Properties : usual ones :

$f, g : U \to R^n$ differentiable at $x \in U$, U open in R^m

$d_x(f+g) = d_x f + d_x g$.

$d_x cf = c \cdot d_x f \qquad c \in R$.

$h : V \to R^n$ diff^{ble} at $f(x) \in V$, f continuous

then $h \circ f$, defined on $U \cap f^{-1}(V)$, is differentiable at x, $d_x(h \circ f) = d_{f(x)}(h) \cdot d_x f$.

Now assume

$(R, +, \cdot, -, 0, 1, <, \dots)$ O-minimal.

So real closed.

<u>Lemma</u> (Rolle) Let $a < b$, and spse the function $f : [a,b]$ is definable, continuous, $f(a) = f(b)$ and f is differentiable at each pt of (a,b). Then there is $c \in (a,b)$ such that $f'(c) = 0$.

<u>Pf</u> Let $c \in (a,b)$ such that $f(c)$ is minimum/maximum

Show $f'(c) = 0$.

(Exercise)

<u>Mean Value theorem</u> $a < b$, $f : [a,b] \to R$ definable, continuous, differentiable on (a,b). Then for some $c \in (a,b)$,

$$f(b) - f(a) = (b-a) f'(c).$$

<u>Pf</u> look at $g(t) : [0,1] \to R$ defined by

$g(t) = f(a + t(b-a)) - t(f(b) - f(a))$

$g(0) = f(a) \qquad = 0$.

$g(1) = f(b) - \qquad - (f(b) - f(a)) = 0$.

$\rightsquigarrow g'(t) = f'(a + t(b-a))(b-a) + f(a) - f(b)$

Lemma

$f : [a,b] \to \mathbb{R}$ continuous, diff^{ble} on (a,b).
If $f'(x) = 0$ for all $x \in (a,b)$ then $f(x)$ is constant.

Goal Let $f : I \to \mathbb{R}$ be definable, $I \subseteq \mathbb{R}$ an interval. Then f is differentiable at all but finitely many points of I.

Need several lemmas

$x \in I$. Define $f(x^+) = \lim_{t \to 0^+} f(x+t) - f(x)/t \in \mathbb{R} \cup \{\pm\infty\}$

$$f(x^-) = \lim_{t \to 0^-} f(x+t) - f(x)/t.$$

f differentiable at x: $f(x^+) = f(x^-) \in \mathbb{R}$.

Lemma Assume f is continuous, $f'(x^+) > 0$ for all $x \in I$. Then f is strictly increasing, $f^{-1} : f(I) \to \mathbb{R}$, satisfies $(f^{-1})'(y^+) = 1/f'(x^+)$ for $x \in I$, $f(x) = y$. $(1/+\infty = 0)$.

Pf: If f were not strictly increasing, then there would be a subinterval J on which f is constant ($f' = 0$), or strictly decreasing, which contradicts $f'(x^+) > 0$.
Give $\varepsilon > 0$
for t sufficiently small >0, we have

$$\lim_{t \to 0^+} \frac{f(x+t) - f(x)}{t} = \lim_{t \to 0^+} \left(\frac{t}{f(x+t) - f(x)} \right)^{-1}$$

$$= \lim_{u \to 0^+} \left(\frac{f^{-1}(y+u) - f^{-1}(u)}{u} \right)^{-1}.$$

<u>Lemma</u> : $f : I \to R$ def., continuous, $x \mapsto f'(x^+)$, and $x' \mapsto f'(x^-)$ are R-valued, continuous on I. Then f is differentiable on I, and f' is continuous

<u>Pf</u> Enough to show : $f'(a^+) = f'(a^-)$ for all $a \in I$.
Otherwise, say $f'(a^+) > f'(a^-)$. Let $c \in R$ be between $f'(a^+), f'(a^-)$, let $J \subset I$ be such that $f'(x^+) > c > f'(x^-)$ on J. Then $g : J \to R$, $g(x) = f(x) - cx$ satisfies : $g'(x^+) > 0$, $g'(x^-) < 0$ for all $x \in J$. So g is both strictly increasing, strictly decreasing on J #
(lemme précédent appliqué à $-g$)

