, say, S_t, of g_t for each $t \in (0,1)$. Now since we are assuming that M is $\aleph_0$-saturated, it follows easily (using the fact that M admits definable Skolem functions) that for some $N \in \mathbb{N}$, S_t may be taken as the set of coordinate functions of some definable function $F_t : (0,1) \to (0,1)^N$, where the family $\{F_t : t \in (0,1)\}$ is also definable. Let S_0 be the limit of this family as $t \to 0^+$, as described at the beginning of this section. By splitting the functions in S_0, we may suppose that they are all either constant or injective and have domains an open subinterval of $(0,1)$. Now throw away the constant functions, and compose each remaining function with a suitable injective linear function (with coefficients in $[-1,1]$), thereby arriving at an $(r-1)$-parameterization, say, S, of a cofinite subset of $(0,1)$.

Now set $V := \big((0,1)^{m+1} \setminus \bigcup_{\phi\in S} I_\phi^{-1}[(0,1)^{m+1}\setminus W]\big) \cap U$ (see Notation 4.2(2)).

Then the injectivity (and continuity) of the ϕ's implies that $V \Subset U$. Clearly, $I_\phi[V] \subset W \subset U$, and so, also, the function f_ϕ is of class $C^{(1)}$ on V (for $\phi \in S$). It remains only to show that if $\phi \in S$ and $\langle \overline{x}_0, y_0\rangle \in V$, then $\partial f_\phi/\partial x_i(\overline{x}_0, y_0)$ is finite for $i = 1, \ldots, m+1$.

Now since $\langle \overline{x}_0, \phi(y_0)\rangle \in W \subset U$, this is clear (by the lemma hypothesis) for $i = 1, \ldots, m$. For the remaining case, we note that there is some linear function λ (with finite coefficients) and some function ψ in S_0 (or, rather, a subfunction of a function in S_0) such that $\phi(y) = \psi(\lambda(y))$ (for all $y \in (0,1)$), and so it is clearly sufficient to show that if $y_1 \in \mathrm{dom}(\psi)$, then $\psi'(y_1)\cdot \partial f/\partial x_{m+1}(\overline{x}_0, \psi(y_1))$ is finite, where we also know that $\langle \overline{x}_0, \psi(y_1)\rangle \in W$. Since W is open, it follows that

(i) $\overline{x}_0 \in W_t(\psi(y_1))$ for all sufficiently small $t \in (0,1)$.

Now by definition of S_0, there is, for each $t \in (0,1)$, (uniformly) a function $\phi_t \in S_t$ such that $\lim_{t\to 0^+} \phi_t(y_1) = \psi(y_1)$, and (as $r \geq 2$) $\lim_{t\to 0^+} \phi_t'(y_1) = \psi'(y_1)$. Hence

(ii) $|\partial f/\partial x_{m+1}(\overline{x}_0, \psi(y_1)) - \partial f/\partial x_{m+1}(\overline{x}_0, \phi_t(y_1))| \leq 1$ (and $\langle \overline{x}_0, \phi_t(y_1)\rangle \in W$) for sufficiently small $t \in (0,1)$ (by the continuity of $\partial f/\partial x_{m+1}$ on W);

(iii) $|\phi_t'(y_1) - \psi'(y_1)| \leq |\partial f/\partial x_{m+1}(\overline{x}_0, \psi(y_1))|^{-1}$ for sufficiently small $t \in (0,1)$; and

(iv) $\overline{x}_0 \in W_t(\phi_t(y_1))$ for all sufficiently small $t \in (0,1)$. (Since if (i) holds for some $t_0 \in (0,1)$, then (iv) holds for any $t < t_0/2$ satisfying $|\psi(y_1) - \phi_t(y_1)| < t_0/2$.)

Thus, if we select some $t \in (0, 1)$ such that (ii) – (iv) all hold simultaneously, we see that

$$\left|\psi'(y_1) \cdot \frac{\partial f}{\partial x_{m+1}}\big(\overline{x}_0, \psi(y_1)\big)\right| \leq |\phi_t'(y_1)| \cdot \left|\frac{\partial f}{\partial x_{m+1}}\big(\overline{x}_0, \psi(y_1)\big)\right| + 1 \quad \text{(by (iii))}$$

$$\leq |\phi_t'(y_1)| \cdot \left|\frac{\partial f}{\partial x_{m+1}}\big(\overline{x}_0, \phi_t(y_1)\big)\right| + |\phi_t'(y_1)| + 1 \quad \text{(by (ii))}$$

$$\leq |\phi_t'(y_1)| \cdot \left|\frac{\partial f}{\partial x_{m+1}}\big(s_t(\phi_t(y_1)), \phi_t(y_1)\big)\right| + |\phi_t'(y_1)| + 1 \quad \text{(by (iv) and } (*)\text{)}.$$

However, $|\phi_t'(y_1)|$ is certainly finite (since $\phi_t \in S_t$), so it suffices to show that

$$\phi_t'(y_1) \cdot \frac{\partial f}{\partial x_{m+1}}\big(s_t(\phi_t(y_1)), \phi_t(y_1)\big)$$

is finite. But since S_t is an r-reparameterization of g_t, it follows that

(v) $(s_t \circ \phi_t)'(y_1)$ is finite, and

(vi) $(d/dy)|_{y=y_1} f(s_t \circ \phi_t(y), \phi_t(y))$ is finite.

Now by (vi), the quantity

$$(s \circ \phi)'(y_1) \cdot \left\langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_m} \right\rangle \big(s_t(\phi_t(y_1)), \phi_t(y_1)\big) + \phi_t'(y_1) \cdot \frac{\partial f}{\partial x_{m+1}}\big(s_t(\phi_t(y_1)), \phi_t(y_1)\big)$$

is finite. Also, the scalar product term here is finite by (v) and by the strong boundedness of the functions $\partial f/\partial x_i$ (for $i = 1, \ldots, m$), as given by the lemma hypothesis. (Note that $\langle s_t(\phi_t(y_1)), \phi_t(y_1)\rangle \in W \subset U$ by (iv) and $(*)$.) Hence the second term is finite, which is what we had to show. □

COROLLARY 4.4

Suppose that $r, n \geq 1$, that $U \Subset (0, 1)^{m+1}$, and that $f : U \to M^n$ is a definable, strongly bounded function. Suppose further that for each $\alpha = \langle \alpha_1, \ldots, \alpha_{m+1}\rangle \in \mathbb{N}^{m+1}$ with $|\alpha| \leq r$ and $\alpha_{m+1} = 0$, $f^{(\alpha)}$ exists, is continuous, and is strongly bounded (on U).

Then for each $k \geq 0$, there exists a set $V_k \Subset U$ and an r-parameterization of a cofinite subset of $(0, 1)$, say, S_k, such that for each $\phi \in S_k$, $I_\phi[V_k] \subset U$, f_ϕ is of class $C^{(r)}$ on V_k, and all its derivatives $f_\phi^{(\alpha)}$ (for $\alpha = \langle \alpha_1, \ldots, \alpha_{m+1}\rangle \in \mathbb{N}^{m+1}$, $|\alpha| \leq r$, $\alpha_{m+1} \leq k$) are strongly bounded (on V_k).

Proof

We may take $V_0 \Subset U$ such that f is a function of class $C^{(r)}$ on V_0 (by o-minimality), and $S_0 = \{\mathrm{id}|_{(0,1)}\}$. So suppose, inductively, that V_k and S_k have been constructed with the required properties.

Let $\Delta := \{\alpha = \langle \alpha_1, \ldots, \alpha_{m+1}\rangle \in \mathbb{N}^{m+1} : |\alpha| \leq r-1, \alpha_{m+1} \leq k\}$, set $\tilde{n} := \#\Delta \cdot \#S_k$, and let $F = \langle F_1, \ldots, F_{\tilde{n}}\rangle : V_k \to M$ be an enumeration of all the functions $f_\phi^{(\alpha)} : V_k \to M$ for $\phi \in S_k$ and $\alpha \in \Delta$. Then the hypotheses of Lemma 4.3 obtain (with F for f, V_k for U, $\tilde{n} \cdot n$ for n, and $r+1$ in place of r)—note the "$r-1$" in the definition of Δ—so that we may choose an r-parameterization, say, S, of a cofinite subset of $(0,1)$ and a set $V_{k+1} \Subset V_k$ such that for each $\psi \in S$, $I_\psi[V_{k+1}] \subset V_k$ (so that, in particular, $f_{\phi\circ\psi} = (f_\phi)_\psi = f_\phi \circ I_\psi$ is of class $C^{(r)}$ on V_{k+1}, being the composition of $C^{(r)}$ functions), and so that each function $(f_\phi^{(\alpha)})_\psi$ is of class $C^{(1)}$ with

$$\frac{\partial}{\partial x_i}\big((f_\phi^{(\alpha)})_\psi\big) \quad \text{strongly bounded on } V_{k+1} \text{ for } i = 1, \ldots, m+1, \alpha \in \Delta, \text{ and } \phi \in S_k. \tag{$\dagger$}$$