<u>Lemma</u> Let $f : I \to R$ be definable. There are only finitely many $x \in I$ at which $f'(x) \in \pm\infty$

<u>Pf</u> Suppose $A = \{x \in I, \ f'(x^+) = +\infty\}$ is infinite
Then A contains an interval, and wlog $A = I$, f is continuous on I. Then f is strictly increasing, and therefore $f'(x^-) \geq 0$ for all $x \in I$.
(We want to reach a contradiction, so we are allowed to shrink I to non-empty subintervals). So after shrinking I, we may assume we are in one of the following two cases

(i) $f'(x^-) = +\infty$ for all $x \in I$

(ii) $f'(x^-) \in R$ for all $x \in I$, and $x \mapsto f'(x^-)$ is continuous on I.

In subcase (i), we have $(f^{-1})'(y^-) = 0 = (f^{-1})'(y^+)$.
i.e, f^{-1} is constant. This contradicts $f'(x^+) > 0$.
In subcase (ii), let $a \in I$, and $c > f'(a^-)$. Then there is a subinterval $J \subset I$ on which $f'(x^-) < c$.

So looking at $g(x) = f(x) - cx$ we have:

$$g'(x^+) = f'(x^+) - c = f'(x^+) = +\infty$$
$$g'(x^-) = f'(x^-) - c < 0.$$

Contradiction. So $|A| < +\infty$.

Replacing $f(x)$ by $f(-x)$, we get that the set of $x \in I$, $f(x^-) = \pm\infty$ is finite.

Proof of the Proposition: if $f: I \to R$ is definable, then there are only finitely many points at which f is not differentiable.

Pf: We saw in the previous lemma that the set of points A such that one of $f'(x^+)$, $f'(x^-)$ is $\pm\infty$, is finite. Furthermore, throwing away finitely many points, we may assume that on each subinterval of $I \setminus A$, the maps $f'(x^+)$ and $f'(x^-)$ are continuous. Hence, at all points of $I \setminus A$, $f'(x^+) = f'(x^-)$ and f is differentiable

<u>Aim</u> Inverse function theorem, and implicit function theorem:

If the Jacobian is invertible at a point $a \in R^m$, then f is locally a homeo around a.

and IFT.

For that we need more lemmas.

Setting definable $(f_1, \dots, f_n) = f : U \to R^n$, $U \subseteq R^m$ open

<u>Def</u> We call f a C^1-map if the partial derivatives $\partial f_i / \partial x_j$ are defined on U and continuous.

One shows easily that:

If f is C^1 then f is differentiable at each point of U, and the map $x \mapsto d_x f \in R^{n \times m} = \mathrm{Lin}(R^m, R^n)$ is continuous. And conversely.

(Usual proof for $\to$).

If $T : R^m \to R^n$ is R-linear (ie $\in R^{n \times m}$) define $|T| = \max\{|Tx| \quad |x| \leq 1, x \in R^m\}$.

Then $|T(x)| \leq |T||x|$.

<u>Lemma</u> Let $f : U \to R^n$ be C^1, $[a,b] = \{(1-t)a + tb \mid 0 \leq t \leq 1\}$ be a line segment contained in U.

Then $|f(b) - f(a)| \leq |b-a| \max_{y \in [a,b]} |d_y f|$

Pf Let $g(t) : [0,1] \to U$, $g(t) = f((1-t)a + tb)$.

Then $g'(t) =$ directional derivative of f at $(1-t)a + tb$, in direction $(b-a)$.

$= d_y f(b-a)$ when $y = (1-t)a + tb$

So $|g'(t)| \leq M$, $M = |b-a| \max_{y \in [a,b]} d_y f$.

By MVT, we have $|f(b) - f(a)| = |g(1) - g(0)| \leq M$.

<u>Lemma</u> Same assumptions, $x \in U$.

$$|f(b) - f(a) - d_x f(b-a)| \leq |b-a| \max_{y \in [a,b]} |d_y f - d_x f|.$$

Pf Consider $h(y) = f(y) - d_x f(y)$.