Thus we define

$$S_{k+1} := \{\phi \circ \psi : \phi \in S_k, \psi \in S\},$$

and it remains to show that if $\alpha = \langle \alpha_1, \ldots, \alpha_{m+1}\rangle \in \mathbb{N}^{m+1}$ with $|\alpha| \leq r$ and $\alpha_{m+1} \leq k+1$ and if $\phi \circ \psi \in S_{k+1}$, then $(f_{\phi\circ\psi})^{(\alpha)}$ is strongly bounded on V_{k+1}.

Now if $\alpha_{m+1} = 0$, then this is clear because $(f_{\phi\circ\psi})^{(\alpha)} = (f_\phi^{(\alpha)})_\psi$ and $f_\phi^{(\alpha)}$ is strongly bounded. If $\alpha_{m+1} > 0$, then $(f_{\phi\circ\psi})^{(\alpha)} = \partial/\partial x_{m+1}(f_{\phi\circ\psi}^{(\beta)})$ for some $\beta \in \Delta$. Furthermore, for $\overline{a} := \langle a_1, \ldots, a_{m+1}\rangle \in V_{k+1}$,

$$(f_{\phi\circ\psi})^{(\beta)}(\overline{a}) = \psi^{(\alpha_{m+1}-1)}(a_{m+1}) \cdot (f_\phi^{(\beta)})_\psi(\overline{a}).$$

Thus

$$(f_{\phi\circ\psi})^{(\alpha)}(\overline{a}) = \psi^{(\alpha_{m+1})}(a_{m+1}) \cdot (f_\phi^{(\beta)})_\psi(\overline{a}) + \psi^{(\alpha_{m+1}-1)}(a_{m+1}) \cdot \frac{\partial}{\partial x_{m+1}}(f_\phi^{(\beta)})_\psi(\overline{a}),$$

which is finite since we have $\alpha_{m+1} \leq |\alpha| \leq r$ and $\beta \in \Delta$ (see ($\dagger$)), and $\psi \in S$, so $\psi^{(\alpha_{m+1}-1)}$ and $\psi^{(\alpha_{m+1})}$ are strongly bounded. □

5. The proofs of Theorems 2.3 and 2.5

For each $m \geq 1$, consider the following two statements.

- $(\mathrm{I})_m$ For all $r, n \geq 1$ and all definable, strongly bounded functions $F : (0,1)^m \to M^n$, there exists an r-reparameterization of F.
- $(\mathrm{II})_m$ For all $r \geq 1$ and every definable, strongly bounded subset $X \subset M^{m+1}$, there exists an r-parameterization of X.

Note that $(\mathrm{I})_1$ holds by Corollary 3.6. Also, $(\mathrm{II})_m$ makes sense for $m = 0$ and clearly holds in this case (via linear functions). We proceed by induction to show that the statements hold for all $m \geq 1$. So suppose that $m \geq 1$, that $(\mathrm{I})_\ell$ holds for all $\ell \leq m$, and that $(\mathrm{II})_\ell$ holds for all $\ell < m$. We now show that $(\mathrm{II})_m$ holds and then that $(\mathrm{I})_{m+1}$ holds.

For $(\mathrm{II})_m$, let $r \geq 1$ and $X \subset M^{m+1}$ be definable and strongly bounded. We may clearly assume that X is a cell in M^{m+1}, and we do the more difficult of the two cases, namely, $X = (f, g)_Y$, where Y is a (strongly bounded) cell in M^m, leaving the other case, $X = \mathrm{graph}(f|Y)$, to the reader.

So let S be an r-parameterization of Y (using $(\mathrm{II})_{m-1}$), and for each $\phi \in S$, let T_ϕ be an r-reparameterization of the function $\langle f \circ \phi, g \circ \phi \rangle : (0,1)^\ell \to M^2$, where $\ell = \dim(Y)$ (using $(\mathrm{I})_\ell$). Then for each $\psi \in T_\phi$, define $\theta_{\phi,\psi} : (0,1)^{\ell+1} \to X$ by

$$\theta_{\phi,\psi}(\overline{x}) := \langle \phi\circ\psi(x_1, \ldots, x_\ell), (1-x_{\ell+1}) f\circ\phi\circ\psi(x_1, \ldots, x_\ell) + x_{\ell+1} g\circ\phi\circ\psi(x_1, \ldots, x_\ell)\rangle,$$

where $\overline{x} = \langle x_1, \ldots, x_{\ell+1}\rangle$. Then the set $\{\theta_{\phi,\psi} : \phi \in S, \psi \in T_\phi\}$ is readily seen to be an r-parameterization of X.