Then $d_y h = d_y f - d_x f$.

<u>Lemma</u> Same assumptions, $m = n$, $a \in U$, and assume that $d_a f$ is invertible. Then there are $\varepsilon > 0$, $C > 0$ in $\mathbb{R}$ s.t

$$|f(x) - f(y)| > C|x-y| \text{ for all } x, y \in U \text{ with } |x-a|, |y-a| < \varepsilon.$$

In particular, f is invertible on a nbhd of a.

<u>Pf</u> Let $\varepsilon > 0$ be small enough so that $B(a, \varepsilon) \subset U$.

By the previous lemma, we have

$$|f(x) - f(y) - d_a f(x-y)| \leq |x-y| \max_{z \in [x,y]} |d_z f - d_a f|$$

$$\geq$$

$$|d_a f(x-y)| - |f(x) - f(y)|$$

$$\rightsquigarrow |f(x) - f(y)| \geq d_a f(x-y) - |x-y| \max_{z \in [x,y]} |d_z f - d_a f|$$

As $d_a f$ is invertible, there is c', not depending on x, y, such that

$$|d_a f(x-y)| \geq c'|x-y|.$$

Indeed, we have $|z| = |d_a f^{-1} d_a f(z)| \leq |d_a f^{-1}| |d_a f(z)|$

So $|d_a f(z)| \geq |z| |d_a f^{-1}|^{-1}$.

Decreasing ε, we may assume that

$$|d_b f - d_a f| < c'/2 \quad \text{for all } b \in B(a, \varepsilon)$$

hence

$$|f(x) - f(y)| \geq c'|x-y| - c'/2\, |x-y| \geq c'/2\, |x-y|.$$

Inverse function theorem

Let $f : U \to R^m$ be a definable C^1 map on a definable open set $U \subseteq R^m$, $a \in U$ s.t $d_a f : R^m \to R^m$ is invertible.

Then there are: a definable open $U' \ni a$, $U' \subseteq U$, and a definable nbhd V' of $f(a)$ such that f maps U' homeomorphically onto V', and $f^{-1} : V' \to U'$ is also C^1.

Pf Since they define the same topology, we may replace $|\ |$ on R^m by $\|\ \|$, $\|(x_1, \dots, x_m)\| = \sqrt{x_1^2 + x_2^2 + \dots + x_m^2}$

We can find $c, \varepsilon > 0$ such that

$$\|x - a\| \leq \varepsilon \to x \in U \text{ and } d_x f \text{ is invertible.}$$

$$\|x - a\|, \|y - a\| \leq \varepsilon \to \|f(x) - f(y)\| \geq c\|x - y\|.$$

<u>Claim</u> $\{y \mid \|y - f(a)\| < \frac{1}{2}c\varepsilon\} \subseteq \{f(x) \mid \|x - a\| \leq \varepsilon\}$

Assume $\|y - f(a)\| < \frac{1}{2}c\varepsilon$

Consider $P(x) = \|f(x) - y\|^2$ on the ball $\|x - a\| \leq \varepsilon$.

As the ball is closed and bounded, $P(x)$ assumes its minimum value on it. However, if $\|x - a\| = \varepsilon$, then

$$P(x) = \|(f(x) - f(a)) - (y - f(a))\|^2$$

$$\geq \Big(\underbrace{\|f(x) - f(a)\|}_{\geq c\varepsilon} - \underbrace{\|y - f(a)\|}_{< \frac{1}{2}c\varepsilon}\Big)^2 > \left(\tfrac{1}{2}c\varepsilon\right)^2 > \|y - f(a)\|^2 = P(a)$$

So the minimum value is attained at b, $\|b - a\| < \varepsilon$.

So $0 = \dfrac{\partial P}{\partial x_j}(b) = \displaystyle\sum_{i=1}^{m} 2(f_i(b) - y_i)\left(\partial f_i / \partial x_j\,(b)\right)$ $\forall j$

ie, $d_b f\,(f(b) - y) = 0$

$d_a f$ invertible implies $f(b) = y$, which proves the claim.
So the image by f of the open set $\{\|x-a\| < \varepsilon\}$ contains the open set $\{\|y - f(a)\| < \frac{1}{2} c\varepsilon\}$.