For $(\mathrm{I})_{m+1}$, we need only do the case $n = 1$ (by Lemma 3.5), so let $r \geq 1$ and $F : (0,1)^{m+1} \to M$ be a definable, strongly bounded function. By $(\mathrm{I})_m$, there exists, for each $u \in (0,1)$, an r-reparameterization, say, S_u, of the function $F_u : (0,1)^m \to M : \overline{x} \mapsto F(\overline{x}, u)$, where $\overline{x} = \langle x_1, \ldots, x_m\rangle$, and by using a saturation and Skolem function argument (just as in the proof of Lemma 4.3), we may suppose that there exist definable families of functions $\{{}^{(1)}\phi_u : u \in (0,1)\}, \ldots, \{{}^{(N)}\phi_u : u \in (0,1)\}$ such that $S_u = \{{}^{(1)}\phi_u, \ldots, {}^{(N)}\phi_u\}$.

Now, for $j = 1, \ldots, N$, define the function ${}^{(j)}F : (0,1)^{m+1} \to M$ by ${}^{(j)}F(\overline{x}, u) := F({}^{(j)}\phi(\overline{x}, u), u)$. Let

$${}^*F := \langle {}^{(1)}\phi, \ldots, {}^{(N)}\phi, {}^{(1)}F, \ldots, {}^{(N)}F\rangle : (0,1)^{m+1} \to M^{mN+N},$$

and notice that the hypotheses of Corollary 4.4 hold with ${}^*F$ for f, $(0,1)^{m+1}$ for U, and $mN+N$ for n. (This is just a restatement of the fact that S_u is an r-reparameterization of F_u, uniformly in u.) So we apply Corollary 4.4 with $k = r$ to obtain $V_r \Subset (0,1)^{m+1}$ and S_r with the properties stated. Now if $V_r = (0,1)^{m+1}$ and S_r were an r-parameterization of all of $(0,1)$, then we could simply take our required r-reparameterization of F to consist of the functions ${}^{(j)}\phi_\psi$ for $j = 1, \ldots, N$ and $\psi \in S_r$. As it is, we at least know that the union of the ranges of these functions (on $(0,1)^{m+1}$) covers $(0,1)^{m+1}$ apart from finitely many planes $\{x_{m+1} = a\}$, and it follows that if we restrict them to the (open) set V_r (where they are all of class $C^{(r)}$ and satisfy the bounding condition for r-reparameterizability), then they still cover a subset of $(0,1)^{m+1}$ of codimension ℓ for some $\ell \leq m$.

Using the (now-proven) $(\mathrm{II})_m$, let T_1 be an r-parameterization of V_r, and let T_2 be an r-parameterization of the ℓ-dimensional set $(0,1)^{m+1} - \bigcup_{1\leq j\leq N} {}^{(j)}\phi_\psi[V_r]$.

For each $\theta \in T_2$, we may apply $(\mathrm{I})_\ell$ to obtain an r-reparameterization, say, U_θ, of the function $F \circ \theta : (0,1)^\ell \to M$. The required r-reparameterization of F is now

given by

$$\{{}^{(j)}\phi_\psi \circ \chi : j = 1 \ldots, N, \psi \in S_r, \chi \in T_1\} \cup \{\hat{\theta} \circ \hat{\lambda} : \theta \in T_2, \lambda \in U_\theta\},$$

where the $\hat{}$ denotes extension of the domain of a function from $(0,1)^\ell$ to $(0,1)^m$ (but leaves its values independent of the last $m - \ell$ variables).

This completes the proof of $(\mathrm{I})_{m+1}$, and the induction is complete. In particular, Theorem 2.3 is now proven. Theorem 2.5 requires one more step, and we leave this to the reader. □

COROLLARY 5.1

Let $m, r \geq 1$, and suppose that $X \subset (0,1)^m$ is a definable set. Then there exists a finite set S of functions, each mapping $(0,1)^{\dim(X)}$ to X and of class $C^{(r)}$, such that

(1) $\bigcup_{\phi \in S} \mathrm{range}(\phi) = X$ *and*

(2) $|\phi^{(\alpha)}(\overline{x})| \leq 1$ *for each $\phi \in S, \alpha \in \mathbb{N}^{\dim(X)}$ with $|\alpha| \leq r$ and all $\overline{x} \in (0,1)^{\dim(X)}$.*

Proof

Let S^* be an r-parameterization of X (as given by Theorem 2.3). Then (1) holds for S^*, and (2) holds with c in place of 1 for some $c \in \mathbb{N}$. Cover $(0,1)^{\dim(X)}$ with $(2c)^{\dim(X)}$ cubes of side $1/c$, and for each such cube K, let $\lambda_K : (0,1)^{\dim(X)} \to K$ be the obvious linear bijection. Then the set of all $\phi \circ \lambda_K$'s, as ϕ varies over S^* and K varies over the cover, is the required S. The details are left to the reader. □

As usual, the existence of definable Skolem functions and a saturation argument imply a uniform version.

COROLLARY 5.2

Let $n, m, r \geq 1$, and suppose that $X \subset (0,1)^n \times M^m$ is a definable family. Then there exists $N \in \mathbb{N}$ and, for each $\overline{y} \in M^m$, a set $S_{\overline{y}}$ of N functions, each mapping $(0,1)^{\dim(X_{\overline{y}})}$ to $X_{\overline{y}}$ and each of class $C^{(r)}$, such that

(1) $\bigcup_{\phi \in S_{\overline{y}}} \mathrm{range}(\phi) = X_{\overline{y}}$ *and*

(2) $|\phi^{(\alpha)}(\overline{x})| \leq 1$ *for each $\phi \in S_{\overline{y}}, \alpha \in \mathbb{N}^{\dim(X_{\overline{y}})}$ with $|\alpha| \leq r$ and all $\overline{x} \in (0,1)^{\dim(X_{\overline{y}})}$.*

Further, the functions comprising $S_{\overline{y}}$ depend definably on $\overline{y}$.

6. The main lemma

We return to the assumption that $\mathscr{S}$ is an o-minimal structure over $\mathbb{R}$.

By a *hypersurface of degree d (in $\mathbb{R}^n$)* we mean a set of the form $\{x \in \mathbb{R}^n : f(x) = 0\}$, where f is a nonzero polynomial over $\mathbb{R}$ of degree d in n variables. If $Z \subset \mathbb{R}^n \times \mathbb{R}^m$ is a family (see Section 1), the *fibre dimension* of Z means the maximum dimension of a fibre of Z (in situations where this makes sense).

The main device in the diophantine part of the argument here, as in [4], [21], [23], and [24], is that the rational points of height at most T in the image of a (sufficiently smooth) map $\phi : [-1, 1]^k \to \mathbb{R}^n$, where $k < n$, reside on few hypersurfaces of prescribed degree d relative to norms of ϕ and its derivatives up to some suitable order (depending on d). A similar result is achieved by p-adic means in the algebraic setting in [13].

Already in [4], where $k = 1$, the dependence of the estimate on these norms was eliminated by the observation that for an algebraic or compact analytic curve, the controlled oscillation implies that intervals on which derivatives are large have to be short and few. (Another manifestation of tameness in [4] is the compactness argument in the proof of [4, Theorem 1].) This device has also been used to obtain bounds for the rational points of a pfaff curve in [25].

Here we use the r-parameterization results of Sections 2–5.

PROPOSITION 6.1

Let $k, n \in \mathbb{N}$ with $k < n$. Then there are, for each $d \in \mathbb{N}, d \geq 1$, a nonnegative integer $r = r(k, n, d)$ and positive constants $\epsilon(k, n, d)$, $C(k, n, d)$ with the following property.

Suppose that $\phi : (0, 1)^k \to \mathbb{R}^n$ is a function of class $C^{(r)}$ with $|\phi^{(\alpha)}(x)| \leq 1$ for all $x \in (0, 1)^k$ and all $\alpha \in \mathbb{N}^k$ with $|\alpha| \leq r$. Let $X = \phi((0, 1)^k)$, and let $T \geq 1$. Then $X(\mathbb{Q}, T)$ is contained in the union of at most

$$C(k, n, d)\, T^{\epsilon(k,n,d)}$$

hypersurfaces of degree at most d. Furthermore, $\epsilon(k, n, d) \to 0$ as $d \to \infty$.