Let $U' = \{x \mid \|x-a\| < \varepsilon\}$. Then, reasoning in the same way with all points $a' \in U'$ we get that f is open on U'. It is also injective, hence it is a homeomorphism between U' and $V' = f(U')$.
It remains to show that $f^{-1} : V' \to U'$ is C^1.

By definition

$$\left(f(b) - f(a) - d_a f(b-a)\right) / \|b-a\| \to 0 \text{ as } b \to a.$$

Also $\|b-a\| \leq c^{-1} \|f(b) - f(a)\|$. Apply $(d_a f)^{-1}$

$$(d_a f)^{-1}\left(f(b) - f(a)\right) - (b-a) / \|f(b) - f(a)\| \to 0$$

as $f(b) \to f(a)$.

ie: f^{-1} is differentiable at $f(a)$, with differential $d_{f(a)} f^{-1} = (d_a f)^{-1}$

This reasoning works at every point $a' \in U'$, and therefore f^{-1} is C^1 ($x \mapsto (d_x f)^{-1}$ is continuous)

<u>Corollary</u> (Implicit function theorem)

Let $U \subseteq R^{m+n}$ be definable open, $f_1, \dots, f_n : U \to R$ be definable C^1. Let $(\underset{m}{x_0}, \underset{n}{y_0}) \in R^{m+n}$ be s.t

$f_1(x_0, y_0) = \dots = f_n(x_0, y_0) = 0$, and the matrix $\left(\partial f_j / \partial y_k (x_0, y_0) \right)$ is invertible. Then there is an open definable nbhd V of x_0 in R^m, and W of y_0 in R^n, and a definable C^1-map $\phi : V \to W$ such that $V \times W \subseteq U$, and for all $(x, y) \in V \times W$ we have

$$f_1(x,y) = \dots = f_n(x,y) = 0 \iff y = \phi(x).$$

Pf Apply the inverse function theorem to the map

$$g : (x, y) \longmapsto (x, f_1(x,y), \dots, f_n(x,y)).$$
$$U \longrightarrow R^{m+n}.$$

Consider the map $\pi g^{-1}(x, 0)$ near x_0.
$\pi : R^{m+n} \to R^m$

What else:

L'Hopital's rule.

$I \subset R$ interval, $f, g : I \to R$ definable
$g'(x) \neq 0$ for all $x \in I$ in a nbhd of a
$\lim_{x \to a} g(x) = 0 = \lim_{x \to a} f(x)$. Then
$\lim f/g(x) = \lim f'/g'(x)$.

Taylor's formula: If $f : I \to R$ is $(n+1)$-times differentiable on I, and $a < b$ are in I. Then

$$f(b) = f(a) + f'(a)(b-a) + \frac{f^{(2)}(a)}{2}(b-a)^2 + \dots$$
$$+ \frac{f^{(n)}(a)}{n!}(b-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}$$

for some $c \in (a, b)$.

Def Let $A \subseteq R^m$, $f: A \to R^n$ a definable map. Then f is a C^1-map if there are a definable open $U \supseteq A$, and a definable C^1-map $F: U \to R^n$, such that $F|_A = f$.

A C^1-cell is a cell in which all defining functions are C^1.

Thm (C^1-cell decomposition)

(I_m) If $A_1, \dots, A_k \subseteq R^m$ are definable, there is a decomposition of R^m into C^1-cells partitioning $A_1, \dots, A_k$.

(II_m) If $A \subseteq R^m$ and $f: A \to R$ is definable, then there is a decomposition $\mathcal{D}$ of R^m into C^1-cells, partitioning A and such that if $C \in \mathcal{D}$, $C \subseteq A$ then $f|_C$ is C^1.