Proof

This follows from [23, Proposition 4.2], with $r(k, n, d)$ taken to be one more than the $b(k, n, d)$ therein. The constant c_{16} in that result corresponding to $C(k, n, d)$ here depends, in addition to k, n, d, on the domain of ϕ and the size of the derivatives up to order r. So the conditions of the hypothesis on those derivatives and fixed domain mean that here it may be taken to depend only on k, n, d. That $\epsilon(k, n, d) \to 0$ as $d \to \infty$ is observed just before the proof of [23, Proposition 4.2]. □

PROPOSITION 6.2 (Main lemma)

Let $Z \subset (0, 1)^n \times M^m$ be a definable family of fibre dimension $k < n$. Let $\epsilon > 0$. There are a $d = d(\epsilon, k, n) \in \mathbb{N}$ and a constant $K(Z, \epsilon)$ with the following property. For any $y \in Y$ and $T \geq 1$, the set $X(\mathbb{Q}, B)$, where $X = X_y$, is contained in the union of at most

$$K(Z, \epsilon)T^{\epsilon}$$

hypersurfaces of degree at most d.

Proof
Take d such that $\epsilon(k, n, d) \leq \epsilon$, and set $r = r(k, n, d)$ as in Proposition 6.1. By Corollary 5.2, there is an $N \in \mathbb{N}$ such that for every $y \in Y$, there is an $r(k, n, d)$-parameterization, say, S_y, of X_y consisting of at most N maps $\phi : (0, 1)^k \to \mathbb{R}^n$ having all derivatives up to order $r(k, n, d)$ of absolute value bounded by 1. To each map $\phi \in S_y$, by Proposition 6.1, we have that $\phi[(0, 1)^k](\mathbb{Q}, T)$ is contained in the union of at most $C(k, n, d)T^\epsilon$ hypersurfaces of degree at most d. This establishes the result with $K(Z, \epsilon) = N \cdot C(k, n, d)$. □

7. Proof of Theorem 1.10

If $X \subset \mathbb{R}^n$ is definable and $k \leq n$, we denote by $\mathrm{reg_k(X)}$ the subset of C^1-smooth points of X of dimension k (see [9, Section 1.8]).

7.1. Proof of Theorem 1.10

Since the rational points of height at most T are stable under the maps $x \mapsto \pm x^{\pm 1}$, as are the algebraic parts of a set, we may suppose that $Z \subset [0, 1]^n \times \mathbb{R}^m$ and so, by a suitable induction on n, that $Z \subset (0, 1)^n \times \mathbb{R}^m$.

Consider first the situation in which $A, B, C \subset (0, 1)^n \times \mathbb{R}^m$ are definable sets with $A \cup B = C$. For any $y \in Y_C$, it is immediate that $X^{\mathrm{alg}}_{A,y} \cup X^{\mathrm{alg}}_{B,y} \subset X^{\mathrm{alg}}_{C,y}$. Therefore, if the theorem holds for A and B and ϵ with sets $W(A, \epsilon)$, $W(B, \epsilon)$ and constants $c(A, \epsilon)$, $c(B, \epsilon)$, then it holds for C with

$$W(C, \epsilon) = W(A, \epsilon) \cup W(B, \epsilon), \qquad c(C, \epsilon) = c(A, \epsilon) + c(B, \epsilon).$$

The proof is by induction on the fibre dimension of Z. If the fibre dimension of Z is zero, then there is a uniform bound c for the number of points in any fibre, and the theorem holds with $c(Z, \epsilon) = c$. Suppose then that $k > 0$, suppose that the theorem holds for all families of fibre dimension at most $k - 1$, let $Z \subset (0, 1)^n \times \mathbb{R}^m$ be a definable family with fibre dimension k, and let $\epsilon > 0$.

Suppose that $k = n$. If $x \in \mathrm{reg_k(X)}$ of any fibre X, then X contains an open ball in $\mathbb{R}^n$ containing x. Therefore $x \in X^{\mathrm{alg}}$. Moreover, for any $k \in \mathbb{N}$, the family

$$R_k(Z) = \left\{\langle x, y\rangle : x \in \mathrm{reg_k(X_y)}\right\}$$

is definable (see [9]). Thus the fibres of $A = R_n(Z)$ are subsets of the algebraic parts of the fibres of Z. So the conclusion for A holds with $W(A, \epsilon) = A$. The fibre dimension of $B = Z - A$ is at most $k - 1$, and so the theorem holds for B by induction. So it may be assumed that $k < n$.

Let $\{x_1, x_2, \ldots, x_n\}$ be the coordinate system in $\mathbb{R}^n$. For a subset $\sigma \subset \{1, 2, \ldots, n\}$, let Π_σ denote the linear coordinate subspace of $\mathbb{R}^n$ with coordinates

$\{x_i : i \in \sigma\}$, and let π_σ be the projection of $\mathbb{R}^n$ onto Π_σ. Let $S = S_{k,n} = \{\sigma \subset \{1, 2, \ldots, n\} : \#\sigma = k + 1\}$, and put $q = \#S$.