If f and A are as in II_m, $p \in \mathrm{Int}(A)$, write

$$\nabla f(p) = \left(\frac{\partial f}{\partial x_1}(p), \dots, \frac{\partial f}{\partial x_m}(p) \right)$$ provided they exist.

$A' = \{ p \in A \mid p \in \mathrm{int}(A) \text{ and } \nabla f \text{ is defined at } p \}$

Then

(III_m) $A \setminus A'$ has empty interior.

[On utilise III_m pour prouver II_m]

We work in $\mathbb{R}$ (or in $\mathbb{C}$).

Let U be an open neighbourhood of a point $a \in \mathbb{R}^n$ and let $f : U \to \mathbb{R}$ be an infinitely differentiable function. The Taylor series of f at a is

$$\sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!} \frac{\partial^\alpha f}{\partial x^\alpha}(a) \cdot (x-a)^\alpha$$

where, if $\alpha = (\alpha_1, \dots, \alpha_n)$ we have:

$$\frac{\partial^\alpha f}{\partial x^\alpha} = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \left(\frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \cdots \right) f \qquad (x-a)^\alpha = \prod_{i=1}^{n} (x_i - a_i)^{\alpha_i}$$

If there is an open nbhd V of a ($V \subseteq U$) such that the Taylor series converges to $f(x)$ for every $x \in V$, then we say that f is analytic at a. If it is analytic at every point of U, we say that f is analytic on U

The implicit function theorem holds for analytic functions. Let $F : U \times V \to \mathbb{R}^m$ be analytic, where $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^m$ are open. Let $(a,b) \in U \times V$ be s.t. $F(a,b) = 0$, and $\det \left(\frac{\partial F_i}{\partial y_j} \right)_{1 \le i,j \le m} \neq 0$. Then there is an open nbhd $U_a \times V_b \subseteq U \times V$ of (a,b), and an analytic map $\Phi : U_a \to V_b$, such that for each $x \in U_a$, $F(x, \Phi(x)) = 0$. Further, $\Phi(x)$ is the unique $y \in V_b$ satisfying $F(x,y) = 0$.

Notation: If $a \in \mathbb{R}^n$, $r \in \mathbb{R}^n_{>0}$ & denote by $\Delta(a;r)$ or $\Delta^n(a;r)$ the box defined by $|x_i - a_i| < r_i$, for $i = 1, \dots, n$, and by $\overline{\Delta}^n(a;r)$ its closure.

We are interested in the following, which a priori seems weaker, but has more or less the same expressive power.

<u>Def</u> Let $\mathbb{R}\{X_1, \dots, X_m\}$ denote the ring of all power series in $X_1, \dots, X_m$ (i.e., $\sum_{\alpha \in \mathbb{N}^n} a_\alpha X^\alpha$, $a_\alpha \in \mathbb{R}$) which converge on a neighbourhood of $[-1,1]^m$.
For $f \in \mathbb{R}\{X_1, \dots, X_m\}$, we define $\tilde{f} : \mathbb{R}^m \to \mathbb{R}$ as follows:

$I = [-1,1]$

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in [-1,1]^m \\ 0 & \text{otherwise.} \end{cases}$$

We call $\tilde{f}(x)$ a <u>restricted analytic function</u>

<u>Note</u> Let $f(x)$ be analytic at $a \in \mathbb{R}^m$, and let $\overline{\Delta}(a;r)$ be a closed box s.t. the Taylor series of f at a converges on some open set containing $\overline{\Delta}(a,r)$ to f. Take a linear map L sending $\overline{\Delta}(a,r)$ to I^m. Then the map $\tilde{\tilde{f}}$ which equals f on $\overline{\Delta}(a,r)$ and 0 outside, is of the form $\tilde{h} \circ L$, for some $h \in \mathbb{R}\{X_1, \dots, X_m\}$.

The structure $\mathbb{R}_{an}$ is the structure

$$(\mathbb{R}, +, -, \times, 0, 1, <, \tilde{f})_{f \in \mathbb{R}\{X_1, \dots, X_m\}}$$

We define $^{-1} : \mathbb{R} \to \mathbb{R}$, by $x^{-1} = 1/x$ if $x \neq 0$, $0^{-1} = 0$.

A bit of history:

Gabrielov (1968) showed the following:

A subset of $\mathbb{R}^n$ is <u>semi-analytic</u> if at the nbhd of each point $x_0 \in \mathbb{R}^n$, it is defined by a finite union of sets of the form $\{f(x)=0, g_1(x)>0, \dots, g_n(x)>0\}$ where f and the g_i are analytic

A <u>P-set</u> is a set of the form $\{x \in \mathbb{R}^n \mid \exists y \in \mathbb{R}^m \ (x,y) \in L\}$ where L is a relatively compact semi-analytic set in $\mathbb{R}^{n+m}$
"bounded

He considers only semi-ana sets of the form $\{x \in I^n \mid f(x)=0, \bigwedge g_i(x)>0\}$ where $I^n \subseteq \mathbb{R}^n$, is a square closed box centered at a point (bounded).

What he shows is that the complement in I^n of a P-set is also a P-set.

$\rightsquigarrow$ $\mathbb{R}_{an}$ is model complete: every formula is eq^t, modulo $Th(\mathbb{R}_{an})$ to an existential formula.

$\rightsquigarrow$ All definable subsets of a bounded box are P-sets.

[Redone by Denef-van den Dries in 1988, showing that $\mathbb{R}_{an}^D$ has quantifier elimination]

Gabrielov actually also shows a similar result, where one considers a restricted class of analytic functions (closed under taking partial derivatives).

In particular this gives model completeness of the following structures

$$(\overline{\mathbb{R}}, \exp|_{[0,1]})$$

$$\mathbb{R}_{exp,trig} = (\overline{\mathbb{R}}, \exp|_{[0,1]}, \sin|_{[0,2\pi]}, \cos|_{[0,2\pi]}).$$

$$\left(\overline{\mathbb{R}} = (\mathbb{R}, +, -, \cdot, 0, 1, <) \right).$$

$\rightsquigarrow$ 0-minimality follows from the description of the def. sets

Structure $\mathbb{R}_{an}$: $(\mathbb{R}, +, -, \cdot, 0, 1, <, \tilde{f})_{\substack{m \in \mathbb{N} \\ f \in \mathbb{R}\{X_1, \dots, X_m\}}}$

$\mathbb{R}_{exp}$: $(\mathbb{R}, +, -, \cdot, 0, 1, <, \exp)$

$\mathbb{R}_{an,exp} = (\mathbb{R}_{an}, \exp)$.

La structure $\mathbb{R}_{an}$ n'élimine pas les quantificateurs, mais elle est modèle-complète : modulo $Th(\mathbb{R}_{an})$, toute formule est équivalente à une formule existentielle. Si on ajoute au langage l'une des deux fonctions suivantes :

$$D(x,y) = \begin{cases} x/y & \text{si } |x| \leq |y| \neq 0 \\ 0 & \text{sinon} \end{cases}$$

$$x^{-1} = \begin{cases} 1/x & \text{si } x \neq 0 \\ 0 & \text{sinon} \end{cases}$$

Alors $Th(\mathbb{R}_{an,D})$ et $Th(\mathbb{R}_{an,-1})$ éliminent les quantificateurs. La structure $\mathbb{R}_{an}$ est 0-minimale.

Wilkie montre en 1992 que $\mathbb{R}_{exp}$ est modèle complète, et qu'elle est 0-minimale.

Van den Dries et Miller montrent en 1994 que $\mathbb{R}_{an,exp}$ est 0-minimale. Puis, van den Dries, Macintyre et Marker montrent que $\mathbb{R}_{an,exp}$ élimine les quantificateurs dans le langage $\mathcal{L}_{an,exp} \cup \{\underset{\sim}{\log}\}$, où

$$\underset{\sim}{\log}(x) = \begin{cases} \log x & \text{si } x > 0 \\ 0 & \text{sinon} \end{cases}$$

(certainement j'écrirai log au lieu de $\underset{\sim}{\log}$).

Applications.