By Proposition 6.2, there are $d \in \mathbb{N}$ and a constant $\alpha(Z, \epsilon)$ such that for any fibre X of Z, any subset $\sigma \in S$, and any $T \geq 1$, $(\pi_\sigma X)(\mathbb{Q}, T)$ is contained in the union of at most

$$\alpha(Z, \epsilon)T^{\epsilon/2q}$$

intersections of $\pi_\sigma X$ with hypersurfaces of degree at most d. (So $X(\mathbb{Q}, T)$ is contained in at most $\alpha(Z, \epsilon)^q T^{\epsilon/2}$ intersections of X with cylinders on hypersurfaces of degree d in each such subspace.)

Let $T \subset \mathbb{R}^p$ parameterize real hypersurfaces of degree d in $\mathbb{R}^{k+1}$. (Note that $T = \mathbb{P}^{\nu(k,d)}(\mathbb{R})$ for suitable ν and so is compact, and we can take $T \subset [-1, 1]^p \subset \mathbb{R}^p$.) Then

$$t = \langle t_\sigma : \sigma \in S\rangle \in \prod T_\sigma \subset (\mathbb{R}^p)^q$$

corresponds to a choice of a hypersurface $L = L(t_\sigma)$ of degree d in each $(k + 1)$-dimensional linear coordinate subspace Π_σ of $\mathbb{R}^n$. We have the definable family

$$\Sigma = \big\{\big\langle x, \langle y, t\rangle\big\rangle : \pi_\sigma(x) \in L(t_\sigma), \text{ all } \sigma \in S\big\} \subset \mathbb{R}^n \times \big(\mathbb{R}^m \times (\mathbb{R}^p)^q\big).$$

Consider a fibre X of Σ. Since any choice of $k + 1$ coordinates is algebraically dependent, X is a closed algebraic set in $\mathbb{R}^n$ of dimension at most k.

Replace Z by

$$\Big\{\big\langle x, \langle y, t\rangle\big\rangle \in \mathbb{R}^n \times \big(\mathbb{R}^m \times (\mathbb{R}^p)^q\big) : \langle x, y\rangle \in Z, t \in \prod T_\sigma\Big\},$$

which has the same fibres. (And so $Z \subset (-1, 1)^n \times \mathbb{R}^{m+pq}$.)

The fibre dimension of $Z \cap \Sigma$ is at most k. If

$$A_1 = \big\{\big\langle x, \langle y, t\rangle\big\rangle \in Z \cap \Sigma : x \notin \mathrm{reg_k}(\mathrm{X}_{Z\cap\Sigma,\langle \mathrm{y,t}\rangle})\big\},$$

then the fibre dimension of A_1 is at most $k - 1$ and, by induction, an estimate

$$c\Big(A_1, \frac{\epsilon}{2}\Big)T^{\epsilon/2}$$

holds for the number of rational points of height at most T on a fibre of A_1 outside (the fibre of) some suitable family $W(A_1, \epsilon/2)$. This includes, in particular, the case of an intersection of a fibre X of Z with cylinders on hypersurfaces of degree d when the intersection has dimension at most $k - 1$.

Similarly, the fibre dimension of

$$A_2 = \big\{\big\langle x, \langle y, t\rangle\big\rangle \in Z \cap \Sigma : x \notin \mathrm{reg_k}(\mathrm{X}_{\Sigma,\langle \mathrm{y,t}\rangle})\big\}$$

is at most $k-1$, and an estimate of the above form holds. It is likewise for

$$A_3 = \left\{\langle x, \langle y, t\rangle\rangle \in Z \cap \Sigma : x \notin \mathrm{reg_k}(X_{Z,\langle y,t\rangle})\right\}.$$

Let B be the subset of $Z \cap \Sigma$ of points that are regular (of dimension k) in their fibres in Z, Σ, $Z \cap \Sigma$. Consider a point $P = \langle x, \langle y, t\rangle\rangle = \langle x, u\rangle \in B$. In some small neighborhood Δ of x in $\mathbb{R}^n$, each of the fibres

$$X_{Z\cap\Sigma,u}, \qquad X_{Z,u}, \qquad X_{\Sigma,u}$$

is a C^1-submanifold of $\mathbb{R}^n$ of dimension k. Since $X_{Z\cap\Sigma,u} \subset X_{Z,u}, X_{\Sigma,u}$, the sets locally coincide. But the intersection $X_{\Sigma,u} \cap \Delta$ is a semialgebraic set of dimension $k \geq 1$ if Δ is taken to be a small ball. Therefore $P \in X_{B,u}^{\mathrm{alg}} \subset X_{Z,u}^{\mathrm{alg}}$. The theorem holds for B with $W(B,\epsilon) = B$.

Now let $y \in Y$, $X = X_{Z,y}$, and $T \geq 1$. Let $P \in X(\mathbb{Q}, T)$. So $\pi_\sigma(P) \in (\pi_\sigma X)(\mathbb{Q}, T)$ for any $\sigma \in S$ and therefore lies on one of the hypersurfaces t_σ. So P lies in one of at most

$$\left(\alpha(Z,\epsilon)\right)^q T^{\epsilon/2}$$

fibres of $Z \cap \Sigma$. Furthermore, either P lies in a family A_1, A_2, A_3 for which an estimate

$$c(A_i,\epsilon)T^{\epsilon/2}$$

holds for the number of rational points of height at most T outside the corresponding fibre of $W(A_i, \epsilon/2)$, or P lies in B. This completes the proof. □

Remark 7.2

In the 1-dimensional case, application of the method to the function $y = e^x$, for which all intersection multiplicities can be precisely controlled, leads to natural proofs of (the real versions of) classical transcendence statements (see [22]). (A similar method was found a little earlier by Laurent [18]; also see [33]). It would be interesting to make the present argument fully quantitative, for example, for the threefold $\log x \log y = \log w \log z$, $x, y, z, w > 0$ associated with the four exponentials conjecture, with a view to showing that there can be only a few solutions in some more general sense than the six exponentials theorem (see [33]).

8. Dilation-integer points

We now take X to be a bounded definable set; otherwise, the conclusion of Theorem 8.1 may fail (e.g., $X = \{\langle x, 2^x\rangle, x \in \mathbb{R}\}$, definable in $\mathbb{R}_{\exp}$). Note that for any X and t, $(tX)^{\mathrm{alg}} = t(X^{\mathrm{alg}})$.

THEOREM 8.1
Let $Z \subset \mathbb{R}^n \times \mathbb{R}^m$ be a bounded, definable family, and let $\epsilon > 0$. There are a definable family $W = W(Z, \epsilon) \subset Z$ and a constant $c(Z, \epsilon)$ with the following property. Let $X = X_{Z,y}$, and put $X_\epsilon = X_{W,y}$. Then $X_\epsilon \subset X^{\mathrm{alg}}$, and

$$\#(tX - tX_\epsilon)(\mathbb{Z}) \leq c(Z, \epsilon)t^\epsilon.$$

Proof
Since Z is bounded, say, $Z \subset [-C, C]^{n+m}$, where $C > 1$, we can assume, in fact, that $Z \subset [-1, 1]^{n+m}$ at the cost of replacing the dilation parameter t by Ct. The proof then follows the method of the proof of Theorem 1.10, using a result for dilation-integer points in the image of a map $\phi : (0, 1)^k \to \mathbb{R}^n$ with suitably bounded derivatives adapted from [23, Proposition 4.1] in the same way that Proposition 6.1 adapts [23, Proposition 4.2] for rational points. □

Remark 8.2
In fact, only the members of the family Z need be bounded (not the parameters), although there is no real added generality here.

Acknowledgments. The authors are very grateful to the referees for their most helpful comments and suggestions. Pila thanks A. Venkatesh and L. van den Dries for assistance. He is grateful to the Mathematical Institute, Oxford, United Kingdom, and, in particular, A. Lauder and R. Heath-Brown, for hospitality during a visit in which parts of this article were written. Wilkie is grateful to all the staff at, and visitors to, the Isaac Newton Institute for Mathematical Sciences, Cambridge, United Kingdom, for making the Programme on Model Theory and Applications to Algebra and Analysis (January 17 – July 15, 2005) so successful. The contents of this article were first presented there, and he thanks, in particular, Zoe Chatzidakis and Sergei Starchenko for extremely helpful comments.

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Pila
School of Mathematics, University of Bristol, University Walk, Bristol BS8 1TW, United Kingdom; j.pila@bristol.ac.uk

Wilkie
Mathematical Institute, University of Oxford, 24-29 St Giles', Oxford OX1 3LB, United Kingdom; wilkie@maths.ox.ac.uk