$\mathbb{C} = \mathbb{R}(i) = \mathbb{R} \oplus i\mathbb{R}$ est interprétable dans le corps $\mathbb{R}$. L'exponentielle complexe ne peut pas être définie dans une structure O-minimale sur $\mathbb{R}$. Cependant des portions bornées le peuvent:

$$\exp(x+iy) = \exp(x)(\cos y + i \sin y).$$

Donc par exemple la restriction de l'exponentielle complexe à $\{z \mid |\mathrm{Im}(z)| \leq 2\pi\}$ est définissable dans $\mathbb{R}_{an,exp}$. En effet ce ruban contient un domaine fondamental de la fonction exp.

D'autres exemples :

Soit $\Lambda \subset \mathbb{C}$ un treillis. Alors $\mathbb{C}/\Lambda$ est un groupe (avec +) et on a une application $(\wp, \wp') : \mathbb{C} \to \mathbb{C}^2 \hookrightarrow (0,0,1) \subset \mathbb{P}^2(\mathbb{C})$ dont l'image est une courbe elliptique dans $\mathbb{P}^2(\mathbb{C})$, $(0,0,1)$ étant le zéro. C'est un homomorphisme de $(\mathbb{C},+)$, avec noyau Λ. Analytique.

Les points de torsion de la courbe elliptique proviennent d'éléments de $\Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$, ie si $\Lambda = \mathbb{Z} \oplus \tau\mathbb{Z}$, alors on a $\mathbb{Q} \oplus \tau\mathbb{Q}$.

$(\wp, \wp')|_{\mathcal{F}}$ interprétable dans $\mathbb{R}_{an}$
$\mathcal{F} \to$ domaine fondamental

Fonction $j : \mathbb{H} \to \mathbb{C}$
$\mathbb{H} \to \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$

fonction analytique, périodique sous l'action de $SL_2(\mathbb{Z})$ sur $\mathbb{H} \subset \mathbb{C}$. domaine fondamental, restriction au domaine fondamental déf. dans $\mathbb{R}_{an}$

Pila – Wilkie : Counting rational points

Let $H : \mathbb{Q} \to \mathbb{R}$ denote the usual height function, if $q = a/b$ where $a, b \in \mathbb{Z}$, $b > 0$, $(a,b) = 1$, then $H(q) = \max\{|a|, b\}$. Extend H to $\mathbb{Q}^n$ by setting $H(q_1, \dots, q_n) = \max\{H(q_i)\}$.

If $X \subseteq \mathbb{R}^n$, let $X(\mathbb{Q}) = X \cap \mathbb{Q}^n$, $X(\mathbb{Z}) = X \cap \mathbb{Z}^n$. For $T \geq 1$, we define

$$X(\mathbb{Q}, T) = \{P \in X(\mathbb{Q}) \mid H(P) \leq T\}$$

and $N(X, T) = \# X(\mathbb{Q}, T)$.

<u>Def</u>

Let $X \subset \mathbb{R}^n$. The algebraic part of X, denoted X^{alg} is the union of all connected semi algebraic subsets of X of positive dimension ← do not allow isolated points (Pila 2005?)

The transcendental part of X is $X \setminus X^{alg}$.

<u>Why</u>: Consider the surface $z = x^y$. It contain many algebraic curves: if $y \in \mathbb{Z}$. Do not want to count their rational points

We work in $\mathcal{R} = (\mathbb{R}, +, -, \times, 0, 1, <, \dots)$, O-minimal

<u>Theorem</u> (V1) Let $X \subset \mathbb{R}^n$ be definable, and $\varepsilon > 0$. There is a constant $c(X, \varepsilon)$ such for any T

$$(*) \qquad N(X \setminus X^{alg}, T) \leq c(X, \varepsilon) T^{\varepsilon}$$

Thm (V2 – uniformity in parameters) Let $Z \subset \mathbb{R}^n \times \mathbb{R}^m$, $\varepsilon > 0$. There is a constant $c(Z, \varepsilon)$ which gives $(*)$ for all fibers $X = Z(a)$, $a \in \mathbb{R}^m$